

PARTICLE MODELS AND NONCOMMUTATIVE GEOMETRY

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We write three particle models in terms of noncommutative gauge theory: the Glashow-Weinberg-Salam model, the Peccei-Quinn model and the standard model.

I. Introduction.

While quantum field theoretic models of pure Yang-Mills type are appealing both physically and geometrically (as the Yang-Mills action has a clear geometric significance) they are unphysical since they only give rise to massless vector bosons. In order to circumvent this problem one adds new fields, the Higgs fields, with a symmetry-breaking mechanism which provides masses for both the vector bosons and the fermions. Our aim in this paper is to show that at the expense of modifying the usual notions of a geometric spacetime, one can recover several models of particle physics, all involving Higgs fields, as pure Yang-Mills models.

It gradually emerges from problems in pure mathematics [Co1] that the class of Riemannian metric spaces is too narrow to encompass some interesting spaces, and that to do so one must reformulate notions of geometry in operator algebraic terms. The basic objects of such a geometry are a (possibly noncommutative) algebra \mathcal{A} , a Hilbert space on which the algebra acts and an operator on the Hilbert space. The algebra can be considered to be generalizing the idea of a manifold, while the operator provides the metric structure. (One finds usual Riemannian geometry as a special case when a manifold M is replaced by its commutative algebra \mathcal{A} of functions, the Hilbert space is that of spinors on M and the Riemannian metric is

replaced by the Dirac operator.) It is then possible to write an action functional on operator-theoretically defined gauge potentials. In the special case of an ordinary Riemannian manifold, the action functional reproduces the standard pure Yang-Mills action.

One can consider doing gauge theory on many types of spaces. (For example, instead of thinking of lattice gauge theory as a simulation of the continuum theory, one can imagine doing a real gauge theory on the lattice.) While the original mathematical motivation was to encompass new situations in which the algebra \mathcal{A} is no longer taken to be commutative, it has wider scope even when the algebra is commutative i.e. when the algebra arises from an ordinary point set. This is because the generalized metric can be taken to be different from a usual metric, by using an operator different from the usual Dirac operator. As was shown in [Co2], a gauge field on an appropriate generalized Riemannian space consists of an ordinary gauge field and a Higgs field. The pure Yang-Mills action on the generalized space will decompose as the sum of the Yang-Mills action for the ordinary gauge field, the kinetic action for the Higgs field and a symmetry-breaking Higgs potential.

The underlying fields in noncommutative gauge theory are the spinor fields. The bosonic fields arise from representing the differential forms, which depend only on

the algebra, on the Hilbert space. Thus to build our particle models we enter the phenomenological data of the fermion representations and the desired symmetry breaking. The fermion representations will determine which algebra we should use, and the vector bundles associated to the algebra. The desired symmetry breaking will determine how the generalized metric will differ from an ordinary metric. As the basic fields are fermionic, the fermionic action can be written in a straightforward way.

Our purpose is to recover models of particle physics by suitable choices of our generalized Riemannian metric space. This paper contains the results of calculations for specific models, along with their geometric interpretation. The outline is as follows. In section II we discuss noncommutative geometry and noncommutative gauge theory. In section III we give the Glashow-Weinberg-Salam (GWS) model for leptons. Our treatment is basically the same as for model II of [Co2], but we show that a defect of that model is corrected by the presence of several generations of leptons with different masses. The corresponding geometric space can be thought of as a spacetime of Kaluza-Klein type, where the fiber is formed of a two-point set. In section IV we add quarks to give the Peccei-Quinn model. The algebras of sections III and IV are both commutative. In section V we show that a quaternionic algebra yields the standard model. Section VI has a discussion of the results.

We wish to emphasize that in this paper we only work at the classical level. Despite the appearance of Dirac operators, which sometimes have a quantum connotation, they are only used to define classical geometries.

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II. Noncommutative Gauge Theory.

We will give a self-contained summary of what we need from noncommutative geometry. At the end of the section we give two point-by-point examples. The first is that of ordinary commutative geometry. The reader may want to follow this example concurrently with the definitions. The second is that of a two-point space. For more motivation and details, we refer to [Co1,Co2] and references therein.

(1) A noncommutative space is given by a $*$ algebra \mathcal{Q} with unit.

Because \mathcal{Q} can be noncommutative, we will have to be careful in distinguishing between left and right actions. Given \mathcal{Q} , the algebra $M_n(\mathcal{Q})$ of $n \times n$ matrices over \mathcal{Q} forms another noncommutative space.

(2) If \mathcal{Q} is a complex commutative algebra, define the character space C , or spectrum, of \mathcal{Q} to be the space of algebra homomorphisms from \mathcal{Q} to \mathbb{C} .

A right (left) module over \mathcal{Q} is a complex vector space on which \mathcal{Q} acts on the right (or left). The tensor product $\mathcal{E}' \otimes_{\mathcal{Q}} \mathcal{E}$ of a left module \mathcal{E}' and a right module \mathcal{E} is generated by

$\{\xi' \otimes \xi: \xi' \in \mathcal{E}', \xi \in \mathcal{E}\}$, with the relation

$\xi'a \otimes \xi - \xi' \otimes a\xi = 0$. If \mathcal{E} and \mathcal{F} are right \mathcal{Q} -modules then we will write $\text{End}_{\mathcal{Q}}(\mathcal{E}, \mathcal{F})$ for

$\{u: \mathcal{E} \rightarrow \mathcal{F}, u \text{ linear}, u(\xi a) = u(\xi)a \text{ for all } \xi \in \mathcal{E}, a \in \mathcal{Q}\}$ and we will write $\text{End}_{\mathcal{Q}}(\mathcal{E})$ for $\text{End}_{\mathcal{Q}}(\mathcal{E}, \mathcal{E})$.

(3) A vector bundle \mathcal{E} associated to \mathcal{Q} is a finite projective right module over \mathcal{Q} . This means that \mathcal{E} is the image $e\mathcal{Q}^n$ of \mathcal{Q}^n under some orthogonal projection

$e \in M_n(\mathcal{Q})$.

The dual space $\mathcal{E}^* = \text{End}_{\mathcal{Q}}(\mathcal{E}, \mathcal{Q})$ is a left module over \mathcal{Q} with the \mathcal{Q} action given by $(a\eta)(\xi) = a\eta(\xi)$ for all

$a \in \mathcal{Q}, \xi \in \mathcal{E}, \eta \in \mathcal{E}^*$. There is a pairing

(.) : $\mathcal{E}^* \times \mathcal{E} \rightarrow \mathcal{A}$ given by $(\eta, \xi) = \eta(\xi)$.

(4) The space of universal k -forms $\Omega^k(\mathcal{A})$ is the bimodule generated by elements $a_0 da_1 \dots da_k$,

with $a_0, a_1, \dots, a_k \in \mathcal{A}$, and the relations

$d(ab) - (da)b - a(db) = 0$ and $d(1) = 0$. The space $\Omega^*(\mathcal{A}) = \bigoplus \Omega^k(\mathcal{A})$ of all differential forms is a graded differential algebra, with the differential given by

$d(a_0 da_1 \dots da_k) = da_0 da_1 \dots da_k$. There is an involution on $\Omega^*(\mathcal{A})$ given by

$$(a_0 da_1 \dots da_k)^* = da_k^* \dots da_1^* a_0^*.$$

(5) A hermitian metric on \mathcal{E} is a map

$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ such that

- $\langle \xi_1 a_1, \xi_2 a_2 \rangle = a_1^* \langle \xi_1, \xi_2 \rangle a_2$ for all $\xi_1, \xi_2 \in \mathcal{E}$, $a_1, a_2 \in \mathcal{A}$

- $\langle \xi_1, \xi_2 \rangle^* = \langle \xi_2, \xi_1 \rangle$ for all $\xi_1, \xi_2 \in \mathcal{E}$

- $\langle \xi, \xi \rangle$ is a positive element of \mathcal{A} for all $\xi \in \mathcal{E}$ and $\langle \xi, \xi \rangle = 0$ iff $\xi = 0$.

(6) The group of unitary gauge transformations of \mathcal{E} is $\mathcal{U} = \{u \in \text{End}_{\mathcal{A}}(\mathcal{E}) : u^*u = uu^* = 1\}$.

(7) A hermitian connection on \mathcal{E} is a linear map

$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that

- $\nabla(\xi a) = (\nabla \xi)a + \xi \otimes da$ for all $\xi \in \mathcal{E}$, $a \in \mathcal{A}$

- $d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle$ for all $\xi, \eta \in \mathcal{E}$.

We can extend ∇ to a differentiation of \mathcal{E} -valued forms, $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^*(\mathcal{A}) \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^*(\mathcal{A})$, by requiring

- $\nabla(\xi \omega) = (\nabla \xi)\omega + \xi \otimes d\omega$ for all $\xi \in \mathcal{E}$, $\omega \in \Omega^*(\mathcal{A})$.

If we write \mathcal{E} in the form $\mathcal{E} = e\mathcal{A}^n$ and ρ is an $n \times n$ matrix of 1-forms, $\rho \in M_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, such that

$e\rho = \rho$, $\rho^* = -\rho$, then we obtain a Hermitian connection by putting $\nabla \xi = e d\xi + \rho \xi$, where $\xi \in \mathcal{A}^n$ and $e\xi = \xi$. Conversely, any Hermitian connection can be written in this form for some ρ . We will define the curvature of ∇ to be $\theta = \nabla^2 \in \text{End}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega^*(\mathcal{A}))$. Acting on \mathcal{E} alone, θ is an element of $\text{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}))$.

