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ANALYSIS OF VARIANCE OF CUSTOMER BALANCES FOR A FAMILY OF STOCHASTIC SERVICE NETWORKS

Richard L. Patterson
School of Natural Resources, The University of Michigan, Ann Arbor, Michigan 48109-1115, USA

Abstract. The coefficient of variation of counts of customers in nodes of all members of an equivalence class of stochastic service networks is computed for three classes of arrival processes: i) Poisson arrivals of individual customers, ii) Poisson arrivals of fixed and random sized batches of customers, and iii) fixed times of arrivals of constant and random sized batches of customers. Time-variable expressions of means, variances, and coefficients of variation are computed in terms of arrival process parameters, nodal linkages within networks, and residence time distributions of customers in nodes. Coefficients of variation are compared and indices of traffic congestion are computed for member networks within an equivalence class. Use of these indices are an efficient means of rapidly evaluating design parameter changes on performance of networks within an equivalence class.

Key Words. network; stochastic; dynamic; queue

Running title. ANOVA of counts in networks.

Let $\mathcal{F} = (\mathcal{S}_2, \mathcal{S}_3, \ldots\ldots, \mathcal{S}_N, \ldots)$ denote a family of equivalence classes of stochastic service networks in which each member network $S \in \mathcal{S}_N$ of each class $\mathcal{S}_N$ contains a finite number N of linked nodes $\alpha_1, \ldots, \alpha_N$ (N=2,3,...). Each node α_j contains a finite number of servers each of unlimited capacity among which customers are routed according to a fixed probabilistic protocol prior to departing α_j for α_k. Each member $S \in \mathcal{S}_N$ for all N is characterized by the following conditions:

i) For each initial node α_i of entry into S by customers from outside the network all other nodes $\alpha_j (j=i)$ are accessible from α_i.

ii) Every customer entering a nonabsorbing node α_j, following a common probabilistic tour among servers within α_j is retained in residence at each server a random length of time such that total nodal residence time is a random variable with conditional cdf $w_{jk}(t)$ whose first two moments are finite and for which $w_{jk}(t)$ is strictly positive for positive t. Residence times in nodes for customers are iid random variables with cdf $w_{jk}(t)$. For absorbing nodes α_j $w_{jk}(t)$ is defined to be zero for finite t and one for t equal to infinity.

iii) Movements of customers among nodes of a member network $S \in \mathcal{S}_N$ are governed by a unique Markov renewal (MR) process, discrete state-continuous time, with N-dimensional conditional residence time distribution function matrix $W = (w_{jk}(t))$ where $w_{jk}(t)$ satisfies conditions given in ii) above. States of (MR) are in one-to-one correspondence with nodes of $S \in \mathcal{S}_N$.

Comments No network contains fewer than two nodes. At least one node of any member $S \in \mathcal{S}_N$ is absorbing which serves purposes of counting departing customers and maintaining records of traffic flow through the network. The requirement of finite mean and variance of $w_{jk}(t)$ rules out absorbing servers within a node which is not an absorbing node. Uniqueness of nodal cdf $w_{jk}(t)$ requires that all customer conditional residence times in node α_j are governed by $w_{jk}(t)$ which, in turn, implies that $w_{jk}(t)$ is independent of a customers point of entry into α_j from a prior node and its point of departure from within α_j to a given destination node α_k.

The Markov renewal process defined by the pair P,W regulating traffic flow within a network generates a semi-Markov process $(X(t)/t>0)$ describing locations of customers at their most recent changes of nodes. $Pr(X(t)=a_j/X(0+)=\alpha_i)$

is the conditional probability that the process (a customer) is in state j (node α_j) at time t, given that it was in state i (initial node α_i) at time t=0+. Conditional probabilities

$f_{ij}(t) = Pr(X(t)=\alpha_j/X(0+)=\alpha_i)$

define an interval transition probability function matrix $F=(f_{ij}(t))$ that is stochastic whenever P is stochastic. Elements of F are computed directly from elements of P and W by conditioning upon the number of changes of state of the process prior to time t. Although pure delays may exist at certain servers within a node alternate routes around such servers exist by assumption so that nodal residence time cdf's are positive for positive t.

