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Renormalization scheme dependence in e^+e^- annihilation and τ-lepton decay at the next-to-next-to-leading order of perturbative QCD

J. Chyla
Institute of Physics, Na Slovance 2, CS-180 40 Prague 8, Czechoslovakia

A. Kataev [1]
Randall Laboratory of Physics, University of Michigan, Ann Arbor, MI 48109-1120, USA

and

S.A. Larin
Institute for Nuclear Research, SU-117312 Moscow, USSR

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We present a detailed investigation of the renormalization scheme dependence of the next-to-next-to-leading order QCD predictions for the processes $e^+e^- \to$ hadrons and $\tau \to \nu_\tau +$ hadrons. Based on it the comparison of the results obtained in three frequently used approaches to resolving the renormalization scheme ambiguities with experimental data is carried out.

1. Introduction

Two years ago the next-to-next-to-leading order (NNLO) QCD calculation of the familiar R-ratio in e^+e^- annihilations

$$R(s) = \frac{\sigma_{\rm tot}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \sum_{i=1}^{f} Q_i^2 [1 + r(s)] \,, \tag{1}$$

for f massless quarks has been reported [1]. In perturbative QCD $r(s)$ can be calculated as an expansion in the renormalized couplant $a = g^2/4\pi^2$

$$r(s) = a(\text{RS}) [1 + r_1(s, \text{RS}) a(\text{RS}) + r_2(s, \text{RS}) a^2(\text{RS}) + \ldots] \,. \tag{2}$$

Defined in a given renormalization scheme (RS) this series, when truncated to any finite order, depends on this RS. In the $\overline{\text{MS}}$ RS [2] the original results [1] yielded very large and positive NNLO corrections which seemed to have serious theoretical and phenomenological consequences. However, after checking these results it has been found by two of us that they are not correct. The revised results [3] change the situation drastically as now the correction in $\overline{\text{MS}}$ RS is moderate and negative. Expressed through the RS invariant ρ_2, introduced in ref. [4], they lead, contrary to ref. [1], to $\rho_2 < 0$! In such circumstances it is important to reanalyse the question of the RS dependence of the NNLO results for the quantity (2) and in particular to investigate the implications in the infrared region. The aim of this paper is twofold. First we discuss the RS dependence of the NNLO results [3]. We start from the $\overline{\text{MS}}$ RS but then concentrate on the discussion within the principle of minimal sensitivity (PMS) introduced in ref. [4] (and studied in the infrared region in refs. [5,6]) and on the effective charges (ECH) approach [7,8], the latter

[1] On leave of absence from the Institute for Nuclear Research, SU-117312 Moscow, USSR.

being essentially equivalent to the scheme invariant perturbation theory of ref. [9]. Our analysis thus replaces those of refs. [10,11], based on ref. [1]. Secondly we investigate the implications of ref. [3] for QCD corrections to the semileptonic τ-lepton decay rate. These corrections have been considered to NLO in ref. [12] and to NNLO (again using ref. [1]) in refs. [13–16].

2. Renormalization scheme dependence at the NNLO

As indicated in (2) both the couplant and the coefficients r_k do depend on the chosen RS. For the discussion of the RS dependence of physical quantities like (2) within the massless QCD (the restriction adopted throughout this paper) this RS may be defined by the set $\{a, r_k;\ k\geqslant 1\}$. The renormalization group (RG) invariance then implies consistency relations among them. These are usually formulated by writing the couplant $a(\mathrm{RS})$ first as a function of a certain scale variable μ, which enters the theory in the process of renormalization

$$\frac{\mathrm{d}a(\mu,\mathrm{RC})}{\mathrm{d}\ln\mu}\equiv\beta(a)=-ba^2(1+ca+c_2a^2+\ldots),\quad(3)$$

where [17]

$$b=\tfrac{1}{6}(33-2f),\quad c=\frac{153-19f}{66-4f}\quad(4)$$

are RG invariants while c_i, $i\geqslant 2$ are free parameters defining the so called renormalization convention (RC). At the NNLO we have therefore two free parameters labeling our RS: c_2 and either the couplant a itself or μ related to it through (3). The value of c_2 corresponding to the $\overline{\mathrm{MS}}$ RC is [18]

