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# Multichannel Nonlinear Scattering for Nonintegrable Equations II. The Case of Anisotropic Potentials and Data*

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The nonlinear scattering and stability results of Soffer and Weinstein (*Comm. Math. Phys.* (1990)) are extended to the case of *anisotropic* potentials and data. The range of nonlinearities for which the theory is shown to be valid is also extended considerably. 

## 1. INTRODUCTION

We consider the initial value problem for the nonlinear Schrödinger equation (NLS)

$$i\frac{\partial\Phi}{\partial t}=(-\Delta+V)\,\Phi+\lambda|\Phi|^{m-1}\,\Phi, \qquad 1<m<\frac{n+2}{n-2}$$
$$\Phi(x,0)=\Phi_0(x)\in H^1(\mathbf{R}^n). \tag{1.1}$$

In [Sof–Wei] we developed a nonlinear scattering and stability theory of NLS with dynamics characterized by the interaction of a single nonlinear bound state channel with a dispersive channel. We showed that a class of small solutions (including any solution in a sufficiently small neighborhood of a small amplitude nonlinear bound state) decomposes into a sum of two terms: (1) a nonlinear bound state with modulating (time-varying) energy,

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$E(t)$, and phase, $\gamma(t)$, and (2) a term, $\phi(t)$, which disperses to zero with free linear dispersion rates. The *collective coordinates* $E(t)$ and $\gamma(t)$ satisfy *modulation equations* and have the form of a system of ordinary differential equations perturbed by interaction terms of the infinite dimensional dispersive channel. In the infinite time limit, $t \to \pm\infty$, the energy and phase tend to asymptotic limits $E^{\pm}$ and $\gamma^{\pm}$. Nonlinear bound states are found to be asymptotically stable in the sense that in response to a small perturbation, the solution will *converge* in $L^p$ $(p>2)$ to a nonlinear bound state of possibly different energy and phase. The results, for $n \geqslant 2$, of [Sof–Wei] required (a) restrictive hypothesis on the range of nonlinearities and (b) that the potential $V$ and the data $\Phi_0$ be symmetric about some common origin of coordinates. In this paper we extend our results for $n \geqslant 3$ to the case of anisotropic potentials, anisotropic data, and remove certain unnatural restrictions on the range of nonlinearities. The main new ingredient in the analysis is the use of recently obtained $L^p$ decay estimates $(p>2)$ for $e^{i(-\Delta+V)t}$ [JSS].

For a discussion of previous results on nonlinear scattering (in the absence of bound states or for completely integrable equations) and stability (in the sense of Lyapunov) with references see the introduction to [Sof–Wei].

*Notation*

All integrals are assumed to be taken over $\mathbf{R}^n$ unless otherwise specified. $\Re(z)$ and $\Im(z)$ denote, respectively, the real and imaginary parts of the complex number $z$

$\Delta =$ Laplacian on $L^2(R^n)$

$\langle x \rangle = (1 + x \cdot x)^{1/2}$ where $x \in R^n$

$\langle f, g \rangle = \int f^* g$, where $f^*$ denotes the complex conjugate of $f$

$L^p = L^p(R^n)$

$H^s = \{f : (I - \Delta)^{s/2} f \in L^2\}$

$C(I; X) =$ the space of functions, $u(t, x)$, which are continuous in $t$, with values in the space $X$.

## 2. Background and Statement of Results

The well-posedness theory of NLS (see, for example, [GV] and others cited in [Sof–Wei]) implies that given initial data $\Phi_0 \in H^1(\mathbf{R}^n)$, there exists a unique local solution $\Phi \in C([0, T_{\max}); H^1(\mathbf{R}^n))$, where either $T_{\max} = +\infty$ or

$$\limsup_{t \uparrow T_{\max}} \|\nabla \Phi(t)\|_{L^2} = +\infty.$$

Furthermore, during the time interval of existence the following functionals remain time-independent on solutions:

$$\mathscr{H}[\varphi] \equiv \int \frac{1}{2} |\nabla\varphi|^2 + \frac{1}{2} V(x) |\varphi|^2 + \frac{1}{m+1} |\varphi|^{m+1} \, dx \tag{2.1}$$

$$\mathscr{N}[\varphi] \equiv \int |\varphi|^2 \, dx. \tag{2.2}$$

For the scattering and stability results of this paper, we shall require $n \geqslant 3$ and the following hypotheses on the linear potential $V(x)$:

(V1) $\langle x \rangle^{\alpha} V(x)$: $H^{\eta} \to H^{\eta}$ for some $\alpha > n+4$ and $\eta > 0$

(V2) $\hat{V} \in L^1$

(V3) 0 is neither an eigenvalue nor a resonance (see, for example, [Je–Ka] or Section 2.3 of [Sof–Wei]).