Distributions of Customer Counts in Nodes

Distributions of counts of customers in nodes are determined by arrival protocols to the network and processing behavior upon entering an initial node. Effects of processing behavior are studied separately by assuming $c_1,\ldots,c_L$ customers arrive at initial (non-absorbing) nodes $\alpha_{i_1},\ldots,\alpha_{i_L}$ at instant $z \geq 0$. Then at time $t > z$ the joint cdf of customers initially in node α_{i_j} that are in nodes $\alpha_1,\ldots,\alpha_N$ is multinomial with parameters $c_j, f_{i_j 1}(t-z),\ldots,f_{i_j N}(t-z)$ ($f_{i_j k}(t-z)>0(k=1,\ldots,N)$). When all L initial nodes are accounted for the marginal cdf of customer count in node α_k is that of the sum of L independently distributed binomial random variables with mean $\sum_{j=1}^{L} c_j f_{i_j k}(t-z)$ and variance $\sum_{j=1}^{L} c_j f_{i_j k}(t-z)(1-f_{i_j k}(t-z))$. The cdf is approximated by a Poisson distribution with parameter β = mean of the sum of the L binomial random variables. The approximation error has been bounded (LeCam (1)). Traffic effects of initial conditions eventually die out although ripple effects through the network may be experienced for some period of time until customers finally enter absorbing nodes.

Three basic cases of arrival protocols are considered: i) Poisson arrivals of individual customers to an initial node; ii) Poisson arrivals of batches of customers to an initial node; iii) Arrivals of batches of customers at fixed intervals $0 < t_1 < t_2 < \ldots < t_k < t$. Case ii) contains three sub-cases: ii.1) batch sizes are iid random variables with common mean m_B and variance v_B where all customers are processed independently upon entering the initial node; ii.2) case ii.1 applies except all batches are processed as single units in which batch service times are modified according to some criterion; ii.3) case ii.1 applies except the cdf of batch size is a two point distribution, ie, batches are of sizes b_0 and b_1 with respective probabilities p and 1-p.

Case i. A network $S \in \mathcal{S}_N$ is initially empty and in an interval (0,t) n customers arrive and enter initial node α_i. The joint distribution of customer counts in the N nodes of S is:

$$Pr(C_1(t)=c_1,\ldots,D_N(t)=c_N) = \prod_{j=1}^{N} \frac{m_{ij}(t)^{c_j}}{c_j!} e^{-m_{ij}(t)} \tag{1}$$

where:

$$m_{ij}(t) = \int_0^t a_i(z).f_{ij}(t-z)dz \quad (j=1,\ldots,N)$$

and:

$a_i(t)$ is the intensity of the Poisson process of customer arrivals to initial node α_i.

Equation (1) was demonstrated by Kelly (2) using generating functions for a system similar to $S \in \mathcal{S}_K$. It was also demonstrated by Harrison and Lemoine (3) using a renewal argument for a system of servers all of which communicate. Equation (1) above is demonstrated simply by conditioning a multinomial probability function of counts in the N nodes *by total arrivals n in the interval* (0,t) *and* averaging over n using a Poisson cdf. Equation (1) shows counts in nodes to be mutually independent. A calculation shows that no count $C_j(t)$ is independent of the cumulative number of arrivals to initial node α_i in (0,t). It is only for this case that the joint cdf of counts in the N nodes is obtained, except for a sub-case in which all arriving batches are processed as single units, remaining intact throughout.

Case ii.1. When customers arrive at initial node α_i in batches of random size where the cdf of batch size has mean m_B and variance v_B (Poisson arrivals of batches) the counts of customers in nodes are no longer mutually independent. The cdf of the marginal count $C_j(t)$ is compound Poisson with mean and variance:

$$E(C_j(t)) = m_B.\int_0^t a_i(z).f_{ij}(t-z)dz \quad (j=1,\dots,N) \quad (2)$$

and:

$$V(C_j(t)) = m_B.\int_0^t a_i(z).f_{ij}(t-z).(1-f_{ij}(t-z))dz +$$

$$(m_B^2 + v_B).\int_0^t a_i(z).(f_{ij}(t-z))^2dz \quad (3)$$

$$(j=1,\dots,N)$$

Assume a cdf of batch size such that b_k customers arrive in a batch with probability p_k $(k=0,1,\dots,M)$. The arrival process to initial node α_i of batches of size b_k only is Poisson with intensity $p_k.a_i(t)$. The mean and variance of numbers of customers in node α_j is obtained from equations 2 and 3 above by setting $m_B=b_k$, $v_B=0$, and replacing $a_i(z)$ by $p_k.a_i(z)$.

