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On pseudo-conservation laws for the cyclic server system with compound Poisson arrivals *

Jhitti Chiarawongse
Center for Information Technology Integration, The University of Michigan, 2901 Hubbard, Ann Arbor, MI 48105, USA

Mandyam M. Srinivasan
Department of Industrial and Operations Engineering, The University of Michigan, Ann Arbor, MI 48109-2117, USA

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Boxma and Groenendijk obtain the pseudo-conservation laws for cyclic server systems, for both the continuous-time system with simple Poisson arrivals, and for the discrete-time system. We extend these laws to the continuous-time cyclic server system with compound Poisson arrivals. In the process we identify an error in Boxma and Groenendijk's analysis of the semi-exhaustive service strategy in the discrete-time cyclic server system.

queueing system; cyclic server; vacation models; conservation law; compound poisson process

1. Introduction

Boxma and Groenendijk [3,4] obtain the pseudo-conservation laws for the cyclic server system (cf. Watson [14]), which expresses a weighted sum of the mean waiting times at the various queues in the system as a function of its traffic characteristics. In [3], they obtain pseudo-conservation laws for a continuous-time cyclic server system where the arrival process at each queue in the system is an independent simple Poisson process, i.e., each arrival consists of only a single customer. The pseudo-conservation laws for a discrete-time cyclic server system are derived in [4] in an analogous manner. In the latter case, time is divided into slots, and the number of arrivals in each slot is an independent and identically distributed (i.i.d.) random variable. By taking the limit as the slot size tends to zero, they obtain the laws for the continuous-time system with compound Poisson arrivals. However, the result for the discrete-time system is incorrect for the semi-exhaustive service strategy. This error is also present in a subsequent paper by Boxma on waiting times in systems with multiple customer classes (see (3.29) of [5]).

This paper uses the approach presented in [3] to derive pseudo-conservation laws for the cyclic server system with compound Poisson arrivals *directly* in the *continuous-time* setting. In the process, we will uncover the error made by Boxma and Groenendijk in their discrete-time system analysis [4]. The notation used in this paper closely follows the notation used in [3].

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2. Preliminaries

Consider a system with N queues attended by a single server. Customers arrive at queue i according to an independent, compound Poisson process with rate γ_i. The number of customers in each batch is a random variable with distribution $X_i(\cdot)$ and probability generating function (p.g.f.) $X_i^*(\cdot)$. It is assumed that $X_i(0)=0$. Let χ_i denote the first moment of $X_i(\cdot)$, and let $\chi_i^{(2)}$ and $\tilde{\chi}_i^{(2)}$ denote, respectively, the second moment and second factorial moment of $X_i(\cdot)$. Customers who arrive at queue i are referred to as type-i customers. The arrival rate, λ_i, of type-i customers is defined as $\lambda_i \equiv \gamma_i \chi_i$.

The server visits the queues in a fixed cyclic order, $1, 2, \ldots, N, 1, \ldots$. If there are any customers waiting at a queue when the server visits it, then he begins to serve that queue. The server adopts one of the following service disciplines (strategies) at each queue, consistently, each time he visits it (see Boxma [5] or Takagi [12] for other possible strategies): Exhaustive (E), Gated (G), Non-Exhaustive (NE), or Semi-Exhaustive (SE). The service times for type-i customers are assumed to be i.i.d. random variables with distribution function $B_i(\cdot)$, with first and second moments β_i and $\beta_i^{(2)}$, respectively. The offered load, ρ_i, at queue i, and the total offered load, ρ, are defined as $\rho_i \equiv \lambda_i \beta_i$ and $\rho \equiv \sum_{i=1}^{N} \rho_i$.

After the server completes service at queue i, he could take a non-zero amount of time to switch to queue $i+1$. The switch-over times between queues i and $i+1$ are independent, identically distributed random variables with first and second moments s_i and $s_i^{(2)}$ respectively. Let S denote the total switch-over time required by the server during a complete scan of all queues. The first and second moments of S are denoted by $s \equiv \sum_{i=1}^{N} s_i$ and $s^{(2)} \equiv s^2 + \sum_{i=1}^{N}(s_i^{(2)} - s_i^2)$, respectively. It is assumed that the arrival processes, the service processes, and the switch-over processes are mutually independent.

