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Quantization of tensor representations and deformation of matrix bialgebras

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Abstract

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The quantum matrix bialgebra $\mathbf{M}_q(2)$ and quantum plane $\mathbf{k}_q^2$ are constructed as *preferred deformations* of the classical matrix bialgebra and plane, that is, the comultiplication for $\mathbf{M}_q(2)$ and the $\mathbf{M}_q(2)$-coaction for $\mathbf{k}_q^2$ remain unchanged on *all* elements (not just generators) during the deformation. The construction of these algebras is obtained by quantizing the standard representations of the Lie algebra $\mathfrak{sl}(2)$ and the appropriate symmetric group on each tensor power of the vector space of coordinate functions on the plane. Analyzing the invariant elements of these representations then leads to the desired deformations.

1. Introduction

This paper contains the main results of [8], the author's Ph.D. Dissertation, and is intended to serve as an introduction to the theory of 'quantum symmetry', announced in [6], in which quantum groups are studied as part of algebraic deformation theory. The present paper applies the general theory of [6] to yield 'preferred presentations' of deformations of the matrix bialgebra $\mathbf{M}(2)$ and of the plane $\mathbf{k}^2$ which are isomorphic to the quantum matrix bialgebra $\mathbf{M}_q(2)$ and the quantum plane $\mathbf{k}_q^2$. The comultiplication for the deformation of $\mathbf{M}(2)$ and the $\mathbf{M}(2)$-coaction for the deformation of $\mathbf{k}^2$ remain unchanged on all elements. The explicit form for the deformed multiplication of the quantum plane, which initially was given by other means, is derived as a straightforward application of the general theory.

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To construct $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$ as deformations we first quantize the representations of the symmetric group, S_n, and the Lie algebra $\mathfrak{sl}(2)$ on the n-fold tensor product $V^{\otimes n}$, where $V=(k^2)^*$ is the (dual) 'fundamental representation'. The basic relationship that these representations are commutants of each other is maintained during the quantization. Classically, this 'double commutant' theorem gives the decomposition of each $V^{\otimes n}$ into simple modules for these representations and we show that the same is true after quantization. For our purposes, the appropriate quantum enveloping algebra is the *Woronowicz quantization* of $U\mathfrak{sl}(2)$ which acts as *semiderivations* of $V^{\otimes n}$, cf. [17]. In place of kS_n we use the group algebra of a (generally) infinite group built from the infinitesimal bialgebra deformations of $\mathbf{M}(2)$. The generators of this group satisfy the relations in the Artin presentation for the symmetric group *except* the braid relation. The mutual commutant property of this group and the Woronowicz quantization on $V^{\otimes n}$ enables us to decompose $V^{\otimes n}$ into simple modules for the quantized action of S_n. This decomposition is 'exactly' the same as for the usual symmetric group which enables us to construct $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$ as deformations as follows: We make use of the isomorphism $M(2)\cong \mathrm{End}_k(k^2)$ to view $\mathbf{k}^2$ and $\mathbf{M}(2)$ as quotients of the tensor algebras TV and $T(V\otimes V^*)$ and define, for each n, a map $T_V^n: V^{\otimes n}\to V^{\otimes n}$ which 'matches' the simple summands for S_n to the corresponding quantized summands. These maps already suffice to give a preferred deformation of $\mathbf{k}^2$. To obtain the associated deformation of $\mathbf{M}(2)$ we tensor these maps with those obtained from the dual decomposition of $V^{*\otimes n}$ to give a coalgebra automorphism of $T(V\otimes V^*)$ which takes the subspace of ordinary skew elements to the standard ideal of relations for $\mathbf{M}_q(2)$. Using this map we define a new multiplication on $T(V\otimes V^*)$ with the properties that it is compatible with the *original* comultiplication and the subspace of ordinary skew elements remains an ideal for this multiplication. There is thus a new product, $*$, on the quotient which is just the underlying vector space of $\mathbf{M}(2)$ and, by construction, $(\mathbf{M}(2),*)\cong\mathbf{M}_q(2)$ as bialgebras.

