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Tight closure and elements of small order in integral extensions*

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1. Introduction

Throughout this paper 'ring' means commutative ring with identity and modules are unital. Given rings are usually assumed to be Noetherian, but there are notable exceptions. The phrase 'characteristic p' always means 'positive prime characteristic p', and the letters q, q', etc. denote p^e, $p^{e'}$, etc., where e, $e' \in \mathbb{N}$, the nonnegative integers.

The authors have recently introduced the notion of tight closure for a submodule N of a finitely generated module M over a Noetherian ring R of characteristic p and in certain equicharacteristic zero cases, including affine algebras over fields of characteristic 0. The theory started with the study of the notion of tight closure for an ideal $I \subseteq R$, i.e. with the case $M = R$, $N = I$, and this is still perhaps the most important case. The notion of tight closure has yielded new proofs, and, in many instances, unexpectedly strong improvements, of the local homological conjectures, of the existence of big Cohen–Macaulay modules, of the Cohen–Macaulay property for subrings which are direct summands of regular rings (where A is a direct summand of R means that A is a direct summand of R as an A-module) in the equicharacteristic case (in fact, tight closure techniques give the first proof of this fact in complete generality in the

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case of equal characteristic zero), and of the Briançon–Skoda theorem on integral closures of ideals in regular rings. The method has so far been limited to the equicharacteristic case. It would be of great importance to extend the notion of tight closure to the mixed characteristic case, where most of the theorems described above are conjectures, and, for this reason, it is of importance to understand what tight closure means in characteristic p from as many points of view as possible. An alternative characterization in characteristic p may well yield a definition that can be used in mixed characteristic.

In this paper we shall explore one such alternative characterization of tight closure (see Theorem 3.1), as well as some related ideas which arise in the proof that it *is* a characterization. In this connection, we prove some new results (see Theorems 3.2, 3.4 and 4.1) about when an element u of 'small order' (essentially, this means with respect to a rather arbitrary $\mathbb{Q}$-valued valuation) in a module-finite extension S of a ring R has the property that $Ru \to S$ splits over R. It turns out that if R is an equicharacteristic complete regular local ring (or a characteristic p complete local F-regular Gorenstein ring), then every u of sufficiently small order has this property in every S. This may be thought of as a generalization of the direct summand theorem (still a conjecture in mixed characteristic), which asserts that $R \to S$ splits when R is an equicharacteristic regular ring and S is a module-finite R-algebra extension; this is the case where $u = 1$. Note that 1 has order 0, which is the smallest possible order and is 'small enough'.

We go quite a bit further, and show that several elements of small order in a module-finite extension of a complete regular ring often generate a free direct summand; this happens, for example, when the orders of the elements are distinct. See Theorem 3.4. Moreover, this result generalizes from regular rings to F-regular Gorenstein rings (Theorem 4.1). We also obtain a parallel result for the case of one element for F-rational rings; see Theorem 5.1.

We refer the reader to [17, 18, 29] for expositions on tight closure, to [20] for the main basic theory, and to [19, 21–25] for the further development of that theory. Background on the local homological conjectures may be found in [39, 40], the papers [5–7, 9–15, 43–47], as well as [21, 24]. For background on direct summands of regular rings (and the original inspiration for the problem, which came from invariant theory) see [3, 16, 26, 33]. Concerning the Briançon–Skoda theorem see [4, 35, 36, 48] as well as [20, 24].

2. Tight closures of ideals in characteristic p

If R is a ring we shall denote by R° the set of elements of R not in any minimal prime of R. Thus, if R is a domain, $R^\circ = R - \{0\}$. We recall that if R is a Noetherian ring of characteristic p, then an element $x \in R$ is the tight closure I^* of an ideal I if there exists an element $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. Here, $I^{[q]}$ denotes the ideal $(i^q : i \in I)R$. The main case is where R is reduced, and then $x \in I^*$ iff there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e$. We mention at once two very important properties of tight closure: in a regular ring every ideal is tightly closed, and the tight closure of any ideal is contained in its integral closure (but is often much smaller); see [20, Sections 4 and 5].

We note that if R is reduced and essentially of finite type over an excellent local ring, and d is any element of R° such that R_d is regular, then d has a power c which is a test element: this means that this single element c can be used in all tight closure tests. A priori, the element c may vary with I and x. (In fact, d has a power c which is a completely stable test element: this means that it can be used in all tight closure tests not only in R, but also in any localization of R, and in the completion of any local ring of R as well.) See [20, Sections 6 and 8], [19], and, especially, [23, Section 5] (where the result on test elements quoted above is proved).