Define the cycle time, C_i, for queue i as the time between two successive arrivals of the server at queue i. Its expected value, $E[C_i]$, is independent of i and is given by (cf. equation (1.1) of [3])

$$E[C_i] = E[C] = s/(1-\rho). \tag{2.1.1}$$

It can be easily shown that the stability conditions for the $\mathrm{M}^X/\mathrm{G}/1$ cyclic server system are the same as for the corresponding $\mathrm{M}/\mathrm{G}/1$ cyclic server system.

3. The analysis

3.1. A stochastic decomposition result

Entirely analogous to the stochastic decomposition result proved in [3] for the $\mathrm{M}/\mathrm{G}/1$ cyclic server system, it is possible to prove the following result for the $\mathrm{M}^X/\mathrm{G}/1$ cyclic server system, stated as Theorem 3.1 below (cf. Theorem 1 of [3]). The proof of Theorem 3.1 is omitted.

Theorem 3.1. *For a single-server cyclic service system with mixed service strategies and independent compound Poisson arrival processes. Suppose the system is ergodic and stationary. Then the amount of work in this system at an arbitrary epoch, V_c^X, is distributed as the sum of the amount of work in the 'corresponding' $\mathrm{M}^X/\mathrm{G}/1$ system at an arbitrary epoch, V^X, and the amount of work in the cyclic server system at an arbitrary epoch during a switching interval, Y^X. In other words,*

$$V_c^X \overset{D}{=} V^X + Y^X \tag{3.1.1}$$

where $\overset{D}{=}$ denotes equality in distribution. Furthermore, V^X and Y^X are independent.

Let W_i denote the waiting time (not including service) at queue i. According to Schrage [11], the expected amount of work, $E[V_c^X]$, in the cyclic server system at an arbitrary epoch consists of two terms: i) the expected amount of work waiting in the queues at an arbitrary epoch, and ii) the expected residual

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amount of work that needs to be completed by the server on the customer currently being served. Hence, from (3.1.1) we obtain:

$$\sum_{i=1}^{N} \rho_i E[W_i] + \sum_{i=1}^{N} \rho_i \frac{\beta_i^{(2)}}{2\beta_i} = E[V^X] + E[Y^X]. \tag{3.1.2}$$

We can show (refer Chiarawongse and Srinivasan [8]) that

$$E[V^X] = \sum_{i=1}^{N} \frac{\lambda_i \beta_i^{(2)}}{2(1-\rho)} + \sum_{i=1}^{N} \frac{\rho_i \beta_i \tilde{\chi}_i^{(2)}}{2(1-\rho)\chi_i}. \tag{3.1.3}$$

So, from (3.1.2) and (3.1.3), we obtain

$$\sum_{i=1}^{N} \rho_i E[W_i] = \rho \sum_{i=1}^{N} \frac{\lambda_i \beta_i^{(2)}}{2(1-\rho)} + \sum_{i=1}^{N} \frac{\rho_i \beta_i \tilde{\chi}_i^{(2)}}{2(1-\rho)\chi_i} + E[Y^X]. \tag{3.1.4}$$

Remark 1. The system with zero switch-over times is the standard $M^X/G/1$ queueing system with multiple customer classes. For this system, (3.1.4) reduces to

$$\sum_{i=1}^{N} \rho_i E[W_i] = \rho \sum_{i=1}^{N} \frac{\lambda_i \beta_i^{(2)}}{2(1-\rho)} + \sum_{i=1}^{N} \frac{\rho_i \beta_i \tilde{\chi}_i^{(2)}}{2(1-\rho)\chi_i}. \tag{3.1.5}$$

Note that (3.1.5) is the conservation law for the standard $M^X/G/1$ queueing system. Takahashi [13] obtained this equation for a system with two classes of customers (i.e. $N = 2$).

