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Dispersion of Small Amplitude Solutions of the Generalized Korteweg–de Vries Equation

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We study the long-time behavior of small solutions of the initial-value problem for the generalized Korteweg–de Vries equation

$$\begin{aligned} \partial_t u + \partial_x^3 u + \partial_x F(u) &= 0 \\ u(x, 0) &= g(x). \end{aligned} \tag{gKdV}$$

For the case where $F(w) = |w|^s$, with $s > (1/4)(23 - \sqrt{57}) \approx 3.8625$, our results imply that if $\|g\|_{L_1^1} + \|g\|_{L_2^2}$ is sufficiently small then $\sup_t (1 + |t|)^{1/3} \|u(t)\|_{L^\infty} < \infty$. In particular, the solution tends to zero in the supremum norm. The proofs make use of Duhamel's formula and dispersion estimates for the linear propagator, as well as chain and Leibniz rules for fractional derivatives of compositions $\|D^\alpha F(u)\|_{L^p}$ and products $\|D^\alpha (fg)\|_{L^p}$, $0 < \alpha < 1$ and $1 < p < \infty$. © 1991 Academic Press, Inc.

INTRODUCTION

In this paper we study the long-time behavior of solutions of the initial-value problem for the generalized Korteweg–de Vries equation (gKdV)

$$\begin{aligned} \partial_t u + \partial_x^3 u + \partial_x F(u) &= 0 \\ u(x, 0) &= g(x). \end{aligned} \tag{1.1}$$

$F(u)$ is a nonlinear function of u, such as $|u|^s$ with $s > 0$. Suitable regularity hypotheses on F and g guarantee the existence of a unique solution for all

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$t \in \mathbb{R}^1$. Our purpose is to show that if g is sufficiently small in some reasonable norm, then u tends to zero in some other norm as t approaches $\pm\infty$. More specifically, when $F(u) = |u|^s$, this dispersion is known to occur provided s is sufficiently large, and we seek to improve the allowed range of s.

For gKdV we obtain all

$$s > s_0 \equiv (23 - \sqrt{57})/4 \approx 3.8625. \tag{1.2}$$

Recently, Ponce and Vega [PV] have obtained all

$$s > (9 + \sqrt{73})/4 \approx 4.39,$$

improving the work of several authors [Str, Kl, Sh, KlPo].

We do not know what the best value of s might be. In the cases $s = 2$ and $s = 3$, (1.1) is known to admit exact solution by the inverse scattering transform. Inverse scattering theory is applied in [AS] to deduce the decay rates of solutions which are assumed to decay to zero in L^∞. They find in the case $s = 3$ that decaying solutions satisfy

$$\|u(t)\|_{L^\infty} = O(t^{-1/3}).$$

For $s = 2$, generically one has

$$\|u(t)\|_{L^\infty} = O(t^{-2/3})$$

but for certain exceptional "resonant data," the rate of decay is

$$\|u(t)\|_{L^\infty} = O\left(\left(\frac{\log t}{t}\right)^{2/3}\right).$$

Our method of proof is to write gKdV as an integral equation, treating the nonlinearity as a small perturbation of the linear part of the equation. We then obtain a priori estimates on time-weighted norms of $u(x, t)$, the time-weights being determined by the decay rates of the linear equation. The cited asymptotics for $s = 2, 3$ indicate that there are fundamental limitations to the perturbative method of proof we employ, for certain s.

We shall use the following notation. Let $u(t)$ denote the function $x \mapsto u(x, t)$. Let $\|\cdot\|_p$ denote the $L^p \equiv L^p(\mathbb{R}, dx)$ norm and $\|\cdot\|_{p,\alpha}$ denote a norm on L^p_α, the subspace of L^p consisting of functions possessing α derivatives in L^p [Ste]. For the case $p = 2$, we set $H^s \equiv L^2_s$. We define $P(D) = (P(\xi)\hat{f}(\xi))^\vee$ where $\hat{f}(\xi) \equiv \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$, is the Fourier transform of f. D^α is the operator with Fourier multiplier

$$(D^\alpha f)^\wedge(\xi) \equiv |\xi|^\alpha \hat{f}(\xi). \tag{1.3}$$

We set $\partial_x f \equiv \partial f/\partial x$ and $\langle y \rangle \equiv (1+y^2)^{1/2}$. $C(\mathbb{R}, H^s)$ and $C_w(\mathbb{R}, H^s)$ denote respectively the spaces of continuous and weakly continuous functions of $t \in \mathbb{R}$ with values in H^s.

We assume throughout that $F(0)=0$. It follows from the work of Kato [Ka] that for $F \in C^2$, if $g \in H^1$ and $\|g\|_{2,1}$ is sufficiently small then (1.1) has a solution $u \in C(\mathbb{R}, L^2) \cap C_w(\mathbb{R}, H^1)$. If $F \in C^3$ and $\|g\|_{2,2}$ is sufficiently small, then (1.1) has a unique solution $u \in C(\mathbb{R}, H^2)$. In [Ka] the existence results are obtained for the case $F \in C^\infty$, but our assertions for F satisfying weaker assumptions follow from his proofs.

Furthermore, one has the following two a priori bounds on solutions:

$$\|D^1 u(t)\|_2 \leqslant C, \qquad t \in \mathbb{R}, \tag{1.4}$$

where C is independent of t, and

$$\|u(t)\|_2 = \|g\|_2, \qquad t \in \mathbb{R}. \tag{1.5}$$

Equation (1.5) is the statement that the L^2 norm is conserved for solutions of gKdV. In addition, gKdV possess a conserved Hamiltonian energy functional [Ka] which together with (1.5) implies (1.4), by Sobolev–Nirenberg–Gagliardo estimates.

We now state results on the dispersion of small data solutions of gKdV in terms of L^∞ decay rates. We have,

THEOREM 1. (A) *Let $s>4$ and $F \in C^2$ be such that $|F'(w)| = O(|w|^{s-1})$ as $|w| \to 0$. Then, there is an $\varepsilon_0 > 0$ such that for every g satisfying*

$$|g|_1 \equiv \|g\|_1 + \|g\|_{2,1} < \varepsilon_0, \tag{1.6}$$

the solution of

$$\begin{aligned} \partial_t u + \partial_x^3 u + \partial_x F(u) &= 0 \\ u(0) &= g \end{aligned} \tag{1.7}$$

in the class $C(\mathbb{R}, H^1)$ satisfies

$$\sup_{t \in \mathbb{R}} \langle t \rangle^{1/3} \|u(t)\|_\infty < \infty. \tag{1.8}$$

(B) *Let $s \in (s_0, 4]$, where $s_0 = (1/4)(23 - \sqrt{57}) \cong 3.8625$, and $F \in C^2$ be such that $|F'(w)| = O(|w|^{s-1})$ and $|F''(w)| = O(|w|^{s-2})$ as $|w| \to 0$. Then, there is an $\varepsilon_0 > 0$ such that for every g satisfying*

$$|g|_2 \equiv \|g\|_{1,1} + \|g\|_{2,2} < \varepsilon_0 \tag{1.9}$$

the unique solution of (1.7) in the class $C(\mathbb{R}, H^2)$ satisfies (1.8).

Remarks. (1) More detailed information on rates of dispersion in various norms is proved on the way to obtaining Theorem 1 (see Sections 4 and 5).

(2) For $F(u)=|u|^s$, gKdV has solitary wave solutions of the form $u(t,x)=w(x-ct;c)$, where $w(\xi;c)$ is an exponentially decaying function of ξ. These solutions are nondecaying with time. Scaling implies $w(\xi;c)=c^{1/(s-1)}w(c^{1/2}\xi;1)$. A simple calculation shows that

$$\|w(\cdot;c)\|_{L^1}+\|w(\cdot;c)\|_{2,1}$$

tends to zero as $c\to 0^+$ for $s<3$. Thus, smallness in the norms of Theorem 1, cannot ensure dispersion for $s<3$ [Str].

