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RESIDENCE PROBABILITY CONTROL

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Abstract—The problem of controlling the residence probability of linear stochastic systems in a bounded domain is considered. Necessary and sufficient conditions for the existence of a controller that makes the residence probability positive (weakly residence probability controllable systems) and arbitrarily close to one (strongly residence probability controllable systems) are derived. The approach is based on the modern large deviations theory for systems perturbed by small white noise.

1. INTRODUCTION

Consider the system

$$dx = (Ax + Bu)\,dt + \epsilon C\,dw, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control, $w(t)$ is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$. There are many problems (see Ref. [1] for examples) where it is desired to maintain $x(t, x_0, u)$, the solution of equation (1), in a given bounded set Ω during a specified time interval $[0, T]$.

A convenient quantity that describes the behaviour of equation (1) in the domain Ω is the first passage time ($x_0 \in \Omega$)

$$\tau^{\epsilon}(x_0, u) = \inf\{t \geqslant 0 \mid x(t, x_0, u) \in \partial\Omega\}, \tag{2}$$

where $\partial\Omega$ is the boundary of Ω. In terms of $\tau^{\epsilon}(x_0, u)$ the above problem formulation becomes

$$\tau^{\epsilon}(x_0, u) \geqslant T. \tag{3}$$

Since $\tau^{\epsilon}(x_0, u)$ is a random quantity expression (3) has to be given some probabilistic meaning. One possibility is to replace expression (3) with the condition

$$\bar{\tau}^{\epsilon}(x_0, u) = \mathrm{E}_{x_0}[\tau^{\epsilon}(x_0, u)] \geqslant T, \tag{4}$$

and another one is to replace expression (3) with

$$\mathrm{P}_{x_0}\{\tau^{\epsilon}(x_0, u) \geqslant T\} \geqslant 1 - \delta, \quad 0 < \delta < 1. \tag{5}$$

In Refs [1, 2] we discussed in detail how to select a control law $u = Kx$ such that inequality (4) is satisfied. In this paper we choose expression (5) as the measure of performance.

The problem of controlling equation (5), i.e. the residence probability, is also not new. It has been described in Ref. [3] and later analyzed in Ref. [4] and more recently in Ref. [5]. However, the conditions under which and to what extent the residence probability can be modified by control remain unknown.

In this paper we show that linear systems of form (1) with linear state feedback control laws can be divided into two classes, *weakly* and *strongly residence probability controllable* systems. Roughly speaking, weakly residence probability controllable are those systems for which condition (5) can be satisfied for *some* $0 < \delta < 1$ and strongly residence probability controllable are systems for which condition (5) can be satisfied for *any* $0 < \delta < 1$.

The paper is organized as follows: in Section 2 we describe the mathematical technique we use; in Section 3 we formulate the control problem and state our main results and in Section 4 we discuss the results. All proofs are given in the Appendix.

2. MATHEMATICAL BACKGROUND

Consider the following linear Ito system

$$\mathrm{d}x = Ax\,\mathrm{d}t + \epsilon C\,\mathrm{d}w, \tag{6}$$

where $x \in \mathbb{R}^n$, $w(t)$ is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$. Let Ω be a bounded domain in $\mathbb{R}^n$ and assume that $x(0) = x_0 \in \Omega$. Let $x(t, x_0)$ be the solution of equation (6) and define the first passage time in Ω as

$$\tau^{\epsilon}(x_0) = \inf\{t \geqslant 0 \,|\, x(t) \in \partial\Omega\},$$

where $\partial\Omega$ is the boundary of Ω. The probability of exit of $x(t, x_0)$ before time T is

$$\mathrm{P}_{x_0}\{\tau^{\epsilon}(x_0) \leqslant T\}. \tag{7}$$

The asymptotic behaviour of $\mathrm{P}_{x_0}\{\tau^{\epsilon}(x_0) \leqslant T\}$ as $\epsilon \to 0$ has been the subject of considerable research for several years. One of the main results is the following [6], [7].

Theorem 2.1

Assume that the boundary $\partial\Omega$ is smooth and (A, C) is disturbable, i.e. rank $[C\,AC \cdots A^{n-1}C] = n$. Then

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \mathrm{P}_{x_0}\{\tau^{\epsilon}(x_0) \leqslant T\} = -\phi(T, x_0), \tag{8}$$

where

$$\phi(T, x_0) = \min_{\substack{y \in \partial\Omega \\ 0 \leqslant t \leqslant T}} \phi(t, x_0, y), \tag{9}$$

$$\phi(t, x_0, y) = \tfrac{1}{2}(y - \mathrm{e}^{At}x_0)^{\mathrm{T}} X^{-1}(t)(y - \mathrm{e}^{At}x_0), \tag{10}$$