From (3.9) of [3], $E[Y^X]$ can be expressed as follows:

$$E[Y^X] = \sum_{i=1}^{N} E\left[M_i^{(1)}\right] + \rho \frac{s^{(2)}}{2s} + \frac{s}{2(1-\rho)}\left(\rho^2 - \sum_{i=1}^{N} \rho_i^2\right) \tag{3.1.6}$$

where $E[M_i^{(1)}]$ is the expected amount of work in queue i at a departure epoch of the server from queue i. The term $E[M_i^{(1)}]$ depends on the service strategy employed at queue i. Using (3.1.4) and (3.1.6), we shall derive the pseudo-conservation laws for a cyclic server system with independent compound Poisson arrivals. To this end, we shall determine $E[M_i^{(1)}]$ for the cyclic server system for each of the four service strategies, namely the Exhaustive, Gated, Non-Exhaustive, and Semi-Exhaustive (Decrementing) service disciplines.

3.2. Determining $E[M_i^{(1)}]$

To determine $E[M_i^{(1)}]$, it will be useful to define V_i as the time that the server spends at queue i per visit to queue i. Let e, g, ne, and se denote the set of Exhaustive, Gated, Non-Exhaustive, and Semi-Exhaustive queues, respectively. The term $E[M_i^{(1)}]$ is easily obtained for the Exhaustive, Gated and Non-Exhaustive service strategies using the same reasoning as in [3].
Exhaustive (cf. (3.11) of [3]).

$$E\left[M_i^{(1)}\right] = 0, \quad i \in e. \tag{3.2.1}$$

Gated (cf. (3.12) of [3]).

$$E\left[M_i^{(1)}\right] = \rho_i^2 \frac{s}{1-\rho}, \quad i \in g. \tag{3.2.2}$$

Non-Exhaustive. Let T_i denote the amount of work left in queue i at the departure epoch of a *customer* from queue i. From (3.13) of [3],

$$E\left[M_i^{(1)}\right] = \frac{\lambda_i s}{1-\rho} E[T_i], \quad i \in ne, \tag{3.2.3}$$

where (cf. Chaudhry and Templeton [7])

$$E[T_i] = \beta_i\left(\lambda_i E[W_i] + \frac{\tilde{\chi}_i^{(2)}}{2\chi_i} + \rho_i\right) = \rho_i E[W_i] + \beta_i\frac{\tilde{\chi}_i^{(2)}}{2\chi_i} + \rho_i\beta_i. \tag{3.2.4}$$

So, from (3.2.3) and (3.2.4), we obtain

$$E\left[M_i^{(1)}\right] = \rho_i\frac{\lambda_i s}{1-\rho}E[W_i] + \frac{\rho_i s\tilde{\chi}_i^{(2)}}{2(1-\rho)\chi_i} + \rho_i^2\frac{s}{1-\rho}, \quad i \in ne. \tag{3.2.5}$$

3.2.1. The Semi-Exhaustive service discipline

This case is not as direct as the others. Let U_i denote the number of customers in queue i at the arrival epoch of the server at queue i. If $U_i \geq 1$, then the server begins service at this queue, and when he departs he leaves behind $U_i - 1$ customers. (Obviously, if $U_i = 0$, he departs immediately.) So, for $i \in se$, we get

$$E\left[M_i^{(1)}\right] = \beta_i E\left[\max(0, U_i - 1)\right] = \beta_i E\left[U_i - 1 \mid U_i \geq 1\right] \Pr\{U_i \geq 1\}. \tag{3.2.6}$$

To determine $E[M_i^{(1)}]$ from (3.2.6) we reason as follows. When $U_i \geq 1$, the server begins service on the customer at the head of the queue. During this service, customers continue to arrive at queue i. Now, consider an alternate service mechanism, SE′, in which the server collects customers who arrive during his sojourn at queue i, and serves only these customers before he departs from queue i (in addition, of course, to the customer at the head of the queue. Then the number of customers present at queue i when he leaves the queue is just $U_i - 1$. Clearly, the queue length process at queue i will be the same whether the server follows the Semi-Exhaustive strategy (SE) or the alternate service mechanism, SE′, described above.

