

LETTERS TO THE EDITOR

DISCUSSION OF "DYNAMIC STABILITY OF CABLES SUBJECTED TO AN AXIAL PERIODIC LOAD"

N. C. PERKINS

Department of Mechanical Engineering and Applied Mechanics, The University of Michigan, Ann Arbor, Michigan 48109-2125, U.S.A.

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1. INTRODUCTION

In the referenced paper [1], the author examines the stability of the planar, linear response of a suspended cable subjected to harmonic axial loading. The study is motivated by a desire to understand the dynamic stability of cables employed in structures such as cable-stayed bridges or guy-towers. In these applications, the harmonic axial load derives from the dynamic reaction of the structure at the connection with the cable.

The problem of interest is illustrated in Figure 1, and consists of a uniform small-sag cable suspended in the vertical plane between two level supports separated by a distance l . In the absence of any excitation, the cable hangs under gravity by an amount f and a static horizontal load H_e develops at both supports. In the author's paper, the cable is then driven by a harmonic "longitudinal" load $H_t \cos \Omega t$ which, in the context of the cited applications, is applied to the cable at one (or perhaps both) supports.

2. DISCUSSION

The author begins his analysis by proposing a model that is based on a result from the small-sag elastic cable theory of Irvine and Caughey [2]. The proposed model, however, represents an improper extension of results in reference [2] and leads to erroneous conclusions regarding dynamic stability.

2.1. Author's model

The author [1] proposes the following equation of motion governing planar, linear response:

$$(H_e + H_t \cos \Omega t)w_{,xx} - (8f/l^2)^2(EA/L_E) \int_0^l w \, dx = mw_{,tt}, \quad x \in (0, l), \quad (1)$$

with $w(0, t) = w(l, t) = 0$. (The limits 0 and l on the integral term in equation (1) are omitted in reference [1].) Here, w is the dependent variable representing the vertical component of the cable displacement (from equilibrium) and x and t are independent space and time variables, respectively. EA is the cable cross-section stiffness and $L_E = l(1 + 8f^2/l^2)$ represents the (approximate) equilibrium cable arc length.

The equation of motion (1) is based on the free vibration theory of small-sag elastic cables developed by Irvine and Caughey [2]. In reference [2], Irvine and Caughey consider a cable suspended between two *fixed* supports and application of the resulting *geometric* boundary conditions ultimately leads to the integral (membrane or stretching) operator

$$L_s[w] = \left(\frac{8f}{l^2}\right)^2 \frac{EA}{L_E} \int_0^l w \, dx, \quad (2)$$

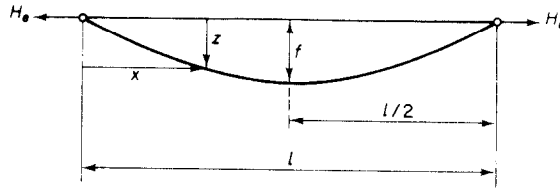


Figure 1. Definition diagram for small-sag cable, reproduced from reference [1].

which also appears in equation (1). This operator captures the possible (quasi-static) stretching of the cable induced by dynamic cable response. However, as shown next, whether such “stretching” exists depends: (1) on the nature of the deflection w , and (2) on the nature of the boundary conditions. For the free response of cables with fixed supports, Irvine and Caughey [2] note from equation (2) that symmetric vibration modes do, while antisymmetric vibration modes do not, induce (first order) stretching.

In developing his model, the author has overlooked the fact that the boundary conditions describing his problem differ from the *geometric* (fixed) conditions considered by Irvine and Caughey [2]. In particular, the condition describing the dynamic force transmitted to the cable is a *natural* boundary condition. Properly accounting for this boundary condition significantly alters the form of the equation of motion from equation (1) assumed by the author. Consider the following two cases.