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t) + X(t)A^{\mathrm{T}} + CC^{\mathrm{T}}. \tag{11}$$

When the distribution of the initial point x_0 is known, the result of Theorem 2.1 can be generalized as follows: let Ω_1 be a subset of Ω such that $\partial\Omega \cap \partial\Omega_1 = \varnothing$ and assume the initial point x_0 has a known distribution with density $f(x_0) > 0$ (independent of ϵ) so that

$$\mathrm{P}\{x_0 \in \Omega_1\} = \int_{\Omega_1} f(x_0)\,\mathrm{d}x_0 = 1. \tag{12}$$

The probability of exit of equation (6) from Ω given $x_0 \in \Omega_1$ is

$$\mathrm{P}\{\tau^{\epsilon}(\Omega_1) \leqslant T\} = \int_{\Omega_1} \mathrm{P}_{x_0}\{\tau^{\epsilon}(x_0) \leqslant T\} f(x_0)\,\mathrm{d}x_0. \tag{13}$$

Theorem 2.2

Under the assumptions of Theorem 2.1 we have

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \mathrm{P}\{\tau^{\epsilon}(\Omega_1) \leqslant T\} = -\phi(T, \Omega_1), \tag{14}$$

where

$$\phi(T, \Omega_1) = \min_{x_0 \in \Omega_1} \phi(T, x_0). \tag{15}$$

Proof. See the Appendix.

The properties of the constant $\phi(T, \Omega_1)$, the logarithmic exit probability from Ω given $x_0 \in \Omega_1$, as described in Theorem 2.2, form the basis for the analysis that follows. Note that $\phi(T, \Omega_1)$ is independent of the distribution of x_0 in Ω_1.

3. RESIDENCE PROBABILITY CONTROL

Consider now a system with control

$$dx = (Ax + Bu)\,dt + \epsilon C\,dw. \tag{16}$$

Assume that $u = Kx$. Then if $(A + BK, C)$ is disturbable it follows from Theorem 2.2 that the *residence probability*

$$P\{\tau^\epsilon(\Omega_1, K) > T\},$$

of equation (16) in the domain Ω given $x_0 \in \Omega_1$ satisfies

$$\lim_{\epsilon \to 0} \epsilon^2 \ln(1 - P\{\tau^\epsilon(\Omega_1, K) > T\}) = -\phi(T, \Omega_1, K),$$

where $\phi(T, \Omega_1, K)$ is given by equations (9)–(11) and (15) with A replaced by $A + BK$. In the remainder of the paper we assume that $\Omega(0 \in \Omega)$ and Ω_1 are fixed given domains and drop the Ω_1 in $\phi(T, \Omega_1, K)$. Based on the above observations we make the following definitions.

Definition 3.1

System (16) is said to be *weakly residence probability controllable* (wrp-controllable) if for any $T > 0$ there exists a control $u = Kx$ such that $\phi(T, K) > 0$.

Definition 3.2

System (16) is said to be *strongly residence probability controllable* (srp-controllable) if for any $T > 0$ and any $\phi > 0$ there exists a control $u = Kx$ such that $\phi(T, K) > \phi$.

The above definitions state, in particular, that system (16) is wrp-controllable if there exists $u = Kx$ such that (for small ϵ)

$$\epsilon^2 \ln(1 - P\{\tau^\epsilon(\Omega_1, K) > T\}) < 0,$$

or equivalently

$$P\{\tau^\epsilon(\Omega_1, K) > T\} > 0;$$

and system (16) is srp-controllable if for any $\phi > 0$ there exists $u = Kx$ such that for small ϵ

$$\epsilon^2 \ln(1 - P\{\tau^\epsilon(\Omega_1, K) > T\}) \cong -\phi$$

or

$$P\{\tau^\epsilon(\Omega_1, K) > T\} \cong 1 - e^{-\phi/\epsilon^2}.$$

In the remainder of the paper we assume that system (16) contains no modes that are both uncontrollable and undisturbable, i.e. $(A, [B\,C])$ is a controllable pair. In this case the pair $(A + BK, C)$ is disturbable for almost any K [8].

The following theorem characterizes the class of wrp-controllable systems.

Theorem 3.1

System (16) is weakly residence probability controllable if and only if there exists a $K^* \in \mathbb{R}^{m \times n}$ such that

$$\min_{x \in \Omega_1} \min_{\substack{y \in \partial\Omega \\ 0 \leq t \leq T}} \| y - e^{(A + BK^*)t} x \|_2 > 0. \tag{17}$$

Proof. See the Appendix.