Thus, with this alternate service mechanism, if $U_i \geq 1$ then the number of customers present in queue i at a departure epoch of an arbitrary customer from this queue consists of two components. One component is the number of customers, $U_i - 1$, who were already present when the server arrived at queue i. The other component consists of the customers, N_i^+, who arrive while the server is present at queue i, and are still waiting to be served. In other words, the expected number of customers left behind by an arbitrary departing customer is $E[U_i - 1 \mid U_i \geq 1] + E[N_i^+]$. Note that this is just equal to $\lambda_i E[W_i] + \tilde{\chi}_i^{(2)}/(2\chi_i) + \rho_i$ (refer (3.2.4)), and so we have

$$E\left[U_i - 1 \mid U_i \geq 1\right] = \lambda_i E[W_i] + \frac{\tilde{\chi}_i^{(2)}}{2\chi_i} + \rho_i - E\left[N_i^+\right]. \tag{3.2.7}$$

Let $\tilde{b}_i$ denote the length of time that the server is present at queue i, conditioned on the fact that $U_i \geq 1$. (Obviously $\tilde{b}_i$ will have the same behavior whether strategy SE or SE′ is followed.) If we can compute $E[\tilde{b}_i]$, then the term $\Pr\{U_i \geq 1\}$ is easily obtained as

$$\Pr\{U_i \geq 1\} = \frac{E[V_i]}{E\left[\tilde{b}_i\right]} = \frac{\rho_i s}{(1-\rho)E\left[\tilde{b}_i\right]}. \tag{3.2.8}$$

Thus, in order to obtain $E[M_i^{(1)}]$, we only need to determine the terms $E[N_i^+]$ and $E[\tilde{b}_i]$. To this end, consider and $M^X/G/1$ queueing system with vacations (see Baba [1]), having the same arrival process and service time requirements as specified for queue i in the cyclic server system. Assume that the server in this vacation system serves the queue exhaustively before he starts his vacation. Furthermore, assume that, in the vacation system, the service time of a customer and the duration of a vacation (which are assumed to be independent of each other) follow the same distribution. In this vacation system, consider the number of customers, N^+, that are present in the system at either the departure epoch of an arbitrary customer, or at the termination of a vacation. Then it is readily observed that N^+ is identically distributed at N_i^+. Also, the time between two successive departure epochs of the server, $\tilde{b}$, is identically distributed as $\tilde{b}_i$. These can be easily visualized by identifying the vacation period of the server in the vacation system with the service period of the customer who is at the head of queue i and, hence, is the first being served when the server arrives at queue i in the cyclic server system. Hence, to determine $E[M_i^{(1)}]$, it is sufficient ot obtain expressions for the terms $E[\tilde{b}]$ and $E[N^+]$. First, note that $\tilde{b}$ is just the cycle time for the $M^X/G/1$ queueing system with vacations. The expression for $E[\tilde{b}]$ and, hence, $E[\tilde{b}_i]$ is, therefore, given by (refer (2.1.1)):

$$E\left[\tilde{b}_i\right] = E\left[\tilde{b}\right] = \frac{\beta_i}{1-\rho_i}. \tag{3.2.9}$$

The expression for $E[N^+]$ is given by Lemma 3.2.

Lemma 3.2. *Consider an ergodic* $M^X/G/1$ *queueing system with vacations and exhaustive service discipline. Customers arrive according to a compound Poisson process with batch arrival rate* γ. *The batch sizes are i.i.d. random variables with distribution* $X(\cdot)$, *and the p.g.f.* $X^*(\cdot)$. *The service times of customers are, also, i.i.d. random variables with distribution* $B(\cdot)$ *whose LST is* $B^*(\cdot)$. *Let* χ *and* $\tilde{\chi}^{(2)}$ *denote the mean and second factorial moment of* $X(\cdot)$, *and let* β *and* $\beta^{(2)}$ *denote the mean and second moment of* $B(\cdot)$. *Define* $\lambda \equiv \gamma\chi$ *as the customer arrival rate, and* $\rho \equiv \lambda\eta$ *as the offered load of the system. If a vacation period of the server (the time between the departure epoch and the arrival epoch of the server) and the service time of a customer follow the same distribution, i.e.* $B(\cdot)$, *then the expected number of customers,* $E[N^+]$, *present in the system at either the departure epoch of an arbitrary customer, or at the vacation termination epoch, is given by*

$$E[N^+] = \frac{\lambda^2\beta^{(2)}}{2(1-\rho)} + \frac{\rho\tilde{\chi}^{(2)}}{2(1-\rho)\chi} + \rho. \tag{3.2.10}$$