These hypotheses ensure the applicability of the linear Schrödinger decay estimates of the following:

THEOREM 2.1 [JSS]. *Let $H = -\Delta + V$ acting on $L^2(\mathbf{R}^n)$, and assume hypotheses* (V1), (V2), (V3) *on $V(x)$. Let $P_c(H)$ denote the projection onto the continuous spectral part of $H$. If $p^{-1} + q^{-1} = 1$, $2 \leqslant q \leqslant \infty$ then*

$$\|e^{itH} P_c(H) \psi\|_{L^q} \leqslant \frac{C_{n,q}}{|t|^{(n/2)-(n/q)}} \|\psi\|_{L^p}. \tag{2.3}$$

A simple and useful consequence is the following *local decay estimate*:

COROLLARY 2.2. *Assume that $H \equiv -\Delta + V$ satisfies the hypotheses* (V1), (V2), *and* (V3), *and let $\sigma > n((1/2)-(1/p))$. Then*

$$\begin{gathered} \|\langle x \rangle^{-\sigma} e^{iHt} P_c(H) \psi\|_{L^2} \leqslant C_{n,q,\sigma} |t|^{(n/p)-(n/2)} \|\psi\|_{L^q}, \\ p^{-1} + q^{-1} = 1, \qquad p \geqslant 2 \end{gathered} \tag{2.4}$$

*Proof.* Let $p_1^{-1} = 2^{-1} - p^{-1}$. Then,

$$\begin{aligned} \|\langle x \}^{-\sigma} e^{iHt} P_c(H) \psi\|_{L^2} &\leqslant \|\langle x \rangle^{-\sigma}\|_{L^{p_1}} \|e^{iHt} P_c(H) \psi\|_{L^p} \\ &\leqslant \frac{C_{n,p} \|\langle x \rangle^{-\sigma}\|_{L^{p_1}}}{|t|^{(n/2)-(n/p)}} \|\psi\|_{L^q}. \end{aligned}$$

The requirement of a "single bound state channel" interacting with dispersion is captured in our last hypotheses on $V(x)$:

(V4) $H \equiv -\Delta + V(x)$ acting on $L^2(\mathbf{R}^n)$ has exactly one negative eigenvalue $E_* < 0$, with corresponding $L^2$ normalized eigenfunction $\psi_*$.

Nonlinear bound states (also termed solitary or standing waves) are finite energy, localized solutions of the form

$$\Phi(x, t) = \psi_E(x)\, e^{-iEt + i\gamma}, \tag{2.5}$$

where

$$(-\Delta + V(x) + \lambda|\psi_E(x)|^{m-1})\, \psi_E(x) = E\psi_E(x). \tag{2.6}$$

A consequence of (V4) is the existence of a two-parameter family of nonlinear bound state solutions to NLS of the form (2.6) which bifurcate from the zero solution at energy $E_*$ (see Section 2.2 in [Sof–Wei] and others cited therein). Its properties are summarized in

THEOREM 2.3. *For $\lambda > 0$, let $E \in (E_*, 0)$ and for $\lambda < 0$, let $E < E_*$. Then, there exists a solution $\psi_E$ of* (2.5) *such that for $s \geqslant 0$*

(a) $\psi_E \in H^s$, $\psi_E > 0$

(b) *The function $E \mapsto \|\psi_E\|_{H^s}$ is smooth for $E \neq E_*$, and*

$$\lim_{E \to E_*} \|\psi_E\|_{H^s} = 0,$$

*i.e., the solution $\psi_E(\cdot)\, e^{i\gamma}$, whose parameters lie in the cylinder $(E_*, 0)_E \times S^1_\gamma$ $(\lambda > 0)$ or $(-\infty, 0)_E \times S^1_\gamma$ $(\lambda < 0)$, bifurcates in $H^s$ from the zero solution at $E = E_*$. Furthermore, if $E$ lies in a sufficiently small neighborhood of $E_*$, we have for $k \in \mathbf{Z}_+$ and $s \geqslant 0$*

$$\|\langle x\rangle^k\, \psi_E\|_{H^s} \leqslant C_{k,s,n} \|\psi_E\|_{H^s}$$

$$\|\langle x\rangle^k\, \partial_E \psi_E\|_{H^s} \leqslant \tilde{C}_{k,s,n} |E - E_*|^{-1}\, \|\psi_E\|_{H^s}.$$

We prove the following:

THEOREM 2.4 (Scattering). *Let $n \geqslant 3$ and assume*

$$2 < m < \frac{n+2}{n-2}.$$

*There is a continuous function $\omega$: $E \mapsto \omega(E)$, $E \in I_0$, (where $I_0$ denotes a real open interval with endpoints $E_*$ and $E_* + \mathrm{sgn}(\lambda)\eta$, and $\eta$ sufficiently small, and an open* cone-like *region of the form*

$$\mathscr{C}_0 = \bigcup_{E \in I_0} \{f \in H^1 \colon \|f - \psi_E\|_{H^1} < \omega(E)\},$$

*such that if $\Phi_0 \in \mathscr{C}_0$, then*

$$\Phi(t) = \exp^{-i\int_0^t E(s)\, ds + i\gamma(t)}(\psi_{E(t)} + \phi(t))$$

*with*

$$\frac{dE}{dt} \in L^1(\mathbf{R}^1) \qquad (\textit{so that } \lim_{t \to \pm\infty} E(t) = E^{\pm} \textit{ exists}),$$

$$\frac{d\gamma}{dt} \in L^1(\mathbf{R}^1) \qquad (\textit{so that } \lim_{t \to \pm\infty} \gamma(t) = \gamma^{\pm} \textit{ exists}),$$

*and* $\|\phi(t)\|_{m+1} = O(\langle t \rangle^{-1-\varepsilon})$ *as* $|t| \to \infty$.