When R is reduced we denote by R^∞ the ring $\bigcup_q R^{1/q}$ obtained by adjoining all qth roots to R, where $q = p^e$. It is shown in [20, Section 6] that one can characterize tight closures of ideals I in R by studying elements of small order in R^∞. First note that if $cx^q \in I^{[q]}R$ for all $q \gg 0$, then, taking qth roots, we have that $c^{1/q}x \in IR^\infty$ for all $q \gg 0$. If one puts a notion of order on R^∞, e.g. by means of a valuation with values in the rational numbers $\mathbb{Q}$, one will have $\operatorname{ord} c^{1/q} = (1/q)(\operatorname{ord} c)$ no matter how one chooses the notion of order, and so it will be true in R^∞ that there are elements of arbitrarily small order that multiply x into IR^∞. A converse, Theorem 6.9, is established in [20, Section 6] for the case where R is generically smooth, torsion-free, and module-finite over a regular domain $A \subseteq R$ (this is another case where one knows that R has a completely stable test element).

If S is a completed local ring of R at a maximal ideal, then S is module-finite over the completion B of A at the contraction of that maximal ideal to A, and as in (6.6) of [20] we have a norm $\mathbf{N}: S \to B$ which extends to a norm $\mathbf{N}: S^\infty \to B^\infty$. Let ord be the valuation on B such that $\operatorname{ord} b \geq r$ precisely if $b \in m_B^r$ (note that (B, m_B) is a regular local ring): ord extends uniquely to a $\mathbb{Q}$-valued valuation on B^∞.

Then Theorem 6.9 of [20] asserts that $x \in I^*$ iff for every completed local ring S of R at a maximal ideal there exists a sequence of elements $\{\varepsilon_n\}_n$ in $(S^\infty)^\circ$ such that for all n we have $\varepsilon_n x \in IS^\infty$ and $\lim_{n\to\infty} \operatorname{ord} \mathbf{N}(\varepsilon_n) = 0$. Notice that this condition is a priori very much weaker than the one given by the original definition of tight closure: here $\{\varepsilon_n\}_n$ may be different for every S. While each ε_n must be the qth root of some element of S for some q, the element of S may vary enormously. All that is required is that $\operatorname{ord} \mathbf{N}(\varepsilon_n) \to 0$. Note that with the original definition one simply takes $\varepsilon_n = c^{1/p^n}$ to get such a sequence, it does not vary with S.

One of our objectives is to find an analogue of the characterization of tight closure given above that makes sense in mixed characteristic. Thus, we want to avoid, as much as possible, the peculiarities of positive characteristic. In characteristic zero and in mixed characteristic the ring R^∞ is not available, but one can still talk about the integral closure of R in an algebraic closure of its fraction field in the domain case. We shall denote this ring by the symbol R^+. (Integrally closed domains with an algebraically closed fraction field are studied in [2], where they are called *absolutely integrally closed.*) It is then natural to define an operation on ideals $I \subseteq R$, let us call it the 'dagger' operation and denote it $I^\dagger$, by letting $x \in I^\dagger$ if there are elements ε_n of $R^+ - \{0\}$ of 'arbitrarily small order' such that $\varepsilon_n x \in IR^+$. We have used quotation marks because it is not immediately clear what 'arbitrarily small order' should mean. In the local case one may fix a $\mathbb{Q}$-valued valuation nonnegative on R and positive on the maximal ideal and use this to give a notion of small order. Of course, a priori, each valuation may give a different dagger operation. (By a theorem of Izumi [32] (see also [42] for an exposition of a generalized version), different valuations centered on the maximal ideal of a complete local domain are each bounded by a positive constant times the other; however, the situation when one extends to R^+ is not clear.)

In the next section we make all this precise, and show that for complete local domains of characteristic p the tight closure of an ideal coincides with the dagger closure. This is of some interest for several reasons. First, on the face of it, the dagger closure might be larger. We are now allowing as our multipliers of small order arbitrary elements from R^+, a much larger ring than R^∞ whose relationship to R is significantly harder to understand.

Second, the dagger closure immediately yields a corresponding notion both in equal characteristic 0 and in mixed characteristic. However, it is quite unclear whether this notion has sufficiently many of the good properties of tight closure to make it useful in solving the many open questions in mixed characteristic that yield to tight closure techniques in characteristic p. We discuss this point further, as well as some alternative ideas, in Section 6.

It is worth mentioning here one of the main results of [22]: If R is an excellent local domain of characteristic p, then R^+ is a (balanced) big Cohen–Macaulay algebra for R; every system of parameters for R is a regular sequence in R^+. This gives a new proof of the existence of big Cohen–Macaulay modules in characteristic p, and provides a surprising new insight into the structure of local rings in characteristic p.