The arguments we use are based on estimates for the free propagator $S(t)=\exp(-i\partial_x^3 t)$. That is, we treat the nonlinearity as a perturbation and write (1.1) as

$$u(t)=S(t)\,g+\int_0^t S(t-\tau)\,\partial_x F(u(\tau))\,d\tau. \tag{1.10}$$

Any solution of gKdV in $C_w(\mathbb{R},H^1)$ satisfies (1.10).

In Section 2 we derive the required estimates for the one parameter group of transformations $g\mapsto S(t)\,g$. In Section 3 chain and Leibniz rules are stated and proved for the purpose of estimating quantities like $\|D^\alpha F(u)\|_p$ and $\|D^\alpha(fg)\|_p$ with $0<\alpha<1$ and $1<p<\infty$.

2. Estimates for the Linear Propagator

The solution of the initial value problem

$$\partial_t u+\partial_x^3 u=0$$
$$u(0)=g$$

is

$$u(t)=S(t)\,g,$$

where

$$(S(t)\,g)^\wedge\,(\xi)=e^{it\xi^3}\hat{g}(\xi).$$

Fortunately for us, practically everything we need to know about the L^p-L^q mapping properties of $S(t)$ has been worked out by previous authors. $S(t)$ is realized by convolution with $(3t)^{-1/3}Ai((3t)^{-1/3}x)$, where

Ai is the well-known Airy function. In particular it is known from the method of stationary phase that $Ai(x)$ is bounded and is $0(|x|^{-1/4})$ as $|x| \to \infty$. This gives

LEMMA 2.1. *For all* $g \in L^1$,

$$\|S(t)\, g\|_\infty \leqslant Ct^{-1/3} \|g\|_1,$$

$$\|S(t)\, g\|_p \leqslant Ct^{-1/4} \|g\|_1, \qquad \textit{for all} \quad p > 4,$$

and

$$\|S(t)\, g\|_p \leqslant C_p t^{-1/3(1-1/p)} \|g\|_1, \qquad \textit{for all} \quad p > 4.$$

The first two estimates follow from the bounds on $Ai(x)$, while the third follows from the first two by interpolation. The next estimate is at the heart of the work of Ponce and Vega [PV].

LEMMA 2.2. *For all* $g \in L^1$,

$$\|D^{1/2} g\|_\infty \leqslant Ct^{-1/2} \|g\|_1.$$

We require a generalization, which is proved by Kenig, Ponce, and Vega [KPV].

LEMMA 2.3. *Let* $p \in [2, \infty)$ *and set* $1/p' = 1 - 1/p$. *Then*

$$\|D^{1/2-1/p} S(t)\, g\|_p \leqslant Ct^{-1/2+1/p} \|g\|_{p'}, \qquad \textit{for all} \quad g \in L^{p'}.$$

If we dualize the third conclusion of Lemma 2.1 and interpolate with Lemma 2.2 we obtain

LEMMA 2.4. *For each* $p \in (1, 4/3)$ *there exist* $\beta > 0$ *and* $\gamma > 0$ *such that*

$$\|D^\beta S(t)\, g\|_\infty \leqslant Ct^{-\gamma} \|g\|_p \qquad \textit{for all} \quad g \in L^p$$

with $\beta \to 1/2$ *and* $\gamma \to 1/2$ *as* $p \to 1^+$. *Moreover* $|\beta - 1/2| + |\gamma - 1/2| = O(p-1)$.

Of course one could give formulae for β and γ, but that won't be necessary.

3. A CHAIN RULE FOR FRACTIONAL DERIVATIVES

PROPOSITION 3.1. *Suppose that* $F \in C^1(\mathbb{C})$, $\alpha \in (0, 1)$, $1 < p, q, r < \infty$, *and* $r^{-1} = p^{-1} + q^{-1}$. *If* $u \in L^\infty(\mathbb{R})$, $D^\alpha u \in L^q$, *and* $F'(u) \in L^p$, *then* $D^\alpha(F(u)) \in L^r$ *and*

$$\|D^\alpha F(u)\|_r \leqslant C\, \|F'(u)\|_p\, \|D^\alpha u\|_q.$$

The hypothesis $u \in L^\infty(\mathbb{R})$ is inessential and serves only to guarantee that $D^\alpha u$ is defined, as a distribution. The proof relies on ideas of Coifman and Meyer [CoMe] and Bony [Bo]; see also Meyer [Me]. Although the result remains valid in $\mathbb{R}^m$ for all m, the proof seems to require an additional ingredient, the Calderón–Zygmund method of rotations [CaZy]. We shall stick to the simpler one-dimensional case.

For the proof some preparation is needed. Introduce $\eta \in C_0^\infty(\mathbb{R})$, nonnegative, supported in $\{1/2 < |\xi| < 2\}$ and satisfying

$$\sum_{j=-\infty}^{\infty} \eta(2^j\xi) \equiv 1 \qquad \text{on } \mathbb{R}\backslash\{0\}.$$

Define Fourier multiplier operators

$$(Q_j f)^\wedge(\xi) = \eta(2^{-j}\xi)\hat{f}(\xi).$$

Then

$$\|D^\alpha f\|_r \sim \left\| \sum_{j=-\infty}^{\infty} 2^{j\alpha} Q_j f \right\|_r \sim \left\| \left(\sum_j 2^{2j\alpha} |Q_j f(\cdot)|^2 \right)^{1/2} \right\|_r$$

for all $f \in L^r$ with $D^\alpha f \in L^r$, by the Hörmander–Mihlin–Marcinkiewicz multiplier theorem and Littlewood–Paley theory (see for instance Stein [Ste]). Moreover, if the right-hand side is finite then $D^\alpha f \in L^r$ in the sense of distributions. Q_j may be realized as a convolution operator $Q_j f = f * \psi_j$ where $\psi_j \in \mathscr{S}$,

$$|\psi_j(x)| + 2^{-j} |\partial_x \psi_j(x)| \leqslant C_N 2^j (1 + 2^j |x|)^{-N} \tag{3.1}$$

for all N, uniformly in $j \in \mathbb{Z}$, and

$$\int \psi_j = 0. \tag{3.2}$$

Construct also $\tilde{\eta} \in C_0^\infty(\{1/2 < |\xi| < 2\})$ but satisfying $\tilde{\eta} \cdot \eta \equiv \eta$. Define

$$(\tilde{Q}_j f)^\wedge(\xi) = \hat{f}(\xi)\, \tilde{\eta}(2^{-j}\xi)$$

so that the identity operator may be resolved as

$$I = \sum Q_j = \sum \tilde{Q}_j Q_j,$$

and $\tilde{Q}_j$ is realized by convolution with a Schwartz function $\tilde{\psi}_j$ satisfying (3.1) and (3.2).