In the proof below, as well as in [Sof–Wei], we work under the hypotheses:

(i) There is a number $\delta_0$ such that $\|\Phi_0\|_{H^1} \leqslant \delta_0$,

(ii) $\min_{E,\Theta} \|\Phi_0 - e^{i\Theta}\psi_E\|_{H^1}$, subject to the constraint $\langle e^{i\Theta}\psi_E, \Phi_0 - e^{i\Theta}\psi_E \rangle = 0$ has a non-zero solution $\psi_{E_0} e^{i\Theta_0}$, i.e., $E_0 \neq E_*$, and

(iii) $V(x) + \lambda |\psi_{E_0}|^{m-1}$ satisfies the nonresonance condition (V3).

As explained in [Sof–Wei], the nonresonance condition is generally satisfied. It is also shown that assumptions (i) and (ii) hold at least for all $\Phi_0$ in an open *cone-like* region with vertex at the origin, described above. Thus, $\omega(E)$ may tend to zero as $E \to E_*$.

The following is a related asymptotic stability result showing that data near a nonlinear bound state of energy $E_0$ and phase $\gamma_0$ decays (via dispersion) as $t \to \pm\infty$ to a nearby nonlinear bound state of energy $E^{\pm}$ and phase $\gamma^{\pm}$.

THEOREM 2.5 (Asymptotic Stability). *Let* $n \geqslant 3$ *and* $2 < m < (n+2)/(n-2)$. *Let* $\Omega_\eta$ *denote the interval of real numbers with endpoints* $E_*$ *and* $E_* + \eta \operatorname{sgn}(\lambda)$, *where* $\eta$ *is positive and sufficiently small. Then, for all* $E_0 \in \Omega_\eta$ *and* $\gamma_0 \in [0, 2\pi)$, *there is a positive number* $\varepsilon(\eta, E_0)$ *such that if* $\Phi(0) = (\psi_{E_0} + \phi(0))\, e^{i\gamma_0}$ *with* $\|\phi(0)\|_{H^1} \leqslant \varepsilon$, *then* $\Phi(t)$ *decomposes into localized and dispersive parts as in Theorem* 2.4.

## 3. ANSATZ AND THE COUPLED CHANNEL EQUATIONS

Following [Sof–Wei] we decompose the solution as follows:

*Ansatz*

$$\begin{aligned} \Phi(t) &= e^{-i\Theta}(\psi_{E(t)} + \phi(t)) \\ \Phi(0) &= e^{i\gamma_0}(\psi_{E_0} + \phi(0)) \\ \Theta &\equiv \int_0^t E(s)\, ds - \gamma(t) \\ E(0) &= E_0, \qquad \gamma(0) = \gamma_0. \end{aligned} \tag{3.1}$$

Here, $\psi_E$ denotes the ground state of (2.5),

$$\begin{gathered} H(E)\,\psi_E=(-\Delta+V+\lambda|\psi_E|^{m-1})\,\psi_E=E\psi_E \\ \psi_E\in H^2, \qquad \psi>0, \end{gathered} \tag{3.2}$$

for $E\in(E_*,0)$ if $\lambda>0$ and $E\in(-\infty,E_*)$ if $\lambda<0$, where

$$E_*\equiv\inf\sigma(-\Delta+V)<0.$$

*Orthogonality Condition*

$$\langle\psi_{E_0},\phi_0\rangle=0 \qquad \text{and} \qquad \frac{d}{dt}\langle\psi_{E_0},\phi(t)\rangle=0. \tag{3.3}$$

Note that this orthogonality condition implies that for all $t$, $\phi(t)\in$ Range($P_c(K)$), the continuous spectral part of a *fixed* Hamiltonian, $K$.

As in [Sof–Wei, Section 3] after substitution of (3.1) into NLS and imposing (3.3) we obtain the coupled system:

$$\begin{aligned} &i\frac{\partial\phi}{\partial t}=(H(E_0)-E_0)\,\phi+(E_0-E(t)+\dot\gamma(t))\,\phi+\mathbf{F}(t), \\ &\phi(0)=\phi_0=P_c(H(E_0))\,\phi_0 \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\text{(a)}\quad \dot E(t)=-\langle\partial_E\psi_E,\psi_{E_0}\rangle^{-1}\,\Im\langle\mathbf{F}_2,\psi_{E_0}\rangle, \\ &\text{(b)}\quad \dot\gamma(t)=\langle\psi_E,\psi_{E_0}\rangle^{-1}\,\Re\langle\mathbf{F}_2,\psi_{E_0}\rangle. \end{aligned} \tag{3.5}$$