3. Tight closure and elements of small order for complete local domains of characteristic p

In good cases (e.g. if R is module-finite and torsion-free over a regular domain or if R is essentially of finite type over an excellent local ring) the issue of whether $x \in R$ is in I^* for a given ideal $I \subseteq R$ is local on the maximal ideals of R and unaffected by completion; this is the case, in fact, whenever R has a completely stable test element (or a completely stable *weak* test element: see [20, Sections 6 and 8]). Therefore, we shall focus in this section on the case where (R, m, K) is a complete local ring of characteristic p. Moreover, although analogues of certain results can be obained when R is equidimensional and reduced, we shall assume for simplicity that R is a domain. To obtain a notion of order we shall fix an arbitrary valuation with values in $\mathbb{Z}$, which we denote ord, which is nonnegative on R and positive on m, and extend it to R^+ so that it takes values in $\mathbb{Q}$; this is always possible. Since R^+ is integral over R, the extension is automatically nonnegative on R^+. Since each element of the unique maximal ideal m^+ of R^+ is nilpotent on mR^+, the extension is automatically positive on m^+. One of our main results is then the following:

Theorem 3.1. *Let (R, m) be a complete local domain of characteristic p, let $x \in R$ and let $I \subseteq R$. Then $x \in I^*$ iff there exists a sequence of elements $\varepsilon_n \in (R^+)^\circ$ such that* ord $\varepsilon_n \to 0$ *as $n \to 0$ and $\varepsilon_n x \in IR^+$ for all n.*

The proof of this result depends heavily on Theorem 3.2 and its corollary Theorem 3.3 below, and is postponed until we have established these results.

Let (A, m, K) be a local Gorenstein ring of dimension d. Before giving the proof of Theorem 3.2 below (and for application in the proofs of Theorems 3.4 and 4.1 as well) we want to discuss a criterion for when a map $g: A^h \to M$ of a finitely generated free module to a finitely generated A-module M has a splitting. Let $x_1, \ldots, x_d$ be a system of parameters for A, let y represent the socle modulo $(\mathbf{x}) = (x_1, \ldots x_d)A$ in $A/(\mathbf{x})$, and let $u_1, \ldots, u_h$ be the images of the standard free basis for A^h in M. A necessary and sufficient condition for the map g to have a splitting is that for all nonzero elements $(\alpha_1, \ldots, \alpha_h) \in K^h$, and for all $t \in \mathbb{N}$,

$$\Big(\sum_{j=1}^{h} \alpha_j u_j\Big)(x_1 \ldots x_d)^t y \not\in (\mathbf{x}^{t+1})M\,,$$

where $(\mathbf{x}^{t+1}) = (x_1^{t+1}, \ldots, x_d^{t+1})A$. This guarantees that the induced map g_t: $(A/(\mathbf{x}^{t+1}))^h \to M/(\mathbf{x}^{t+1})M$ obtained from g by applying $(A/(\mathbf{x}^{t+1})) \otimes_A$ is injective for each t, because the socle in $A/(\mathbf{x}^{t+1})$ is generated by the image of $(x_1 \cdots x_d)^t y$, and so this condition implies that a typical nonzero socle element does not map to zero. Since each map g_t is injective, taking the direct limit we find that the map $g \otimes_A \mathrm{id}_E : E^h \to M \otimes_A E$ is injective, where $E = \varinjlim_t A/(\mathbf{x}^t)$ is an injective hull for K over A. Applying $\mathrm{Hom}_A(\ , E)$ yields that the induced map from $\mathrm{Hom}_A(M \otimes_A E, E) \cong M \otimes_A \mathrm{Hom}_A(E, E) \cong M \otimes_A \hat{A}$ to $\mathrm{Hom}_A(A^h \otimes_A E, E) \cong A^h \otimes_A \hat{A}$ is surjective, which implies the existence of a splitting after completion. But this implies the existence of a splitting over A: cf. [9, Lemma 1].

Theorem 3.2. *Let (A, m) be a complete regular local ring of equal characteristic and let* ord *be a valuation from A^+ to $\mathbb{Q}$ which is nonnegative on A and positive on m. Let $x_1, \ldots, x_d$ be minimal generators of m and let $\nu = \min_j$ ord x_j. Let $u \in A^+$ be any element such that* ord $u < \nu$. *Then the map $Au \to A^+$ splits over A.*

Proof. It suffices to show that if $A[u] \subseteq R \subseteq A^+$ with R module-finite over A, then $A \to R$ splits, for then $A \to A^+$ is pure and splits by a result of M. Auslander (cf. [20, Section 6, the second paragraph of the proof of Corollary 6.24]) since A is complete. Let $I_t = (x_1^{t+1}, \ldots, x_d^{t+1})A$.

When A is regular, to show that $Au \to M$ splits for a finitely generated A-module M, it suffices to show that $(x_1 \cdots x_d)^t u \not\in I_t M$ for all $t \in \mathbb{N}$. Thus, it will suffice to show that $u \not\in I_t R :_R (x_1 \cdots x_d)^t$ for all t. By part (a) of Theorem 7.15 of [20] for the characteristic p case and by the results of [25] for the equicharacteristic zero case (see also the comments in the paragraph immediately following this proof), the colon ideal on the right-hand side is contained in $((x_1, \ldots, x_d)R)^*$, and hence we are done unless u is in the tight closure of $(x_1, \ldots, x_d)R$. But the condition on the valuation shows that u is not even in the integral closure of this ideal, a contradiction. □