Let M denote the maximal function of Hardy and Littlewood:

$$Mf(x) \equiv \sup_{r>0} (2r)^{-1} \int_{-r}^{r} |f(x-s)| \, ds.$$

LEMMA 3.2. *For any* g,

$$|\tilde{Q}_j g(y) - \tilde{Q}_j g(x)| \leqslant C \begin{cases} 2^j |x-y| \, Mg(x) & \text{if } |x-y| \leqslant C2^{-j} \\ Mg(x) + Mg(y) & \text{for all } x, y. \end{cases}$$

Proof.

$$|\tilde{Q}_j g(y) - \tilde{Q}_j g(x)| \leqslant \int |\tilde{\psi}_j(y-z) - \tilde{\psi}_j(x-z)| \cdot |g(z)| \, dz.$$

For any x, $|\tilde{Q}_j g(x)| \leqslant CMg(x)$ because of the bounds (3.1). If $|x-y| \leqslant C2^{-j}$ then

$$|\tilde{\psi}_j(y-z) - \tilde{\psi}_j(x-z)| \leqslant C2^{2j} |x-y| \cdot (1 + 2^j |x-z|)^{-2},$$

again by (3.1). By a standard calculation this implies the desired estimate; see Stein [Ste, pp. 62–63]. ∎

Proof of Proposition 3.1.

$$\begin{aligned} Q_j F(u)(x) &= \int F(u)(y) \, \psi_j(x-y) \, dy \\ &= \int [F(u)(y) - F(u)(x)] \, \psi_j(x-y) \, dy \\ &= \int \left[\int_0^1 F'(tu(y) + (1-t) \, u(x)) \, dt \right] (u(y) - u(x)) \\ &\quad \cdot \psi_j(x-y) \, dy. \end{aligned} \tag{3.3}$$

Note that

$$\left| \int_0^1 F'(tu(y) + (1-t) \, u(x)) \, dt \right| \leqslant 2M(F'(u))(x).$$

To estimate (3.3) decompose $u = \sum Q_k u = \sum \tilde{Q}_k Q_k u$ to obtain

$$\begin{aligned} |Q_j F(u)(x)| \leqslant CM(F'(u))(x) \cdot \sum_{k=-\infty}^{\infty} \int |\tilde{Q}_k Q_k u(y) - \tilde{Q}_k Q_k u(x)| \\ \cdot |\psi_j(x-y)| \, dy. \end{aligned} \tag{3.4}$$

Break the sum over k into the cases $k<j$ and $k \geqslant j$. Then

$$\begin{aligned}
&\sum_{k<j} \int |\tilde{Q}_k Q_k u(y) - \tilde{Q}_k Q_k u(x)| \cdot |\psi_j(x-y)|\, dy \\
&\quad \leqslant C \sum_{k<j} \int_{|x-y| \leqslant 2^{-k}} 2^k |x-y| \cdot (MQ_k u)(x) \\
&\qquad \cdot 2^j (1+2^j|x-y|)^{-3}\, dy \\
&\qquad + C \sum_{k<j} \int_{|x-y|>2^{-k}} [MQ_k u(x) + MQ_k u(y)] \\
&\qquad \cdot 2^j (1+2^j|x-y|)^{-3}\, dy \\
&\quad \leqslant C \sum_{k<j} 2^{k-j} MQ_k u(x) + C \sum_{k<j} 2^{k-j} \\
&\qquad \cdot (MQ_k u(x) + M^2 Q_k u(x)) \\
&\quad \leqslant C \sum_{k<j} 2^{k-j} M^2 Q_k u(x),
\end{aligned} \tag{3.5}$$

where of course $M^2 = M \circ M$.

Likewise

$$\begin{aligned}
&\sum_{k \geqslant j} \int |\tilde{Q}_k Q_k u(y) - \tilde{Q}_k Q_k u(x)| \cdot |\psi_j(x-y)|\, dy \\
&\quad \leqslant C \sum_{k \geqslant j} \int (MQ_k u(x) + MQ_k u(y))\, 2^j (1+2^j|x-y|)^{-2}\, dy \\
&\quad \leqslant C \sum_{k \geqslant j} M^2 Q_k u(x).
\end{aligned} \tag{3.6}$$

Putting (3.5) and (3.6) into (3.4), we have

$$\begin{aligned}
&\left(\sum_{j=-\infty}^{\infty} 2^{2j\alpha} |Q_j F(u)(x)|^2 \right)^{1/2} \\
&\quad \leqslant CM(F'(u))(x) \left(\sum_j 2^{2j\alpha} \left(\sum_{k<j} 2^{k-j} M^2 Q_k u(x) \right.\right. \\
&\qquad \left.\left. + \sum_{k \geqslant j} M^2 Q_k u(x) \right)^2 \right)^{1/2} \\
&\quad \leqslant C \cdot M(F'(u))(x) \\
&\qquad \cdot \sum_{m=-\infty}^{\infty} 2^{-\varepsilon|m|} \left(\sum_{k=-\infty}^{\infty} 2^{2k\alpha} (M^2 Q_k u(x))^2 \right)^{1/2},
\end{aligned}$$

by substituting $j=k-m$ after applying Minkowski's inequality, where

$$\varepsilon \equiv 2 \cdot \operatorname{Min}(\alpha, 1-\alpha) > 0.$$

To conclude,

$$\begin{aligned}
\|D^\alpha F(u)\|_r &\leqslant C \left\| \left(\sum 2^{2j\alpha} |Q_j F(u)|^2 \right)^{1/2} \right\|_r \\
&\leqslant C \left\| M(F'(u)) \cdot \left(\sum 2^{2k\alpha} (M^2 Q_k u)^2 \right)^{1/2} \right\|_r \\
&\leqslant C \|MF'(u)\|_p \cdot \left\| \left(\sum 2^{2k\alpha} (M^2 Q_k u)^2 \right)^{1/2} \right\|_q \\
&\leqslant C \|F'(u)\|_p \cdot \left\| \left(\sum 2^{2k\alpha} |Q_k u|^2 \right)^{1/2} \right\|_q \\
&\leqslant C \|F'(u)\|_p \|D^\alpha u\|_q.
\end{aligned}$$

In the second-to-last inequality we have invoked the vector-valued maximal theorem of Fefferman and Stein [FeSt]:

$$\left\| \left(\sum (Mh_k)^2 \right)^{1/2} \right\|_q \leqslant C_q \left\| \left(\sum |h_k|^2 \right)^{1/2} \right\|_q$$

for all $\{h_k\}$ and all $q \in (1\ \infty)$. ∎

Analogous to the chain rule is a version of Leibniz's rule. Its proof is a straightforward application of ideas of Coifman and Meyer.

PROPOSITION 3.3. *Let* $\alpha \in (0, 1)$, $1 < r, p_1, p_2, q_1, q_2 < \infty$, *and suppose* $r^{-1} = p_i^{-1} + q_i^{-1}$, *for* $i = 1, 2$. *Suppose that* $f \in L^{p_1}$, $D^\alpha f \in L^{p_2}$, $g \in L^{q_2}$, $D^\alpha g \in L^{q_1}$. *Then* $D^\alpha(fg) \in L^r$ *and*

$$\|D^\alpha(fg)\|_r \leqslant C \|f\|_{p_1} \|D^\alpha g\|_{q_1} + C \|D^\alpha f\|_{p_2} \|g\|_{q_2}.$$

Proof. Alter $\tilde{\eta}$ so as to be identically one on $[1/4, 4]$ and supported on $(1/8, 8)$. Define

$$P_k f = \sum_{j \leqslant k-3} Q_j f.$$

Since the Fourier transform of a product is the convolution of the Fourier transforms of the factors,

$$Q_k g \cdot P_k f = \tilde{Q}_k (Q_k g \cdot P_k f)$$

for all f, g. Write

$$\begin{aligned} fg &= \sum_k Q_k g \cdot P_k f + \sum_k Q_k f \cdot P_k g + \sum_{|i-j| \leqslant 2} Q_i f \cdot Q_j g \\ &= \sum_k \tilde{Q}_k(Q_k g \cdot P_k f) + \sum_k \tilde{Q}_k(Q_k f \cdot P_k g) \\ &\quad + \sum_{|i-j| \leqslant 2} (Q_i f \cdot Q_j g). \end{aligned}$$

For the first term,

$$\begin{aligned} &\left\| \left(\sum_k 2^{2k\alpha} \, |\tilde{Q}_k(Q_k g \cdot P_k f)|^2 \right)^{1/2} \right\|_r \\ &\quad \leqslant C \left\| \left(\sum_k (M(2^{k\alpha} Q_k g \cdot P_k f))^2 \right)^{1/2} \right\|_r \\ &\quad \leqslant C \left\| \left(\sum_k 2^{2k\alpha} \, |Q_k g|^2 \cdot (Mf)^2 \right)^{1/2} \right\|_r \\ &\quad \leqslant C \, \|Mf\|_{p_1} \cdot \left\| \left(\sum_k 2^{2k\alpha} \, |Q_k u|^2 \right)^{1/2} \right\|_{q_1} \\ &\quad \leqslant C \, \|f\|_{p_1} \, \|D^\alpha g\|_{q_1}. \end{aligned}$$