$$c_2(\overline{\mathrm{MS}})=\frac{77139-15099f+325f^2}{9504-576f}.\quad(5)$$

For phenomenologically interesting cases $f=5(3)$ used in the following we thus have $b=1.26(1.78)$ and $c_2(\overline{\mathrm{MS}})=1.475(4.47)$. At the NNLO the explicit dependence of r_1, r_2 on μ and c_2 is determined by the equations

$$\frac{\mathrm{d}r_1(\mu)}{\mathrm{d}c_2}=0,\quad\frac{\mathrm{d}r_1(\mu)}{\mathrm{d}\ln\mu}=b,$$
$$\frac{\mathrm{d}r_2(\mu,c_2)}{\mathrm{d}\ln\mu}=b(c+2r_1),\quad\frac{\mathrm{d}r_2(\mu,c_2)}{\mathrm{d}c_2}=-1.\quad(6)$$

The solution of (3), (6) can be written in terms of the RG invariants ρ, ρ_2 as [4]

$$r_1=b\ln(\mu/\Lambda)-\rho,\quad(7)$$

$$r_2=\rho_2-c_2+(r_1+\tfrac{1}{2}c)^2.\quad(8)$$

Up to the NNLO the dependence of the couplant on μ is given implicitly as

$$b\ln\left(\frac{\mu}{\tilde{\Lambda}}\right)=\frac{1}{a}+c\ln\left(\frac{ca}{\sqrt{1+ca+c_2a^2}}\right)+f(a,c_2),\quad(9)$$

$$f(a,c_2)=\frac{2c_2-c^2}{d}\left(\arctan\frac{2c_2a+c}{d}-\arctan\frac{c}{d}\right),$$
$$d=\sqrt{4c_2-c^2},\quad 4c_2>c^2,\quad(10)$$

$$f(a,c_2)=\frac{2c_2-c^2}{d}\left(\ln\left|\frac{2c_2a+c-d}{2c_2a+c+d}\right|-\ln\left|\frac{c-d}{c+d}\right|\right),$$
$$d=\sqrt{c^2-4c_2},\quad 4c_2<c^2.\quad(11)$$

In this notation $\tilde{\Lambda}=\Lambda(2c/b)^{-c/b}=1.15\Lambda$ (for $f=5$) where Λ is the conventional definition of the Λ-parameter, is held fixed and varying the RS means varying μ. Although this convention of labelling the RS singles out one RS as referential this is merely a matter of bookkeeping. Combining (7), (9) we get

$$r_1=\frac{1}{a}+c\ln\left(\frac{ca}{\sqrt{1+ca+c_2a^2}}\right)+f(a,c_2)-\rho,\quad(12)$$

and putting all together we obtain r^{NNLO} as a function of RS invariants ρ, ρ_2 and the RS-dependent quantities a, c_2. The energy dependence of r^{NNLO} enters entirely through the RG invariant ρ which can be written as

$$\rho=b\ln(\sqrt{s}/\Lambda)-r_1(\mu=\sqrt{s})\equiv b\ln(\sqrt{s}/\tilde{\Lambda}_{\mathrm{eff}}),\quad(13)$$

where Λ_{eff} is the RS invariant but process dependent parameter [8].

In the following we shall, when using μ, take for the referential RS the $\overline{\mathrm{MS}}$ one. We prefer to label the RS by means of a, c_2 because there is then no need to introduce any referential RS at all. The usefulness of using the RG invariant ρ instead of s to describe the energy dependence rests on the fact that the theoretical expression for $r^{\rm NNLO}(\rho)$ is unique while that for $r^{\rm NNLO}(s)$ depends on the value of $\Lambda_{\overline{\rm MS}}$ which must be determined from data. The RS dependence of $r^{\rm NNLO}(a, c_2; \rho, \rho_2)$ can therefore be represented by a two dimensional surface in three dimensions. In this picture each point on such a surface represents uniquely one RS. Recall that at the NLO

$$r^{\rm NLO}=a\left[2+ca\ln\left(\frac{ca}{1+ca}\right)-\rho a\right], \qquad (14)$$

and the corresponding curve was close to a parabola [4]. The RS ambiguity of (14) is simple and its energy dependence obvious: both PMS and FAC blow up at $\rho=0$ while the $\overline{\rm MS}$ RS does so somewhat later at $\rho=-2.411$.