Proof. Let $P(\cdot)$ and $P^*(\cdot)$ denote the distribution and the p.g.f. of N^+. Employing the imbedded Markov Chain analysis technique [9], we examine the exhaustive service $M^X/G/1$ system with vacations at epoch $t_1, t_2, \ldots,$ of vacation termination or service completion. The state space of the system is (N_t^+, τ_t) where N_t^+ is the number of the customers in the system at epoch t, and $\tau_t = 1$ if t is the vacation termination instant, otherwise $\tau_t = 2$ if t is the service completion instant of a customer. Define

$$P_1(n) \equiv \Pr\{N^+ = n, \tau_t = 1\}, \text{ and } P_2(n) \equiv \Pr\{N^+ = n, \tau_t = 2\}.$$

Then

$$P_1(n) = (P_1(0) + P_2(0)) \sum_{k=0}^{\infty} \left(\int_0^{\infty} \frac{(\gamma\xi)^k \, e^{-\gamma\xi}}{k!} X^{(k)}(n) \, dB(\xi) \right), \quad n = 0, 1, \ldots, \tag{3.2.11}$$

and

$$P_2(n) = \sum_{j=1}^{n+1} (P_1(j) + P_2(j)) \sum_{k=0}^{\infty} \left(\int_0^{\infty} \frac{(\gamma\xi)^k \, e^{-\gamma\xi}}{k!} X^{(k)}(n-j+1) \, dB(\xi) \right), \quad n = 0, 1, \ldots, \tag{3.2.12}$$

where $X^{(k)}(\cdot)$ is the k-fold convolution of $X(\cdot)$, the batch size distribution, with itself. Since

$$P(n) \equiv \Pr\{N^+ = n\} = P_1(n) + P_2(n), \tag{3.2.13}$$

we get

$$P(n) = P(0) \sum_{k=0}^{\infty} \left(\int_0^{\infty} \frac{(\gamma\xi)^k \, e^{-\gamma\xi}}{k!} X^{(k)}(n) \, dB(\xi) \right) + \sum_{j=1}^{n+1} P(j) \sum_{k=0}^{\infty} \left(\int_0^{\infty} \frac{(\gamma\xi)^k \, e^{-\gamma\xi}}{k!} X^{(k)}(n-j+1) \, dB(\xi) \right), \quad n = 0, 1, \ldots. \tag{3.2.14}$$

Multiplying (3.2.14) by z^n and taking the summation over $n = 0, 1, \ldots$, we obtain

$$P^*(z) = \frac{P(0)(1-z)B^*(\gamma - \gamma X^*(z))}{B^*(\gamma - \gamma X^*(z)) - z}. \tag{3.2.15}$$

Taking the limit as $z \uparrow 1$ in (3.2.15), it follows that $P(0) = 1 - \rho$. Hence,

$$E[N^+] = \lim_{z \uparrow 1} \frac{\mathrm{d}P^*(z)}{\mathrm{d}z} = \frac{\lambda^2 \beta^{(2)}}{2(1-\rho)} + \frac{\rho \tilde{\chi}^{(2)}}{2(1-\rho)\chi} + \rho. \qquad \square \tag{3.2.16}$$

Remark 2. In (4.16) of [4], Boxma and Groenendijk concluded that N_i^+ is the amount of work left behind by a departing customer in the discrete-time M/G/1 system with the same arrival and service processes as queue i in the cyclic-server system. However, this is true only when the busy period in the discrete-time M/G/1 system is initiated when there is only one customer present in the system. Since there can be more than one arrival during a slot in the discrete-time system considered by Boxma and Groenendijk, it may be noted that the busy period will not always be initiated with only one customer present in the system.