The second term is the same, but with the roles of f and g reversed. For the third, when $|i-j| \leqslant 2$, $Q_k(Q_i f \cdot Q_j g) \equiv 0$ unless $k \leqslant \max(i, j) + 4$. Thus

$$\begin{aligned} &\left(\sum_k 2^{2k\alpha} \left| Q_k \left(\sum_{|i-j| \leqslant 2} Q_i f \cdot Q_j g \right) \right|^2 \right)^{1/2} \\ &\quad \leqslant C \left(\sum_k 2^{2k\alpha} \left| Q_k \sum_{\substack{|i-j| \leqslant 2 \\ \operatorname{Max}(i,j) \geqslant k-4}} (Q_i f \cdot Q_j g) \right|^2 \right)^{1/2} \\ &\quad \leqslant C \sum_{s \geqslant -6} \sum_{|t| \leqslant 2} \left(\sum_j 2^{2(j-s)\alpha} \, |Q_{j-s}(Q_{j-t} f \cdot Q_j g)|^2 \right)^{1/2} \\ &\quad = C \sum_{s \geqslant -6} \sum_{|t| \leqslant 2} 2^{-s\alpha} \left(\sum_j |Q_{j-s}(Q_{j-t} f 2^{j\alpha} Q_j g)|^2 \right)^{1/2}. \end{aligned}$$

The L^r norm is then estimated as above. ∎

Almost exactly the same lemma appears in Kato and Ponce [KaPo], but with $(I + D^2)^{\alpha/2}$ in place of D^α. See also Kenig, Ponce, and Vega [KPV].

4. Proof of Theorem 1(A)

The point of departure is the integral equation (see also (1.10))

$$u(t) = S(t)\,g + \int_0^t S(t-\tau)\,\partial_x F(u(\tau))\,dt, \tag{4.1}$$

where $(S(t)\,g)^{\wedge}\,(\xi) = e^{it\xi^3}\hat{g}(\xi)$. The strategy is to derive a pair of estimates, for the L^∞ and L^6 norms of $u(t)$. Roughly speaking each is proved by using (4.1) plus both the L^∞ and L^6 estimates for $u(\tau)$, $|\tau| < |t|$. Earlier work on the problem relied principally on one estimate, which was used to prove itself in the same fashion; the coupled system allows greater flexibility. The exponent 6 has been chosen simply because it appears to give the best results, not for any conceptual reason.

To formulate the main estimates let us introduce the notation

$$M_\infty(t) = \sup_{|\tau| \leqslant t} \langle \tau \rangle^{1/3} \|u(\tau)\|_\infty \tag{4.2}$$

$$M_6(t) = \sup_{|\tau| \leqslant t} \langle \tau \rangle^{5/18} \|u(\tau)\|_6 \tag{4.3}$$

and

$$|g|_1 = \|g\|_1 + \|g\|_{2,1}. \tag{4.4}$$

The exponents 1/3 and 5/18 correspond exactly to the decay estimates of Lemma 2.1 for the linear propagator.

Proposition 4.1. *Let $s > 4$ and suppose that $F \in C^2(\mathbb{R})$ satisfies $F(0) = 0$ and $|F'(u)| = O(|u|^{s-1})$ as $|u| \to 0$. Then there exists $C < \infty$ such that for all $g \in L^1 \cap L^2_1$, any solution u of* (gKdV) *in the class $C_w([0,T]; H^1)$ with initial data g satisfies for $|t| < T$*

$$M_\infty(t) \leqslant C\,|g|_1 + C' M_6(t)^3\, M_\infty(t)^{s-4} \tag{4.5}$$

$$M_6(t) \leqslant C\,|g|_1 + C' M_6(t)^3\, M_\infty(t)^{s-(11/3)} \tag{4.6}$$

where $C' = C'(F, s, \|g\|_{2,1}) \to 0$ as $\|g\|_{2,1} \to 0$.

The notation in the proof will be greatly simplified by a convention. Let δ be a positive number which will eventually be chosen small enough, depending on s, for a certain inequality to hold. For any exponent α, $\alpha+$ will denote an exponent $\beta \geqslant \alpha$ of the form $\beta = \alpha + O(\delta)$. Similarly we write $\alpha-$ and $\alpha\pm$. Thus $\|\cdot\|_{p+}$ denotes the L^q norm, $q = p + O(\delta)$, $q > p$. This inconvenience is forced on us by two technical nuisances: $\partial_x \circ D^{-1}$ is not bounded on L^1, though it is on L^{1+}; likewise our chain rule for fractional derivatives is unavailable for L^1.

We first derive Proposition 4.1, then deduce the theorem.

Proof of (4.5). Write

$$\partial_x = D^{1/2-} \circ (\partial_x D^{-1}) \circ D^{1/2+}.$$

By the integral equation (4.1),

$$\begin{aligned}
\|u(t)\|_\infty &\leqslant \|S(t)\, g\|_\infty + \int_0^t \|S(t-\tau)\, \partial_x F(u(\tau))\|_\infty \, d\tau \\
&\leqslant C\langle\tau\rangle^{-1/3} |g|_1 \\
&\quad + C \int_0^t \|D^{1/2-} S(t-\tau)(\partial_x D^{-1})\, D^{1/2+} F(u(\tau))\|_\infty \, d\tau \\
&\leqslant C\langle\tau\rangle^{-1/3} |g|_1 \\
&\quad + C \int_0^t |t-\tau|^{-1/2+} \|D^{1/2+} F(u(\tau))\|_{1+} \, d\tau,
\end{aligned}$$

using Lemma 2.4 in the last inequality plus the boundedness of $\partial_x \circ D^{-1}$—the Hilbert transform times a numerical factor—on L^{1+}. By applying the chain rule we may continue

$$\begin{aligned}
&\leqslant C\langle\tau\rangle^{-1/3} |g|_1 \\
&\quad + C \int_0^t |t-\tau|^{-1/2+} \|D^{1/2+} u(\tau)\|_{2+} \, \| \, |u(\tau)|^{s-1} \|_2 \, d\tau \\
&\leqslant C\langle\tau\rangle^{-1/3} |g|_1 \\
&\quad + C \|g\|_{2,1} \int_0^1 |t-\tau|^{-1/2+} \|u(\tau)\|_6^3 \, \|u(\tau)\|_\infty^{s-4} \, d\tau
\end{aligned}$$

by the conservation law, since $\|D^{1/2+} u(\tau)\|_{2+} \leqslant C \|u(\tau)\|_2 + C \|\partial_x u(\tau)\|_2 \leqslant C \|g\|_{2,1}$ provided δ is sufficiently small. Multiplying through by $\langle\tau\rangle^{1/3}$ we get

$$\begin{aligned}
&\langle\tau\rangle^{1/3} \|u(\tau)\|_\infty \\
&\quad \leqslant C |g|_1 + C' \langle\tau\rangle^{1/3} \int_0^t |t-\tau|^{-1/2+} \langle\tau\rangle^{-15/18} \langle\tau\rangle^{15/18} \|u(\tau)\|_6^3 \\
&\qquad \cdot \langle\tau\rangle^{-(1/3)(s-4)} \langle\tau\rangle^{(1/3)(s-4)} \|u(\tau)\|_\infty^{s-4} \, d\tau \\
&\quad \leqslant C' |g|_1 + C' M_6(t)^3 \, M_\infty(t)^{s-4} \langle\tau\rangle^{1/3} \\
&\qquad \cdot \int_0^t |t-\tau|^{-1/2+} \langle\tau\rangle^{-(15/18)-(1/3)(s-4)} \, d\tau.
\end{aligned}$$