We now fix some notation and conventions which will be used throughout the paper. Let $\mathbf{M}(n)$ be the ring of polynomial functions on $M(n)$, the $n\times n$ matrices with entries in some field of characteristic zero (usually the complex numbers). As an algebra, $\mathbf{M}(n)$ is the commutative polynomial ring $k[x_{11},\dots,x_{nn}]$, where x_{ij} is dual to the matrix unit $e_{ij}\in M(n)$. The multiplication in $M(n)$ induces an algebra morphism $\Delta:\mathbf{M}(n)\to\mathbf{M}(n)\otimes\mathbf{M}(n)$, called the comultiplication, which on the generators is given by $\Delta x_{ij}=\sum_p x_{ip}\otimes x_{pj}$. Evaluation of functions at the identity element of $M(n)$ gives an algebra map $\varepsilon:\mathbf{M}(n)\to k$, the counit, and so $\mathbf{M}(n)$ is a bialgebra. The ring of polynomial functions on k^n, (affine) n-dimensional space, is $\mathbf{k}^n=k[x_1,\dots,x_n]$, where x_i is the ith coordinate function on k^n. The standard action of $M(n)$ on k^n makes $\mathbf{k}^n$ a left $\mathbf{M}(n)$-comodule algebra. This means that there is an algebra morphism $\lambda:\mathbf{k}^n\to\mathbf{M}(n)\otimes\mathbf{k}^n$ which gives $\mathbf{k}^n$ the structure of a left $\mathbf{M}(n)$-comodule. On the generators we have that $\lambda x_i=\sum_p x_{ip}\otimes x_p$. The foregoing, of course, holds much more generally and enables one to study affine algebraic groups G and their representations in terms of the Hopf algebra $\mathcal{O}(G)$ of polynomial functions on G and its comodules (cf. [16]).

The standard quantum matrix bialgebra $\mathbf{M}_q(n)$ has a succinct description in terms of the Faddeev–Reshetikhin–Takhtajan [FRT] construction for a quantum Yang–Baxter matrix R. Set X to be the vector space generated by x_{ij} and let TX be the tensor algebra of X. The FRT construction associates in a canonical way a varying subspace of $X \otimes X$ depending on a parameter $q \in k^{\times}$; these are the commutation relations for $\mathbf{M}_q(n)$. The quotient of TX by the two-sided ideal generated by this subspace is defined to be $\mathbf{M}_q(n)$. (Actually, the FRT construction uses the noncommutative polynomial ring $k\langle x_{11}, \ldots, x_{nn}\rangle$ instead of the tensor algebra TX; for our purpose it will be more convenient to work with TX which is isomorphic to $k\langle x_{11}, \ldots, x_{nn}\rangle$.) There is a bialgebra structure on $\mathbf{M}_q(n)$ in which the comultiplication is defined on the generators to be the same as that of $\mathbf{M}(n)$ and then is extended to all of $\mathbf{M}_q(n)$ to be an algebra morphism (the counit is defined similarly). Associated to $\mathbf{M}_q(n)$ is the quantum n-dimensional space, $\mathbf{k}_q^n$, which is a left $\mathbf{M}_q(n)$-comodule algebra. The coaction on generators is the same as the $\mathbf{M}(n)$-coaction for $\mathbf{k}^n$ and then is extended to make $\mathbf{k}_q^n$ a left $\mathbf{M}_q(n)$-comodule algebra.