The proof of Theorem 3.2 does not use the full strength of the results on operations on ideals generated by monomials in a system of parameters obtained in [20]; the only result we really need is that

$$(x_1^{t+1}, \ldots, x_d^{t+1})R :_R (x_1 \cdots x_d)^t R$$

is contained in the integral closure of the ideal $(x_1, \ldots, x_d)R$ for all t. However, so far as we know, the proof of this result in equal characteristic 0, even when stated only for integral closure, requires tight closure techniques. Following [28], one can use Artin approximation to reduce to studying the local ring of an affine algebra at a maximal ideal, and then pass to characteristic p. The fact that the colon is contained in the *tight* closure in equal characteristic zero cases where tight closure is defined and, more generally, in the *regular* closure (see Section 6) can also be proved using Artin approximation [1], but the argument is not straightforward, one must find precisely the right statement to which to apply Artin approximation. Details will appear in forthcoming manuscripts, still in preparation, on tight closure in characteristic zero, beginning with [25].

We note that there are related results on controlling colons of parameter ideals and Koszul homology of parameters using integral closure: see [28, 30, 31, 41]. However, without ideas related to tight closure it has so far not been possible to control

$$(x_1^{t+1}, \ldots, x_d^{t+1})R :_R (x_1 \cdots x_d)^t R .$$

From Theorem 3.2 we can deduce the following:

Theorem 3.3. *Let (R, m, K) be a complete local domain. Let* ord *be a $\mathbb{Q}$-valued valuation on R^+ nonnegative on R (and, hence, on R^+) and positive on m (and, hence, on m^+). Then there exist a fixed real number $\nu > 0$ and a fixed postive* *integer r such that for every element u of R^+ of order $<\nu$ there is an R-linear map $\phi: R^+ \to R$ such that $\phi(u) \not\in m^r$.*

Proof. Fix a complete regular local ring (A, Q, K) with $A \subseteq R$ such that R is module-finite over A. Let $Q = (x_1, \ldots, x_d)A$ and let $\nu = \min_j \operatorname{ord} x_j$. Let $\omega = \operatorname{Hom}_A(R, A)$, which is a torsion-free rank one R-module, and choose an isomorphism $\omega \cong J$ with an ideal J of R. By the Artin–Rees lemma we can choose r so large that $m^r \cap J \subseteq QJ$ (m and QR have the radical, and so $m^t J \subseteq QJ$ for $t \gg 0$). We shall show that these choices of ν and r will give the conclusion of the theorem.

Note that we may identify R^+ with A^+. By Theorem 3.2, we can fix an A-linear map $\theta: R^+ \to A$ such that $\theta(u) = 1$. This gives an R-linear map $\psi: R^+ \to \operatorname{Hom}_A(R, A)$ by letting $\psi(s)$ be the functional λ_s defined by $\lambda_s(r) = \theta(rs)$. Now compose ψ with $\operatorname{Hom}_A(R, A) \cong J \subseteq R$ to get ϕ. Note that the value of $\psi(u)$ on $1 \in R$ is $\theta(1 \cdot u) = \theta(u) = 1$. It follows that $\psi(u) \not\in Q \operatorname{Hom}_A(R, A)$, and so $\phi(u) \not\in QJ$. Since $m^r \cap J \subseteq QJ$, it follows that $\phi(u) \not\in m^r$. □

Proof of Theorem 3.1. Choose $\nu > 0$ and r as in Theorem 3.3. Fix $q = p^e > 0$. Choose n so large that $\operatorname{ord} \varepsilon_n < \nu/q$. Let $\varepsilon = \varepsilon_n^q$. Then $\varepsilon u^q \in I^{[q]}R^+$ and $\operatorname{ord} \varepsilon < \nu$. Applying an R-linear map ϕ as in Theorem 3.3 we find that $c_q u^q \in I^{[q]}$, with $c_q = \phi(\varepsilon) \in R - m^r$. Thus, $c_q u^q \in (I^{[q]})^*$ for all q, and so $c_q \in I_q = (I^{[q]})^* :_R u^q R$ for all q.

The sequence I_q is nonincreasing, for if $yu^{pq} \in (I^{[pq]})^*$ then $c'(yu^{pq})^{q'} \in (I^{[pq]})^{[q']} = I^{[qpq']}$ for all $q' \gg 0$ and some $c' \neq 0$. But then $c'(yu^q)^{pq'} \in (I^{[q]})^{[pq']}$ for all $q' \gg 0$, and so $yu^q \in (I^{[q]})^*$, as required.

Since the sequence $\{I_q\}_q$ is nonincreasing, if it had intersection (0), Chevalley's theorem would force $I_q \subseteq m^r$ for large q. Since $c_q \in I_q - m^r$ for all q, we can choose a nonzero element d in $\bigcap_q I_q$. But then $du^q \in (I^{[q]})^*$ for all q. If c is a test element for R, we then have $cdu^q \in I^{[q]}$ for all q, which proves that $u \in I^*$. □

We conclude this section with a generalization of Theorem 3.2 which shows that when (A, m, K) is an equicharacteristic complete regular local ring the ring A^+ splits off many copies of A.