It follows from (3.2.6)–(3.2.9), and Lemma 3.2, that

$$E[M_i^{(1)}] = \rho_i \frac{\lambda_i s(1-\rho_i)}{1-\rho} E[W_i] - \frac{\lambda_i^2 \beta_i^{(2)} \rho_i s}{2(1-\rho)} + \frac{\rho_i(1-2\rho_i) s \tilde{\chi}_i^{(2)}}{2(1-\rho)\chi_i}. \tag{3.2.17}$$

3.3. *The pseudo-conservation law*

Combining the results from previous subsections, we obtain the following theorem:

Theorem 3.4. *Consider an ergodic cyclic server system with independent compound Poisson arrival processes and mixed service strategies. Then*

$$\begin{aligned} &\sum_{i \in e,g} \rho_i E[W_i] + \sum_{i \in ne} \rho_i \left(1 - \frac{\lambda_i s}{1-\rho}\right) E[W_i] + \sum_{i \in se} \rho_i \left(1 - \frac{\lambda_i s(1-\rho_i)}{1-\rho}\right) E[W_i] \\ &\quad = \rho \sum_{i=1}^{N} \frac{\lambda_i \beta_i^{(2)}}{2(1-\rho)} + \sum_{i=1}^{N} \frac{\rho_i \beta_i \tilde{\chi}_i^{(2)}}{2(1-\rho)\chi_i} + \rho \frac{s^{(2)}}{2s} + \frac{s}{2(1-\rho)} \left(\rho^2 - \sum_{i=1}^{N} \rho_i^2\right) \\ &\quad\quad + \frac{s}{2(1-\rho)} \left(\sum_{i \in g,ne} 2\rho_i^2 + \sum_{i \in ne} \frac{\rho_i \tilde{\chi}_i^{(2)}}{\chi_i} - \sum_{i \in se} \lambda_i^2 \beta_i^{(2)} \rho_i + \sum_{i \in se} \frac{\rho_i(1-2\rho_i)\tilde{\chi}_i^{(2)}}{\chi_i} \right). \end{aligned} \tag{3.3.1}$$

Remark 3. If the arrival processes are Poisson, all the $\tilde{\chi}_i^{(2)}$'s will be equal to zero, and (3.3.1) reduces to equation (3.22) of [3].

References

[1] Y. Baba, "On the $M^X/G/1$ queue with vacation time", *Oper. Res. Lett.* **5**, 93–98 (1986).

[2] O.J. Boxma and B. Meister, "Waiting-time approximations for cyclic service systems with switch-over times", *Performance '86 and ACM Sigmetrics 1986 Proceedings*, Raleigh, North Carolina, 264–272 (1986).

[3] O.J. Boxma and W.P. Groenendijk, "Pseudo-conservation laws in cyclic service systems", *J. Appl. Prob.* **24**, 949–964 (1987).

[4] O.J. Boxma and W.P. Groenendijk, "Waiting times in discrete-time cyclic service systems", *IEEE Trans. Comm.* **36**, 164–170 (1988).

[5] O.J. Boxma, "Workloads and waiting times in single-server systems with multiple customer classes", *Queueing Syst.* **5**, 185–214 (1989).

[6] P.J. Burke, "Delays in single-server queues with batch input", *Oper. Res.* **23**, 830–833 (1975).
[7] M.L. Chaudhry and J.G.C. Templeton, *A First Course in Bulk Queues*, John Wiley & Sons, New York, 1983.
[8] J. Chiarawongse, and M.M. Srinivasan, "Pseudo-conservation laws for cyclic server systems with compound Poisson arrivals", Tech. Report CITI-TR-89-4, Center for Information Technology Integration, University of Michigan, 1989.
[9] R.B. Cooper, *Introduction to Queueing Theory*, 2nd ed., Elsevier, North-Holland, Amsterdam, 1981.
[10] B. Doshi, "Queueing systems with vacations – a survey", *Queueing Syst.* **1**. 29–66 (1986).
[11] L. Schrage, "An alternative proof of a conservation law for the queue G/G/*l*", *Oper. Res.* **18**, 185–187 (1970).
[12] H. Takagi, "Queueing analysis of polling models", *ACM Computing Surveys* **20**, 5–28 (1988).
[13] Y. Takahashi, "On the relationship between work load and waiting time in single server queues with batch inputs", *Oper. Res. Lett.* **7**, 51–56 (1988).
[14] K.S. Watson, "Performance evaluation of cyclic service strategies – a survey", in: E. Gelenbe (ed.), *Performance'84*, North-Holland, Amsterdam, 521–533 (1984).
[15] R.W. Wolff, "Poisson arrivals see time averages", *Oper. Res.* **30**, 223–231 (1982).