For $n = 2$, let $a = x_{11}$, $b = x_{12}$, $c = x_{21}$, and $d = x_{22}$ and again let X be the vector space they generate. The commutation relations for $\mathbf{M}_q(2)$ are given by

$$\begin{aligned} &a \otimes b = qb \otimes a\,, && a \otimes c = qc \otimes a\,, \\ &b \otimes d = qd \otimes b\,, && c \otimes d = qd \otimes c\,, \\ &b \otimes c = c \otimes b\,, && a \otimes d - d \otimes a = (q - q^{-1}) b \otimes c \end{aligned}$$

and $\mathbf{M}_q(2)$ is then the quotient of TX modulo the ideal generated by these relations. The associated linear space of $\mathbf{M}_q(2)$ is the quantum plane, $\mathbf{k}_q^2$. If $x = x_1$ and $y = x_2$ are the coordinate functions on k^2 and V is the vector space they generate, then $\mathbf{k}_q^2$ is the quotient of TV subject to the ideal generated by the commutation relation $x \otimes y = qy \otimes x$. Note that when $q = 1$, the algebras $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$ reduce to the classical algebras $\mathbf{M}(2)$ and $\mathbf{k}^2$ with their usual structure. Thus for $q \neq 1$, one has ‘quantizations’ of $\mathbf{M}(2)$ and $\mathbf{k}^2$.

Another approach to quantization of $\mathbf{M}(2)$ is through algebraic deformation theory. In [7], Gerstenhaber and Schack developed a cohomology and deformation theory for arbitrary bialgebras. If μ and Δ are the multiplication and comultiplication of a bialgebra A, then a formal *deformation*, $A_t = A(\mu_t, \Delta_t)$, of A, is a $k[\![t]\!]$-bialgebra structure on the $k[\![t]\!]$-module $A[\![t]\!]$ with $\mu_t = \mu + \mu_1 t + \mu_2 t^2 + \cdots$ and $\Delta_t = \Delta + \Delta_1 t + \Delta_2 t^2 + \cdots$, where $\mu_i \in \mathrm{Hom}_k(A \otimes A, A)$ and $\Delta_i \in \mathrm{Hom}_k(A, A \otimes A)$ are extended to be $k[\![t]\!]$-bilinear. The existence of a unit and counit is preserved under deformation; in fact, every deformation is equivalent to one in which the unit and counit do not change. It is proved in [7] that *every* bialgebra deformation of $\mathbf{M}(n)$ is equivalent or isomorphic to one in which $\Delta_t = \Delta$, the original comultiplication on $\mathbf{M}(n)$. Thus every deformation of $\mathbf{M}(n)$ is isomorphic to one in which *only* the algebra structure varies. A deformation of $\mathbf{M}(n)$ in which $\Delta_t = \Delta$ is called a *preferred presentation* of the deformation. The same is true for deformations of $\mathcal{O}(G)$ when G is a reductive group. Similarly, if N is any comodule algebra over $\mathbf{M}(n)$ or $\mathcal{O}(G)$, then every deformation of N is isomorphic to one in which the coaction remains strictly unchanged on all elements.

As with the classical case of deformations of algebras [5], the deformations of bialgebras are 'controlled' by a cohomology theory, this being the theory introduced in [7]. While we shall not describe (or use) cohomology in this paper, we should mention that the pair (μ_1, Δ_1) is the 'infinitesimal' of the deformation and must be a two-cocycle for $(\mu + \mu_1 t, \Delta + \Delta_1 t)$ to define a bialgebra structure on $A[t]/t^2$. There are then obstructions (all of which are three-cocycles) to lifting $(\mu + \mu_1 t, \Delta + \Delta_1 t)$ to a bialgebra structure on $A[\![t]\!]$.

A virtue of this approach is that the new bialgebra structure is defined *directly* on the underlying vector space of A (base extended to $A[\![t]\!]$). Although it has many significant applications, the FRT construction of $\mathbf{M}_q(n)$ is not of this form; nor is it obvious that it is isomorphic to a deformation of $\mathbf{M}(n)$. A significant question, then, is to determine whether $\mathbf{M}_q(n)$ is a deformation of $\mathbf{M}(n)$. If so, then it follows that it is isomorphic to a preferred presentation. In [7], formulas for the products $x_{ij} * x_{kl}$ of generators of $\mathbf{M}(n)$ are given which give the same commutation relations as for $\mathbf{M}_q(n)$, but do not give an associative product when applied to arbitrary elements of $\mathbf{M}(n)$. (Takhtajan has, independently, given the same formulas [15].)