There is a dual connection $\nabla^* : \mathcal{E}^* \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}^*$ defined by

$$d(\eta, \xi) = (\nabla^* \eta, \xi) + (\eta, \nabla \xi) \text{ for all } \eta \in \mathcal{E}^*, \xi \in \mathcal{E}.$$

For any element u of \mathcal{U} , we obtain a new Hermitian connection, the gauge transform of ∇ , by $\nabla^u = u \nabla u^{-1}$. The curvature of ∇^u is given by $\theta^u = u \theta u^{-1}$.

(8) In order to proceed further, we will want a K -cycle on \mathcal{A} , that is, a $*$ representation of \mathcal{A} by bounded operators on a Hilbert space \mathcal{H} and a (possibly unbounded) self-adjoint operator D on \mathcal{H} such that

- $[D, a]$ is a bounded operator for all $a \in \mathcal{A}$

- $(1 + D^2)^{-1}$ is a compact operator on \mathcal{H} .

We will also assume a \mathbb{Z}_2 grading operator Γ on \mathcal{H} such that $\Gamma D + D \Gamma = 0$ and $a \Gamma = \Gamma a$ for all $a \in \mathcal{A}$.

We will write $B(\mathcal{H})$ for the space of bounded operators on \mathcal{H} and $B(\mathcal{H}, \mathcal{H}')$ for the space of bounded operators between two Hilbert spaces \mathcal{H} and \mathcal{H}' .

(9) If \mathcal{A} is a complex commutative algebra then any K -cycle gives a metric d on the character space C by

$$d(p, q) = \sup \{|p(f) - q(f)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\}.$$

(If \mathcal{A} is a general C^* -algebra then the same definition gives a metric on the state space of \mathcal{A} .)

(10) Define a map $\pi : \Omega^*(\mathcal{A}) \rightarrow B(\mathcal{H})$ by

$$\pi(a^0 da^1 \dots da^k) = a_0 i[D, a_1] \dots i[D, a_k].$$

This is an algebra homomorphism (but not a homomorphism of differential algebras). There are extensions of π , which we will also denote by π , to a (right) module homomorphism

$$\pi: \mathfrak{E} \otimes_{\mathcal{Q}} \Omega^*(\mathcal{Q}) \rightarrow B(\mathfrak{H}, \mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{H}) \text{ and to a linear map}$$

$$\pi: \text{End}_{\mathcal{Q}}(\mathfrak{E}, \mathfrak{E} \otimes_{\mathcal{Q}} \Omega^*(\mathcal{Q})) \rightarrow B(\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{H}).$$

(11) Given a hermitian connection ∇ , we obtain a self-adjoint operator D_{∇} on the Hilbert space $\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{H}$ by

$$D_{\nabla}(\xi \otimes \eta) = \xi \otimes D\eta - i\pi(\nabla\xi)\eta \text{ for all } \xi \in \mathfrak{E}, \eta \in \mathfrak{H}.$$

There is an action of \mathcal{U} on $\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{H}$ for which D_{∇} is gauge covariant in the sense that $D_{\nabla}^u = u D_{\nabla} u^{-1}$.

(12) We will define the spinor action as

$$I_{\psi} = \langle \psi, D_{\nabla} \psi \rangle \text{ for } \psi \in \mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{H}.$$

In order to define the Yang-Mills action of a connection, we will need the notion of the Dixmier trace of a compact operator [Di]. Although we will only need this in a special case, where the Dixmier trace is effectively computable, we will give the general definition. For an infinite-dimensional Hilbert space \mathfrak{H} , define an ideal by

$$\mathfrak{L}^{1+}(\mathfrak{H}) = \{T \in B(\mathfrak{H}) : T \text{ is compact, } \sum_{i=0}^N \mu_i(T) = O(\log(N+1))\},$$

where $\{\mu_i\}$ are the eigenvalues of $|T|$. If ω is a mean on the space $l^{\infty}(\mathbb{Z}^+)$ of bounded sequences on \mathbb{Z}^+ , and T is a positive element of $\mathfrak{L}^{1+}(\mathfrak{H})$, define

$$\text{Tr}_{\omega}(T) = \omega\left(\sum_{i=0}^N \mu_i(T) / \log(N+1)\right).$$

What is nontrivial is that if ω is chosen to be scale-invariant in an appropriate way then Tr_{ω} is a linear functional and extends to a positive trace on $\mathfrak{L}^{1+}(\mathfrak{H})$, the Dixmier trace. (If \mathfrak{H} is finite-dimensional then we will define Tr_{ω} to be the usual trace.)

We will only need the Dixmier trace of pseudodifferential operators. Suppose that Z is n -dimensional and that P is a pseudodifferential operator of order $-n$ acting on sections of a vector bundle over Z . Then $P \in \mathfrak{L}^{1+}(\mathfrak{H})$. Let σ_P denote the principal symbol of P , a matrix-valued function on T^*Z . Then regardless of the mean ω used to define the Dixmier trace, one has

$$\text{Tr}_{\omega}(P) = (n(2\pi)^n)^{-1} \int_{S^*Z} \text{trace}(\sigma_P) d\mu,$$

where S^*Z is the cosphere bundle of Z (the subset of the cotangent bundle consisting of unit covectors) with the standard measure $d\mu$ [Co3]. In this case one sees that the Dixmier trace is the integral of a local expression over Z .

(13) Let us consider the Dixmier trace on

$$\mathfrak{L}^{1+}(\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{H}).$$

Suppose that D is such that $D^{-2k} \in \mathfrak{L}^{1+}(\mathfrak{H})$ for some integer k . (We will want to choose the smallest such k .) Then we will define the Yang-Mills action to be $I_{\nabla} = \text{Tr}_{\omega}((\pi(\theta))^2 D_{\nabla}^{-2k})$. One can also define a topological action, for which we refer to [Co2].

(14) One can take the product of noncommutative spaces as follows. Let $(\mathcal{Q}_1, \mathfrak{H}_1, D_1, \Gamma_1)$ and $(\mathcal{Q}_2, \mathfrak{H}_2, D_2, \Gamma_2)$ be two noncommutative spaces with K -cycles. Form a new noncommutative space with K -cycle by putting

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2, \mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2, D = D_1 \otimes I + \Gamma_1 \otimes D_2 \text{ and } \Gamma = \Gamma_1 \otimes \Gamma_2.$$

The character space for \mathcal{Q} is $C = C_1 \times C_2$. The space of differential forms $\Omega^*(\mathcal{Q})$ maps to $\Omega^*(\mathcal{Q}_1) \otimes \Omega^*(\mathcal{Q}_2)$. Given vector bundles \mathfrak{E}_1 (associated to \mathcal{Q}_1) and \mathfrak{E}_2

(associated to \mathcal{Q}_2) with Hermitian metrics, there is a vector bundle $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ associated to \mathcal{Q} with Hermitian metric.

Examples

A. Commutative Geometry

(1) Let Z be an even-dimensional smooth compact spin manifold and take $\mathcal{Q} = C^\infty(Z) \otimes \mathbb{C}$.

(2) $\mathcal{C} = Z$, with $z \in Z$ corresponding to the homomorphism $f \rightarrow f(z)$.

(3) Take E to be a smooth finite-dimensional vector bundle over Z and put $\mathcal{E} = C^\infty(E)$, the space of smooth cross-sections of E . \mathcal{E}^* is the space of smooth cross-sections of E^* , the vector bundle whose transition functions are the inverse transpose of those of E .

(4) $\Omega^k(\mathcal{Q})$ can be identified with the space of smooth functions $g: Z^{k+1} \rightarrow \mathbb{C}$ such that

$g(z_1, \dots, z_{i-1}, z_i, z_i, z_{i+2}, \dots, z_{k+1}) = 0$ for all $1 \leq i \leq k$. The left action of \mathcal{Q} is given by

$(fg)(z_1, \dots, z_{k+1}) = f(z_1) g(z_1, \dots, z_{k+1})$ and the right action of \mathcal{Q} is given by

$$(gf)(z_1, \dots, z_{k+1}) = g(z_1, \dots, z_{k+1}) f(z_{k+1}).$$

Note that the right and left actions are not the same, even though \mathcal{Q} is commutative. The involution on $\Omega^k(\mathcal{Q})$ is given by $g^*(z_1, \dots, z_{k+1}) = (-1)^k g(z_{k+1}, \dots, z_1)$.

The product of a k -form g and a k' -form g' is given by $(gg')(z_1, \dots, z_{k+k'+1}) =$

$$g(z_1, \dots, z_{k+1}) g'(z_{k+1}, \dots, z_{k+k'+1}).$$

The differential on $\Omega^*(\mathcal{Q})$ is given by $(dg)(z_1, \dots, z_{k+2}) =$

$$\sum_{i=1}^{k+2} (-1)^{i+1} g(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+2}).$$

(5) Any smooth hermitian inner product (\cdot, \cdot) on E gives a hermitian metric on \mathcal{E} by sending the pair $\xi_1, \xi_2 \in \mathcal{E}$ to the function $\langle \xi_1, \xi_2 \rangle \in \mathcal{Q}$ defined by

$$\langle \xi_1, \xi_2 \rangle(z) = (\xi_1(z), \xi_2(z)).$$

(6) \mathcal{U} is the group of smooth unitary gauge transformations of E .