Theorem 3.4. *Let (A, m, K) be a complete equicharacteristic regular local ring with regular system of parameters $x_1, \ldots, x_d$ and let* ord *be a valuation from A^+ to $\mathbb{Q}$ that is nonnegative on A and positive on m. Let $\nu = \min_j \operatorname{ord} x_j$. Choose a coefficient field $K \subseteq A$.*

Let $u_1, \ldots, u_h$ be elements of A^+ linearly independent over K such that in the K-vector space $V = \sum_i Ku_i$ every nonzero element has order less than ν. (This holds, in particular, if the u_i have distinct *orders all less than ν.) Then the A-module $G = \sum_{i=1}^h Au_h$ is free and $G \to A^+$ splits over A.*

Proof. Let $R \subseteq A^+$ be module-finite over A and contain the u_i. It suffices to show that the map ϕ of A^h to R sending $(a_1, \ldots, a_d)$ to $\sum_i a_i u_i$ splits for each such R, for then the map $A^h \twoheadrightarrow G \subseteq A^+$ is pure, and consequently, splits, since A is complete. By the discussion preceding the proof of Theorem 3.2, this is equivalent to showing that one cannot choose a nonzero element $(\alpha_1, \ldots, \alpha_h) \in K^h$ such that $\sum_{i=1}^h \alpha_i (x_1 \cdots x_d)^t u_i$ is in $I_t R$, i.e. such that

$$\sum_{i=1}^{h} \alpha_i u_i \in I_t R :_R (x_1 \cdots x_d)^t R ,$$

which, as in the proof of Theorem 3.2, is contained in the integral closure of $(x_1, \ldots, x_d)R$. But the order of $\sum_{i=1}^h \alpha_i u_i \in V - \{0\}$ is, by hypothesis, too small for it to be in the integral closure of $(x_1, \ldots, x_d)R$. □

Remark 3.5. The argument also shows that if $u_1, \ldots, u_h \in R$, a module-finite extension of the regular local ring A, then the u_i are a free basis for an A-module that is a direct summand of R over A provided that the u_i are linearly independent over K and the K-vector space $\sum_{i=1}^h K u_i$ meets the tight closure (or the integral closure, which is larger) of $(x_1, \ldots, x_d)R$ only in 0. Provided that we know that A has a coefficient field we do not need to assume that A is complete when the result is stated in this form.

In the next section we shall show that Theorems 3.2 and 3.4 hold when A is an F-regular Gorenstein complete local domain of characteristic p.

4. Splitting theorems over F-regular Gorenstein rings

Recall that a Noetherian ring of characteristic p is called *weakly F-regular* if every ideal is tightly closed (this implies that every submodule of every finitely generated module is tightly closed, see [20, Section 8]), and *F-regular* if all of its localizations are weakly F-regular. We have not been able to show that the two notions coincide in general, although we believe that this should be true under mild conditions on the ring. However, the two notions do coincide when the ring is Gorenstein, see [23, Section 4]. A weakly F-regular ring is normal, and is Cohen-Macaulay if the ring is a homomorphic image of a Cohen–Macaulay ring.

We mention that it is shown in [24, Section 5] that a weakly F-regular ring of charateristic p is a direct summand of every module-finite extension, and in [24, Section 6] that this characterizes weakly F-regular rings of characteristic p in the Gorenstein case, provided that the ring is locally excellent. We do not know whether the property of being a direct summand of every module-finite extension (also studied in [37]) is sufficient to characterize weak F-regularity in general in the excellent case (without the Gorenstein hypothesis), but it is shown in [22] that the property of being a direct summand of every module-finite extension ring is sufficient to guarantee that a ring of characteristic p is Cohen–Macaulay. This is very different from the situation in equal characteristic 0, where every normal ring is a direct summand of every module-finite extension.

In [19] (see also [23, Section 5]) we introduced the notion of a strongly F-regular ring: R is called *strongly F-regular* if it is reduced, $R^{1/p}$ is module-finite over R, and for every element $d \in R^{\circ}$, the map $Rd^{1/q} \subseteq R^{1/q}$ splits over R for all sufficiently large q. It turns out that strong F-regularity is equivalent to F-regularity (and to weak F-regularity) when the ring is Gorenstein and $R^{1/p}$ is module-finite over R. This point of view for F-regularity is evidently closely related to the splitting results we obtain here. We do not know whether a weakly F-regular ring R such that $R^{1/p}$ is module-finite over R must be strongly F-regular.

Our main objective in this section is to extend Theorems 3.2 and 3.4 to the case where the complete local ring A is assumed only to be weakly F-regular Gorenstein (instead of regular). The discussion is limited to characteristic p. The critical tool is Theorem 3.1.