In [6], it is shown that $\mathbf{M}_q(n)$ *is* a deformation of $\mathbf{M}(n)$ and it is constructed as a preferred presentation. Similarly, quantum n-space is exhibited as a preferred presentation and the formulas for its deformed multiplication are given. Other deformations of $\mathbf{M}(n)$ are constructed in [6] including the multiparameter family appearing in [1], [13], and [14]. Also, a previously unknown family of deformations at the 'boundary' of the deformation space is constructed in [6]; for $n = 2$ this is the Jordan quantization of $\mathbf{M}(2)$, cf. [2].

The deformations in [6] are all constructed in a uniform way starting with unobstructed infinitesimal bialgebra deformations of $\mathbf{M}(n)$, which according to the cohomology theory of [7] correspond to elements $\gamma \in \wedge^2 M(n)$ whose *Schouten bracket* $[\gamma, \gamma] \in \wedge^3 M(n)$ is invariant under the adjoint action of $M(n)$. In Section 2, we associate to each $\gamma \in \wedge^2 M(n)$ a subgroup of $\mathrm{GL}(V^{\otimes m})$ for each m which we denote as $S_m(\gamma)$. This group is used to quantize the standard representation of the symmetric group on $V^{\otimes m}$. Analyzing the decomposition of $V^{\otimes m}$ under $S_m(\gamma)$ then leads to deformations of $\mathbf{M}(n)$ and of $\mathbf{k}^n$. To obtain $\mathbf{M}_q(n)$ as a deformation in this way, we make use of the Woronowicz quantization of $U\mathfrak{sl}(n)$ to obtain the necessary information about the invariant elements of $S_m(\gamma)$.

This paper is a complete treatment of the theory of quantum symmetry as it applies to $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$. All necessary proofs are given, whereas in [6] some are only sketched; a more thorough account of the results of [6] will appear in a separate note. In addition, there are some technical difficulties in the general theory that are absent when $n = 2$, thus making this case more accessible.

2. Quantized representations of S_n

The quantum matrix bialgebra $\mathbf{M}_q(2)$, the quantum plane $\mathbf{k}_q^2$, and many other 'quantum' algebras are deformations, at least intuitively, of polynomial rings. A polynomial ring in n variables may be viewed as the symmetric algebra of an n-dimensional vector space. After reviewing this basic construction, we introduce the notion of a 'quantum' symmetric algebra and show how to realize the quantizations $\mathbf{k}_q^2$ and $\mathbf{M}_q(2)$ as quantum symmetric algebras. Using these constructions, it is possible to obtain canonical preferred deformations of $\mathbf{M}(2)$ and $\mathbf{k}^2$ which are isomorphic to $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$, respectively. For the quantum plane, it is possible to exhibit explicitly the deformed multiplication for its preferred presentation.

Let W be an n-dimensional vector space. The polynomial ring $k[x_1, \dots, x_n]$ or symmetric algebra SW is naturally a quotient of the tensor algebra TW. To describe SW we make use of the action of the nth symmetric group S_n on $W^{\otimes n}$. The left S_n-module structure on $W^{\otimes n}$ is given by $\sigma(w_1 \otimes \cdots \otimes w_n) = w_{\sigma^{-1}1} \otimes \cdots \otimes w_{\sigma^{-1}n}$ for each $\sigma \in S_n$. Let $\mathrm{sk}(TW)$ be the two-sided ideal in TW generated by all elements of the form $a \otimes b - b \otimes a$ for $a,b \in W$. The symmetric algebra is the quotient $TW/\mathrm{sk}(TW)$. Since k has characteristic 0, the exact sequence