Case ii.2. When all batches are processed as single units throughout every node with residence time cdfs $w_{ijB}(t)$ the cdf of the number of batches resident in node α_j at time t is Poisson with mean:

$$\int_0^t a_i(z).f_{ijB}(t-z)dz$$

where:

$f_{ijB}(t)$ is the interval transition probability function for batches whose size distribution follows that defined in case ii.1 above.

The number $C_j(t)$ of customers in node α_j has a compound Poisson cdf with mean and variance:

$$E(C_j(t)) = m_B.\int_0^t a_i(z).f_{ijB}(t-z)dz \quad (j=1,\dots,N)(4)$$

and:

$$V(C_j(t)) = (m_B^2 + v_B).\int_0^t a_i(z).f_{ijB}(t-z)dz \quad (5)$$

$$(j=1,\dots,N)$$

If batches of fixed size b_k only are considered equations 4 and 5 are modified in a manner analogous to that shown in case ii.1. above.

Case ii.3. When the cdf of batch size is a two point distribution, a subcase of case ii.1. above, assume batches of sizes b_0 and b_1 arrive at initial node α_i with probabilities p_0 and $p_1=1-p_0$, respectively. The mean and variance of batch size are, respectively:

$$b_0.p_0 + b_1.p_1 = m_B$$

and:

$$(b_0-m_B)^2.p_0 + (b_1-m_B)^2.p_1 = v_B$$

For fixed p_0 batch sizes b_0 and b_1 can be computed which give predetermined mean and variance of batch size. In particular, the mean m_B can be small and the variance can be large or vice versa. The mean and variance of customer count $C_j(t)$ in node α_j at time t are given by equations 2 and 3. If $b_0=0$ case ii.3. is equivalent to case ii.1. where $a_i(t)$ is replaced by $a_i(t).p_1$, $v_B=0$, and $m_B=b_1$.

Case iii. Batches arrive at fixed instants $t_1,\dots,t_k$ $(0< t_1 < \dots<t_k < t)$ at initial node α_i. For each arriving batch of size $B_1,\dots,B_k$ all customers are processed independently throughout. The mean and variance of the marginal cdf of $C_j(t)$, customer count in node α_j at time t, are:

$$E(C_j(t)) = m_B.\sum_{r=1}^k f_{ij}(t-t_r) \quad (j=1,\dots,N) \quad (6)$$

and:

$$V(C_j(t)) = m_B.\sum_{r=1}^k f_{ij}(t-t_r).(1-f_{ij}(t-t_r)) +$$

$$v_B.\sum_{r=1}^k f_{ij}^2(t-t_r) \quad (j=1,\dots,N) \quad (7)$$

cable to cases ii.1., ii.2., ii.3., and iii. developed above. The count $C_j(t)$ of the number of customers in node α_j at time t is the sum of independent but not identically distributed Bernoulli random variables, conditioned on batch sizes and times of arrival to initial node α_i.

By averaging over those two parameter sets inequalities analogous to those presented in Theorem 4 of Hoeffding's paper are obtained which give upper and lower bounds on $Pr(C_j(t) \leq c_j)$ $(c_j=0,1,2,...)$.

The probability that a customer entering initial node α_i at time z>0 is in node α_j at time t>z is $f_{ij}(t-z)$. The mean amount of time a customer is in residence in nonabsorbing node α_j prior to entering an absorbing node is therefore:

$$T_j = \int_z^{\infty} f_{ij}(t-z)dt \qquad (12)$$

The mean proportion of time in residence in nonabsorbing node α_j relative to all nonabsorbing nodes is:

$$T_j / \sum_k T_k$$

where:

k denotes indices of all nonabsorbing nodes of $S \epsilon \mathcal{S}_N$.