The integral is bounded uniformly as $|t| \to \infty$, provided that δ is chosen sufficiently small and

$$\frac{1}{2}+\frac{15}{18}+\frac{s-4}{3}-\frac{1}{3}>1,$$

which holds if and only if $s>4$. Then we've proved

$$M_\infty(t) \leqslant C\,|g|_1 + C' M_6(t)^3\, M_\infty(t)^{s-4}$$

as desired. ∎

Proof of (4.6). By Lemma 2.3, $D^{1/3}S(t)$ maps $L^{6/5}$ to L^6 with a bound $O(t^{-1/3})$. Therefore since $\partial_x D^{-1}$ is bounded on L^6, by the integral equation

$$\begin{aligned}\|u(t)\|_6 &\leqslant \|S(t)\,g\|_6 + C\int_0^t \|D^{1/3}S(t-\tau)\,D^{2/3}F(u(\tau))\|_6\,d\tau\\ &\leqslant C\langle t\rangle^{-5/18}\,|g|_1 + C\int_0^t |t-\tau|^{-1/3}\,\|D^{2/3}\,F(u(\tau))\|_{6/5}\,d\tau\\ &\leqslant C\langle t\rangle^{-5/18}\,|g|_1 + C\int^t |t-\tau|^{-1/3}\,\|D^{2/3}u(\tau)\|_q\,\|u(\tau)\|_p^{s-1}\,d\tau, \qquad (4.7)\end{aligned}$$

where $(s-1)/p+1/q=5/6$. Choose $q^{-1}=7/18$ so that $(s-1)/p=4/9$. The choice is dictated by the exponent 2/3 on D, for $7/18=(2/3)\cdot(1/2)+(1/3)\cdot(1/6)$, whence an interpolation gives

$$\begin{aligned}\|D^{2/3}u(\tau)\|_{18/7} &\leqslant C\,\|Du(\tau)\|_2^{2/3}\,\|u(\tau)\|_6^{1/3}\\ &\leqslant C\,\|g\|_{2,1}^{2/3}\,\langle\tau\rangle^{-(5/18)\cdot(1/3)}\,M_6(t)^{1/3} \qquad (4.8)\end{aligned}$$

provided $|\tau| \leqslant |t|$.

To estimate the L^p norm in (4.7) write $p^{-1}=\theta/6+(1-\theta)/\infty$ and estimate

$$\begin{aligned}\|u(\tau)\|_p &\leqslant \|u(\tau)\|_6^\theta\,\|u(\tau)\|_\infty^{1-\theta}\\ &\leqslant \langle\tau\rangle^{-(1-p^{-1})/3}\,M_6(t)^{6/p}\,M_\infty(t)^{1-6/p} \qquad (4.9)\end{aligned}$$

whence $p=9(s-1)/4>27/4>6$ as long as $s>4$. Putting (4.8) and (4.9) together gives

$$\|D^{2/3}u(\tau)\|_q\,\|u(\tau)\|_p^{s-1} \leqslant C'\langle\tau\rangle^{-\alpha}\,M_6(t)^3\,M_\infty(t)^{s-11/3}, \qquad (4.10)$$

where

$$\alpha = \frac{5}{18}\frac{1}{3} + \frac{s-1}{3}(1 - p^{-1})$$

$$= \frac{5}{54} + \frac{s-1}{3} - \frac{1}{3}\frac{4}{9}$$

$$= \frac{s}{3} - \frac{7}{18}.$$

Inserting (4.10) into (4.7) yields

$$\|u(t)\|_6 \leqslant C\langle\tau\rangle^{-5/18} |g|_1$$
$$+ C'M_6(t)^3 M_\infty(t)^{s-11/3} \cdot \int_0^t |t-\tau|^{-1/3} \langle\tau\rangle^{-s/3+7/18}\, d\tau.$$

The integral on the right is $0(\langle\tau\rangle^{-\beta})$, $\beta = 1/3 + s/3 - 7/18 - 1 = (6s-19)/18$. We need $\beta > 5/18$, which means $6s > 24$, or $s > 4$ once again. Thus we've proved that if $s > 4$,

$$M_6(t) \leqslant C\,|g|_1 + C'M_6(t)^3 M_\infty(t)^{s-11/3},$$

which is (4.6). So Proposition 4.1 is established. ∎

To deduce Theorem 1(A) define

$$M(t) = M_6(t) + M_\infty(t).$$

We've proved that there exist $C < \infty$ and $\alpha, \beta > 0$ such that for any $\varepsilon_0 > 0$, if $|g|_1$ is sufficiently small then for all t,

$$M(t) \leqslant \varepsilon_0(1 + M(t)^\alpha + M(t)^\beta).$$

By the assumption that $u \in C(\mathbb{R}; L^2) \cap C_w(\mathbb{R}; H^1)$ plus interpolation and Sobolev embedding, $\|u(\tau)\|_6$ and $\|u(\tau)\|_\infty$ are continuous functions of τ, hence M is also continuous. Therefore $M(t)$ remains in the connected component of $\{y \geqslant 0\colon y \leqslant C\varepsilon_0(1 + y^\alpha + y^\beta)\}$ containing the origin, for all time. But if ε_0 is chosen small enough, which means $|g|_1$ is small enough, then that connected component is bounded. ∎

5. Proof of Theorem 1(B)

More is now required of the initial data, namely $g \in L^1$, $\partial_x g \in L^1$, and $g \in L_2^2$, and we define

$$|g|_2 = \|g\|_{L^1} + \|\partial_x g\|_{L^1} + \|g\|_{2,2}. \tag{5.1}$$

Now control of the H^2 norm will allow us to bound $\partial_x u(t)$ in various norms, and to use the fact that for the linear propagator, $\partial_x S(t)g$ decays faster in L^p norms than $S(t)g$ (assuming g has one derivative), as suggested by Lemma 2.2. In the proof of Theorem 1(A) our only estimate on $\partial_x u$ was the conservation law, but now we shall prove various decay estimates which in turn can be used to control u itself for a better range of nonlinearities. The idea of exploiting the faster decay of derivatives of u is already present in the work of Ponce and Vega [PV].

Henceforth let $F \in C^3$ satisfy $F(0) = F'(0) = 0$ and $|F''(u)| = O(|u|^{s-2})$. Let $g \in L^1 \cap H^2$ with $\partial_x g \in L^1$, and let u be the unique solution of (gKdV) with initial data g, in the class $C(\mathbb{R}, H^2)$; this solution exists by Kato [Ka].

The notation will be simpler if we restrict ourselves to the case $s_0 < s < 4$. If $F''(u) = O(|u|^{s-2})$ as $u \to 0$ then the same holds for smaller s, so there is no loss whatever in taking $s < 4$. Unfortunately our strategy is to estimate each of the following:

$$M_4(u, t) \equiv \sup_{|\tau| \leqslant t} \langle \tau \rangle^{1/4 - \delta} \|u(\tau)\|_4$$

$$M_\infty(u, t) \equiv \sup_{|\tau| \leqslant t} \langle \tau \rangle^{1/3} \|u(\tau)\|_\infty$$

$$M_\infty(\partial_x u, t) \equiv \sup_{|\tau| \leqslant t} \langle \tau \rangle^{1/3 + \varepsilon} \|\partial_x u(\tau)\|_\infty$$

$$M_4(D^{1/2}u, t) \equiv \sup_{|\tau| \leqslant t} \langle \tau \rangle^{1/4} \|D^{1/2}u(\tau)\|_4$$

$$M_\infty(D^{1/2}u, t) \equiv \sup_{|\tau| \leqslant t} \langle \tau \rangle^{1/3 + \varepsilon} \|D^{1/2}u(\tau)\|_\infty$$

$$M_2(\partial_x^2 u, t) \equiv \sup_{|\tau| \leqslant t} \|\partial_x^2 u(\tau)\|_2,$$

where $\delta = \delta(s)$ is positive but will be chosen sufficiently small for various inequalities to work out; it should be regarded as a technical nuisance. On the other hand ε is positive and is essential to the argument. It must satisfy

$$\varepsilon > \frac{4 - s}{3}. \tag{5.2}$$

We write $\varepsilon = (4 - s)/3 + \delta'$. We will prove that for each $s \in (s_0, 4)$ there exists $\varepsilon > (4 - s)/3$ such that for all $\delta > 0$, the quantities $M_4(u, t), ..., M_2(\partial_x^2 u, t)$ remain uniformly bounded as $|t| \to \infty$. Of course it suffices to prove that for small δ. When it should cause no confusion we write $M_4(u), ..., M_2(\partial_x^2 u)$ to simplify the notation.