$$0 \to \mathrm{sk}(TW) \to TW \to SW \to 0$$

has a canonical linear splitting identifying SW with the symmetric or invariant elements, $\mathrm{sym}(TW)$, in TW. An element $\alpha \in W^{\otimes n}$ is *symmetric* if $\sigma\alpha = \alpha$ for all $\sigma \in S_m$. If $\bar{\alpha} \in SW$, then the corresponding symmetric element of $\mathrm{sym}(TW)$ is given by $\frac{1}{n!}\sum_{\sigma \in S_n} \sigma(\alpha)$, where α is a preimage of $\bar{\alpha}$ in TW. For example, the element $xy \in k[x, y]$ corresponds to the symmetric tensor $\frac{1}{2}(x \otimes y + y \otimes x)$.

Our main interest in this paper will be the vector spaces associated with $M(2)$ and k^2. To describe these we need some notation and conventions. Let $\mathscr{X} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\mathscr{V} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ be the matrices of coordinate functions on $M(2)$ and k^2 and denote the vector spaces their entries generate by X and V, respectively. Using the language of symmetric algebras we have that $\mathbf{M}(2) = SX$ and $\mathbf{k}^2 = SV$. Since $M(2) = \mathrm{End}_k(k^2) \cong k^2 \otimes (k^2)^*$, we can make the identifications $X = V \otimes V^*$ and $\mathscr{X} = \mathscr{V} \widehat{\otimes} \mathscr{V}^*$, where

$$\mathscr{V} \widehat{\otimes} \mathscr{V}^* = \begin{pmatrix} x \otimes x^* & x \otimes y^* \\ y \otimes x^* & y \otimes y^* \end{pmatrix}.$$

With this the algebra $\mathbf{M}(2)$ becomes $SX = S(V \otimes V^*) = SV \otimes SV^*$.

To quantize $\mathbf{M}(2)$ and $\mathbf{k}^2$ we view $SV \otimes SV^* \subset TV \otimes TV^* = TX$ and consider certain quantizations of the standard representation of S_n on $X^{\otimes n}$, $V^{\otimes n}$, and $V^{*\otimes n}$. These quantizations are built from the infinitesimal bialgebra deformations of $\mathbf{M}(2)$. Recall that these infinitesimals can naturally be identified with $\wedge^2 M(2)$. We view $\wedge^2 M(2)$ as the submodule of $M(2) \otimes M(2)$ generated by elements of the form $\frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)$ for $\alpha, \beta \in M(2)$. We also use the Kronecker product of matrices to identify $M(2) \otimes M(2)$ with $M(4)$. Recall that the Kronecker product $M \otimes M'$ is defined to be

$$\begin{pmatrix} aM' & bM' \\ cM' & dM' \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\mathcal{V} \widehat{\otimes} \mathcal{V}$ be the column vector

$$\begin{pmatrix} x \otimes x \\ x \otimes y \\ y \otimes x \\ y \otimes y \end{pmatrix}$$

whose entries generate $V \otimes V$ and view $\gamma \in \wedge^2 M(2)$ as an element of $M(4)$ under the identifications $\wedge^2 M(2) \subset M(2) \otimes M(2) = M(4)$ discussed earlier. Set $Q = \exp(t\gamma)$, and let E be the matrix representing the interchange $(12)_V : V \otimes V \to V \otimes V$, defined by $(12)_V(u \otimes v) = v \otimes u$. The matrix E is block diagonal with blocks (1), $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and (1).