Theorem 4.1. *Let (A, m, K) be a complete, local, weakly F-regular Gorenstein ring of characteristic p, and suppose that a coefficient field $K \subseteq A$ has been fixed. Let A^+ denote the integral closure of A in an algebraic closure of its fraction field and let* ord *denote a $\mathbb{Q}$-valued valuation on A^+ nonnegative on A and positive on m. Then there exists a positive real number ν such that if $u_1, \ldots, u_h \in A^+$ are linearly independent over K and every nonzero element of $V = \sum_{i=1}^{h} Ku_i$ has order less than ν (which is true if the u_i have distinct orders less then ν), then $G = \sum_{i=1}^{h} Au_i$ has the u_i as a free basis and $G \subseteq A^+$ splits over A. In particular, if $u \in A^+$ and* ord $u < \nu$, *then Au is a direct summand of A^+.*

Proof. Let $x_1, \ldots, x_d$ denote a system of parameters for A and let $y \in A$ be an element whose image generates the socle in $A/(x_1, \ldots, x_d)$. Let $I_t = (x_1^{t+1}, \ldots, x_d^{t+1})A$. Then the image of $(x_1 \cdots x_d)^t y$ in A/I_t generates the socle there. Since $(x_1, \ldots, x_d)A$ is tightly closed and y is not in the ideal, we can choose $\nu > 0$ so that if $c \in A^+$ and $\operatorname{ord} c < \nu$, then $cy \not\in (x_1, \ldots, x_d)A^+$; if we could not choose such a ν, Theorem 3.1 would imply that y is in the tight closure of $(x_1, \ldots, x_d)A$.

We can now follow the lines of the argument for Theorem 3.4. We must check that if $A \subseteq R \subseteq A^+$ with R module-finite over A, then the induced map $(A/I_t)^h \rightarrow R/I_tR$ is injective, which is equivalent to showing that if $v = \sum_{i=1}^{h} \alpha_i u_i$ is a nonzero element of V, then $(x_1 \cdots x_d)^t yv$ does not map to 0 in R/I_tR, i.e. we must check that

$$vy \not\in I_tR :_R (x_1 \cdots x_d)^t \subseteq (x_1, \ldots, x_d)^* .$$

But if $vy \in (x_1, \ldots, x_d)^*$, then there are elements w of arbitrarily small order in

$R^+ = A^+$ multiplying vy into $(x)R^+$, and we can choose w such that $\operatorname{ord}(vw) < \nu$. This contradicts our choice of ν. $\square$

We do now know whether corresponding results hold for complete weakly F-regular rings without the Gorenstein hypothesis. There is a parallel for Theorem 4.1 for the case where $h = 1$ (or a parallel for Theorem 3.2) if one limits attention to ideals generated by parameters. One is then led to consider F-rational rings rather than weakly F-regular rings. This is pursued in Section 5.

In the remark following the proof of Theorem 3.2 it is indicated that one is only using that a certain colon is always in the *integral* closure of $(x_1, \ldots, x_d)$, although it really is in the *tight* closure (but the proof that it is in the integral closure seems to require tight closure techniques). No such remark applies to the proof of Theorem 4.1: however, tight closure ideas appear to be even more innately necessary for the proof.

5. The F-rational case

A local ring R of characteristic p is called *F-rational* if one ideal generated by a system of parameters is tightly closed, in which case every ideal generated by part of any system of parameters is tightly closed. If R is F-rational, then it is normal, and if it is a homomorphic image of a Cohen–Macaulay ring, then it must be Cohen–Macaulay. Henceforth, we shall consider only F-rational Cohen–Macaulay rings. The property passes to all local rings of R. If R is not local, we define it to be F-rational if all its local rings are. We refer the reader to [23, Section 4] and to [8] for details.

In the Gorenstein case, F-rational and F-regular coincide. However, there are examples of F-rational rings which are not F-pure (cf. [27]) and, hence, not F-regular, see [24, Section 4]. These examples were constructed by Watanabe [49] for a different purpose. Until quite recently we did not know whether F-rationality together with F-purity implies F-regularity. We are indebted to K.-i. Watanabe for the following counterexample, which uses a result of [50].

Let K be a field of characteristic p, with $p \equiv 1 \bmod 3$, and suppose that $\omega \in K$ is a primitive cube root of unity. Let

$$S = K[X, Y, Z]/(X^3 - YZ(Y+Z)) = K[x, y, z].$$

Let $G = \{1, \omega, \omega^2\}$ act on S K-linearly so as to send the elements x, y, z to x, ωy, ωz respectively. Let R be the fixed ring S^G, which is generated over K by x, y^3, y^2z, z^3 (note that $yz^2 = x^3 - y^2z$). Because $R \to S$ splits over R and S is known to be F-pure, R is F-pure. Moreover, R is F-rational. However, S is not F-regular, and, consequently, neither is R, by [50, Theorem (2.6)].

For isolated singularities and in certain other cases in equal characteristic 0, F-rationality implies rational singularities, and it is possible that the characteristic 0 notion of F-rationality coincides with rational singularity, this is the reason for the name.