Proposition 5.1. *Let $s \in (s_0, 4)$, $F \in C^3$, and suppose $F(0) = F'(0) = 0$ and $|F''(u)| = O(|u|^{s-2})$. Let $g \in L^1 \cap H^2$ with $\partial_x g \in L^1$ and let u be a* *solution of* (gKdV) *with initial data g in the class $C(\mathbb{R}, H^2)$. Let $\varepsilon = (4-s)/3+\delta'$ where $\delta' > 0$. If δ', δ are chosen sufficiently small, depending only on s, F, then for all times t, the following six inequalities are valid:*

$$M_2(\partial_x^2 u) \leqslant C' \exp[CM_\infty(u)^{s-2}\, M_\infty(\partial_x u)]$$

$$M_\infty(u) \leqslant C' + C'M_4(u)^{s-1}\, M_\infty(D^{1/2}u)^{(s-3)/2+}$$

$$M_\infty(\partial_x u) \leqslant C' + C'M_4(u)^{s-1}\, M_\infty(\partial_x u)^{(s-3)/2}\, M_2(D_2 u)^{1/2+} + C'M_4(u)^{s-2}\, M_4(D^{1/2}u)^{1-}\, M_\infty(\partial_x u)^{(s-3)/2}$$

$$M_\infty(D^{1/2}u) \leqslant C' + C'M_4(u)^{s-1}\, M_\infty(\partial_x u)^{(s-3)/2+}$$

$$M_4(u) \leqslant C' + C'M_\infty(u)^{s-7/2}\, M_4(u)^{5/2}\, M_4(D^{1/2}u)^{1/2} \cdot M_\infty(\partial_x u)^{1/2}$$

$$M_4(D^{1/2}u) \leqslant C' + CM_2(D^2u)^{1/2}\, M_\infty(\partial_x u)^{3/4}\, M_\infty(u)^{s-7/2}\, M_4(u)^{5/2} + C'M_\infty(\partial_x u)^{(s-2)/2}\, M_4(D^{1/2}u)^{1/2}\, M_4(u)^{s-3/2},$$

where $C' = C'(F, s, \delta, \delta', |g|_2) \to 0$ as $|g|_2 \to 0$.

As in section 4, an exponent $\alpha\pm$ means $\alpha \pm O(\delta)$.

Proof for $M_2(\partial_x^2 u)$. By Eq. (3.10) of [PV],

$$\|\partial_x u(t)\|_2 \leqslant C\, \|g\|_{2,2} \exp\left(C \int_0^t \|u(\tau)\|_\infty^{s-2}\, \|\partial_x u(\tau)\|_\infty\, d\tau\right)$$

$$\leqslant C' \exp\left(CM_\infty(u,t)^{s-2}\, M_\infty(\partial_x u, t) \cdot \int_0^t \langle\tau\rangle^{-(s-2)/3}\, \langle\tau\rangle^{-1/3-\varepsilon}\, d\tau\right)$$

$$\leqslant C' \exp(CM_\infty(u,t)^{s-2}\, M_\infty(\partial_x u, t))$$

provided $(s-2)/3 + 1/3 + \varepsilon > 1$, which means $\varepsilon > (4-s)/3$.

Proof for $M_\infty(u)$. By the integral equation (4.1),

$$\|u(t)\|_\infty \leqslant C\, \|S(t)\, g\|_\infty + \int_0^t \|S(t-\tau)\, \partial_x F(u(\tau))\|_\infty\, d\tau$$

$$\leqslant C\langle t\rangle^{-1/3}\, |g|_2 + C\int_0^t \|D^{1/2-}S(t-\tau)\circ(\partial_x D^{-1}) \circ D^{1/2+}F(u(\tau))\|_\infty\, d\tau$$

$$\leqslant C\langle t\rangle^{-1/3}\, |g|_2 + C\int_0^t |t-\tau|^{-1/2+}\, \|D^{1/2+}F(u(\tau))\|_{1+}\, d\tau,$$

using Lemma 2.4.

The chain rule for fractional-order derivatives, interpolation, and the uniform boundedness of $u(\tau)$ in H^1 imply

$$\begin{aligned}\|u(t)\|_\infty &\leqslant C\langle t\rangle^{-1/3}|g|_2 + C\int_0^t |t-\tau|^{-1/2+}\\ &\quad\cdot \|D^{1/2+}u(\tau)\|_{4/(5-s)+}\,\|\,|u(\tau)|^{s-1}\|_{4/(s-1)}\,dt\\ &\leqslant C\langle t\rangle^{-1/3}|g|_2 + C\int_0^t |t-\tau|^{-1/2+}\,\|u(\tau)\|_4^{s-1}\\ &\quad\cdot\|D^{1/2+}u(\tau)\|_{4/(5-s)+}\,d\tau\\ &\leqslant C\langle t\rangle^{-1/3}|g|_2 + C\int_0^t |t-\tau|^{-1/2+}\,\|u(\tau)\|_4^{s-1}\\ &\quad\cdot\|D^{1/2}u(\tau)\|_\infty^{(s-3)/2+}\,\|D^{1/2+}u(\tau)\|_2^{(5-s)/2-}\,d\tau\\ &\leqslant C\langle t\rangle^{-1/3}|g|_2 + C'M_4(u)^{s-1}\,M_\infty(D^{1/2}u)^{(s-3)/2+}\\ &\quad\cdot\int_0^t |t-\tau|^{-1/2+}\,\langle\tau\rangle^{-(s-1)/4+}\,\langle\tau\rangle^{-(1/3+\varepsilon)(s-3)/2}\,d\tau\\ &\leqslant C'\langle t\rangle^{-1/3}\left(1+M_4(u)^{s-1}\,M_\infty(D^{1/2}u)^{(s-3)/2+}\right)\end{aligned}$$

provided

$$1/2+\frac{s-1}{4}+(1/3+\varepsilon)\frac{s-3}{2}>1+1/3$$

and δ is chosen sufficiently small. Using the restriction $\varepsilon>(4-s)/3$, the requirement becomes $-2s^2+19s-43>0$, which (for $s<4$) holds for $s>(19-\sqrt{17})/4$. The lower bound is $<15/4=3.75$, which is less than $s_0\approx 3.86$. That takes care of $M_\infty(u)$.

Proof for $M_\infty(\partial_x u)$. Since g and $\partial_x g\in L^1$ it follows that $\partial_x D^{-1/2}g\in L^1$. Hence

$$\begin{aligned}\|\partial_x u(t)\|_\infty &\leqslant \|\partial_x S(t)\,g\|_\infty + \int_0^t \|\partial_x S(t-\tau)\,\partial_x F(u(\tau))\|_\infty\,d\tau\\ &\leqslant C'\langle t\rangle^{-1/2}\\ &\quad + C\int_0^t |t-\tau|^{-1/2+}\,\|D^{1/2+}\partial_x F(u(\tau))\|_{1+}\,d\tau. \qquad (5.3)\end{aligned}$$

To estimate the integrand, apply the chain and Leibniz rules:

$$\begin{aligned}&\|D^{1/2+}\,\partial_x F(u(\tau))\|_{1+}\\ &\quad = \|D^{1/2+}(F'(u)\,\partial_x u)\|_{1+}\\ &\quad \leqslant C\,\|F'(u)\|_p\,\|D^{3/2+}u\|_{q+} + C\,\|F''(u)\|_{p_1}\,\|D^{1/2+}u\|_{p_2+}\,\|\partial_x u\|_r, \qquad (5.4)\end{aligned}$$

where $p^{-1}+q^{-1}=1=p_1^{-1}+p_2^{-1}+r^{-1}$ and all exponents belong to $(1,\infty)$ but are otherwise at our disposal.