2.2. Case A: cable driven by harmonic end-load

Using reference [3] as a convenient starting point, the non-linear equations of motion describing the planar response of an elastic cable about a planar equilibrium with arbitrary sag are, for the tangential component U_1 ,

$$[EA\varepsilon + (P + EA\varepsilon)(U_{1,S} - KU_2)]_{,S} - K(P + EA\varepsilon)(U_{2,S} + KU_1) = mU_{1,tt}, \tag{3}$$

and for the normal component U_2 ,

$$[(P + EA\varepsilon)(U_{2,S} + KU_1)]_{,S} + K[EA\varepsilon + (P + EA\varepsilon)(U_{1,S} - KU_2)] = mU_{2,tt}. \tag{4}$$

In this formulation, the planar response of the cable is resolved into the components $U_1(S, t)$ and $U_2(S, t)$ aligned, respectively, with the tangential and normal directions defined by the equilibrium curve; see Figure 2. The arc length co-ordinate of the equilibrium curve is denoted by $S \in (0, L_e)$ and

$$\varepsilon = U_{1,S} - KU_2 + \frac{1}{2}[(U_{1,S} - KU_2)^2 + (U_{2,S} + KU_1)^2] \tag{5}$$

is the dynamic component of the Lagrangian strain of the cable centerline. The (non-constant) coefficients $P(S)$ and $K(S)$ appearing in equations (3) and (4) denote the tension

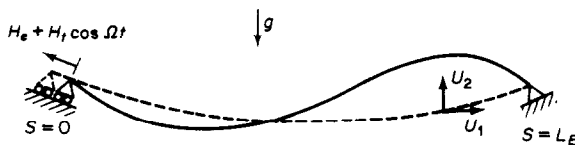


Figure 2. Small-sag cable subjected to harmonic end-load. The cable forms the equilibrium (dashed curve) under self-weight and static end-load H_e . Dynamic end-load $H_e \cos \Omega t$ leads to planar motion about equilibrium, which is resolved into tangential $U_1(S, t)$ and normal $U_2(S, t)$ displacement components.

and curvature distributions of the equilibrium cable. For cables with arbitrary sag, these are given by reference [3]:

$$P(S) = [H_e^2 + (mgS)^2]^{1/2}, \quad K(S) = mgH_e/[H_e^2 + (mgS)^2], \quad (6)$$

where H_e will again denote the horizontal component of the cable tension. The equilibrium solution (6) describes the classical catenary cable of elementary statics.

Following references [4, 5], the general theory (3)–(6) may be specialized to linear small-sag cable theory [2] as follows. For small-sag cables, a parabolic approximation to the catenary is used and, to leading order, the coefficients (6) are constant [5] (this approximation is based on the fact that, for small-sag cables, $H_e \gg mgL_e$):

$$P(S) \approx H_e, \quad K(S) \approx mg/H_e = K_e. \quad (7)$$

K_e is identified as a small (curvature) parameter which is used to order the terms in equations (3)–(5). For small-sag cables, the curvature K_e is related to the sag f and span l through $K_e = 8f/l^2$.

Next, the inertia term on the right side of equation (3) is dropped on the assumption that the cable stretches in a quasi-static manner. This assumption is based on the fact that $EA \gg H_e$. After doing so, and linearizing, equation (3) becomes, to leading order,

$$EA\varepsilon_{,s} = 0, \quad (8)$$

where, again, $EA \gg H_e$ is used. Upon integration, this result states that the dynamic component of the cable tension T_d is a function of time alone,

$$T_d(t) = EA(U_{1,S} - K_e U_2). \quad (9)$$

Indeed, for Case A, $T_d(t)$ is *prescribed* by the (natural) boundary condition governing the transmitted load at the left support:

$$T_d(t) = EA(U_{1,S} - K_e U_2) = H_t \cos \Omega t. \quad (10)$$

Integration of equation (10) and application of the remaining geometric boundary condition $U_1(L_e, t) = 0$ leads to

$$U_1(S, T) = K_e \left[\int_0^S U_2(\eta, t) d\eta - \int_0^{L_e} U_2(\eta, t) d\eta \right] + (S - L_e)(H_t/EA) \cos \Omega t \quad (11)$$

and the tangential co-ordinate is eliminated from the formulation. Linearizing the remaining equation for transverse motion (4) about the solution (10) and retaining terms to order K_e leads to the final result:

$$[H_e + H_t \cos \Omega t]U_{2,SS} + K_e H_t \cos \Omega t = mU_{2,tt}, \quad (12)$$

with $U_2(0, t) = U_2(L_e, t) = 0$.