The main result of this section is the following:

Theorem 5.1. *Let (A, m, K) be a complete local F-rational ring of characteristic p and let* ord *be a $\mathbb{Q}$-valued valuation on A^+ nonnegative on A and positive on m. Then there exists $\nu > 0$ such that if u is any element of A^+ with* ord $u < \nu$, *then for every ideal I of A generated by part of a system of parameters, $IA^+ \cap Au = Iu$.*

The proof is postponed until we have established Lemma 5.2 below. Note that when A is Gorenstein, the contractedness of all ideals I generated by parameters implies that $Au \subseteq A^+$ is pure and, hence, since A is complete, splits. Thus, Theorem 5.1 is a generalization of Theorem 3.2 (and Theorem 4.1 for the case $h = 1$). We first prove the following:

Lemma 5.2. *Let V be a finite-dimensional vector space over a field K Let $\eta: V - \{0\} \to (0, \infty)$ be a function from the nonzero elements of V to the positive real numbers satisfying*

(1) *if $v \in V - \{0\}$ and $\lambda \in K - \{0\}$, then $\eta(\lambda v) = \eta(v)$,*

(2) *if $v, w, v + w \in V - \{0\}$, then $\eta(v + w) \leq \eta(v) + \eta(w)$.*

Then there exists $\nu > 0$ such that for all $v \in V - \{0\}$, $\eta(v) > \nu$.

Proof. We use induction on $\dim V = h + 1$. If $h = 0$, the result follows from (1). If $h > 0$, pick $v \neq 0$ in V and let W be the subspace of V spanned by $S = \{w \in V - \{0\}: \eta(w) < \eta(v)/h\}$. Then $v \notin W$, or else $v = \sum_{j=1}^{k} \lambda_j w_j$ with $k \leq h$, independent $w_1, \ldots, w_k \in S$, and every $\lambda_j \in K - \{0\}$. Repeated application of (1) and (2) then yields $\eta(v) < (k/h)\eta(v) \leq \eta(v)$, a contradiction. Since $\dim W < \dim V$ the induction hypothesis implies η is bounded away from 0 on $W - \{0\}$, and hence on $V - \{0\}$. □

Proof of Theorem 5.1. Fix one system of parameters $x_1, \ldots, x_d$ for A and let V denote the socle in $A/(x_1, \ldots, x_d)A$. We define a function from $V - \{0\}$ to the positive reals as follows: for each $v \in V$, if $a \in A$ represents v, let $\eta(v)$ denote $\inf\{\text{ord}\, u: u \in A^+ - \{0\} \text{ and } ua \in (x_1, \ldots, x_d)A^+\}$. The set of u such that $ua \in (\mathbf{x})A^+$, is independent of the choice of representative a, so that η is a function of v. Its value is strictly positive because $(\mathbf{x})A$ is tightly closed, and we may apply Theorem 3.1. It is clear that $\eta(\lambda v) = \eta(v)$ if $\lambda \in K - \{0\}$. Moreover, if $ua, u'a' \in (\mathbf{x})A^+$, then $(uu')(a + a') \in (\mathbf{x})A^+$, from which it is easy to see that $\eta(v + v') \leq \eta(v) + \eta(v')$ when $v, v', v + v' \in V - \{0\}$. We may therefore apply Lemma 5.2 to conclude that there exists $v > 0$ such that if $u \in A^+$, and $a \in (\mathbf{x})A :_A m$ (i.e. a represents an element of V modulo $(\mathbf{x})A$) then if ord $u < \nu$, $ua \notin (\mathbf{x})A^+$. We shall show that any ν with this property satisfies the conclusion of the theorem.

We first consider the case where $I = (x_1^{t+1}, \ldots, x_d^{t+1})A$. The socle in A/I consists of all elements represented by $(x_1 \cdots x_d)^t a$, where a represents an element of the socle in $A/(\mathbf{x})A$. If the result were false, we could choose $a \in A$ representing an element of V and $u \in A^+$ with $\operatorname{ord} u < \nu$ such that $(x_1 \cdots x_d)^t au \in IA^+$, i.e. such that $au \in IA^+ : (x_1 \cdots x_d)^t$. This will also hold when A^+ is replaced by a suitable subring R module-finite over A containing u, and it will then follow as before that $au \in ((\mathbf{x})R)^*$. But then there are elements w of arbitrarily small order in $R^+ = A^+$ multiplying au into $(\mathbf{x})A^+$, and we can choose w such that $\operatorname{ord}(uw) < \nu$. This contradicts our choice of ν.

Since $(\mathbf{x}^t)Au$ is contracted from A^+ for all t, the induced map of local cohomology $H^d_{(\mathbf{x})A}(Au) \to H^d_{(\mathbf{x})A}(A^+)$ is injective, and this is the same as $H^d_m(Au) \to H^d_m(A^+)$. It follows that for any system of parameters $y_1, \ldots, y_d$, $H^d_{(\mathbf{y})A}(Au) \to H^d_{(\mathbf{y})A}(A^+)$ is injective, which implies that $(\mathbf{y})A^+ \cap Au = (\mathbf{y})u$.