(7) If $\mathcal{E} = \mathcal{Q}$ then a connection is specified by a skew element $\rho = \sum_j a_j db_j$ of $\Omega^1(\mathcal{Q})$. The curvature is

$\theta = d\rho + \rho^2 \in \Omega^2(\mathcal{Q})$. Note that the ρ^2 term does not vanish, unlike in the usual calculus of differential forms.

(8) Let S denote the vector bundle of spinors on Z and put $\mathcal{H} = L^2(S)$, the Hilbert space of square-integrable sections of S . Let D be the Dirac operator and let Γ be the Hermitian chirality operator on \mathcal{H} .

(9) This reproduces the geodesic distance on Z .

(10) If we note by da the operator $\sum_\mu \gamma^\mu \partial_\mu a$ on \mathcal{H} then $\pi(a^0 da^1 \dots da^k) = a^0 da^1 \dots da^k$.

(11) If $\mathcal{E} = \mathcal{Q}$ then $D_\nabla = \not{D} - i \sum_j a_j db_j$, an operator on $L^2(S)$, which is recognized to be the Dirac operator coupled to a $U(1)$ gauge field.

(12) This gives the usual action for a spinor field in a $U(1)$ background.

(13) If Z is four-dimensional and E is a \mathbb{C}^N vector bundle over Z then one finds

$$I_{\nabla} = (8\pi^2)^{-1} \int_Z [|\mathbb{F}|^2 + X^2] \text{dvol},$$

the sum of the usual Yang-Mills action and a Gaussian term which decouples. The topological action is proportionate to the usual topological charge $p_1(E)$.

(14) If Z_1 and Z_2 are ordinary spaces then \mathcal{Q} is the algebra corresponding to $Z = Z_1 \times Z_2$. \mathfrak{H} is the Hilbert space of square-integrable spinors on Z , D is the Dirac operator on Z and Γ is the Hermitian chirality operator for Z . If \mathfrak{E}_1 and \mathfrak{E}_2 arise from ordinary vector bundles E_1 (on Z_1) and E_2 (on Z_2) then \mathfrak{E} arises from the ordinary vector bundle $\pi_1^* E_1 \otimes \pi_2^* E_2$ on Z .

B. Two-Point Space

(1) Take $\mathcal{Q} = \mathbb{C} \oplus \mathbb{C}$, with

$$(\lambda_1, \lambda_1') + (\lambda_2, \lambda_2') = (\lambda_1 + \lambda_2, \lambda_1' + \lambda_2')$$
 and

$$(\lambda_1, \lambda_1') (\lambda_2, \lambda_2') = (\lambda_1 \lambda_2, \lambda_1' \lambda_2').$$

(2) C is a two-point space, $C = \{p, p'\}$, with

$$p: (\lambda, \lambda') \rightarrow \lambda \text{ and } p': (\lambda, \lambda') \rightarrow \lambda'.$$

(3) Take $\mathfrak{E} = \mathbb{C}^k \oplus \mathbb{C}^{k'}$, with the \mathcal{Q} action given by $(\xi, \xi')(\lambda, \lambda') = (\xi\lambda, \xi'\lambda')$ for all $(\xi, \xi') \in \mathfrak{E}$ and

$(\lambda, \lambda') \in \mathcal{Q}$. If we assume that $k \leq k'$ then we can write

$\mathfrak{E} = e\mathcal{Q}^{k'}$ where e is a diagonal $k' \times k'$ matrix with entries consisting of k (1,1)'s and $k'-k$ (0,1)'s.

(4) $\Omega^*(\mathcal{Q})$ has the same interpretation as in the previous example. In particular, $\Omega^0(\mathcal{Q}) = \mathcal{Q} = \mathbb{C}^2$ and $\Omega^1(\mathcal{Q})$ can be identified as \mathbb{C}^2 , the values of g at (p, p') and (p', p) . The involution on $\Omega^1(\mathcal{Q})$ is given by $(r, s)^* = -(s^*, r^*)$ for all $(r, s) \in \mathbb{C}^2$. The differential $d: \Omega^0(\mathcal{Q}) \rightarrow \Omega^1(\mathcal{Q})$ is given by $d(v, v') = (v' - v, v - v')$.

(5) We can take

$$\langle (\xi_1, \xi_1'), (\xi_2, \xi_2') \rangle = (\xi_1^* \xi_2, \xi_1'^* \xi_2') \in \mathcal{Q}$$

for all $(\xi_1, \xi_1'), (\xi_2, \xi_2') \in \mathfrak{E}$.

(6) \mathcal{U} is $U(k) \times U(k')$.

(7) If $\mathfrak{E} = \mathcal{Q}$ then a connection is specified by a skew element $\rho \in \Omega^1(\mathcal{Q})$, which we can write as

$$\rho = (r, r^*) \in \mathbb{C}^2.$$

(8) Take $\mathfrak{H} = \mathbb{C}^N \oplus \mathbb{C}^N$, with the action of \mathcal{Q} given by $(\lambda, \lambda')(\eta, \eta') = (\lambda\eta, \lambda'\eta')$

for all $(\lambda, \lambda') \in \mathcal{Q}$ and $(\eta, \eta') \in \mathfrak{H}$.

If M is an $N \times N$ matrix, let D be the operator $\begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}$ and

let Γ be the operator $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(9) Let L denote the square root of the inverse of the largest eigenvalue of M^*M . Then the metric on C is given by $d(p, p') = L$.

(10) If for $a = (\lambda, \lambda') \in \mathcal{Q}$ we denote by da the operator

$$i(\lambda - \lambda') \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix} \text{ then}$$

$\pi(a^0 da^1 \dots da^k) = a^0 da^1 \dots da^k$. In particular, for a skew

1-form $\rho = \sum_j a_j^0 da_j^1$, if we write ρ in the form

$(\phi^*-1, \phi-1) \in \mathbb{C}^2$ then $\pi(\rho)$ is the operator

$$\begin{pmatrix} 0 & i(\bar{\phi}-1)M^\dagger \\ i(\phi-1)M & 0 \end{pmatrix}. \text{ The image } \pi(\theta) \text{ of the curvature}$$

$$\text{of } \rho \text{ is } \pi(d\rho + \rho^2) = \pi\left(\sum_j da_j^0 da_j^1\right) + \pi(\rho^2) =$$

$\sum_j \pi(da_j^0) \pi(da_j^1) + \pi(\rho)^2$, which turns out to be the

$$\text{operator } (1 - |\phi|^2) \begin{pmatrix} M^\dagger M & 0 \\ 0 & MM^\dagger \end{pmatrix}.$$

(11) With the connection given by ρ as above, D_∇ is the

$$\text{operator } \begin{pmatrix} 0 & \bar{\phi}M^\dagger \\ \phi M & 0 \end{pmatrix}.$$

(12) The fermionic action is $I_\psi = \psi^\dagger D_\nabla \psi$ for $\psi \in \mathfrak{H}$.

(13) The Yang-Mills action is

$I_\nabla = 2(1 - |\phi|^2)^2 \text{Tr}(M^\dagger M)^2$. When $\mathfrak{E} = \mathfrak{Q}$, the topological action vanishes. (More generally, the topological action will be proportionate to $k - k'$.)

(14) We can take the product of noncommutative geometries corresponding to two two-point spaces to get a noncommutative geometry for a four-point space, in a straightforward way.

III. Glashow-Weinberg-Salam Model.

Let Z be a closed Riemannian spin 4-manifold. (We will make the transition later to a Lorentzian spacetime). Let S denote the spinor bundle and let $\gamma \in B(L^2(S))$

denote the Hermitian chirality operator. Let N_G be the number of generations and let M be an $N_G \times N_G$ matrix, the mass matrix.

In order to build our models, the general strategy will be to think of the algebra \mathfrak{Q} and the vector bundle \mathfrak{E} together as specifying the gauge group, and the action of \mathfrak{Q} on the Hilbert space \mathfrak{H} as specifying the fermionic representation of the gauge group. For example, suppose that we take \mathfrak{Q} to be $C^\infty(Z)$ and \mathfrak{E} to be $\mathbb{C}^N \otimes C^\infty(Z)$. Suppose that the action of $f \in \mathfrak{Q}$ on \mathfrak{E} is given by $I_N \otimes f$, that is,

$(v \otimes g) f = v \otimes gf$. Then the unitary gauge group is $\text{Map}(Z, U(N))$. Suppose that \mathfrak{H} is $L^2(S) \otimes \mathbb{C}^M$ and that the action of $f \in \mathfrak{Q}$ on \mathfrak{H} is given by $f \otimes I_M$, that is,

$f(\eta \otimes w) = f\eta \otimes w$. Then the space of fermions is

$\mathfrak{E} \otimes_{\mathfrak{Q}} \mathfrak{H} = \mathbb{C}^N \otimes L^2(S) \otimes \mathbb{C}^M$. In other words, we obtain M distinct fermions that are in the fundamental representation of $U(N)$.