At the NNLO the situation is more complicated as the surface representing $r^{\rm NNLO}$ depends nontrivially on the mutual relation of the two RG invariants ρ, ρ_2. In order to determine the value of the invariant ρ_2 for the quantity (2) r_1, r_2 must be calculated to the NNLO in some RS. In massless QCD this is most conveniently done in the $\overline{\rm MS}$ RS where one finds [19]

$$r_1=1.986-0.115f, \qquad (15)$$

and [3]

$$r_2=-6.637-1.2f-0.005f^2$$
$$-1.239\left(\sum_f Q_f\right)^2\Big/\left(3\sum_f Q_f^2\right). \qquad (16)$$

Combining (5), (8), (15), (16) we get $\rho_2=\rho_2(f)$ with the following values for $f=5$, 3:

$$\rho_2(f=5)=-15.5\,, \quad \rho_2(f=3)=-12.2\,, \qquad (17)$$

i.e. ρ_2 turns out to be negative in both phenomenologically interesting cases! This invalidates the conclusions of the recent phenomenological analyses [10,11], based on ref. [1]. In the following we therefore reanalyze the NNLO corrections to (2) using the values in (17). The negative value of the corresponding ρ_2 invariant has recently been found also in the analysis of the NNLO corrections to the total hadronic decay width of the Higgs boson [20,21]. A detailed discussion of the case $\rho_2>0$ will be given elsewhere. A quantitative idea of the shape of $r^{\rm NNLO}$ as a function of a, c_2 and ρ can be obtained by looking for stationary points with respect to the variation of a, given by the equation

$$\frac{{\rm d}r^{\rm NNLO}}{{\rm d}a}=0\,, \qquad (18)$$

or, explicitly,

$$c_2+2r_1(c+c_2a)+3r_2(1+ca+c_2a^2)=0\,. \qquad (19)$$

These points from a curve in the plane a, c_2 along which we look for stationary points with respect to c_2 as well

$$\frac{{\rm d}r^{\rm NNLO}}{{\rm d}c_2}=0\,. \qquad (20)$$

This leads to the condition

$$a=I(a,c,c_2)\,[1+(c+2r_1a)]\,,$$
$$I(a,c,c_2)$$
$$=\frac{1}{a^2}\left(\frac{a[cc_2a-(2c_2-c^2)]}{1+ca+c_2^2a}-4c_2f(a,c_2)\right). \qquad (21)$$

The system of coupled equations (19), (21) is too complicated to be solved analytically but there is no problem to solve them numerically. The complication of the case $\rho_2<0$ with respect to $\rho_2>0$ comes basically from the fact that in the former case the solution to (19), (21), which turns out to be the saddle point, lies for each ρ in the quadrant $a>0$, $c_2<0$ where there is a boundary line defining the region of the physical, i.e. positive, couplant. Its equation

$$1+ca+c_2a^2=0 \qquad (22)$$

implies that for a given $c_2<0$: $a<a^*(c_2)=(-c-\sqrt{c^2-4c_2})/2c_2$. For $\rho_2<0$ this saddle point, defining the PMS "optimized" RS, is moreover infrared stable, i.e. has a finite limit as $\rho\to-\infty$ [4].

In the effective charges (ECH) approach [7,8], which is is based on the requirement that to any order of perturbation theory

$$r(s)=a_{\rm eff}\,, \qquad (23)$$

we get at the NNLO the condition

$$r_1({\rm eff})+r_2({\rm eff})a_{\rm eff}=0\,. \qquad (24)$$

Assuming $r_1=r_2=0$ separately defines what we call the "canonical" ECH approach. The μ-dependence of $a_{\rm eff}$ is governed by the effective β-function:

$$\frac{\mathrm{d}a_{\mathrm{eff}}}{\mathrm{d}\ln\mu}\equiv\beta_{\mathrm{eff}}(a)$$
$$=-ba_{\mathrm{eff}}^2(1+ca_{\mathrm{eff}}+c_2^{\mathrm{eff}}a_{\mathrm{eff}}^2+...)\ . \tag{25}$$

Due to the fact that $c_2^{\mathrm{eff}}=\rho_2+(\frac{1}{2}c)^2<0$ (for $r_1=r_2=0$) this effective β-function has an IR fixed point at

$$a_{\mathrm{eff}}=\frac{-1}{2c_2^{\mathrm{eff}}}\left(c+\sqrt{c^2-4c_2^{\mathrm{eff}}}\right)\ , \tag{26}$$

and therefore $r_{\mathrm{eff}}^{\mathrm{NNLO}}=a_{\mathrm{eff}}$ is IR stable as in the PMS approach discussed above (numerically the respective IR limits are different, see below). As for the PMS approach the physical relevance of this IR fixed point is an interesting problem in its own, lying, however, beyond the scope of this paper. We merely stress that the IR behaviour of the series (2) is closely related to the divergence thereof and can be answered only once a way of summing it is found.