Here

$$\begin{gathered} \mathbf{F}\equiv\mathbf{F}_1+\mathbf{F}_2, \\ \mathbf{F}_1\equiv\dot\gamma\psi_E-i\dot E\,\partial_E\psi_E, \end{gathered} \tag{3.6}$$

and

$$\mathbf{F}_2\equiv\mathbf{F}_{2,\mathrm{lin}}+\mathbf{F}_{2,\mathrm{nl}}.$$

$\mathbf{F}_{2,\mathrm{lin}}$ is a term which is linear in $\phi$,

$$\mathbf{F}_{2,\mathrm{lin}}(\phi,\psi)=\lambda\left(\frac{m+1}{2}\psi_E^{m-1}-\psi_{E_0}^{m-1}\right)\phi+\frac{m-1}{2}\psi_E^{m-1}\phi^*, \tag{3.7}$$

and $\mathbf{F}_{2,\mathrm{nl}}$ is a term that is nonlinear in $\phi$ such that

$$|\mathbf{F}_{2,\mathrm{nl}}(\phi,\psi)|\leqslant|\lambda|\,c[A(\psi)\,|\phi|^2+|\phi|^m], \tag{3.8}$$

where $|A(s)|$ is bounded for $s$ bounded, $|A(s)| \to 0$ as $s \to 0$, and $c$ is independent of $\psi$ and $\phi$. Equations (3.4)–(3.5) comprise a coupled system for the interacting dispersive, (3.4), and bound state, (3.5), channels.

To prove Theorem 2.4 and Theorem 2.5 it will suffice to obtain a priori bounds on $\dot{E}$, $\dot{\gamma}$ and $\phi(t)$ in suitable norms. This is carried out in the next section.

## 4. A Priori Estimates of Solutions to the Coupled Channel Equations

With a view toward obtaining à priori estimates on the solution of (3.4)–(3.5) we first rewrite (3.4) as an integral equation:

$$\phi(t) = U(t, 0)\, \phi_0 - i \int_0^t U(t, s)\, P_0(H(E_0))\, F(s)\, ds. \tag{4.1}$$

Here, $U(t, s)$ denotes the propagator associated with the flow

$$\begin{aligned} \frac{\partial u}{\partial t} &= (H(E_0) - E_0)\, u + (E_0 - E(t) - \dot{\gamma}(t))\, u(t) \\ u(s) &= f, \end{aligned} \tag{4.2}$$

that is,

$$\begin{aligned} U(t, s) = \exp &\left( -i \int_s^t (E_0 - E(\tau))\, d\tau - i(\gamma(t) - \gamma(s)) \right) \\ &\exp(i(H(E_0) - E_0)(t - s)) \end{aligned} \tag{4.3}$$

(see [Sof–Wei]).

In this section we estimate the coupled channel equations and show that for $E$ near $E_*$ and $\phi_0$ small, in a precise sense, $\phi(t)$ is dispersive and $E(t)$, $\gamma(t)$ have asymptotic limits as $t \to \pm\infty$.

Theorem 4.1. *Let $n \geqslant 3$, and $m_*(n) < m < (n+2)/(n-2)$. Assume*

$$\|\phi_0\|_2 + \|\phi_0\|_1 + \|\phi_0\|_{m+1} \qquad \text{and} \qquad |E_0 - E_*|$$

*are sufficiently small. Then, $\phi(t)$ is dispersive, in the sense that*

$$\sup_{t \in \mathbf{R}} \langle t \rangle^{\zeta} \| \langle x \rangle^{-\sigma} \phi(t) \|_2 \leqslant C'(\|\phi_0\|_1 + \|\phi_0\|_2),$$

*where* $\zeta \equiv \min((n/2), 1+\varepsilon)$, $\varepsilon > 0$ *and*

$$\sup_{t \in \mathbf{R}} \langle t \rangle^{(n/2) - n/(m+1)} \|\phi(t)\|_{m+1} \leq C''(\|\phi_0\|_1 + \|\phi_0\|_2).$$

*The positive number* $\varepsilon$ *is defined below. Furthermore,* $\dot{E}$, $\dot{\gamma} \in L^1$, *and in particular*

$$\sup_t \langle t \rangle^{\zeta} (|\dot{\gamma}(t)| + |\dot{E}(t)|) \leq C'''(\|\phi_0\|_1 + \|\phi_0\|_2).$$

Our aim is to obtain a closed system for the local decay norm $\|\langle x \rangle^{-\sigma} \phi(t)\|_2$ and $\|\phi(t)\|_q$, for some $q > 2$.

In the proof the following dimension-dependent number arises. Let $m_*(n)$ denote the positive root of the quadratic equation

$$f_n(m) = m^2 - \left(1 + \frac{2}{n}\right) m - \frac{2}{n} = 0,$$

i.e.,

$$m_*(n) = \frac{1}{2} + \frac{1}{n} + \frac{1}{2n} (n^2 + 12n + 4)^{1/2}.$$

Note that $m_*(3) = 2$, and that $m_*(n)$ is decreasing with increasing $n$, for $n \geq 3$.