To write the GWS model, we will first take the structure group to be $U(1) \times U(2)$ and later restrict to $U(1) \times SU(2)$. We have seen in example A of the previous section that when one specializes noncommutative gauge theory to the usual commutative case, one recovers the standard formalism of gauge fields and Yang-Mills actions. In example B we saw that when one does gauge theory on a two-point space, with an off-diagonal Dirac operator, one obtains a variable ϕ , which we can interpret as a Higgs field. The Yang-Mills action in example B was exactly a symmetry-breaking Higgs potential. In order to write the GWS model, we will simply take the product of the noncommutative spaces of examples A and B. Let us write our two factor spaces as noncommutative spaces with K-cycles, $(\mathfrak{Q}_1, \mathfrak{H}_1, D_1, \Gamma_1)$ and $(\mathfrak{Q}_2, \mathfrak{H}_2, D_2, \Gamma_2)$.

$(\mathfrak{Q}_1, \mathfrak{H}_1, D_1, \Gamma_1)$ will correspond to the usual Riemannian geometry on Z :

$$\mathfrak{Q}_1 = C^\infty(Z), \mathfrak{H}_1 = L^2(S), D_1 = \not{D} \text{ and } \Gamma_1 = \gamma.$$

For $(\mathcal{Q}_2, \mathfrak{K}_2, D_2, \Gamma_2)$, we will take

$$\mathcal{Q}_2 = \mathbb{C} \oplus \mathbb{C}, \mathfrak{K}_2 = (\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^{N_G}, D_2 = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}$$

$$\text{and } \Gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we write an element of \mathfrak{K}_2 in the form (η, η') with

$\eta, \eta' \in \mathbb{C}^{N_G}$ then the action of $(\lambda, \lambda') \in \mathcal{Q}_2$ on

$(\eta, \eta') \in \mathfrak{K}_2$ will be taken to be

$(\lambda, \lambda') (\eta, \eta') = (\lambda\eta, \lambda'\eta')$. For the product geometry

$(\mathcal{Q}, \mathfrak{K}, D, \Gamma)$, we have

$$\mathcal{Q} = C^\infty(Z) \oplus C^\infty(Z), \mathfrak{K} = L^2(S) \otimes \mathbb{C}^2 \otimes \mathbb{C}^{N_G},$$

$$D = \not{D} \otimes I_2 \otimes I_{N_G} + \gamma \otimes \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} \text{ and}$$

$$\Gamma = \gamma \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_{N_G}.$$

The character space C_1 of $(\mathcal{Q}_1, \mathfrak{K}_1, D_1, \Gamma_1)$ is Z and the character space C_2 of $(\mathcal{Q}_2, \mathfrak{K}_2, D_2, \Gamma_2)$ is a two-point space. Thus the character space C of the product geometry is the disjoint union of two copies of Z , say Z and Z' . Let us denote the geodesic distance on Z by d_Z . Let L denote the square root of the inverse of the largest eigenvalue of $M^\dagger M$. One can show that the distance d on C satisfies

$$d(p, q) = d_Z(p, q) \text{ for } p, q \in Z,$$

$$d(p', q') = d_Z(p', q') \text{ for } p', q' \in Z',$$

$$(d_Z(p, p')^2 + L^2)^{1/2} \leq d(p, p') \leq d_Z(p, p') + L$$

for $p \in Z, p' \in Z'$.

Thus the picture is of two copies of Z separated by a distance L . (As one loses information in passing to the character space, this picture should not be taken too literally.)

For the vector bundles, let us take $\mathfrak{E}_1 = \mathcal{Q}_1$ and

$\mathfrak{E}_2 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, with the action of $(\lambda, \lambda') \in \mathcal{Q}_2$ on $(\xi_1, \xi_2, \xi_3) \in \mathfrak{E}_2$ given by

$(\xi_1, \xi_2, \xi_3)(\lambda, \lambda') = (\xi_1\lambda, \xi_2\lambda', \xi_3\lambda')$. Then the vector bundle \mathfrak{E} is $C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z)$. This can be

written more succinctly as $\mathfrak{E} = e\mathcal{Q}^2$, with $e = \begin{pmatrix} (1,1) & 0 \\ 0 & (0,1) \end{pmatrix}$

. The unitary gauge group of \mathfrak{E} is $\text{Map}(Z, U(1) \times U(2))$.

In terms of the character space, \mathfrak{E} can be thought of as a \mathbb{C}^1 bundle over Z and a \mathbb{C}^2 bundle over Z' . The

fermion space is given by $\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{K} = L^2(S) \otimes \mathbb{C}^3 \otimes \mathbb{C}^{N_G}$.

We can write elements of $\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{K}$ in the form $\begin{pmatrix} e_R \\ e_L \\ v_L \end{pmatrix}$,

where $e_R, e_L, v_L \in L^2(S) \otimes \mathbb{C}^{N_G}$. In this way, e_R is in the fundamental representation of $U(1)$ and (e_L, v_L) is in the fundamental representation of $U(2)$.

Let us write a Hermitian connection ∇ on $\mathfrak{E} = e\mathcal{Q}^2$ in the form $\nabla\xi = ed\xi + \rho\xi$,

where $\xi \in \mathcal{Q}^2$, $e\xi = \xi$, $\rho \in M_2(\mathcal{Q}) \otimes_{\mathcal{Q}} \Omega^1(\mathcal{Q})$, $e\rho = \rho$.

The matrix ρ is a 2×2 matrix of one forms, the components of which can be written as

$$\rho_{kl} = \sum_j a_{klj} db_{klj}, a_{klj} = (f_{klj}, f'_{klj}), b_{klj} = (g_{klj}, g'_{klj}),$$

$$1 \leq k, l \leq 2.$$

The condition $e\rho = \rho$ becomes $f'_{21j} = f'_{22j} = 0$.

Let us fix k and l for the moment. Each a_{klj} and b_{klj} acts on \mathfrak{K} as a 2×2 diagonal matrix. Then

$\pi(\rho_{kl}) = \sum_j a_{klj} i [D, b_{klj}]$ is a 2×2 matrix with components

$$\pi(\rho_{kl})_{11} = \sum_j f_{klj} \mathbf{d}g_{klj} \otimes I_{N_G}$$

$$\pi(\rho_{kl})_{12} = \sum_j -if_{klj} (g_{klj} - g'_{klj}) \gamma \otimes M^\dagger$$

$$\pi(\rho_{kl})_{21} = \sum_j if'_{klj} (g_{klj} - g'_{klj}) \gamma \otimes M$$

$$\pi(\rho_{kl})_{22} = \sum_j f'_{klj} \mathbf{d}g'_{klj} \otimes I_{N_G}$$

(We are using the notation that for a function g ,

$$\mathbf{d}g = i [D, g] = \sum_\mu \gamma^\mu \partial_\mu g.)$$

We have that the 4×4 matrix $\pi(\rho)$, an operator on

$\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{K}$, has vanishing third row. By its skew-Hermiticity, it also has vanishing third column. Thus we can write $\pi(\rho)$ as a 3×3 matrix. One finds that it can be written in the form

$$\pi(\rho) = \begin{pmatrix} A \otimes I_{N_G} & i(\bar{\phi}_1 - 1) \gamma \otimes M^\dagger & i\bar{\phi}_2 \gamma \otimes M^\dagger \\ i(\phi_1 - 1) \gamma \otimes M & A' \otimes I_{N_G} & -W^* \otimes I_{N_G} \\ i\phi_2 \gamma \otimes M & W \otimes I_{N_G} & Z \otimes I_{N_G} \end{pmatrix}.$$

Here $A = \sum_j f_{11j} \mathbf{d}g_{11j}$, $A' = \sum_j f'_{11j} \mathbf{d}g'_{11j}$,

$$W = \sum_j f'_{21j} \mathbf{d}g'_{21j}, Z = \sum_j f'_{22j} \mathbf{d}g'_{22j},$$

$$\phi_1 - 1 = \sum_j f'_{11j} (g_{11j} - g'_{11j}) \text{ and}$$

$\phi_2 = \sum_j f'_{21j} (g_{21j} - g'_{21j})$. In terms of their behaviour under gauge transformations, A is an ordinary $U(1)$

gauge field, $\begin{pmatrix} A' & -W^* \\ W & Z \end{pmatrix}$ is a $U(2)$ gauge field and

$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a Higgs doublet with covariant derivative

$$D\Phi = d\Phi + \begin{pmatrix} A' - A & -W^* \\ W & Z - A \end{pmatrix} \Phi.$$

(In terms of the intuitive picture of the geometry given by the character space, the $U(1)$ gauge field is locally given by a differential form on Z and the $U(2)$ gauge field is locally given by differential forms on Z' . The Higgs fields involve the differences of functions on Z and Z' , and so can be thought of as differential forms where the differential is replaced by a difference operator.)