In the case of the ECH approach there appears at the NNLO a further complication connected with its very definition as (23) does not in general have a unique solution. In the case $r_1/r_2<0$ there is an infinite number of them corresponding to the intersection of the surface $r^{\mathrm{NNLO}}(a, c_2)$ with the plane $r=a$. As with the solutions to (18) there are two, one or no intersections for any given c_2, depending on the value of ρ. In the PMS approach an additional condition and namely the requirement $\mathrm{d}r^{\mathrm{NNLO}}/\mathrm{d}c_2=0$ could be used to find a unique point. Here on the other hand, there seems to be no reason to single out any of the solutions of (24). Note that even for $c_2=c_2^{\mathrm{eff}}=\rho_2+(\frac{1}{2}c)^2$ there is besides the trivial solution $r_2=0$ (defining the canonical ECH) another solution to the equation

$$r_2+cr_2a-r_2^2a^2=r_2(1+ca-r_2a^2)=0\ , \tag{27}$$

implied by (23) and (8). In the following we shall use these two different solutions to estimate the ambiguity of the ECH approach at the NNLO.

3. Comparison with experimental data

We shall now present the phenomenological analysis of r^{NNLO} for the quantity (2) in all three above mentioned approaches using the value $\rho_2=-15.5$. As the comparison is done in the energy range $\sqrt{s}>$ 10 GeV, we take $f=5$ and have $c=1.26$, $r_1=1.411$, $r_2=-12.8$. The world average value of $\Lambda_{\overline{\mathrm{MS}}}^{(5)}=150$ MeV [22] (based mainly on the deep inelastic scattering data which determines primarily $\Lambda_{\overline{\mathrm{MS}}}^{(4)}=230$ MeV) [23] translates to our $\bar{\Lambda}=172$ MeV and means that the corresponding region in ρ lies safely inside the interval $\rho\in(12, 23)$.

We point out that the recent NLO analyses [24–26] of (2) in the energy range spanned by the PETRA, PEP and TRISTAN experiments give higher values of $\Lambda_{\overline{\mathrm{MS}}}$ which cannot, even taking into account the large associated error, be made compatible with the cited world average. Even for so large values of $\Lambda_{\overline{\mathrm{MS}}}$ we have, however, still $\rho\in(11, 23)$. Fig. 1a displays the dependence of $r^{\mathrm{NNLO}}(\rho)$ on ρ in a wider interval in order to see where the various approaches differ really significantly. Fig. 1b shows the same in detail for the interval $\rho\in(11, 21)$. Included for comparison are also the NLO results. From fig. 1 we conclude the following:

(1) The upper ECH curve, corresponding to the nontrivial solution of (27) is far above all the other ones, while the canonical ECH results are very close to PMS results in most of the allowed range of the couplant.

(2) The difference between the PMS and canonical ECH results is practically negligible down to ρ of about 3 and remains at the level of 20% even in the IR limit! This feature has been previously theoretically suggested in refs. [11,27].

(3) In the region where NLO and NNLO approximations can be trusted, i.e. roughly $\rho\geqslant 12$ the inclusion of the NNLO contributions decreases the differences between the $\overline{\mathrm{MS}}$ and PMS/ECH results. This welcome feature indicates the importance of their being taken into account. The numerical difference between the NLO and NNLO results can be considered as measure of the theoretical uncertainties of the current perturbative calculations of the quantity (2).

(4) There is a qualitative difference between some of the curves for low values of ρ and in particular in the IR region. This pattern, observed for other physical quantities [20,21] as well will be discussed in detail elsewhere.