The mean amount of time nonabsorbing node α_j contains n customers is:

$$C_{jn} = \int_0^{\infty} Pr(C_j(t)=n)dt \qquad (13)$$

The mean proportion of time nonabsorbing node α_j contains n customers is consequently:

$$C_{jn} / \sum_{k=0}^{\infty} C_{jk}$$

The mean time of residence of a customer in $S \epsilon \mathcal{S}_N$ exclusive of absorbing nodes is:

$$\sum_j T_j = \sum_j \int_z^{\infty} f_{ij}(t-z)dt \qquad (14)$$

where:
index j ranges over all nonabsorbing nodes in the network.

The cdf of residence time of a customer in the network is:
Pr(residence time in $S \epsilon \mathcal{S}_N \leq t-z$/customer entered S at time z) = $(\sum_j f_{ij}(t-z))$ (15)

where:
index j ranges over all absorbing nodes in S.

Network residence times for customers and counts of customers in a network are related by the formula:

$$\sum_k E(C_k(t)) = \sum_k \int_0^t a_i(z).f_k(t-z)dz =$$

$$\sum_j \int_0^t a_i(z).(1-f_{ij}(t-z)dz =$$

$$\int_0^t a_i(z)dz - \int_0^t a_i(z).(\sum_j f_{ij}(t-z))dz \qquad (16)$$

where:
i) index k ranges over nonabsorbing nodes;
ii) index j ranges over absorbing nodes.

Indices for Network Design

Equations (12) - (16) may be used to compare different structural configurations of networks within a given class $\mathcal{S}_N$. Additionally, the squared C.V. and ratios of squared C.V.'s may be used for making the same kinds of comparisons. The above indices may be used to make comparisons at both a nodal and network wide level. Interval transition probability functions need not necessarily be computed exactly although a catalog of $f_{ij}(t)$ functions may be compiled for a wide variety of network configurations. Qualitative behavior of $f_{ij}(t)$'s may be used to make comparisons by assuming monotonicity, convexity or concavity, maximum values, and zeros of the first time derivative of $f_{ij}(t)$ within an interval (0,T). An advantage of reducing a network to membership in an equivalence class of minimal dimension prior to evaluation lies in efficiency of analysis and an ability to compare networks which may contain widely differing numbers of component service systems, each of infinite capacity.

Networks containing N nodes some or all of which have finite capacity behave the same as

Continuous approximations to equations 6 and 7 are:

$$E(C_j(t)) \simeq m_B . \int_0^{t_k} f_{ij}(t-z)dz \quad (j=1,\ldots,N) \qquad (8)$$

and:

$$V(C_j(t)) \simeq m_B . \int_0^{t_k} f_{ij}(t-z).(1-f_{ij}(t-z))dz + v_B . \int_0^{t_k} f_{ij}^2(t-z)dz \quad (j=1,\ldots,N) \qquad (9)$$

Coefficients of Variation of $C_j(t)$

The squared coefficients of variation of $C_j(t)$ are used to compare variability among cases. Let

$$I_1 = \int_0^t a_i(z).f_{ij}(t-z)dz \text{ and } I_2 = \int_0^t a_i(z).f_{ij}^2(t-z)dz$$

The squared C.V. is shown in Table 1 for all cases. By dividing the squared C.V. in each case except ii.2. by I_1^{-1} a comparison of the coefficient of variation relative to that of the standard case of individual Poisson arrivals is obtained and shown in Table 2. Adequacy of approximation of models of cases ii.1., ii.3., and iii. by the standard model of Poisson arrivals of single customers is shown in terms of magnitudes of ratios of squared C.V.s. Effects of variability of times between successive arrivals of batches are estimated by comparing cases ii.1. and iii. in either Tables 1 or 2, representing completely random arrivals and completely determined times of arrivals. As shown in Table 2 the difference in ratios is approximately $I_2.I_1^{-1}$, a fraction always less than unity. The corresponding difference computed from Table 1 is approximately $I_2.I_1^{-2}$, a quantity that does not exceed I_1^{-1} in magnitude, the reciprocal of the mean number of customers in node α_j arriving from initial node α_i in (0,t). Processes describing batch arrivals to initial node α_i for which the variance of independent interarrival times is less than that of the Poisson process with intensity $a_i(t)$ will yield squared C.V.'s of $C_j(t)$ which differ from either case ii.1 or case iii. by at most I_1^{-1}.

