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# The three-loop QED photon vacuum polarization function in the $\overline{\mathrm{MS}}$-scheme and the four-loop QED $\beta$-function in the on-shell scheme

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We present the three-loop analytical QED results for the asymptotic expression of the photon vacuum polarization function $\Pi$ in the $\overline{\mathrm{MS}}$-scheme. Using the recently obtained four-loop results for the QED $\beta$-function in the MS-like and MOM schemes ($\beta_{\mathrm{MS}}$- and $\Psi$-functions) and for $(g-2)_\mu$ we get the four-loop QED $\beta$-function in the on-shell scheme.

**1.** One of the most important quantities, entering various QED and standard model investigations and applications, is the photon vacuum polarization function $\Pi$. In QED its asymptotic three-loop expression in the on-shell (OS) scheme has been known for many years (see e.g. ref. [1]). However, up to recently the three-loop non-logarithmic term of $\Pi$ remained unknown. At present its total value is known numerically in the OS-scheme from the unification of the four-loop numerical calculations of the muon anomalous magnetic moment $(g-2)_\mu$ [2] with the corresponding renormalization group (RG) theoretical studies [3]. Moreover, recently the part of this term (namely, the contribution to $\Pi$ of the three-loop diagrams with two fermion loops) was calculated in the OS-scheme even analytically [4].

In this work we present the analytical three-loop expression for $\Pi$ in the $\overline{\mathrm{MS}}$-scheme. These results have been previously obtained by us in the course of the work [5] and recently confirmed during the four-loop QED studies [6,7]. All results were obtained with the help of the original program [8] written for the previous version of the system [9]. We also use the recent numerical [2] and analytical [4] three-loop results for $\Pi$ in the OS-scheme and the four-loop analytical QED results for the QED $\beta$-function in the MS-like and momentum (MOM) subtractions schemes [7] (i.e. for the $\beta_{\mathrm{MS}}$- and $\Psi$-functions) to determine the expression of the four-loop coefficient in the $\beta$-function in the OS-scheme ($\beta_{\mathrm{OS}}$-function) for the case of QED with $N$ types of leptons. The comparison of the behavior of the four-loop perturbative series for the $\beta_{\mathrm{OS}}$-function with the ones for the $\beta_{\mathrm{MS}}$ and $\Psi$-functions is given. Other possible applications of the results obtained are briefly discussed.

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2. Let us start from the definition of the QED invariant charge

$$\bar{E}(x,\{y_i\},E)=\frac{E}{1+E\Pi(x,\{y_i\},E)}, \tag{2.1}$$

where the dimensionless parameters $x$, $y$ govern the energy and massive dependence and $E=\alpha/4\pi$ is the renormalized coupling constant.

In the MS-like schemes one has $x=Q^2/\mu^2$ and $y_i=m_i^2/Q^2$, $i=1,2,...,N$ where the $m_i$ are the renormalized lepton masses. The corresponding RG relation for the inverse photon propagator then has the form

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta_{\rm MS}(E)\frac{\partial}{\partial E}\right.$$
$$\left.-\gamma_{\rm m}(E)\sum_{j=1}^{N}m_j\frac{\partial}{\partial m_j}-\beta_{\rm MS}(E)\right)$$
$$\times(1+E\Pi(x,\{y_i\},E))$$
$$=0, \tag{2.2}$$

where the QED $\beta$-function and the anomalous dimension function $\gamma_{\rm m}$ are defined as

$$\beta_{\rm MS}(E)=\mu^2\frac{\partial E}{\partial\mu^2}=\sum_{i\geq 0}\beta_i E^{i+1}, \tag{2.3}$$

$$\gamma_{\rm m}(E)=-\mu^2\frac{\partial\ln m_i}{\partial\mu^2}=\sum_{i\geq 0}\gamma_i E^i, \tag{2.4}$$