Definition 2.1. The *quantum interchange* $\tau_V(\gamma)$ is the linear transformation defined on the column vector of generators by

$$\tau_V(\gamma)(\mathcal{V} \widehat{\otimes} \mathcal{V}) = Q^{-2}E(\mathcal{V} \widehat{\otimes} \mathcal{V}).$$

Note that if M and M' are matrices in $M(2)$, then $E(M \otimes M') = (M' \otimes M)E$, so $E\gamma = -\gamma E$ since $\gamma \in \wedge^2 M(2)$, and thus $EQ = Q^{-1}E$. With this, it follows that $(\tau_V(\gamma))^2 = Q^{-2}EQ^{-2}E = EQ^2Q^{-2}E = \mathrm{Id}_{V \otimes V}$. For each n, let $S_n(\gamma)$ be the subgroup of $\mathrm{GL}(V^{\otimes n})$ generated by $\tau_1, \ldots, \tau_{n-1}$, where τ_i acts as $\mathrm{Id}_V^{\otimes i-1} \otimes \tau_V(\gamma) \otimes \mathrm{Id}_V^{\otimes n-i-1}$. The generator τ_i is the ith quantum interchange. For $S_2(\gamma)$ we have that $\tau_1 = \tau_V(\gamma)$ and $S_2(\gamma) \cong S_2$. For $S_n(\gamma)$, each τ_i satisfies $\tau_i^2 = \mathrm{Id}_{V^{\otimes n}}$ and $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j| > 1$. The braid relation, $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$, however, is generally *not* satisfied and so $S_n(\gamma)$ usually is infinite for $n > 2$. For certain choices of γ, the quantum interchanges $\tau_1 = \tau_V(\gamma) \otimes \mathrm{Id}_V$ and $\tau_2 = \mathrm{Id}_V \otimes \tau_V(\gamma)$ which generate $S_3(\gamma)$ may satisfy the braid relation which gives, for each n, an evident isomorphism $S_n(\gamma) \cong S_n$ as subgroups of $\mathrm{GL}(V^{\otimes n})$. This is not the case for the γ giving $\mathbf{M}_q(2)$.

Now $V^{*\otimes n}$ becomes a left $S_n(\gamma)$-module where $\sigma \in S_n(\gamma)$ acts as $\sigma^{-*} \in \mathrm{GL}(V^{*\otimes n})$, the inverse dual to $\sigma \in \mathrm{GL}(V^{\otimes n})$. The quantum interchanges for $V^{*\otimes n}$ are defined using $(\tau_V(\gamma))^{-*}$ which we denote by $\tau_{V^*}(\gamma)$. Viewing $\mathcal{V}^* \widehat{\otimes} \mathcal{V}^*$ as the dual row vector to $\mathcal{V} \widehat{\otimes} \mathcal{V}$, $\tau_{V^*}(\gamma)$ is given by right multiplication by the matrix $(Q^{-2}E)^{-1} = EQ^2$. Using the representations of $S_n(\gamma)$ on $V^{\otimes n}$ and $V^{*\otimes n}$ we define the associated left $S_n(\gamma)$-module structure on $X^{\otimes n}$ by setting $\tau_X(\gamma) =$

$\tau_V(\gamma)\otimes\tau_{V^*}(\gamma)\in \mathrm{GL}((V\otimes V)\otimes(V^*\otimes V^*))=\mathrm{GL}(X\otimes X)$. On the matrix

$$\mathscr{X}\widehat{\otimes}\mathscr{X}=\begin{pmatrix} a\otimes a & a\otimes b & b\otimes a & b\otimes b\\ a\otimes c & a\otimes d & b\otimes c & b\otimes d\\ c\otimes a & c\otimes b & d\otimes a & d\otimes b\\ c\otimes c & c\otimes d & d\otimes c & d\otimes d\end{pmatrix}$$

whose entries generate $X\otimes X$, the action of $\tau_X(\gamma)$ is given by

$$\begin{aligned}\tau_X(\gamma)(\mathscr{X}\widehat{\otimes}\mathscr{X}) &= \tau_V(\gamma)\otimes\tau_{V^*}(\gamma)\{(\mathscr{V}\widehat{\otimes}\mathscr{V})\widehat{\otimes}(\mathscr{V}^*\widehat{\otimes}\mathscr{V}^*)\}\\ &=(Q^{-2}E\mathscr{V}\widehat{\otimes}\mathscr{V})\widehat{\otimes}(\mathscr{V}^*\widehat{\otimes}\mathscr{V}^*EQ^2)\\ &=Q^{-2}E(\mathscr{X}\widehat{\otimes}\mathscr{X})EQ^2\,.\end{aligned}$$