*Estimation of* $\|\phi(t)\|_{L^{m+1}}$

Using (4.1), the specific form of $F(s)$ as detailed in (3.6)–(3.7), and the decay estimates of Section 2 we have, for $q^{-1} + p^{-1} = 1$, $q > 2$:

$$\begin{aligned}
\|\phi(t)\|_q &\leq \|U(t,0)\,\phi_0\|_q + \int_0^t \|U(t,s)\,P_c(H(E_0))\,F(s)\|_q\,ds \\
&\leq C\langle t \rangle^{-(n/2)+(n/q)} (\|\phi_0\|_p + \|\phi_0\|_2) \\
&\quad + \int_0^t |t-s|^{-(n/2)+(n/q)} \{|\dot{\gamma}|\,\|\psi_E\|_p + |\dot{E}|\,\|\psi_E\|_p \\
&\quad + \|\psi_E^{m-1}\phi\|_p + \|A(\psi)\,|\phi|^2\|_p + \||\phi|^m\|_p\}\,ds \\
&= C\langle t \rangle^{-(n/2)+(n/q)} (\|\phi_0\|_p + \|\phi_0\|_2) \\
&\quad + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}),
\end{aligned} \tag{4.4}$$

where terms (I)–(V) denote the integral terms in (4.4). We estimate the integrals individually.

*Estimation of* (I)

$$
\begin{aligned}
|\dot{\gamma}|\ \|\psi_E\|_p &\leqslant C|\langle\psi_E,\psi_{E_0}\rangle|^{-1}\ \|\psi_E\|_p\ |\langle F_2,\psi_{E_0}\rangle| \text{ by (3.5b)}\\
&\leqslant C|\langle\psi_E,\psi_{E_0}\rangle|^{-1}\ \|\psi_{E_0}\|_p\ [\,|\langle\psi_E^{m-1}|\phi|,\psi_{E_0}\rangle|\\
&\quad +|\langle|A(\psi)|\ |\phi|^2,\psi_{E_0}\rangle|+|\langle|\phi|^m,\psi_{E_0}\rangle|\,], \qquad \text{by (3.6)(3.7)},\\
&\leqslant C|\langle\psi_E,\psi_{E_0}\rangle|^{-1}\ \|\psi_{E_0}\|_p\\
&\quad\times[\,\|\langle x\rangle^\sigma\psi_E^{m-1}\psi_{E_0}\|_2\ \|\langle x\rangle^{-\sigma}\phi(t)\|_2)\\
&\quad+\|\langle x\rangle^{2\sigma}A(\psi)\,\psi_{E_0}\|_\infty\ \|\langle x\rangle^{-\sigma}\phi(t)\|_2^2+\|\psi_{E_0}\|_q\ \|\phi\|_{pm}^m]. \qquad (4.5)
\end{aligned}
$$

Similarly, we have by (3.5a) instead of (3.5b), the following estimate of (II):

$$
\begin{aligned}
|\dot{E}|\ \|\partial_E\psi_E\|_p \leqslant C|\langle\partial_E\psi_E,\psi_{E_0}\rangle|^{-1}\ &\|\partial_E\psi_E\|_p\\
&+[\,\|\langle x\rangle^\sigma\psi_E^{m-1}\psi_{E_0}\|_2\ \|\langle x\rangle^{-\sigma}\phi(t)\|_2\\
&+\|\langle x\rangle^{2\sigma}A(\psi)\,\psi_{E_0}\|_\infty\ \|\langle x\rangle^{-\sigma}\phi(t)\|_2^2\\
&+\|\psi_{E_0}\|_q\ \|\phi(t)\|_{pm}^m]. \qquad (4.6)
\end{aligned}
$$

*Estimation of* (III)

$$\|\psi_E^{m-1}\phi(t)\|_p\leqslant\|\langle x\rangle^\sigma\psi_E^{m-1}\|_r\ \|\langle x\rangle^{-\sigma}\phi(t)\|_2, \qquad (4.7)$$

where

$$r^{-1}+2^{-1}=p^{-1}.$$

*Estimation of* (IV)

$$\|A(\psi)\ |\phi|^2\|_p\leqslant\|A(\psi)\|_{p_1}\ \|\phi\|_q^2, \qquad (4.8)$$

where

$$p_1^{-1}+\left(\frac{q}{2}\right)^{-1}=p^{-1}.$$

*Estimation of* (V)

$$\|\,|\phi|^m\|_p=\|\phi\|_{pm}^m. \qquad (4.9)$$

It is clear from our estimates of (I)–(V) that a natural choice of $L^q$ norm is one for which $q=mp$, or since $p^{-1}+q^{-1}=1$, we have $q=m+1$. Note that with this choice of $q$, the exponents $r$ and $p$ above are well defined and positive so long as $m\geqslant 2$.

We shall proceed by deriving à priori decay estimates for $\|\phi(t)\|_q$ and $\|\langle x\rangle^{-\sigma}\phi(t)\|_2$, on some finite time interval $|t|\leqslant T$. To obtain a closed set of coupled inequalities we must estimate $\|\phi(t)\|_2$, as well. The decay rates of $\|\phi(t)\|_q$ and $\|\langle x\rangle^{-\sigma}\phi(t)\|_2$ are those suggested by linear theory, except for a small range of values of $m$ where the local decay rate is taken to be weaker, though integrable. (This constraint appears to be of a technical nature.)