Although the curves in fig. 1 are unique, independent of any Λ-parameter, to transform them into

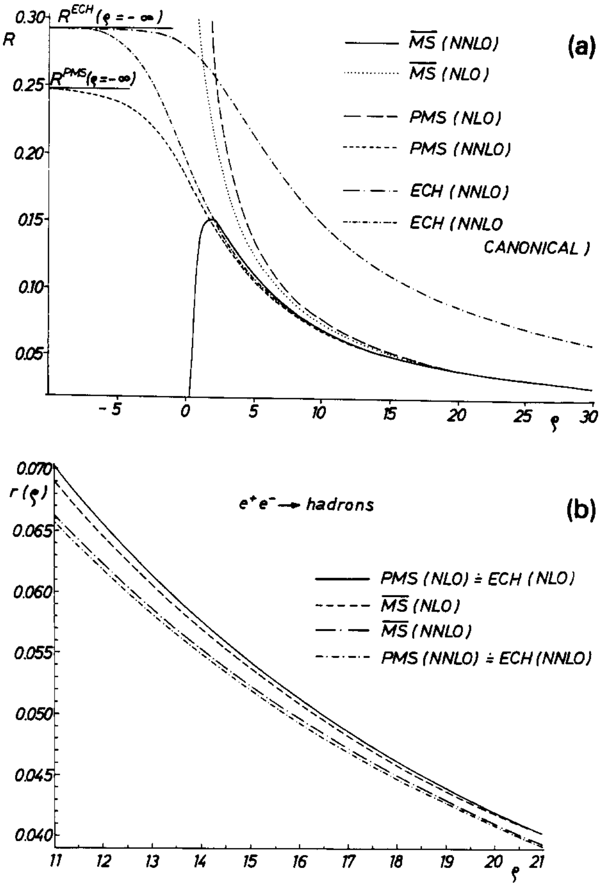

Fig. 1. (a) The shape of the function $r(\rho)$ for various choices of the renormalization scheme. Indicated are also the IR limits of this quantity in PMS and ECH approaches. (b) The same as (a), but in a narrower interval of ρ.

curves describing the measured energy dependence of $r(s)$ requires the knowledge of $\Lambda_{\overline{\rm MS}}$ and moreover means that each such $\Lambda_{\rm MS}$ will be associated with a different curve describing $r^{\rm NNLO}(s)$. Fitting these curves to experimental data will then fix $\Lambda_{\overline{\rm MS}}$. A simple way how to do this graphically is as follows. We first plot the experimental data as a function of $b\ln(\sqrt{s})=1.9165\ln(s)$ and then shift the chosen curve in fig. 1 horizontally and in negative direction by the amount Δ so that it fits the data best. In view of (13) we find $\Delta=b\ln(\Lambda_{\overline{\rm MS}})+r_1(\overline{\rm MS})$ and thus finally $\Lambda_{\overline{\rm MS}}=\exp\{[r_1(\overline{\rm MS})-\Delta]/b\}$. To make this procedure straightforward we have fitted the curves of fig. 1a in the interval $\rho\in(11,23)$ by a simple analytical formula of the form

$$r(\rho)=r_0+r_1(\rho-14)+r_2(\rho-14)^2+r_3(\rho-14)^3\,. \tag{28}$$

Within the whole mentioned interval (28) reproduces all the curves of fig. 1a with accuracy better than 1%, which is quite sufficient for the purposes of their mutual comparison and the extraction of $\Lambda_{\overline{\rm MS}}$. The coefficients r_k are given in table 1. Although very small, the coefficient r_3 is needed in order to have an accurate description of these curves up to $\rho=23$. They can not be obtained from one another by horizontal shift and thus the differences between them are not expressible as differences between the resulting values of $\Lambda_{\overline{\rm MS}}$.

Instead of making our own fit to experimental data along the lines sketched above we take as the basic experimental input the values for $r(\sqrt{s}=34)$ coming from two recent extensive analyses of the quantity (2) in the PETRA, PEP and TRISTAN energy range namely $r(\sqrt{s}=34)=0.056\pm0.008$ [24] and $r(\sqrt{s}=34)=0.051\pm0.007$. The values of ρ and $\Lambda_{\overline{\rm MS}}$ corresponding to these values of $r(s)$ in various approaches discussed above are given in table 2. Clearly, the inclusion of the NNLO corrections increase the value of $\Lambda_{\overline{\rm MS}}$ by a factor of 1.2 while the differences between $\overline{\rm MS}$ and PMS or ECH are, at both NLO and NNLO, small. Note that the values of $\Lambda_{\overline{\rm MS}}$ extracted from the data of ref. [25] are closer to the world average than those based on ref. [24].