This establishes the result for any ideal generated by a full system of parameters. But if $I = (y_1, \ldots, y_s)A$ where $y_1, \ldots, y_s, y_{s+1}, \ldots, y_d$ is a system of parameters we have that for all t, if $J_t = (y_1, \ldots, y_s, y_{s+1}^t, \ldots, y_d^t)$ then $IA^+ \cap Au \subseteq J_t A^+ \cap Au \subseteq J_t u$, and the result now follows from the fact that $\bigcap_t J_t = I$. $\square$

6. What is tight closure?

We have seen here that tight closure in characteristic p can be defined using the 'multipliers of small order' idea. The definition the authors use in [25] in the case of affine algebras over a field of characteristic 0 is quite different, it uses reduction to characteristic p. The advantage of this rather complicated technique is that we can prove that the resulting closure operation has good properties comparable to the original characteristic p operation; for example (and this is a crucial example), if $x_1, \ldots, x_i$ are locally parameters in an affine domain R, one has

$$(x_1, \ldots, x_{i-1})R :_R x_i R \subseteq ((x_1, \ldots, x_{i-1})R)^* .$$

While the dagger closure arising from a valuation is immediately well defined for complete local domains in both equal characteristic zero and in mixed characteristic, we do not know that it has this property of 'capturing the colon'. Indeed, it may be far 'easier' for an ideal to be dagger closed in characteristic zero and mixed characteristic, so that it is possible that this kind of operation only gives useful information in characteristic p. We feel that it is important to raise (and answer) this question.

There are many other possible approaches to tight closure in characteristic p that might generalize to mixed characteristic. In good cases (when there is a test element) the tight closure of an ideal I in a local ring (R, m, K) is the intersection of the tight closures of the m-primary ideals containing I. This spotlights the problem of characterizing the tight closure of an m-primary ideal in characteristic

p. One can get such a characterization by studying the asymptotic behavior of $l(R/I^{[q]})$, where l denotes length; this is the *Hilbert–Kunz function* (see [34, 38]). This function is known to have the form $\gamma_I q^d + O(q^{d-1})$ where γ_I is a positive real constant and $d = \dim R$. It turns out [20, Theorem 8.17] that if R is analytically unramified, formally equidimensional, and has a test element (e.g. if R is complete, reduced, and equidimensional), then the tight closure of an m-primary ideal I is the largest ideal J containing I such that $\gamma_J = \gamma_I$. Said otherwise, if $I \subseteq J \subseteq m$, then $J \subseteq I^*$ iff $\lim_{q\to\infty} l(J^{[q]}/I^{[q]})/q^d = 0$. This raises the question of whether a useful notion of tight closure in mixed characteristic can be obtained by studying $l(R/I_q)$ for some sequence of ideals I_q canonically associated with I (these ideals might conceivably be defined in a sophisticated manner placing them somewhere between analogues of ordinary powers and analogues of Frobenius powers of I).

Finally, there is the possibility of defining notions corresponding to tight closure by using maps to various classes of rings. It is worth noting that in characteristic p, if $R \subseteq S$ is module-finite and I is any ideal of R, then $IS \cap R \subseteq I^*$ (cf. [24, Section 5]). It is possible that in characteristic p one can characterize the notion of tight closure utilizing this idea. We have taken some small steps in this direction in [24, Section 6], which are dependent on characterizing when elements are in the tight closure of an ideal using finitely many equations that must be satisfied. The problem is that the results we have at the moment are limited to the case where the ideal is generated by a regular sequence consisting of test elements.

It should be emphasized, however, that it is certainly impossible to obtain a useful notion of tight closure in equal characteristic zero by looking at contractions from module-finite extensions in a naive way, because in a normal ring containing $\mathbb{Q}$ every ideal is contracted from every module-finite extension.

Finally, we note that both in characteristic p and for affine algebras over a field of characteristic 0, if R is a domain, then $I^* \subseteq I^{\mathrm{reg}}$, where I^{reg}, the *regular closure* of I, is the set of elements $x \in R$ such that $x \in IS$ for every injective map from R to a regular ring S (see [21, Section 5]). In general, I^* is strictly contained in I^{reg}. The interesting facts that we have been able to prove about I^{reg} are consequences of tight closure theory. Nonetheless, there is some possibility that a variant of regular closure will yield a useful notion of tight closure in mixed characteristic. It would be important to answer, for example, the following question: if $x_1, \ldots, x_{i+1}$ is part of a system of parameters in a complete local domain R of mixed characteristic, is

$$(x_1, \ldots, x_i) : x_{i+1}R \subseteq (x_1, \ldots, x_i)^{\mathrm{reg}}\ ?$$

This is true in the equicharacteristic case using tight closure techniques (and Artin approximation in equal characteristic 0).

The theory of tight closure has produced some startlingly strong and unexpected results, while simultaneously generating a seemingly endless progression of difficult problems.

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