The three-loop approximation of the $\gamma_{\rm m}$-function has been calculated in the MS-like scheme in ref. [10]. The results of these calculations read [10]

$$\gamma_0=3,\quad \gamma_1=(\tfrac{3}{2}-\tfrac{10}{3}N),$$
$$\gamma_2=\{\tfrac{129}{2}-[46-48\zeta(3)]N-\tfrac{140}{27}N^2\}. \tag{2.5}$$

The coefficients of the $\beta_{\rm MS}$-function are known even at the four-loop level [7]:

$$\beta_0=\tfrac{4}{3}N,\quad \beta_1=4N,\quad \beta_2=-2N-\tfrac{44}{9}N^2,$$
$$\beta_3=-46N+[\tfrac{760}{27}-\tfrac{832}{9}\zeta(3)]N^2-\tfrac{1232}{243}N^3. \tag{2.6}$$

The corresponding asymptotic three-loop approximation of the photon vacuum polarization function $\Pi$ can be defined as

$$E\Pi(x,\{y_i\},E)$$
$$=\sum_{k=1}^{3}[A_k+B_k\ln(x)]E^k+C_3\ln^2(x)E^3+...$$
$$+\sum_{j\geq 1}y_j\left(\sum_{k\geq 1}^{3}A_{kj}^{(2)}E^k+\sum_{k\geq 2}^{3}B_k^{(2)}\ln(x)E^k\right.$$
$$\left.+C_3^{(2)}\ln^2(x)E^3+...+O(y_j)\right). \tag{2.7}$$

In the $\overline{\rm MS}$-scheme the results of our three-loop calculations read

$$A_1=\tfrac{20}{9}N,\quad A_2=[\tfrac{55}{3}-16\zeta(3)]N,$$
$$A_3=[-\tfrac{286}{9}-\tfrac{296}{3}\zeta(3)+160\zeta(5)]N$$
$$+[-\tfrac{7402}{81}+\tfrac{608}{9}\zeta(3)]N^2,$$
$$B_1=-\tfrac{4}{3}N,\quad B_2=-4N,$$
$$B_3=2N+[\tfrac{88}{3}-\tfrac{64}{3}\zeta(3)]N^2,$$
$$C_3=-\tfrac{8}{3}N^2, \tag{2.8}$$

and

$$A_{1j}^{(2)}=-8,\quad A_{2j}^{(2)}=-64,$$
$$A_{3j}^{(2)}=[-\tfrac{3334}{3}+48\zeta(3)+\tfrac{1120}{3}\zeta(5)]+\tfrac{760}{3}N$$
$$+\sum_{i=1}^{N}\frac{m_i^2}{m_j^2}[\tfrac{1024}{3}-256\zeta(3)],$$
$$B_2^{(2)}=48,\quad B_3^{(2)}=408-\tfrac{416}{3}N,$$
$$C_3^{(2)}=-144+32N, \tag{2.9}$$

where the $m_i$ are the masses of leptons propagating in the internal loops of the corresponding three-loop diagrams. The results (2.9) have been obtained in the course of similar QCD calculations [11]. The correction of the numerically insignificant misprint in the $\zeta(3)$-term in $A_{3j}^{(2)}$ obtained in ref. [11] was done in ref. [12] with the help of the program of ref. [13]. In the course of the calculations the method of the dimensional regularization [14] and the integration-by-parts algorithm [15] inspired by the ideas proposed in ref. [14] were used.