4. τ-lepton decay

The results of ref. [3] can be easily converted [12] into the NNLO predictions for

$$R^\tau=\frac{\Gamma(\tau\to\nu_\tau+\text{hadrons})}{\Gamma(\tau\to\nu_\tau+e^-\bar{\nu}_e)}=3(1+r_\tau)\,, \tag{29}$$

which is anomalous to $r(s)$ of (2) with $s=M_\tau^2$ and can be written in exactly the same form. In the $\overline{\rm MS}$ RS and taking $f=3$ which is appropriate for the considered process, we have [3] $r_1(\overline{\rm MS})=5.2$, $r_2(\overline{\rm MS})=26.3$. Note that like in the case of the Higgs boson decay [20,21] the NNLO corrections are in the $\overline{\rm MS}$ RS positive contrary to the case of $r^{\rm NNLO}$ in e^+e^- annihilations. Despite this difference in r_2 coefficients we get for (29) as well as for the Higgs boson decay rate, negative values of $\rho_2{:}\rho_2=-6.27$! The

Table 1
The values of the coefficients r_k, $k=0, 1, 2, 3$, parametrising $r(\rho)$ in the interval $\rho\in(14, 21)$ as $r(\rho)=r_0+r_1(\rho-14)+r_2(\rho-14)^2+r_3(\rho-14)^3$.

Approach	r_0	r_1	r_2	r_3
PMS(NNLO)	0.0548	−0.00305	0.000162	−0.0000062
PMS(NLO)	0.0575	−0.00354	0.000212	−0.0000083
$\overline{\text{MS}}$(NNLO)	0.0551	−0.00301	0.000138	−0.0000042
$\overline{\text{MS}}$(NLO)	0.0569	−0.00338	0.000175	−0.0000042

Table 2
The values of ρ and the extracted $\Lambda^{(5)}_{\overline{\text{MS}}}$ in various RS at both NLO and NNLO and using data from refs. [24,25].

Approach		ref. [24]		ref. [25]	
		ρ	$\Lambda^{(5)}_{\overline{\text{MS}}}$ [MeV]	ρ	$\Lambda^{(5)}_{\overline{\text{MS}}}$ [MeV]
NLO	PMS	$14.43^{+2.80}_{-2.10}$	474^{+344}_{-246}	$16.07^{+2.96}_{-2.21}$	309^{+241}_{-166}
	$\overline{\text{MS}}$	$14.27^{+2.74}_{-2.17}$	489^{+378}_{-252}	$15.93^{+2.99}_{-2.25}$	321^{+256}_{-174}
NNLO	PMS	$13.62^{+2.92}_{-2.24}$	587^{+466}_{-313}	$15.30^{+3.06}_{-2.33}$	375^{+314}_{-206}
	$\overline{\text{MS}}$	$13.77^{+2.89}_{-2.30}$	572^{+463}_{-298}	$15.45^{+3.03}_{-2.37}$	363^{+311}_{-198}

whole analysis of the previous section therefore applies, with simple numerical differences, to this case as well. The principal question is, of course, whether it makes sense to use perturbation theory for the quantity (30), which may be dominated by nonperturbative effects and which, moreover, corresponds within the perturbation theory to a low value of ρ. However, it was shown in ref. [14] (see also ref. [28]) that this quantity is rather exceptional in the sense that even for so small values of $\sqrt{s}=M_\tau$ the nonperturbative power corrections are expected to be quite small with respect to the purely perturbative ones. For a discussion of other nonperturbative uncertainties see ref. [29]. In such circumstances the question of taking into account higher order perturbative corrections to (29) and investigating their relevance in various RS is clearly worth pursuing.

The results on $r_\tau(\rho)$ are displayed in fig. 2 in a wide interval of ρ although the values relevant for the decay of the τ-lepton lie somewhere in the range $\rho\in(1.5, 5)$. This interval follows if we convert the world average on $\Lambda^{(4)}_{\overline{\text{MS}}}=230$ MeV [23] to $\Lambda^{(3)}_{\overline{\text{MS}}}=280$ MeV, relevant for (29), and evaluate besides the value

$$\rho_\tau=b\ln(M_\tau/\Lambda^{(3)}_{\overline{\text{MS}}})-r_1(\overline{\text{MS}})+c\ln(2c/b)$$
$$=2.72\,, \quad (30)$$