The curvature $\theta = \nabla^2: \mathfrak{E} \rightarrow \mathfrak{E} \otimes_{\mathcal{Q}} \Omega^2(\mathcal{Q})$ of the connection is $\theta = e de de + e dp + \rho^2$. Its image $\pi(\theta)$, a self-adjoint operator on $\mathfrak{E} \otimes_{\mathcal{Q}} \mathfrak{K}$, can be written as a 3×3 matrix T of operators. The only subtlety in computing T comes from the fact that T is locally an operator with values in the even part of the Clifford algebra. It does not consist only of Clifford algebra elements of degree two, as one might expect for a curvature. For example, T_{11} has a term $A^2 \otimes I_{N_G}$. Because A is an operator of the form

$A = \sum_{\mu} A_{\mu} \gamma^{\mu}$, instead of vanishing, A^2 is actually a scalar

field $\sum_{\mu} A_{\mu}^2$ of degree zero. In general, the product of two

of our operator 1-forms can be decomposed in the Clifford algebra into the sum of a 2-form and a 0-form. Keeping this in mind, when decomposed into its various degrees we can write T as

$$T_{11} = F_A \otimes I_{N_G} + X \otimes I_{N_G} - (|\Phi|^2 - 1) \otimes M^{\dagger} M$$

$$\begin{pmatrix} T_{21} \\ T_{31} \end{pmatrix} = i(D\Phi) \gamma \otimes M$$

$$\begin{pmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{pmatrix} = F \begin{pmatrix} A' & -w^* \\ w & Z \end{pmatrix} \otimes I_{N_G} + \begin{pmatrix} X' & (X'')^* \\ X'' & X''' \end{pmatrix} \otimes I_{N_G} - (\Phi \Phi^{\dagger} - I) \otimes M M^{\dagger}.$$

Here X, X', X'' and X''' are new scalar fields.

In order to obtain a $U(1) \times SU(2)$ gauge theory, we must relate the $U(1)$ gauge field to the $U(1)$ part of the $U(2)$ gauge field. In terms of the hypercharge generator Y , we want $A = -2Y$ and $A' + Z = -2Y$. Thus we will require $A = A' + Z$.

The Yang-Mills action $I_{\nabla} = \text{Tr}_{\omega}((\pi(\theta))^2 D_{\nabla}^{-4})$ is the sum of three terms:

$$S_1 = (8\pi^2)^{-1} \int_Z [N_G |F_A|^2 +$$

$$N_G (X - N_G^{-1} (\text{Tr} M^{\dagger} M) (|\Phi|^2 - 1))^2 +$$

$$(|\Phi|^2 - 1)^2 \text{Tr}(M^{\dagger} M - N_G^{-1} (\text{Tr} M^{\dagger} M) I_{N_G})^2] \text{dvol}$$

$$S_2 = 2 (8\pi^2)^{-1} \text{Tr} M^{\dagger} M \int_Z D\Phi^{\dagger} D\Phi \text{dvol}$$

$$S_3 = (8\pi^2)^{-1} \int_Z [N_G |F_{U(2)}|^2 +$$

$$N_G \text{Tr} (X_{U(2)} - N_G^{-1} (\text{Tr} M^{\dagger} M) (\Phi \Phi^{\dagger} - I))^2 + ((|\Phi|^2 - 1)^2 + 1) \text{Tr}(M^{\dagger} M - N_G^{-1} (\text{Tr} M^{\dagger} M) I_{N_G})^2] \text{dvol}.$$

Because the auxiliary fields X and $X_{U(2)}$ appear quadratically and algebraically, one can immediately minimize over them. If there is more than one generation, and not all of the electron-like fermions in the various generations have the same mass, then we obtain a symmetry-breaking Higgs potential. (If there is one generation then $M^{\dagger} M - N_G^{-1} (\text{Tr} M^{\dagger} M) I_{N_G}$ vanishes and we do not have a Higgs potential). We will discuss the question of relations among the coupling constants in section VI.

The fermionic action is

$$I_{\psi} = \overline{(e_R e_L v_L)} \left(\begin{pmatrix} \not{D} \otimes I_{N_G} \gamma \otimes M^{\dagger} & 0 \\ \gamma \otimes M \not{D} \otimes I_{N_G} & 0 \\ 0 & 0 & \not{D} \otimes I_{N_G} \end{pmatrix} - i \pi(\rho) \right) \begin{pmatrix} e_R \\ e_L \\ v_L \end{pmatrix}.$$

In order to write an action for a Lorentzian spacetime, we will do the obvious changes for the bosons, and require that the fermion field $\psi \in \mathcal{E} \otimes_{\mathcal{Q}} \mathcal{H}$ satisfy $\Gamma \psi = \psi$. In this way we obtain the GWS model [Gl, We1, Sa].

IV. The Peccei-Quinn Model.

We now want to add the quarks to the GWS model. This will lead to the Peccei-Quinn model. The idea is that

the triplets $\begin{pmatrix} e_R \\ e_L \\ \nu_L \end{pmatrix}$ and $\begin{pmatrix} d_R \\ d_L \\ u_L \end{pmatrix}$ differ in their $U(1) \times SU(2)$

properties by a total hypercharge. The u_R quark is left over. This suggests having an algebra \mathcal{A} with vector bundle \mathcal{E} , with gauge group $U(1) \times U(1) \times U(2)$. The first $U(1)$ will act on u_R , the second $U(1)$ will act on e_R and d_R , and the $U(2)$ will act on (e_L, ν_L) and (d_L, u_L) . We also have to add the color degrees of freedom to the quarks. If we can do this then the gluons are automatically generated in our model. In order to provide the color we will add a new algebra \mathfrak{B} which acts on the Hilbert space, and a vector bundle \mathcal{F} associated to \mathfrak{B} . If the gauge group of \mathcal{F} were taken to just be $U(3)$, then all of the physical fermions would have to have color. Thus we will take \mathfrak{B} and \mathcal{F} so that the gauge group of \mathcal{F} is $U(1) \times U(3)$. The $U(1)$ will act on the leptons and the $U(3)$ will act on the quarks. The total gauge group is now $U(1) \times U(1) \times U(2) \times U(1) \times U(3)$. When the various $U(1)$ factors are related, the final gauge group will be $U(1) \times SU(2) \times SU(3)$. There will be no Higgs fields associated to the gauge group of \mathcal{F} , and so the $SU(3)$ gauge group will be unbroken.

Let us first describe the geometry of our model over a point in Z . \mathcal{A} will be the algebra of a three-point space:

$\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Let $\mathfrak{B} = \mathbb{C} \oplus \mathbb{C}$ be an auxiliary algebra, which one can think of as taking the place of \mathbb{C} as a ground ring. Take \mathfrak{K} to be $(\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2) \otimes \mathbb{C}^{N_G}$. (One can think that the first factor in \mathfrak{K} corresponds to u_R , the second factor corresponds to e_R and d_R , and the third factor corresponds to the four left-handed fermions.) The (left) action of $(f, f', f'') \in \mathcal{A}$ on \mathfrak{K} is given by

$(fI_1 \oplus f'I_2 \oplus f''I_2) \otimes I_{N_G}$, and the (right) action of (g, g')

$\in \mathfrak{B}$ on \mathfrak{K} is given by $\left(g'I_1 \oplus \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \oplus \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \right) \otimes I_{N_G}$.

The operator D will be

$$D = \begin{pmatrix} 0 & 0 & M_2^\dagger \\ 0 & 0 & M_3^\dagger \\ M_2 & M_3 & 0 \end{pmatrix}, \text{ where } M_2 = \begin{pmatrix} 0 \\ M_u \end{pmatrix} \text{ is a } 2N_G \times N_G$$

matrix and $M_3 = \begin{pmatrix} M_e & 0 \\ 0 & M_d \end{pmatrix}$ is a $2N_G \times 2N_G$ matrix.

Note that requiring that D commutes with the action of \mathfrak{B} gives M_2 and M_3 their special forms. The \mathbb{Z}_2 -grading operator will be $\Gamma = (I_1 \oplus I_2 \oplus (-I_2)) \otimes I_{N_G}$.

Now let us give the full model. Let \mathcal{A} be

$C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z)$, \mathfrak{B} be $C^\infty(Z) \oplus C^\infty(Z)$ and

\mathfrak{K} be $L^2(S) \otimes (\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2) \otimes \mathbb{C}^{N_G}$. The (left) action of $(f, f', f'') \in \mathcal{A}$ on \mathfrak{K} is given by

$(fI_1 \oplus f'I_2 \oplus f''I_2) \otimes I_{N_G}$, and the (right) action of

$(g, g') \in \mathfrak{B}$ on \mathfrak{K} is given by

$\left(g'I_1 \oplus \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \oplus \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \right) \otimes I_{N_G}$. The Dirac operator is

$$D = \not{D} \otimes (I_1 \oplus I_2 \oplus I_2) \otimes I_{N_G} + \gamma \otimes \begin{pmatrix} 0 & 0 & M_2^\dagger \\ 0 & 0 & M_3^\dagger \\ M_2 & M_3 & 0 \end{pmatrix},$$

where $M_2 = \begin{pmatrix} 0 \\ M_u \end{pmatrix}$ is a $2N_G \times N_G$ matrix and

$M_3 = \begin{pmatrix} M_e & 0 \\ 0 & M_d \end{pmatrix}$ is a $2N_G \times 2N_G$ matrix. The \mathbb{Z}_2 -grading operator is $\Gamma = \gamma \otimes (I_1 \oplus I_2 \oplus (-I_2)) \otimes I_{N_G}$.