Variability of customer loading in a node α_j at time t expressed as a percentage of the mean may be maximized in some cases. As shown in Table 1 for cases ii.1., ii.3., and iii. the squared C.V. is maximized for a fixed t by setting the first derivative of squared C.V. with respect to I_1^{-1} equal to zero and solving for I_1^{-1}. For cases ii.1. and ii.3. maximization occurs when:

$$I_1 = 2I_2.(1-m_B-\frac{v_B}{m_B}) \qquad (10)$$

For case iii. maximization occurs when:

$$I_1 = 2I_2.(1-\frac{v_B}{m_B}) \qquad (11)$$

For given arrival intensity $a_i(t)$ equations 10 and 11 are integral equations in the unknown interval transition probability function $f_{ij}(z)$ defined on (0,z).

Effects of Dependencies in Customer Routing

Without specification of service times of batches considered as single units a direct comparison of cases ii.1. and ii.2. cannot be made. In the special case where service time of a batch of any size is the same as service time of a customer the difference between squared C.V. for those cases is:

$$\frac{1}{m_B}.I_1^{-1}.(1-I_2I_1^{-1})$$

from which bounds on the difference are:

$$0 \leq \frac{1}{m_B}.I_1^{-1}.(1-I_2I_1^{-1}) \leq \frac{1}{m_B}.I_1^{-1} .$$

Cases ii.1. and ii.2. represent extremes of dependencies among customer routings among nodes of $S\epsilon\mathcal{S}_N$. Squared C.V. for case ii.1. is never smaller than that for ii.2. and it is only when $f_{ij}(z)$ is identically 1 over the interval (0,t) that they are equal.

Bounds on $C_j(t)$

Hoeffding (4) developed bounds on probabilities of sums of independently but not identically distributed Bernoulli random variables, appli-

$\mathcal{S}_N$ so long as no customer is interrupted during its normal processing sequence through the infinite capacity network. Therefore, if interruptions due to blockages, waits for service, etc.. occur only occasionally a finite capacity network may also be evaluated in an approximate sense using the indices developed above. For the majority of customers that encounter no interruptions the above indices apply exactly.

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TABLE 1 Squared Coefficient of Variation of $C_j(t)$ for Five Cases

Case Number	Squared C.V.
i.	I_1^{-1}
ii.1.	$\frac{1}{m_B}.I_1^{-1} + (1 + \frac{v_B}{m_B^2} - \frac{1}{m_B}).I_2.I_1^{-2}$
ii.2.	$(1 + \frac{v_B}{m_B^2} + 0).I_2.I_1^{-2}$
ii.3.	$\frac{1}{m_B}.I_1^{-1} + (1 + \frac{v_B}{m_B^2} - \frac{1}{m_B}).I_2.I_1^{-2}$
iii.	$\frac{1}{m_B}.I_1^{-1} + (0 + \frac{v_B}{m_B^2} - \frac{1}{m_B}).I_2.I_1^{-2}$

TABLE 2 Ratio of Squared Coefficients of Variation Using Case i. as a Standard of Comparison

Case Number	Ratio of Squared C.V.
i.	1
ii.1.	$\frac{1}{m_B} + (1 + \frac{v_B}{m_B^2} - \frac{1}{m_B}).I_2.I_1^{-1}$
ii.3.	$\frac{1}{m_B} + (1 + \frac{v_B}{m_B^2} - \frac{1}{m_B}).I_2.I_1^{-1}$
iii.	$\frac{1}{m_B} + (0 + \frac{v_B}{m_B^2} - \frac{1}{m_B}).I_2.I_1^{-1}$