The equation of motion (12) reveals that the tangential end-loading creates *simultaneous parametric* and *external* excitation for the transverse response. The excitation in the normal direction derives from the tangential end-load due to the non-zero equilibrium cable curvature. The same effect also arises when the tangential motion of the left support is a prescribed function of time [6]. Furthermore, the dynamic cable tension is determined solely by the applied tangential end-load and the integral (stretching) term in equation (1) is now absent in equation (12). Indeed, under conditions of free response ($H_t = 0$), the left support is free to move along the tangent direction (under the action of the static support load H_e). In the case of constrained supports considered by Irvine and Caughey, the integral (stretching) term is present, as demonstrated next.

2.3. Case B: cable with fixed supports

The same procedure as above is followed to re-derive the linear cable theory of Irvine and Caughey [2], in order to illustrate the key role played by the boundary conditions. Integrating equation (9) again and replacing the natural boundary condition (10) with the geometric boundary condition $U_1(0, t) = 0$, and again using $U_1(L_e, t) = 0$, leads to

$$T_d(t) = -K_e(EA/L_e) \int_0^{L_e} U_2(\eta, t) d\eta, \quad (13)$$

and

$$U_1(S, t) = K_e \left[\int_0^S U_2(\eta, t) d\eta - (S/L_e) \int_0^{L_e} U_2(\eta, t) d\eta \right]. \quad (14)$$

Thus, for case B, the dynamic cable tension develops from first order cable stretching. Linearizing equation (4) about equation (13) and retaining terms to order K_e leads to the final result:

$$H_e U_{2,SS} - K_e^2(EA/L_e) \int_0^{L_e} U_2(\eta, t) d\eta = m U_{2,tt}, \quad (15)$$

with $U_2(0, t) = U_2(L_e, t) = 0$. The associated eigenvalue problem governing the free response [5] is the same as that derived by Irvine and Caughey [2].

3. OBSERVATIONS ON DYNAMIC STABILITY

In his paper [1], the author employs the symmetric eigenfunctions for a cable with fixed supports [2] as the basis for an eigenfunction expansion to equation (1). These eigenfunctions, however, do not satisfy the natural boundary condition (10) for the problem of interest. In addition, they are not orthogonal with respect to the operator $(H_t \cos \Omega t) w_{,xx}$ appearing in equation (1). As a result, the author obtains a set of coupled Mathieu-Hill equations governing the response of each "mode"; refer to equation (6) of reference [1]. The author proceeds to determine the stability boundaries employing the harmonic balance method and, from the particular coupling, concludes that sum-type combination resonances exist.

The opposite conclusion is reached upon considering an eigenfunction expansion solution to equation (12). For Case A, the associated eigenfunctions, which are those of a simple taut string, are orthogonal with respect to *all* operators in equation (12) and lead to the *decoupled* equations

$$\ddot{a}_n + \omega_n^2 \left(1 + \frac{H_t}{H_e} \cos \Omega t \right) a_n + \bar{v}_n K_e H_t \cos \Omega t = 0, \quad n = 1, 2, \dots \quad (16)$$

Here, $a_n(t)$ governs the response of the n th mode having natural frequency $\omega_n = (n\pi/L_e)(H_e/m)^{1/2}$, and for which

$$\bar{v}_n = \frac{1}{n\pi} \sqrt{\frac{2L_e}{m}} [(-1)^n - 1]. \quad (17)$$

Thus, no combination resonances exist. Moreover, for antisymmetric modes ($n = 2, 4, 6, \dots$), the external excitation vanishes ($\bar{v}_n = 0$) and equation (16) reduces to the classical Mathieu equation. For symmetric modes ($n = 1, 3, 5, \dots$), however, the constant

\bar{v}_n does not vanish and these modes experience simultaneous parametric and external excitation. The combined effect of these two excitations is the subject of numerous studies; see, for example, references [7, 8].

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