We introduce the quantities

$$M_1(T)=\sup_{|t|\leqslant T}\langle t\rangle^{\zeta}\|\langle x\rangle^{-\sigma}\phi(t)\|_2$$

where

$$\zeta\equiv\min\left(\frac{n}{2},1+\varepsilon\right),$$

and where $\varepsilon>0$ is defined below.

$$M_2(T)\equiv\sup_{|t|\leqslant T}\langle t\rangle^{(n/2)-(n/q)}\|\phi(t)\|_q,$$

where

$$q\equiv m+1,$$

and

$$M_3(T)\equiv\sup_{|t|\leqslant T}\|\phi(t)\|_2.$$

Here, $n\geqslant 3$. When convenient, we shall write $M_j$ instead of $M_j(T)$. Using the above estimates in (4.4) we obtain

$$\begin{aligned}\|\phi(t)\|_q\leqslant{}& C\langle t\rangle^{-(n/2)+(n/q)}(\|\phi_0\|_p+\|\phi_0\|_2)\\&+C_q\int_0^t|t-s|^{-(n/s)+(n/q)}(\langle s\rangle^{-n/2}+\langle s\rangle^{m((n/q)-(n/2))}\,ds\\&\times(M_1(T)+M_1^2(T)+M_2^m(T)).\end{aligned}\tag{4.10}$$

Convergence of the time integral in (4.10) is controlled by the condition $m(n/(m+1)-(n/2))>1$ which holds if $m>m_*(n)$. Here, $C_q$ depends on suprema over $|t|\leqslant T$ of $\psi_{E(t)}$ and $\psi_{E_0}$ dependent coefficients appearing in the above estimates of (I)–(V). $C_q$ has the property that $C_q\downarrow 0$ as $E\to E_*$. Multiplication of both sides of (4.10) by $\langle t\rangle^{(n/2)-(n/q)}$ and taking supremum over $|t|\leqslant T$ yields

$$\begin{aligned}M_2(T)\leqslant{}& C(\|\phi_0\|_2+\|\phi_0\|_{(m+1)/m})\\&+C_q'(M_1(T)+M_1^2(T)+M_2^m(T)),\end{aligned}\tag{4.11}$$

where $C_q'\downarrow 0$ as $E\to E_*$.

*Estimation of* $\|\langle x\rangle^{-\sigma}\phi(t)\|_2$

As before, we begin with (4.1):

$$\begin{aligned}\|\langle x\rangle^{-\sigma}\phi(t)\|_2 \leqslant \|\langle x\rangle^{-\sigma} U(t,0)\,\phi_0\|_2 &+ \int_0^{t-1} \|\langle x\rangle^{-\sigma} U(t,s)\,P_c F(s)\|_2\,ds \\ &+ \int_{t-1}^{t} \|\langle x\rangle^{-\sigma} U(t,s)\,P_c F(s)\|_2\,ds \\ &\equiv A + B + C. \end{aligned} \tag{4.12}$$

The time integral has been split up this way to handle the non-integrable time singularity of $\langle x\rangle^{-\sigma} U(t,s)\,P_c$ at $t=s$ (see (2.4)). In the following estimate of $A$, $B$, and $C$, we use Corollary 2.2.

*Estimation of A*

$$\|\langle x\rangle^{-\sigma} U(t,0)\,\phi_0\|_2 \leqslant C\langle t\rangle^{-n/2}\,(\|\phi_0\|_1 + \|\phi_0\|_2).$$

*Estimation of B*

$$\begin{aligned}&\int_0^{t-1} \|\langle x\rangle^{-\sigma} U(t,s)\,P_c F(s)\|_2\,ds \\ &\leqslant \int_0^{t-1} \|\langle x\rangle^{-\sigma} U(t,s)\,P_c(F(s)-\lambda|\phi|^{m-1}\phi)\|_2\,ds \\ &\quad + \int_0^{t-1} \|\langle x\rangle^{-\sigma} U(t,s)\,P_c\lambda|\phi|^{m-1}\,\phi\|_2\,ds \\ &\leqslant C\int_0^{t-1} |t-s|^{-n/2}\,\|F(s)-\lambda|\phi|^{m-1}\,\phi\|_1\,ds \\ &\quad + \int_0^{t-1} C|t-s|^{(n/r)-(n/2)}\,\|\phi\|_{m\tilde{r}}^m\,ds, \\ &\quad (r^{-1}+\tilde{r}^{-1}=1,\ r>2), \\ &\leqslant C\int_0^{t-1} |t-s|^{-n/2}\,\{\|\psi_E\|_1\,|\dot{\gamma}| + \|\partial_E\psi_E\|_1\,|\dot{E}| \\ &\quad + \|\psi_E^{m-1}\phi\|_1 + \|A(\psi)\,|\phi|^2\|_1\}\,ds \\ &\quad + \int_0^{t-1} C|t-s|^{(n/r)-(n/2)}\,\|\phi\|_{m\tilde{r}}^m\,ds, \\ &= B1 + B2 + B3 + B4 + B5.\end{aligned}$$