To handle the first term in (5.4) take $p=4/(s-1)$ and therefore $q=4/(5-s)$ to obtain

$$\begin{aligned}
&\|F'(u)\|_p \, \|D^{3/2+}u\|_{q+}\\
&\quad\leqslant C\,\|u\|_4^{s-1}\,\|D^{3/2+}u\|_{4/(5-s)+}\\
&\quad\leqslant C\,\|u\|_4^{s-1}\,\|D^{3/2+}u\|_2^{4-s-}\,\|D^{3/2}u\|_4^{s-3}
\end{aligned}$$

by interpolation since $2<4/(5-s)<4$. Two more interpolations give

$$\|D^{3/2}u\|_4^{s-3}\leqslant C\,\|\partial_x u\|_\infty^{(s-3)/2}\,\|\partial_x^2 u\|_2^{(s-3)/2}$$

and

$$\|D^{3/2+}u\|_2\leqslant \|\partial_x u\|_2^{1/2}\,\|\partial_x^2 u\|_2^{1/2+}\leqslant C'\,\|\partial_x^2 u\|_2^{1/2+}.$$

In all, the first term in (5.4) is

$$\begin{aligned}
&\leqslant C\,\|u\|_4^{s-1}\,\|\partial_x u\|_\infty^{(s-3)/2}\,\|\partial_x^2 u\|_2^{(s-3)/2}\cdot(C'\,\|\partial_x^2 u\|_2^{1/2+})^{4-s-}\\
&\leqslant C'\,\|u\|_4^{s-1}\,\|\partial_x u\|_\infty^{(s-3)/2}\,\|\partial_x^2 u\|_2^{1/2\pm}.
\end{aligned}\tag{5.5}$$

Now, turn to the second term in (5.4). Taking $p_1=4/(s-2)$, $p_2=4$, and $r=4/(5-s)$ and applying the hypothesis that $F''(u)=O(|u|^{s-2})$ leads to

$$\begin{aligned}
&\|F''(u)\|_{p_1}\leqslant C\,\|u\|_4^{s-2},\\
&\|\partial_x u\|_r\leqslant C\,\|\partial_x u\|_2^{(5-s)/2}\,\|\partial_x u\|_\infty^{(s-3)/2}\\
&\qquad\leqslant C'\,\|\partial_x u\|_\infty^{(s-3)/2}
\end{aligned}$$

and

$$\begin{aligned}
\|D^{1/2+}u\|_{p_2+}&\leqslant C\,\|D^{1/2}u\|_4^{1-}\,\|D^{1/2+}u\|_6^{0+}\\
&\leqslant C'\,\|D^{1/2}u\|_4^{1-};
\end{aligned}$$

the exponent 6 is of no particular significance and any finite exponent larger than 4 would do as well. The last inequality follows from the H^1 a priori bound (1.11) and Sobolev embedding. Combining the last three estimates, our bound for the second term in (5.4) is

$$C'\,\|u\|_4^{s-2}\,\|D^{1/2}u\|_4^{1-}\,\|\partial_x u\|_\infty^{(s-3)/2}.\tag{5.6}$$

Majorizing (5.4) by (5.5) + (5.6) and inserting the result into (5.3), we obtain the elegant estimate

$$
\begin{aligned}
&\langle t\rangle^{1/3+\varepsilon}\,\|\partial_x u(\tau)\|_\infty\\
&\quad\leqslant C' + C' M_4(u)^{s-1}\, M_\infty(\partial_x u)^{(s-3)/2}\, M_2(\partial_x^2 u)^{1/2\pm}\\
&\qquad\cdot\langle t\rangle^{1/3+\varepsilon}\int_0^t |t-\tau|^{-1/2+}\,\langle\tau\rangle^{-\alpha\pm}\,d\tau\\
&\qquad + C' M_4(u)^{s-2}\, M_\infty(\partial_x u)^{(s-3)/2}\, M_4(D^{1/2}u)^{1-}\\
&\qquad\cdot\langle t\rangle^{1/3+\varepsilon}\int_0^t |t-\tau|^{-1/2+}\,\langle\tau\rangle^{-\beta\pm}\,d\tau,
\end{aligned}
$$

where

$$\alpha = \frac{s-1}{4} + (1/3+\varepsilon)\,\frac{s-3}{2}$$

and

$$\beta = \frac{s-2}{4} + \frac{1}{4}\,(1/3+\varepsilon)\,\frac{s-3}{2} = \alpha.$$

We want to know that there exists $\varepsilon > (4-s)/3$ such that the two integrals, times $\langle t\rangle^{1/3+\varepsilon}$, remain bounded as $|t| \to \infty$. This happens, provided that δ, δ' are chosen sufficiently small, if

$$\frac{1}{2} + \frac{s-1}{4} + \left(\frac{1}{3} + \frac{4-s}{3}\right)\left(\frac{s-3}{2}\right) < \frac{4}{3} + \frac{4-s}{3}.$$

Some computation reduces this to $-2s^2 + 23s - 59 > 0$, which means

$$(23 - \sqrt{57})/4 < s < (23 + \sqrt{57})/4.$$

The lower bound is our s_0. That finishes the estimate for $M_\infty(\partial_x u)$; it is a pivotal step in that it entails the most stringent limitation on s. ∎

Proof for $M_\infty(D^{1/2}u)$. In the usual manner we estimate

$$
\begin{aligned}
\|D^{1/2}u(t)\|_\infty &\leqslant C\,\|D^{1/2}S(t)\,g\|_\infty\\
&\quad + C\int_0^t \|D^{1/2}S(t-\tau)\,\partial_x F(u(\tau))\|_\infty\,d\tau\\
&\leqslant C'\langle t\rangle^{-1/2}\\
&\quad + C\int_0^t |t-\tau|^{-1/2}\,\|F'(u(\tau))\,\partial_x u(\tau)\|_1\,d\tau.
\end{aligned}
$$

By Hölder's inequality and interpolation,

$$\begin{aligned}
&\|F'(u(\tau))\,\partial_x u(\tau)\|_1 \\
&\quad\leqslant C\,\|u(\tau)\|_4^{s-1}\,\|\partial_x u(\tau)\|_{4/(5-s)} \\
&\quad\leqslant C\,\|u(\tau)\|_4^{s-1}\,\|\partial_x u(\tau)\|_2^{(5-s)/2}\,\|\partial_x u(\tau)\|_\infty^{(s-3)/2} \\
&\quad\leqslant C'\,\|u(\tau)\|_4^{s-1}\,\|\partial_x u(\tau)\|_\infty^{(s-3)/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\langle t\rangle^{1/3+\varepsilon}\,\|D^{1/2}u(t)\|_\infty \\
&\quad\leqslant C' + CM_4(u)^{s-1}\,M_\infty(\partial_x u)^{(s-3)/2} \\
&\qquad\cdot\langle t\rangle^{1/3+\varepsilon}\int_0^t |t-\tau|^{-1/2}\,\langle t\rangle^{-\alpha+}\,d\tau,
\end{aligned}$$

where this time

$$\alpha=\frac{s-1}{4}+\frac{s-3}{2}\left(\frac{1}{3}+\frac{4-s}{3}+\delta'\right).$$