Condition (17) is, in a sense, a stability condition on $[0, T]$ for the deterministic system

$$\frac{dx}{dt} = (A + BK^*)x, \quad x(0) = x_0. \tag{18}$$

Indeed, if condition (17) is satisfied then no trajectory of equation (19) starting in Ω_1 can reach the boundary of Ω during the time interval $[0, T]$. Thus, wrp-controllability is equivalent to "stabilizability" of system (16) on the interval $[0, T]$.

For srp-controllability we have the following result.

Theorem 3.2

Assume that system (16) is wrp-controllable. Then system (16) is strongly residence probability controllable if and only if

$$\operatorname{Im} C \subseteq \operatorname{Im} B. \tag{19}$$

Proof. See the Appendix.

4. DISCUSSION

In this paper we formulated and analysed the problem of residence probability control of linear systems with small additive white noise disturbances. The main results are the following: a system is wrp-controllable if and only if it is stabilizable (in a certain sense) on the interval $[0, T]$ and it is srp-controllable if and only if the image of the noise input matrix is contained in the image of the control input matrix.

In Ref. [1] we analysed an analogous problem for the residence time (4). It was shown that linear systems are divided into two classes, wrt- and srt-controllable. Furthermore, it was shown that wrt-controllability is equivalent to stabilizability of the pair (A, B) and a system is srt-controllable if and only if it is stabilizable and the image of the noise input matrix is contained in the image of the control input matrix. In view of the above discussion we conclude that the results of this note parallel the results of Ref. [1] with the only difference being that here stabilizability is only needed on the bounded interval $[0, T]$ whereas in Ref. [1] stabilizability was needed on the infinite interval $[0, \infty)$.

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APPENDIX

Proof of Theorem 2.2 (Outline)

Note that by Theorem 2.1 we have

$$\mathrm{P}_{x_0}\{\tau^\epsilon(x_0) \leqslant T\} = C(\epsilon, x_0)\exp[-\phi(T, x_0)/\epsilon^2](1 + o_{x_0}(1)) \tag{A.1}$$

where $C(\epsilon, x_0)$ grows no faster than polynomially in ϵ. Therefore,

$$\mathrm{P}\{\tau^\epsilon(\Omega_1) \leqslant T\} = \int_{\Omega_1} C(\epsilon, x_0)\exp[-\phi(T, x_0)/\epsilon^2](1 + o_{x_0}(1))f(x_0)\,\mathrm{d}x_0. \tag{A.2}$$

Now equality (14) follows immediately by Laplace integration in equation (A.2). Q.E.D.

Proof of Theorem 3.1

Note that

$$\phi(t, x, y, K) = \tfrac{1}{2}(y - \exp[(A + BK)t]\,x)^{\mathrm{T}}X^{-1}(t, K)(y - \exp[(A + BK)t]\,x) \leqslant \|X^{-1}(t, K)\|_2\,\|y - \exp[(A + BK)t]\,x\|_2^2. \tag{A.3}$$

Thus,

$$\phi(T,K)=\min_{x\in\Omega_1}\min_{\substack{y\in\partial\Omega\\0\leqslant t\leqslant T}}\phi(t,x,y,K)\leqslant\tfrac{1}{2}\|X^{-1}(t,K)\|_2\min_{x\in\Omega_1}\min_{\substack{y\in\partial\Omega\\0\leqslant t\leqslant T}}\|y-\exp[(A+BK)t]x\|_2^2. \tag{A.4}$$

The necessity of condition (17) follows from relationship (A.4). Next note that

$$\phi(t,x,y,K)\geqslant\tfrac{1}{2}\|y-\exp[(A+BK)t]x\|_2^2\,\lambda_{\min}(X^{-1}(t,K))$$

$$=\frac{\|y-\exp[(A+BK)t]x\|_2^2}{2\lambda_{\max}(X(t,K))}=\frac{\|y-\exp[(A+BK)t]x\|_2^2}{2\|X(t,K)\|_2}\geqslant\frac{\|y-\exp[(A+BK)t]x\|_2^2}{2\|X(T,K)\|_2}. \tag{A.5}$$

The sufficiency part of the theorem follows directly from relationship (A.5). Q.E.D.