The integrands of $B_1$ and $B_2$ satisfy the estimates (4.5)–(4.6) with $\|\psi_E\|_p$ replaced by $\|\psi_E\|_1$ and $\|\partial_E \psi_E\|_p$ replaced by $\|\partial_E \psi_E\|_1$.

$$B3\colon \|\psi^{m-1}\phi\|_1 \leqslant \|\langle x\rangle^{\sigma} \psi_E^{m-1}\|_2 \, \|\langle x\rangle^{-\sigma} \phi\|_2$$

$$B4\colon \|A(\psi)\,|\phi|^2\|_1 \leqslant \|\langle x\rangle^{2\sigma} A(\psi)\|_\infty \, \|\langle x\rangle^{-\sigma} \phi\|_2^2$$

$$B5\colon \|\phi\|_{m\bar{r}}^m = \|\phi\|_{mr/(r-1)}^m$$

We set $(n/2)-(n/r)=1+\varepsilon$, and observe that $\varepsilon$ is positive provided $r>2n/(n-2)$. We let $r=r(\delta)\equiv 2n/(n-2)+\delta$, with $\delta>0$. Thus, $\varepsilon(\delta)\downarrow 0$, and $r(\delta)\downarrow 2n/(n-2)$ as $\delta\downarrow 0$. Using further that

$$\frac{mr(\delta)}{r(\delta)-1}=\frac{2nm}{n+2}\left(1+\frac{n-2}{2n}\delta\right)\left(1+\frac{n-2}{n+2}\delta\right)^{-1}=\frac{2nm}{n+2}(1-\mathcal{O}(\delta)),$$

where $\mathcal{O}(\delta)\downarrow 0$ as $\delta\downarrow 0$, the integral (B5) now becomes

$$\int_0^{t-1} C|t-s|^{-1-\varepsilon(\delta)} \, \|\phi\|_{2nm(1-\mathcal{O}(\delta))/(n+2)}^m \, ds.$$

We next interpolate the norm appearing in the previous integral between $L^2$ and $L^{m+1}$ as

$$\|\phi\|_{2nm(1-\mathcal{O}(\delta))/(n+2)}^m \leqslant \|\phi\|_{m+1}^{m(m+1)/(m-1)(m-1-(2/n)-\mathcal{O}(\delta))} \|\phi\|_2^{\beta(m,n,\delta)},$$

where $\beta>0$.

Combining these estimates, we obtain

$$\begin{aligned} B &\equiv \int_0^{t-1} \|\langle x\rangle^{-\sigma} U(t,s)\, P_c F(s)\|_2 \, ds \\ &\leqslant C_q'' \int_0^{t-1} \langle s\rangle^{-\zeta} \, |t-s|^{-n/2} \, ds (M_1(T)+M_1^2(T)+M_2(T)) \\ &\quad + \int_0^{t-1} \langle s\rangle^{-\zeta} \, |t-s|^{-1-\varepsilon} \, ds \, M_3^{\beta}(T) \, M_2^{m(m+1)/(m-1)(m-1-(2/n)-\mathcal{O}(\delta))}(T), \end{aligned}$$

where

$$\zeta=\min\left(\frac{n}{2}, 1+\varepsilon\right).$$

It follows that

$$B\leqslant C_q'''\langle t\rangle^{-\zeta}\,(M_1+M_1^2+M_2+M_3^{\beta} M_2^{m(m+1)/(m-1)(m-1-(2/n)-\mathcal{O}(\delta))}), \tag{4.13}$$

where $C_q''' \downarrow 0$ as $E \to E_*$ provided $\zeta > 1$. This can be done for $\delta$ and therefore $\varepsilon$ sufficiently small provided $n(1/2 - 1/m+1) \times m(m+1)/(m-1)(m-1-(2/n))$ is larger than 1, or equivalently $m > m_*(n)$.

*Estimation of C*

$$\begin{aligned}
&\int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s)\, P_c F(s)\|_2\, ds \\
&\quad\leqslant \int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s)\, P_c F_{\mathrm{Lin}}(s)\|_2\, ds \\
&\qquad + \int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s)\, P_c F(_{\mathrm{N\,Lin}}(s)\|_2\, ds, \\
&\quad\equiv C1 + C2,
\end{aligned} \tag{4.14}$$

where $\varepsilon_0$ is small and to be chosen. $F_{\mathrm{Lin}}$ and $F_{\mathrm{N\,Lin}}$ denote respectively the parts of $F(s)$ which have linear and nonlinear functional dependence on $\phi$.