Take $\mathfrak{E} = C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z) = e \mathbb{C}^2$,

with $e = \begin{pmatrix} (1,1,1) & 0 \\ 0 & (0,0,1) \end{pmatrix}$, and

$\mathfrak{F} = C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z) = \mathfrak{R}^3 f$, with

$f = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (0,1) & 0 \\ 0 & 0 & (0,1) \end{pmatrix}$. The gauge group of \mathfrak{E} is

$\text{Map}(Z, U(1) \times U(1) \times U(2))$ and the gauge group of \mathfrak{F} is $\text{Map}(Z, U(1) \times U(3))$.

An element of the physical Hilbert space

$\mathcal{P} = \mathfrak{E} \otimes_{\mathbb{C}} \mathfrak{H} \otimes_{\mathfrak{R}} \mathfrak{F}$ can be written as

$$(u_R, e_R, d_R, e_L, d_L, v_L, u_L) \in L^2(S) \otimes (\mathbb{C}^3 \oplus (\mathbb{C}^1 \oplus \mathbb{C}^3) \oplus (\mathbb{C}^1 \oplus \mathbb{C}^3) \oplus (\mathbb{C}^1 \oplus \mathbb{C}^3)) \otimes \mathbb{C}^{N_G}.$$

A connection on \mathfrak{F} will provide a $U(1)$ gauge field and a $U(3)$ gauge field, and no Higgs fields. If a connection on \mathfrak{E} is given by the matrix ρ then one computes that $\pi(\rho)$, an operator on $\mathfrak{E} \otimes_{\mathbb{C}} \mathfrak{H}$, is a 4×4 matrix of operators

$$\pi(\rho) = \begin{pmatrix} A \otimes I_{N_G} & 0 & i(\bar{\phi}_2 - 1) \gamma \otimes M_2^\dagger & i\bar{\phi}_4 \gamma \otimes M_2^\dagger \\ 0 & A' \otimes I_{N_G} & i(\bar{\phi}_3 - 1) \gamma \otimes M_3^\dagger & i\bar{\phi}_5 \gamma \otimes M_3^\dagger \\ i(\phi_2 - 1) \gamma \otimes M_2 & i(\phi_3 - 1) \gamma \otimes M_3 & A'' \otimes I_{N_G} & -W^* \otimes I_{N_G} \\ i\phi_4 \gamma \otimes M_2 & i\phi_5 \gamma \otimes M_3 & W \otimes I_{N_G} & Z \otimes I_{N_G} \end{pmatrix}.$$

(We are thinking of an element of $\mathfrak{E} \otimes_{\mathbb{C}} \mathfrak{H}$ as

$u_R \oplus (e_R, d_R) \oplus (e_L, d_L) \oplus (v_L, u_L)$.) A and A' are $U(1)$

gauge fields and $\begin{pmatrix} A'' & -W^* \\ W & Z \end{pmatrix}$ is a $U(2)$ gauge field. There

are two Higgs doublets, $\Phi_1 = \begin{pmatrix} \phi_2 \\ \phi_4 \end{pmatrix}$ and $\Phi_2 = \begin{pmatrix} \phi_3 \\ \phi_5 \end{pmatrix}$, with

covariant derivatives $D\Phi_1 = d\Phi_1 + \begin{pmatrix} A'' - A & -W^* \\ W & Z - A' \end{pmatrix} \Phi_1$,

$D\Phi_2 = d\Phi_2 + \begin{pmatrix} A'' - A' & -W^* \\ W & Z - A' \end{pmatrix} \Phi_2$.

The image $\pi(\theta)$ of the \mathfrak{E} -curvature, a self-adjoint operator on $\mathfrak{E} \otimes_{\mathbb{C}} \mathfrak{H}$, can be written as a 4×4 matrix T of operators, with

$$T_{11} = F_A \otimes I_{N_G} + X \otimes I_{N_G} - (|\Phi_1|^2 - 1) \otimes M_2^\dagger M_2$$

$$T_{22} = F_{A'} \otimes I_{N_G} + X' \otimes I_{N_G} - (|\Phi_2|^2 - 1) \otimes M_3^\dagger M_3$$

$$\begin{pmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{pmatrix} = F \begin{pmatrix} A'' & -W^* \\ W & Z \end{pmatrix} \otimes I_{N_G} + \begin{pmatrix} X'' & (X''')^* \\ X''' & X'''' \end{pmatrix} \otimes I_{N_G}$$

$$- (\Phi_1 \Phi_1^\dagger - I) \otimes M_2 M_2^\dagger - (\Phi_2 \Phi_2^\dagger - I) \otimes M_3 M_3^\dagger$$

$$T_{21} = (\phi_1 - \Phi_2^\dagger \Phi_1) \otimes M_3^\dagger M_2$$

$$\begin{pmatrix} T_{31} \\ T_{41} \end{pmatrix} = i(D\Phi_1) \gamma \otimes M_2$$

$$\begin{pmatrix} T_{32} \\ T_{42} \end{pmatrix} = i(D\Phi_2) \gamma \otimes M_3.$$

Here X, X', X'', X''', X'''' and ϕ_1 are new auxiliary scalar fields.

We now have a $U(1) \times U(1) \times U(2) \times U(1) \times U(3)$ gauge theory. In order to reduce this to $U(1) \times SU(2) \times SU(3)$, let us put that the first $U(1)$ part A of the connection on \mathfrak{E} vanishes, the second $U(1)$ part A' of the connection on \mathfrak{E} is $-2Y$, the trace of the $U(2)$ part of the connection on \mathfrak{E} is $-2Y$, the $U(1)$ part of the connection on \mathfrak{F} vanishes and the trace of the $U(3)$ part of the connection on \mathfrak{F} is $4Y$.

Upon computing the action as in section III, we obtain quartic potentials for Φ_1 and Φ_2 . There is also a mixed

term coming from $\begin{pmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{pmatrix}$ which is proportionate to $|\Phi_1 \uparrow \Phi_2|^2$. Putting everything together, we obtain the Peccei-Quinn model [PQ]. This model has an axion [We2,Wi] which seems to be ruled out experimentally [EG].

Let us note that because the first U(1) gauge field arising from \mathfrak{S} and the U(1) part of the gauge field arising from \mathfrak{F} both vanish, we could have taken the algebras to be $\mathfrak{Q} = \mathbb{C} \oplus C^\infty(Z) \oplus C^\infty(Z)$ and $\mathfrak{B} = \mathbb{C} \oplus C^\infty(Z)$. The only effect that this would have would be to enforce the vanishing from the beginning.

V. The Standard Model.

In the standard model the Yukawa term which gives mass to the up quark involves not the Higgs doublet Φ , but the field $\Phi' = J \bar{\Phi}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Φ' has the same isospin as Φ , but opposite hypercharge. One can use Φ' to form a gauge-invariant action because the fundamental representation of SU(2) is unitarily equivalent to its complex conjugate. That is,

$$(*) \quad J \bar{g} = g J$$

for all $g \in \text{SU}(2)$. Our choice of the algebra \mathfrak{Q} for the standard model is based on the observation that (*) defines a subalgebra of $M_2(\mathbb{C})$, namely the quaternion algebra \mathbb{H} .

Let us first consider an example with the algebra $\mathfrak{Q} = \mathbb{H}$ and a vector bundle $\mathfrak{S} = \mathbb{H}$, with \mathfrak{Q} acting on \mathfrak{S} by right multiplication. Then $\text{End}_{\mathfrak{Q}}(\mathfrak{S})$ equals \mathbb{H} , acting on \mathfrak{S} by left multiplication. The unitary gauge group is the group of unitary quaternions, namely SU(2).

This suggests taking the algebra \mathfrak{Q} for the standard model to be $C^\infty(Z) \otimes (\mathbb{C} \oplus \mathbb{H})$. We will add the gluons later. We will first work with complex vector spaces and describe directly the full model. We will later show how to describe the geometry of the full model as the product of ordinary spacetime geometry and a finite geometry, both of these factor geometries being expressed most simply using quaternions.

Let us first give our conventions for charge conjugation in Euclidean space. For any representation of the Clifford algebra on \mathbb{C}^4 , charge conjugation is given by the operator $R: \mathbb{C}^4 \rightarrow \mathbb{C}^4$ defined by $R(v) = C^{-1} \bar{v}$, where the matrix C satisfies $C \gamma^\mu C^{-1} = -(\gamma^\mu)^T$, $C^T = -C$ and $C \bar{C} = -1$. Also $C \gamma C^{-1} = \bar{\gamma}$. The operator R preserves chirality.