Since $E\mathscr{X}\widehat{\otimes}\mathscr{X}E$ represents the ordinary interchange $(12)_X$ the quantum interchange has the form

$$\tau_X(\gamma)\mathscr{X}\widehat{\otimes}\mathscr{X}=Q^{-2}\{(12)_X\mathscr{X}\widehat{\otimes}\mathscr{X}\}Q^2\,.$$

Setting $t=0$ gives the usual representations of S_n on $V^{\otimes n}$, $V^{*\otimes n}$ and $X^{\otimes n}$. In this construction we are tacitly assuming that coefficients are extended to the power series modules over TX, TV, and TV^* and all maps are extended to be $k[\![t]\!]$-bilinear. With this the quantum interchanges can be expressed as power series whose leading terms are the ordinary interchanges.

Definition 2.2. If there is an action of $S_n(\gamma)$ on $W^{\otimes n}$, then an element $\alpha\in W^{\otimes n}$ is *quantum symmetric* if it is invariant under this action, that is, $\tau_i\alpha=\alpha$ for all i.

(Here, $W^{\otimes n}$ is used to represent $V^{\otimes n}$, $V^{*\otimes n}$, or $X^{\otimes n}$.) The subspace of quantum symmetric elements of $W^{\otimes n}$ will be denoted by $\mathrm{sym}_q(W^{\otimes n})$ and when $S_n(\gamma)$ operates on $W^{\otimes n}$ for all n their direct sum is denoted $\mathrm{sym}_q(TW)$. The two-sided ideal in TW generated by elements of $W\otimes W$ of the form $\alpha-\tau_W(\gamma)\alpha$ are the *quantum skew* elements, $\mathrm{sk}_q(TW)$. When the symmetric group acts in the usual way on $W^{\otimes n}$ these spaces are denoted by $\mathrm{sym}(W^{\otimes n})$, $\mathrm{sym}(TW)$, and $\mathrm{sk}(TW)$. The quantum skew elements in TX are then generated by the quadratic relations $\mathscr{X}\widehat{\otimes}\mathscr{X}=Q^{-2}\{(12)_X\mathscr{X}\widehat{\otimes}\mathscr{X}\}Q^2$ or, equivalently, $Q^2\mathscr{X}\widehat{\otimes}\mathscr{X}=((12)_X\mathscr{X}\widehat{\otimes}\mathscr{X})Q^2$. This construction of the quantum skew elements in thus the Faddeev–Reshetikhin–Takhtajan construction [4] for the matrix Q^2.

We shall show how to use these representations to produce *deformations* of $\mathbf{k}^2$ and $(\mathbf{k}^2)^*$ by showing that the quantum skew elements coincide with the ideal of relations for $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$ for a particular choice of γ. Analogous to the classical situation, $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$ are viewed as the subspaces of quantum symmetric elements associated to the appropriate infinitesimal.

The infinitesimal used to construct $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$, henceforth denoted γ_q, is $e_{12} \wedge e_{21} \in \wedge^2 M(2)$. Using γ_q, we have that