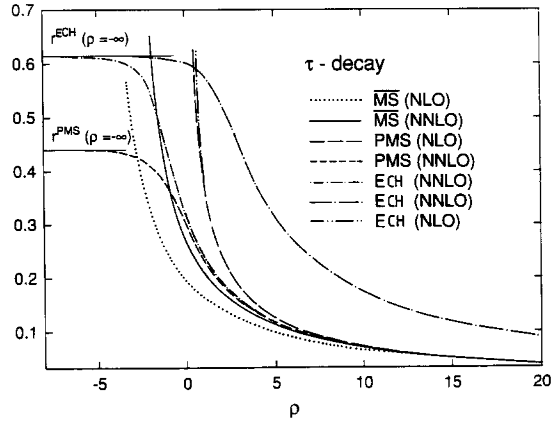

Fig. 2. The same as fig. 1a but for the quantity r_τ of (29).

also those corresponding to the variation of $\Lambda_{\overline{\text{MS}}}$ in the range 50–600 MeV. To facilitate the easy use of our results for the quantity (29) we have parametrised the curves of fig. 2 in the range $\rho\in(1.5, 5)$ by a simple second order polynomial with the coefficients r_k given in table 3.

For $\rho\in(1.5, 5)$ the differences between some of the curves are large, but as for (2) the results of PMS and ECH remain, at both NLO and NNLO, for our pur-

Table 3
The values of the coefficients r_k, $k=0, 1, 2, 3$, parametrising $r(\rho)$ in the interval $\rho \in (1.5, 5)$ as $r(\rho)=r_0+r_1\ (\rho-5)+r_2\ (\rho-5)^2$.

Approach	r_0	r_1	r_2
PMS(NNLO)	0.1147	−0.01182	0.00412
PMS(NLO)	0.1258	−0.00612	0.01182
$\overline{\text{MS}}$(NNLO)	0.1100	−0.01033	0.00333
$\overline{\text{MS}}$(NLO)	0.0976	−0.0075	0.00210

poses essentially indistinguishable. For ρ given by (30) we find

$$\text{NLO: } r_\tau(\text{PMS})=0.202, \quad r_\tau(\overline{\text{MS}})=0.126\,, \tag{31}$$

$$\text{NNLO: } r_\tau(\text{PMS})=0.163, \quad r_\tau(\overline{\text{MS}})=0.151\,. \tag{32}$$

From the above numbers we see how important it is in this case to include the NNLO contributions and how different results we may get in different RS. Most important the difference between the PMS/ECH and $\overline{\text{MS}}$ results which is 50% at the NLO shrinks to a mere 10% at the NNLO! Thus for $\Lambda^{(3)}_{\overline{\text{MS}}}=280$ MeV and taking into account the three RS discussed above we estimate the NNLO prediction for R_τ to lie within the interval

$$R_\tau^{\text{NNLO}}=3.39-3.49\,. \tag{33}$$

Turning the argument around, (33) could be used to determine the value of $\Lambda_{\overline{\text{MS}}}$ at both NLO and NNLO once a choice of a a particular RS is made. The ratio of the extracted Λ-parameters

$$x=\frac{\Lambda^{(3)}_{\overline{\text{MS}}}\ (\text{NLO})}{\Lambda^{(3)}_{\overline{\text{MS}}}\ (\text{NNLO})} \tag{34}$$

depends moderately on the value of r_τ and sensitively on the RS where the calculation has been performed. We find $x>1$ in the $\overline{\text{MS}}$ RS (contrary to the case of (2)) and $x<1$ in PMS/ECH approaches. The accurate measurement of R^τ could thus provide a good opportunity to test perturbative QCD in the region where higher order corrections are important. The numbers (31), (32) suggest that while the change due to going from NLO to NNLO is substantial it is small enough for the NNLO approximations to r_τ to give a reasonable description of the experimental data. This is a positive message as there is no hope to go in the foreseable future to still higher order calculations for quantities like (2), (29).

5. Conclusions

Corrected results of the NNLO perturbative calculations of the R-ratio in e^+e^- annihilations and τ-lepton decay were shown to improve considerably the situation as far as the difference between QCD predictions in various RS ar concerned. In particular we have shown that the inclusion of the NNLO corrections is of vital importance for the application of these calculations to the latter process. A precise measurement of the τ-lepton decay rate would thus provide a valuable and sensitive test of perturbative QCD.

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