Take $\mathfrak{Q} = C^\infty(Z) \otimes (\mathbb{C} \oplus \mathbb{H})$ and

$\mathfrak{H} = L^2(S) \otimes (\mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2) \otimes \mathbb{C}^{N_G}$. (The first \mathbb{C}^2 factor corresponds to (e_R, d_R) , the \mathbb{C} factor corresponds to u_R , the second \mathbb{C}^2 factor corresponds to (e_L, d_L) and the third \mathbb{C}^2 factor corresponds to (v_L, u_L) .) We will identify \mathfrak{H} with \mathbb{C}^2 , where the action of an element (α, β) of $\mathfrak{H} =$

\mathbb{C}^2 on \mathbb{C}^2 is given by a matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. The action of

$(f, (\alpha, \beta)) \in \mathcal{U}$ on \mathfrak{H} is given by

$(f I_2 \oplus f I_1 \oplus \begin{pmatrix} \alpha I_2 & \beta I_2 \\ -\beta I_2 & \alpha I_2 \end{pmatrix}) \otimes I_{N_G}$. The operator D is

given by

$$D(v) = (\not{D} \otimes (I_2 \oplus I_1 \oplus I_2 \oplus I_2) \otimes I_{N_G}) v +$$

$$\gamma \otimes \begin{pmatrix} 0 & 0 & M_1^\dagger & 0 \\ 0 & 0 & 0 & 0 \\ M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v +$$

$$\gamma \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -M_2^T \\ 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \end{pmatrix} R(v),$$

where $M_1 = \begin{pmatrix} M_e & 0 \\ 0 & M_d \end{pmatrix}$ is a $2N_G \times 2N_G$ matrix and

$M_2 = \begin{pmatrix} 0 \\ M_u \end{pmatrix}$ is a $2N_G \times N_G$ matrix. D is a symmetric operator with respect to the underlying real structure on \mathfrak{H} . The \mathbb{Z}_2 -grading is given by

$$\Gamma = \gamma \otimes (I_2 \oplus I_1 \oplus (-I_2) \oplus (-I_2)) \otimes I_{N_G}.$$

We will take the vector bundle \mathfrak{E} to be \mathcal{U} . The unitary gauge group of \mathfrak{E} is $\text{Map}(Z, U(1) \times SU(2))$. The Hilbert space $\mathfrak{E} \otimes_{\mathcal{U}} \mathfrak{H}$ is the same as \mathfrak{H} . A connection on \mathfrak{E} is given by a skew element ρ of $\Omega^1(\mathcal{U})$. A computation gives that $\pi(\rho)$ can be written as an operator on \mathfrak{H} in the form

$$\pi(\rho)(v) =$$

$$\begin{pmatrix} A \otimes I_{N_G} & 0 & i(\bar{\phi}_1 - 1) \gamma \otimes M_1^\dagger & i(\bar{\phi}_2) \gamma \otimes M_1^\dagger \\ 0 & A \otimes I_{N_G} & 0 & 0 \\ i(\bar{\phi}_1 - 1) \gamma \otimes M_1 & 0 & A' \otimes I_{N_G} & -W \otimes I_{N_G} \\ i \bar{\phi}_2 \gamma \otimes M_1 & 0 & W \otimes I_{N_G} & -A' \otimes I_{N_G} \end{pmatrix} v +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i \bar{\phi}_2 \gamma \otimes M_2^T & -i(\bar{\phi}_1 - 1) \gamma \otimes M_2^T \\ 0 & -i \bar{\phi}_2 \gamma \otimes M_2 & 0 & 0 \\ 0 & i(\bar{\phi}_1 - 1) \gamma \otimes M_2 & 0 & 0 \end{pmatrix} R(v).$$

A is a $U(1)$ gauge field, $\begin{pmatrix} A' & -W^* \\ W & -A' \end{pmatrix}$ is an $SU(2)$ gauge field

and $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a Higgs doublet with covariant derivative

$$D\Phi = d\Phi + \begin{pmatrix} A'-A & -W^* \\ W & -A'-A \end{pmatrix} \Phi. \text{ The image } \pi(\theta) \text{ can be}$$

written as an operator on \mathfrak{H} in the form

$\pi(\theta)(v) = S v + T \gamma R(v)$, where S is a 4×4 self-adjoint matrix of operators with

$$S_{11} = F_A \otimes I_{N_G} + X \otimes I_{N_G} + (1 - |\Phi|^2) \otimes M_1^\dagger M_1$$

$$S_{22} = F_A \otimes I_{N_G} + X \otimes I_{N_G} - (1 - |\Phi|^2) \otimes M_2^T \overline{M_2}$$

$$\begin{pmatrix} S_{33} & S_{34} \\ S_{43} & S_{43} \end{pmatrix} = F \begin{pmatrix} A' & -W^* \\ W & -A' \end{pmatrix} \otimes I_{N_G} + X' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes I_{N_G} +$$

$$1/2 (1 - |\Phi|^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (M_1 M_1^\dagger - M_2 M_2^\dagger) +$$

$$\begin{pmatrix} Y & Y'^* \\ Y' & -Y \end{pmatrix} \otimes [M_1 M_1^\dagger + M_2 M_2^\dagger]$$

$$\begin{pmatrix} S_{31} \\ S_{41} \end{pmatrix} = i(D\Phi) \gamma \otimes M_1$$

and T is a 4x4 antisymmetric matrix of operators with

$$\begin{pmatrix} T_{32} \\ T_{42} \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{D\Phi} \otimes M_2.$$

Here X, X', Y and Y' are auxiliary scalar fields.

In order to provide the color, we will add an auxiliary algebra $\mathfrak{B} = C^\infty(Z) \oplus C^\infty(Z)$ and a left \mathfrak{B} -module

$$\mathfrak{F} = C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z) \oplus C^\infty(Z) = \mathfrak{B}^3 f, \text{ with}$$

$$f = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (0,1) & 0 \\ 0 & 0 & (0,1) \end{pmatrix}. \text{ The gauge group of } \mathfrak{F} \text{ is}$$

$\text{Map}(Z, U(1) \times U(3))$. Let us write the Hilbert space

$$\mathfrak{K} = L^2(S) \otimes (\mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2) \otimes \mathbb{C}^{N_G} \text{ as}$$

$\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \oplus \mathfrak{K}_3 \oplus \mathfrak{K}_4$. We will let \mathfrak{B} act on the right on $\mathfrak{K}_1, \mathfrak{K}_3$ and \mathfrak{K}_4 , and on the left on \mathfrak{K}_2 . The

action of $(g, g') \in \mathfrak{B}$ on $\mathfrak{K}_1, \mathfrak{K}_3$ and \mathfrak{K}_4 will be by $\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}$

$\otimes I_{N_G}$, and the action on \mathfrak{K}_2 will be by $g I_1 \otimes I_{N_G}$. We will then take the physical fermion space to be

$$(\mathfrak{K}_1 \otimes_{\mathfrak{B}} \mathfrak{F}) \oplus (\mathfrak{F}^* \otimes_{\mathfrak{B}} \mathfrak{K}_2) \oplus (\mathfrak{K}_3 \otimes_{\mathfrak{B}} \mathfrak{F}) \oplus (\mathfrak{K}_4 \otimes_{\mathfrak{B}} \mathfrak{F}).$$

This has the effect of adding the color to the quarks.

A connection on \mathfrak{F} will give rise to a U(1) gauge field and a U(3) gauge field. The total gauge group is now U(1) x SU(2) x U(1) x U(3). In order to reduce this to U(1) x SU(2) x SU(3), let us denote the U(1) part of the connection on \mathfrak{E} by -Y. Let us require that the U(1) part of the connection on \mathfrak{F} be -Y, and that the trace of the U(3) part of the connection on \mathfrak{F} be Y. Upon computing the action as in section III, we obtain a Euclidean version of the standard model with Y as the hypercharge generator. We will discuss the possible relations among the coupling constants in section VI.

When written in Minkowski space, there are some slight differences due to the different properties of charge conjugation. Using the same matrices C and γ as above, when we rotate to a (+, -, -, -) Minkowski space the charge conjugation operator becomes $R(v) = \gamma^0 C^{-1} \bar{v}$, which changes chirality. We will again take

$$\mathfrak{Q} = C^\infty(Z) \otimes (\mathbb{C} \oplus \mathbb{H}) \text{ and}$$

$\mathfrak{K} = L^2(S) \otimes (\mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2) \otimes \mathbb{C}^{N_G}$, with the same action of \mathfrak{Q} on \mathfrak{K} . The operator D will be given by

$$D(v) = (\not{D} \otimes (I_2 \oplus I_1 \oplus I_2 \oplus I_2) \otimes I_{N_G}) v +$$

$$i\gamma \otimes \begin{pmatrix} 0 & 0 & M_1^\dagger & 0 \\ 0 & 0 & 0 & 0 \\ M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v +$$

$$i\gamma \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2^T \\ 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \end{pmatrix} R(v).$$

(Our pure Dirac operator satisfies $\mathcal{D}^2 = -\partial_0^2 + \partial_j^2$.) The \mathbb{Z}_2 -grading will be given by

$\Gamma = \gamma \otimes (I_2 \oplus (-I_1) \oplus (-I_2) \oplus (-I_2)) \otimes I_{N_G}$. Of the various factors of \mathfrak{K} , the first \mathbb{C}^2 factor corresponds to (e_R, d_R) , the \mathbb{C} factor corresponds to u_R^c , the second \mathbb{C}^2 factor corresponds to (e_L, d_L) and the third \mathbb{C}^2 factor corresponds to (v_L, u_L) . The rest of the discussion extends straightforwardly to Minkowski space. We will make the physical fermions chiral by requiring $\Gamma\psi = \psi$.