Using the corresponding RG equations one can show that the coefficients in eqs. (2.4), (2.6), (2.7) satisfy the following relations:

$$\beta_0=-B_1,\quad \beta_1=-B_2,\quad \beta_2=-A_2B_1-B_3,$$
$$2C_3=-B_1B_2. \tag{2.10}$$

The similar equations for the coefficients of the $O(m^2/Q^2)$-terms contain the contributions from the anomalous mass dimension terms and read

$$B_2^{(2)} = -2\gamma_0 A_{1j}^{(2)} ,$$
$$B_3^{(2)} = -2\gamma_1 A_{1j}^{(2)} - (2\gamma_0 - \beta_0) A_{2j}^{(2)} ,$$
$$C_3^{(2)} = (2\gamma_0^2 - \gamma_0\beta_0) A_{1j}^{(2)} . \quad (2.11)$$

**3.** It was shown in ref. [7] that using the RG transformation equations between the $\beta_{\mathrm{MS}}$-function and the $\Psi$-function, namely

$$\Psi(E_{\mathrm{MOM}}) = \frac{\partial E_{\mathrm{MOM}}(E_{\mathrm{MS}})}{\partial E_{\mathrm{MS}}} \beta_{\mathrm{MS}}(E_{\mathrm{MS}}) , \quad (3.1)$$

one can obtain the corresponding four-loop expression of the $\Psi$-function (see ref. [7]):

$$\Psi(E_{\mathrm{MOM}}) = \sum_{i \geq 0} \psi_i E_{\mathrm{MOM}}^{i+1} ,$$
$$\psi_0 = \tfrac{4}{3} N , \quad \psi_1 = 4N ,$$
$$\psi_2 = -2N + [\tfrac{64}{3}\zeta(3) - \tfrac{189}{9}] N^2 ,$$
$$\psi_3 = -46N + [104 + \tfrac{512}{3}\zeta(3) - \tfrac{1280}{3}\zeta(5)] N^2 + [128 - \tfrac{256}{3}\zeta(3)] N^3 . \quad (3.2)$$

The $\Psi$-function is connected with the $\beta_{\mathrm{OS}}$-function by the following relation:

$$\Psi(E_{\mathrm{MOM}}) = \frac{\partial E_{\mathrm{MOM}}(E_{\mathrm{OS}})}{\partial E_{\mathrm{OS}}} \beta_{\mathrm{OS}}(E_{\mathrm{OS}}) , \quad (3.3)$$

where $E_{\mathrm{MOM}}$ coincides with the invariant charge $\bar{E}$ of eq. (2.1) at $Q^2 = \mu^2$.

The corresponding three-loop coefficients of the photon vacuum polarization function $\Pi_{\mathrm{OS}}$, namely $A_i^{\mathrm{OS}}$ ($i = 1, 2$), $B_i^{\mathrm{OS}}$ ($i = 1, 2, 3$) and $C_3^{\mathrm{OS}}$, have been previously presented in a number of works on the subject (see e.g. refs [1,6]). The non-logarithmic coefficients $A_i$ enter the RG relation derived in ref. [6]

$$\beta_3^{\mathrm{OS}} = \psi_3 - 2\beta_2^{\mathrm{OS}} A_1^{\mathrm{OS}} - \beta_1^{\mathrm{OS}} (A_1^{\mathrm{OS}})^2 + 2\beta_0^{\mathrm{OS}} (A_1^{\mathrm{OS}} A_2^{\mathrm{OS}} + A_3^{\mathrm{OS}}) . \quad (3.4)$$

This equation allows us to deduce that to find the four-loop coefficient $\beta_3^{\mathrm{OS}}$ of the $\beta_{\mathrm{OS}}$-function it is necessary to know the coefficient $A_3^{\mathrm{OS}}$ which can be presented in the following form:

$$A_3^{\mathrm{OS}} = A_{3,1} N + A_{3,2} N^2 . \quad (3.5)$$

Its numerical value has been obtained from the combination of the results of the numerical computations of the four-loop corrections to $(g-2)_\mu$ [2] with the asymptotic formulae of ref. [1] and read [#1]

$$A_3^{\mathrm{OS}} = 66.624(2.624) . \quad (3.6)$$

Using now the corrected analytical expression of $A_{3,2}$ obtained in ref. [4], namely

$$A_{3,2} = -\tfrac{614}{27} - \tfrac{128}{3}\zeta(2) + \tfrac{545}{9}\zeta(3) = -20.133 , \quad (3.7)$$