Proof of Theorem 3.2

Assume first that system (16) is srp-controllable. Then there exists a sequence $\{K_i\}$ such that $\phi(T,K_i)\to\infty$ as $i\to\infty$. Note that

$$\phi(T,K)=\min_{x\in\Omega_1}\min_{\substack{y\in\partial\Omega\\0\leqslant t\leqslant T}}\phi(t,x,y,K)\leqslant\min_{\substack{y\in\partial\Omega\\0\leqslant t\leqslant T}}\phi(t,0,y,K)$$

$$=\min_{\substack{y\in\partial\Omega\\0\leqslant t\leqslant T}}\tfrac{1}{2}y^{\mathrm{T}}X^{-1}(t,K)y=\min_{y\in\partial\Omega}\tfrac{1}{2}y^{\mathrm{T}}X^{-1}(T,K)y\leqslant\min_{y\in\partial\mathrm{B}(R)}\tfrac{1}{2}y^{\mathrm{T}}X^{-1}(T,K)y$$

$$=\tfrac{1}{2}\lambda_{\min}(X^{-1}(T,K))R^2=\frac{R^2}{2\lambda_{\max}(X(T,K))}=\frac{R^2}{2\|X(T,K)\|_2}, \tag{A.6}$$

where $\mathrm{B}(R)$ is the ball with centre at zero and radius R and

$$R^2=\max_{y\in\partial\Omega}y^{\mathrm{T}}y.$$

It follows from relationship (A.6) that $\|X(T,K_i)\|_2\to 0$ as $i\to\infty$. Therefore, $\operatorname{Tr}X(T,K_i)\to 0$ as $i\to\infty$ and by Fatou's lemma we get

$$0=\lim_{i\to\infty}\int_0^T\operatorname{Tr}\exp[(A+BK_i)t]\,CC^{\mathrm{T}}\exp[(A+BK_i)^{\mathrm{T}}t]\,\mathrm{d}t\geqslant\int_0^T\liminf_{i\to\infty}\operatorname{Tr}\exp[(A+BK_i)t]\,CC^{\mathrm{T}}\exp[(A+BK_i)^{\mathrm{T}}t]\,\mathrm{d}t. \tag{A.7}$$

Thus, since $\exp[(A+BK_i)t]\,CC^{\mathrm{T}}\exp[(A+BK_i)^{\mathrm{T}}t]\geqslant 0$, it follows from inequality (A.7) that

$$\liminf_{i\to\infty}\operatorname{Tr}\exp[(A+BK_i)t]\,CC^{\mathrm{T}}\exp[(A+BK_i)^{\mathrm{T}}t]=0, \tag{A.8}$$

for almost all (a.a.) $t\in[0,T]$. Furthermore, there exists a subsequence $\{K_j\}$ such that

$$\lim_{j\to\infty}\exp[(A+BK_j)t]\,CC^{\mathrm{T}}\exp[(A+BK_j)^{\mathrm{T}}t]=0,\quad \text{a.a. } t\in[0,T]. \tag{A.9}$$

Next note that $X(t,K_j)$ satisfies the equation

$$(A+BK_j)X(t,K_j)+X(t,K_j)(A+BK_j)^{\mathrm{T}}+CC^{\mathrm{T}}=\exp[(A+BK_j)t]\,CC^{\mathrm{T}}\exp[(A+BK_j)^{\mathrm{T}}t]. \tag{A.10}$$

Combining equations (A.9) and (A.10) gives

$$\lim_{j\to\infty}[(A+BK_j)X(t,K_j)+X(t,K_j)(A+BK_j)^{\mathrm{T}}]+CC^{\mathrm{T}}=0,\quad \text{a.a. } t\in[0,T]. \tag{A.11}$$

Thus, since $X(t,K_j)\to 0$ as $j\to\infty$, a.a. $t\in[0,T]$, we get

$$\lim_{j\to\infty}[BK_jX(t,K_j)+X(t,K_j)K_j^{\mathrm{T}}B^{\mathrm{T}}]+CC^{\mathrm{T}}=0,\quad \text{a.a. } t\in[0,T]. \tag{A.12}$$

Therefore, the exists a $Q(t)$ such that

$$BQ(t)+Q^{\mathrm{T}}(t)B^{\mathrm{T}}+CC^{\mathrm{T}}=0,\quad \text{a.a. } t\in[0,T]. \tag{A.13}$$

Finally, equation (A.13) can be true only if condition (19) is satisfied. This completes the necessity part of the proof.

Assume now that conditions (19) is satisfied. Then condition (19) and the controllability of $(A,[BC])$ imply that (A,B) is controllable. It was shown in Ref. [1] that there exists a sequence $\{K_i\}$ such that $A+BK_i$ is Hurwitz for each i and

$$\lim_{i\to\infty}X(K_i)=0, \tag{A.14}$$

where

$$(A+BK_i)X(K_i)+X(K_i)(A+BK_i)^{\mathrm{T}}+CC^{\mathrm{T}}=0. \tag{A.15}$$

Note that for any $T\geqslant 0$

$$X(T,K_i)\leqslant X(K_i). \tag{A.16}$$

Thus, it follows from condition (A.5) that

$$\lim_{i\to\infty}\phi(T,K_i)=\infty. \tag{A.17}$$