$$\begin{aligned}
C2\colon &\int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s)\, P_c F_{\mathrm{N\,Lin}}(s)\|_2\, ds \\
&\leqslant \int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma}\|_r\, \|U(t,s)\, P_c F_{\mathrm{N\,Lin}}(s)\|_q\, ds, \\
&\qquad r^{-1} + q^{-1} = 2^{-1} \\
&\leqslant C \int_{t-\varepsilon_0}^{t} |t-s|^{-(n/2)+(n/q)}\, \|F_{\mathrm{N\,Lin}}(s)\|_{L^p}\, ds, \\
&\qquad p^{-1} + q^{-1} = 1 \\
&\leqslant CC_q \int_{t-\varepsilon_0}^{t} |t-s|^{-(n/2)+(n/q)} \left(\langle s\rangle^{-n} + \langle s\rangle^{-m((n/2)-n/(m+1))}\right) ds \\
&\qquad \times (M_1^2(T) + M_2^m(T)), \qquad \text{(using the estimates of (I)–(V))} \\
&\leqslant CC_q \langle t\rangle^{-\eta} \int_{t-\varepsilon_0}^{t} |t-s|^{-(n/2)+(n/q)}\, ds (M_1^2(T) + M_2^m(T)),
\end{aligned}$$

where $\eta \equiv \min[n, m((n/2) - n/(m+1))]$. Thus,

$$\begin{aligned}
&\int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s)\, P_c F_{\mathrm{N\,Lin}}(s)\|_2\, ds \\
&\qquad \leqslant CC_q \langle t\rangle^{-\eta} (M_1^2(T) + M_2^m(T)).
\end{aligned} \tag{4.15}$$

$$C1: \int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s) P_c F_{\text{Lin}}(s)\|_2 \, ds$$
$$\leqslant C \int_{t-\varepsilon_0}^{t} \frac{1}{|t-s|^{(n/2)-n/(n+1)}} \|\langle x\rangle^{-\sigma} \phi(s)\|_2 \, ds$$
$$\leqslant C \int_{t-\varepsilon_0}^{t} \frac{1}{|t-s|^{(n/2)-n/(m+1)}} \frac{1}{\langle s\rangle^{n/2}} \, ds \, M_1(T)$$
$$\leqslant C\langle t\rangle^{-n/2} M_1(T)\, \varepsilon_0^{1-(n/2)+n/(m+1)}. \tag{4.16}$$

Combining estimates (4.15) and (4.16) we have

$$C \equiv \int_{t-\varepsilon_0}^{t} \|\langle x\rangle^{-\sigma} U(t,s) P_c F(s)\|_2 \, ds$$
$$\leqslant \tilde{C} C_q \langle t\rangle^{-3} (M_1^2(T) + M_2^m(T))$$
$$+ C\varepsilon_0^{1-(n/2)+n/(m+1)} \langle t\rangle^{-n/2} M_1(T). \tag{4.17}$$

Note that $1-(n/2)+n/(m+1) > 0$ if $m < (n+2)/(n-2)$.

Finally, using our estimates of $A$, $B$, and $C$ in (4.12) we get

$$\|\langle x\rangle^{-\sigma} \phi(t)\|_2 \leqslant C\langle t\rangle^{-n/2} (\|\phi_0\|_1 + \|\phi_0\|_2)$$
$$+ C_q''' \langle t\rangle^{-\zeta} (M_1 + M_1^2$$
$$+ M_2 + M_3^\beta M_2^{m(m+1)/(m-1)(m-1-(2/n)-\mathcal{O}(\delta))})$$
$$+ \tilde{C} C_q \langle t\rangle^{-\eta} (M_1^2 + M_2^m)$$
$$+ C\varepsilon_0^{1-(n/2)+n/(m+1)} \langle t\rangle^{-n/2} M_1.$$

Multiplication by $\langle t\rangle^{-\zeta}$, taking supremum over $|t| \leqslant T$, $\varepsilon_0$ sufficiently small and $E$ sufficiently near $E_*$ (so that $C_q'''$ can be made small) we have

$$M_1 \leqslant \tilde{C}(\|\phi_0\|_1 + \|\phi_0\|_2)$$
$$+ \tilde{C}'''(M_1^2 + M_2 + M_3^\beta M_2^{m(m+1)/(m-1)(m-1-(2/n)-\mathcal{O}(\delta))} + M_2^m) \tag{4.18}$$

*Estimation of* $\|\phi(t)\|_2$

We estimate $\|\phi(t)\|_2$ directly from the equation for $\phi(t)$, (3.4). To do this, we multiply (3.4) by $\phi^*$, take the imaginary part of the resulting equation and then integrate over all space. This gives,

$$\frac{d}{dt} \|\phi(t)\|_\infty^2 \leqslant \int |\phi(t)| \, |F(t)|.$$

The form of $F(t)$, as displayed in (3.6)–(3.8), and previous estimates of this section can be seen to imply, for $|t| \leqslant T$,

$$\frac{d}{dt} \|\phi(t)\|_2^2 \leqslant C\langle t\rangle^{-\zeta} (M_2^{m+1}(T) + M_1^2(T)).$$

Integration gives

$$M_3^2(T) \leqslant C'(M_2^{m+1}(T) + M_1^2(T)) \tag{4.19}$$

Therefore, (4.12), (4.18), and (4.19) comprise a closed coupled system of inequalities for $M_j(T)$ $j = 1, 2, 3$. $M_3(T)$ can be eliminated from (4.18), using (4.19). By an argument presented in Section 5.3 of [Sof–Wei] we now can obtain the assertions of Theorem 4.1 and the proof is complete.

The proofs of Scattering Theorem 2.4 and Asymptotic Stability Theorem 2.5 are a consequence of Theorem 4.1 and the discussion of scattering theory in section 6 of [Sof–Wei].

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