and taking into account that $\zeta(2) = \pi^2/6 = 1.6449...$ and $\zeta(3) = 1.2020...$ one can get the following numerical expression of the coefficient $A_{3,1}$:

$$A_{3,1} = 86.757(2.624) . \quad (3.8)$$

Substituting now eqs. (3.8), (3.7), (3.5) and the results from refs. [1,6] and taking into account the expressions for the coefficients $\psi_3$ (see eq. (3.2)) and $\beta_2^{\mathrm{OS}}$ (see ref. [1]), namely

$$\beta_2^{\mathrm{OS}} = -2N - \tfrac{224}{9} N^2 , \quad (3.9)$$

we arrive at the following (partly numerical) expression for the coefficient $\beta_3^{\mathrm{OS}}$:

$$\beta_3^{\mathrm{OS}} = -46N + [\tfrac{1520}{27} - \tfrac{1664}{9}\zeta(3) - \tfrac{1280}{3}\zeta(5) + 231(7)] N^2 + [\tfrac{14416}{81} - \tfrac{1024}{9}\zeta(2) - \tfrac{56}{3}\zeta(3)] N^3 . \quad (3.10)$$

The order $N$-term in eq. (3.10) coincides with the ones in eqs. (2.6), (3.2) due to the property of universality which is fulfilled within the MS-like, MOM- and the OS-schemes. The analytical $N^3$-contribution is known from the results of refs. [4,6]. Thus, the analytical calculation of the $N^2$-contribution to the $\beta_3^{\mathrm{OS}}$-function is on the agenda. This problem is closely related to the analytical evaluation of the $A_{3,1}$-term in the OS-scheme (e.g. starting from the results of eq. (2.8) in the $\overline{\mathrm{MS}}$-scheme) and therefore to the analytical calculation of certain four-loop diagrams contributing to $(g-2)_\mu$ [3]. We are going to consider this problem in the future. The results of eq. (3.10) can also be used to find, with the help of the RG relations, the value of certain five-loop $\ln(m_\mu/m_e)$

[#1] Note that within our normalization conditions one has $A_3 = 64 a_3$, where $a_3$ is presented in ref. [2].

contributions to $(g-2)_\mu$ in order to compare them with the results of the numerical computations presented in ref. [2].

**4.** Let us now compare the obtained numerical four-loop approximation for the $\beta_{OS}$-function, namely

$$\beta_{OS}(\alpha)=0.0833\left(\frac{\alpha}{\pi}\right)^2+0.0625\left(\frac{\alpha}{\pi}\right)^3-0.1050\left(\frac{\alpha}{\pi}\right)^4-0.4443\left(\frac{\alpha}{\pi}\right)^5, \quad (4.1)$$

with similar ones for the $\beta_{MS}$-function and for the $\Psi$-function obtained in ref. [7], namely

$$\beta_{MS}(\alpha)=0.0833\left(\frac{\alpha}{\pi}\right)^2+0.0625\left(\frac{\alpha}{\pi}\right)^3-0.0269\left(\frac{\alpha}{\pi}\right)^4-0.1309\left(\frac{\alpha}{\pi}\right)^5,$$

$$\Psi(\alpha)=0.0833\left(\frac{\alpha}{\pi}\right)^2+0.0625\left(\frac{\alpha}{\pi}\right)^3+0.0125\left(\frac{\alpha}{\pi}\right)^4-0.1502\left(\frac{\alpha}{\pi}\right)^5. \quad (4.2)$$

Note that in all three cases the four-loop coefficients are negative and relatively sizeable (comparatively to the previous three coefficients). These results also indicate that in spite of the asymptotic enhancement of the absolute values of the corresponding coefficients the possible sign-alternating character of the QED perturbative series [16] does not manifest itself for the four-loop approximations of the QED $\beta$-functions in the different renormalization schemes. This conclusion is not surprising since the status of the QED estimates [16] remains unclear.

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