Now $1/3+\varepsilon<1/2$ (provided δ' is sufficiently small) since $\varepsilon=(4-s)/3+\delta'$ and $s>s_0>7/2$, so for the integral to remain bounded as $|t|\to\infty$ we need only have $\alpha+1/2>1+1/3+\varepsilon$. This is exactly the restriction we encountered in estimating $M_\infty(\partial_x u)$. ∎

Proof for $M_4(u)$.

$$\begin{aligned}
\|u(t)\|_4 &\leqslant C'\langle t\rangle^{-1/4+} + C\int_0^t \|D^{1/4}S(t-\tau)\,D^{3/4}F(u(\tau))\|_4\,d\tau \\
&\leqslant C'\langle t\rangle^{-1/4+} + C\int_0^t |t-\tau|^{-1/4}\,\|D^{3/4}F(u(\tau))\|_{4/3}\,d\tau
\end{aligned}$$

by Lemma 2.3. By the chain rule and interpolation,

$$\begin{aligned}
\|D^{3/4}F(u(\tau))\|_{4/3} &\leqslant C\,\|F'(u)\|_{8/5}\,\|D^{3/4}u\|_8 \\
&\leqslant C\,\|u\|_\infty^{s-(7/2)}\,\|u\|_4^{5/2}\cdot\|\partial_x u\|_\infty^{1/2}\,\|D^{1/2}u\|_4^{1/2}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\langle t\rangle^{1/4-}\,\|u(t)\|_4 \\
&\quad\leqslant C' + CM_\infty(u)^{s-(7/2)}\,M_4(u)^{5/2}\,M_4(D^{1/2}u)^{1/2}\,M_\infty(\partial_x u)^{1/2} \\
&\qquad\cdot\langle t\rangle^{1/4-}\int_0^t |t-\tau|^{-1/4}\,\langle t\rangle^{-\alpha+}\,d\tau,
\end{aligned}$$

where

$$\alpha = (\tfrac{1}{3} + \varepsilon)(s - \tfrac{7}{2}) + \tfrac{1}{4}\tfrac{5}{2} + \tfrac{1}{4}\tfrac{1}{2} + (\tfrac{1}{3} + \varepsilon)\tfrac{1}{2}$$

and we need $\alpha + 1/4 > 1 + 1/4$. When $\varepsilon = (4 - s)/3$, this happens if and only if $7/2 < s < 9/2$, by the quadratic formula.

Proof for $M_4(D^{1/2}u)$.

$$\begin{aligned}
&\|D^{1/2}u(t)\|_4 \\
&\quad \leqslant \|D^{1/2}S(t)\,g\|_4 + C\int_0^t \|D^{1/4}S(t-\tau)\,D^{1/4}\,\partial_x F(u(\tau))\|_4\,d\tau \\
&\quad \leqslant C'\langle t\rangle^{-1/4} + C\int_0^t |t-\tau|^{-1/4}\,\|D^{1/4}(F'(u(\tau))\,\partial_x u(\tau))\|_{4/3}\,d\tau.
\end{aligned}$$

Let us gather our strength for the last time to estimate the integrand. By the chain and Leibniz rules,

$$\begin{aligned}
\|D^{1/4}(F'(u)\,\partial_x u)\|_{4/3} \leqslant{} & C\,\|F'(u)\|_p\,\|D^{5/4}u\|_{q_1} \\
& + C\,\|F''(u)\|_{r_1}\,\|D^{1/4}u\|_{r_2}\,\|\partial_x u\|_{q_2},
\end{aligned} \tag{5.7}$$

where $p^{-1} + q_1^{-1} = 3/4 = r_1^{-1} + r_2^{-1} + q_2^{-1}$ and all are in $(1, \infty)$.

To estimate the first term on the right in (5.7) choose $q_1 = 8$ and $p = 8/5$. Then since $s > 7/2$,

$$\begin{aligned}
\|F'(u)\|_{8/5} &\leqslant C\,\|\,|u|^{s-1}\|_{8/5} \\
&= C\left(\int |u|^{(s-1)8/5}\right)^{5/8} \\
&\leqslant C\,\|u\|_4^{5/2}\,\|u\|_\infty^{(s-1)-5/2} \\
&= C\,\|u\|_4^{5/2}\,\|u\|_\infty^{s-7/2}.
\end{aligned}$$

And

$$\|D^{5/4}u\|_8 \leqslant C\,\|\partial_x^2 u\|_2^{1/4}\,\|\partial_x u\|_\infty^{3/4}$$

by interpolation. So for the first term in (5.7) we have the bound

$$C\,\|u\|_4^{5/2}\,\|u\|_\infty^{s-7/2}\,\|\partial_x u\|_\infty^{3/4}\,\|\partial_x^2 u\|_2^{1/4}. \tag{5.8}$$

For the second term in (5.7), take $r_2 = 4$, $r_1 = 4/(s-2)$, and therefore $q_2 = 4/(4-s)$.

$$\begin{aligned}
\|F''(u)\|_{r_1} &\leqslant C\,\|\,|u|^{s-2}\|_{4/(s-2)} \\
&= C\,\|u\|_4^{s-2} \\
\|D^{1/4}u\|_{r_2} &= \|D^{1/4}u\|_4 \\
&\leqslant C\,\|D^{1/2}u\|_4^{1/2}\,\|u\|_4^{1/2},
\end{aligned}$$

and

$$\begin{aligned}\|\partial_x u\|_{q_2} &= \|\partial_x u\|_{4/(4-s)} \\ &\leqslant C\,\|\partial_x u\|_2^{(4-s)/2}\,\|\partial_x u\|_\infty^{(s-2)/2} \\ &\leqslant C'\,\|\partial_x u\|_\infty^{(s-2)/2}.\end{aligned}$$

Combining the last three estimates with (5.8) we obtain a bound for (5.7):

$$\begin{aligned}&C\,\|u\|_4^{5/2}\,\|u\|_\infty^{s-7/2}\,\|\partial_x^2 u\|_2^{1/4}\,\|\partial_x u\|_\infty^{3/4} \\ &\quad + C'\,\|u\|_4^{s-3/2}\,\|D^{1/2}u\|_4^{1/2}\,\|\partial_x u\|_\infty^{(s-2)/2}.\end{aligned}$$

So we have

$$\begin{aligned}&\langle t\rangle^{1/4}\,\|D^{1/2}u(t)\|_4 \\ &\quad\leqslant C' + CM_4(u)^{5/2}\,M_\infty(u)^{s-(7/2)}\,M_2(\partial_x^2 u)^{1/4}\,M_\infty(\partial_x u)^{3/4} \\ &\qquad\cdot\langle t\rangle^{1/4}\int_0^t |t-\tau|^{-1/4}\,\langle\tau\rangle^{-\alpha+}\,d\tau \\ &\qquad + C'M_4(u)^{s-(3/2)}\,M_4(D^{1/2}u)^{1/2}\,M_\infty(\partial_x u)^{(s-2)/2} \\ &\qquad\cdot\langle t\rangle^{1/4}\int_0^t |t-\tau|^{-1/4}\,\langle\tau\rangle^{-\beta+}\,d\tau,\end{aligned}$$

where

$$\alpha = \frac{5}{2}\frac{1}{4} + \left(s-\frac{7}{2}\right)\frac{1}{3} + \frac{3}{4}\left(\frac{1}{3} + \frac{4-s}{3} + \delta'\right)$$

and

$$\beta = \left(s-\frac{3}{2}\right)\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{s-2}{2}\left(\frac{1}{3} + \frac{4-s}{3} + \delta'\right).$$

We need both exponents to be larger than 1. Now

$$\beta > \frac{s}{4} - \frac{1}{4} + \frac{s-2}{6} = \frac{5}{12}s - \frac{7}{12}.$$

This last quantity is >1 if and only if $s > 19/5 = 3.8 < s_0$. And

$$\alpha = \frac{2s-7}{24} + 1,$$

so all is well since $s > s_0 > 7/2$. ∎

Theorem 1(B) follows directly from Proposition 5.1 by the standard argument at the close of Section 4.

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