As with the previous models, it is possible to see our noncommutative spacetime geometry as the product of an ordinary spacetime geometry and the geometry over a point. We will do this for the electroweak geometry, for which it is convenient to use quaternions throughout and take tensor products over the quaternions. In the Euclidean case we use the fact that the spinor representation of $\text{Spin}(4)$ is quaternionic, that is, the representation commutes with the generators of a quaternion algebra. This can be seen by noting that the double covering of the embeddings

$$\text{SO}(4) \rightarrow \text{SO}(5) \rightarrow \text{SO}(5,1) \text{ is}$$

$(\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)) \rightarrow \text{U}(2, \mathbb{H}) \rightarrow \text{SL}(2, \mathbb{H})$. The spinor bundle S has fiber \mathbb{H}^2 , and we will take the Clifford algebra to act on the right. Left multiplication by i defines a complex structure on \mathbb{H}^2 , and the operator L_j of left multiplication by j then defines charge conjugation. L_j is complex-antilinear, and because the chirality operator γ is right multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, L_j preserves chirality. The ordinary spacetime geometry is then given by

$\mathcal{A}_2 = C^\infty(Z)$ (a real algebra), $\mathfrak{K}_2 = L^2(S)$ with the right \mathcal{A}_2 action,

$$D_2 = \begin{pmatrix} 0 & \partial_0 + i\partial_1 + j\partial_2 + k\partial_3 \\ -\partial_0 + i\partial_1 + j\partial_2 + k\partial_3 & 0 \end{pmatrix}$$

(a real-symmetric operator acting on the right) and $\Gamma_2 = \gamma$.

Over a point, we will take the algebra to be

$\mathcal{A}_1 = \mathbb{C} \oplus \mathbb{H}$, where we will think of $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ as a subalgebra of \mathbb{H} . The (real) Hilbert space will be

$\mathfrak{K}_1 = (\mathbb{H}^2 \oplus \mathbb{H}^1 \oplus \mathbb{H}^2 \oplus \mathbb{H}^2)^{N_G}$ with the action of

$(f, q) \in \mathcal{A}_1$ given by

$$(f, q) v = \left(\begin{pmatrix} fI_2 & 0 & 0 & 0 \\ 0 & fI_1 & 0 & 0 \\ 0 & 0 & qI_2 & 0 \\ 0 & 0 & 0 & qI_2 \end{pmatrix} \otimes I_{N_G} \right) v. \text{ The relation}$$

between this quaternionic description of the action of the algebra and the complex description is given by a change of basis, using the identity

$$\frac{1}{2} \begin{pmatrix} -i & -k \\ -1 & j \end{pmatrix} \begin{pmatrix} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ -\beta_R + i\beta_I & \alpha_R - i\alpha_I \end{pmatrix} \begin{pmatrix} i & -1 \\ k & -j \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_R + i\alpha_I + j\beta_R + k\beta_I & 0 \\ 0 & \alpha_R + i\alpha_I + j\beta_R + k\beta_I \end{pmatrix}$$

The operator D_1 is $\begin{pmatrix} 0 & \mathcal{Z}^\dagger \\ \mathcal{Z} & 0 \end{pmatrix}$, with

$$\mathcal{Z} = \begin{pmatrix} -M_{1I} + iM_{1R} & M_{2R} + iM_{2I} \\ M_{1R} + iM_{1I} & -M_{2I} + iM_{1R} \end{pmatrix}$$

Here M_{1R} and M_{1I} are the real and imaginary parts of M_1 , and similarly for M_2 .

We will take Γ_1 to be $\begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & -I_2 \end{pmatrix} \otimes I_{N_G}$.

D_1 takes its particular form because it anticommutes with

Γ_1 , commutes with $\begin{pmatrix} iI_2 & 0 & 0 & 0 \\ 0 & iI_1 & 0 & 0 \\ 0 & 0 & iI_2 & 0 \\ 0 & 0 & 0 & iI_2 \end{pmatrix} \otimes I_{N_G}$, which is the

translation of the fact that in the complex description D acts complex linearly on the down quarks and complex-antilinearly on the up quarks, and commutes with

$\begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & -I_1 & 0 & 0 \\ 0 & 0 & 0 & iI_2 \\ 0 & 0 & -iI_2 & 0 \end{pmatrix} \otimes I_{N_G}$, which is the translation of the

fact that D does not mix the up and down quarks.

The product geometry is now given by

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{K} = \mathcal{K}_1 \otimes_{\mathbb{H}} \mathcal{K}_2,$$

$$D = D_1 \otimes \Gamma_2 + I \otimes D_2 \text{ and } \Gamma = \Gamma_1 \otimes \Gamma_2.$$

In the Minkowski case one can perform a similar construction using the fact that with signature $(+, -, -, -)$, the spinor representation is again quaternionic. This can be seen by noting that the double covering of the embeddings $SO(3,1) \rightarrow SO(4,1) \rightarrow SO(5,1)$ is

$(\text{Spin}(3,1) = SL(2, \mathbb{C})) \rightarrow U(1,1, \mathbb{H}) \rightarrow SL(2, \mathbb{H})$. The spinor bundle S again has fiber \mathbb{H}^2 , with the Clifford algebra acting on the right. Left multiplication by i defines a complex structure on \mathbb{H}^2 , and the operator L_j of left multiplication by j defines charge conjugation, a complex-antilinear operator. The (antisymmetric) chirality operator γ is right multiplication by i , which commutes with charge conjugation. The ordinary spacetime geometry is then given by

$\mathcal{A}_2 = C^\infty(\mathcal{Z})$ (a real algebra), $\mathcal{K}_2 = L^2(S)$ with the right \mathcal{A}_2 action,

$$D_2 = \begin{pmatrix} j\partial_3 + k\partial_4 & j\partial_0 + j\partial_1 \\ -j\partial_0 + j\partial_1 & j\partial_3 + k\partial_4 \end{pmatrix}$$

(a real operator acting on the right) and $\Gamma_2 = \gamma$.

For the point geometry, we take the same \mathcal{A}_1 and \mathcal{K}_1 as

in the Euclidean case, and we take $D_1 = \begin{pmatrix} 0 & -\mathcal{Z}^\dagger \\ \mathcal{Z} & 0 \end{pmatrix}$,

$\Gamma_1 = \begin{pmatrix} iI_2 & 0 & 0 & 0 \\ 0 & -iI_1 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \end{pmatrix} \otimes I_{N_G}$. The product geometry

is given by

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{K} = \mathfrak{K}_1 \otimes_{\mathbb{H}} \mathfrak{K}_2,$$

$$D = D_1 \otimes \Gamma_2 + I \otimes D_2 \text{ and } \Gamma = \Gamma_1 \otimes \Gamma_2.$$

VI. Discussion

When one considers the renormalizability of our models, it will be important to know how restricted our actions are. First, it is possible that there are other operator-theoretically defined bosonic actions which reproduce the Yang-Mills action in the ordinary case. We will only consider here the simplest possibility, based on the Dixmier trace, as given in section II, (13). One can then consider two possibilities. The first uses the pure Dixmier trace to define the action, as in II (13). The second uses the fact that one can break I_{∇} into its various gauge-invariant pieces. That is, if $\{P_i\}$ are projections from $\mathfrak{B}(\mathfrak{E} \otimes_{\mathcal{A}} \mathfrak{K})$ to itself which satisfy $P_i(uTu^{-1}) = P_i(T)$ for all $u \in \mathcal{U}$ and $T \in B(\mathfrak{E} \otimes_{\mathcal{A}} \mathfrak{K})$, then there is a gauge-invariant action given by

$$I = \sum c_j \text{Tr}_{\omega}((P_i(\pi(\theta)))^2 D_{\nabla}^{-2k}) \text{ for arbitrary constants } \{c_j\}.$$

Let us illustrate this for the standard model of section V. Even if, to give the first possibility, we use the pure Dixmier trace, there is an arbitrary constant in front. For the standard model there are two distinct connections, coming from \mathfrak{E} and \mathfrak{F} , and two unrelated Hilbert spaces corresponding to the leptons and quarks. There is no reason why the four ensuing Dixmier traces should be related, and so we have four arbitrary constants. This leads to one relation among the masses and coupling constants. In the limit in which the other fermion masses are negligible compared to the top quark mass, we find that the relation becomes that the Higgs mass is $2^{1/2}$ times the top mass. On the other hand, with the second possibility we find the same amount of arbitrariness for the constants as in the usual standard model.

If it turns out that the mass relation is satisfied, it might indicate that our action arises from a more unified operator-theoretic action. One could speculate that the special action

obtained from a pure constant times the Dixmier trace is singled out among others as the restriction of an action with a much larger and simple invariance group of unitary operators.

We should stress two unsatisfactory features of our version of the standard model. The first is the need to relate the various $U(1)$ factors in order to get the right gauge group and hypercharge assignments. The second has to do with the different role played by the \mathfrak{B} algebra and the chromodynamics sector. There may be some hint to the meaning of this from looking at what happens over a point in ordinary spacetime. There the role of the \mathfrak{B} algebra can be seen as a change of the ground ring \mathbb{C} to $\mathbb{C} \oplus \mathbb{C}$, which is well understood as a feature of Kasparov's bivariant KK theory [Ka].

Let us note that as all of the fermion representations for our models arise from representations of algebras, the fermions must be in fundamental representations of the gauge groups. Thus it is impossible to get GUTs by our methods. We also need nonsimple gauge groups in order to have Higgs fields.

In conclusion, one way to look at this paper is the following. In the same way that Minkowski space arises naturally from Maxwell's theory, we look for the modification of spacetime arising from the electroweak unification. We want to find a theory on a (possibly) noncommutative spacetime which is comparable in simplicity to that of electrodynamics on an ordinary spacetime. Our noncommutative spacetime geometry can be considered as phenomenological, in that it is obtained unambiguously from the phenomenological action of the standard model.

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