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t/2) & \sin(t/2) & 0 \\ 0 & -\sin(t/2) & \cos(t/2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Letting $S(t) = \sin(t)$ and $C(t) = \cos(t)$, the quantum interchange $\tau_V(\gamma_q)$ on $V \otimes V$ operates on the basis elements in the following way:

$$\tau_V(\gamma_q)(x \otimes x) = x \otimes x ,$$

$$\tau_V(\gamma_q)(x \otimes y) = C(t) y \otimes x - S(t) x \otimes y ,$$

$$\tau_V(\gamma_q)(y \otimes x) = C(t) x \otimes y + S(t) y \otimes x ,$$

$$\tau_V(\gamma_q)(y \otimes y) = y \otimes y .$$

In $V \otimes V$ it is easy to show that $x \otimes x$, $y \otimes y$, and $qx \otimes y + y \otimes x$ are quantum symmetric (fixed by $\tau_V(\gamma_q)$) while $x \otimes y - qy \otimes x$ is quantum skew (sent to its negative under $\tau_V(\gamma_q)$), where $q = \sec(t) - \tan(t)$.

For $X \otimes X$ the quantum interchange acts in the following way on the basis elements:

$$\tau_X(\gamma_q)(a \otimes a) = a \otimes a , \qquad \tau_X(\gamma_q)(b \otimes b) = b \otimes b ,$$

$$\tau_X(\gamma_q)(c \otimes c) = c \otimes c , \qquad \tau_X(\gamma_q)(d \otimes d) = d \otimes d ,$$

$$\tau_X(\gamma_q)(a \otimes b) = C(t) b \otimes a - S(t) a \otimes b ,$$

$$\tau_X(\gamma_q)(b \otimes a) = C(t) a \otimes b + S(t) b \otimes a ,$$

$$\tau_X(\gamma_q)(a \otimes c) = C(t) c \otimes a - S(t) a \otimes c ,$$

$$\tau_X(\gamma_q)(c \otimes a) = C(t) a \otimes c + S(t) c \otimes a ,$$

$$\tau_X(\gamma_q)(b \otimes d) = C(t) d \otimes b - S(t) b \otimes d ,$$

$$\tau_X(\gamma_q)(d \otimes b) = C(t) b \otimes d + S(t) d \otimes b ,$$

$$\tau_X(\gamma_q)(c \otimes d) = C(t) d \otimes c - S(t) c \otimes d ,$$

$$\tau_X(\gamma_q)(d \otimes c) = C(t) c \otimes d + S(t) d \otimes c ,$$

$$\tau_X(\gamma_q)(a \otimes d) = C^2(t) d \otimes a + S^2(t) a \otimes d - C(t)S(t)(b \otimes c + c \otimes b) ,$$

$$\tau_X(\gamma_q)(d \otimes a) = C^2(t) a \otimes d + S^2(t) d \otimes a + C(t)S(t)(b \otimes c + c \otimes b) ,$$

$$\tau_X(\gamma_q)(b \otimes c) = C^2(t) c \otimes b - S^2(t) b \otimes c + C(t)S(t)(d \otimes a - a \otimes d) ,$$

$$\tau_X(\gamma_q)(c \otimes b) = C^2(t) b \otimes c - S^2(t) c \otimes b + C(t)S(t)(d \otimes a - a \otimes d) .$$

The following elements in $X \otimes X$ form a basis for the subspace of quantum symmetric elements:

$$a \otimes a\,,\quad b \otimes b\,,\quad c \otimes c\,,\quad d \otimes d\,,$$
$$qa \otimes b + b \otimes a\,,\qquad qa \otimes c + c \otimes a\,,$$
$$qb \otimes d + d \otimes b\,,\qquad qc \otimes d + d \otimes c\,,$$
$$(1 + q^4)a \otimes d + 2q^2 d \otimes a - (q - q^3)(b \otimes c + c \otimes b)\,,$$
$$2q(b \otimes c + c \otimes b) + (1 - q^2)(d \otimes a - a \otimes d)\,,$$

while the following comprise a basis for the subspace of quantum skew elements:

$$a \otimes b - qb \otimes a\,,\qquad a \otimes c - qc \otimes a\,,$$
$$b \otimes d - qd \otimes b\,,\qquad c \otimes d - qd \otimes c\,,$$
$$2q(a \otimes d - d \otimes a) + (1 - q^2)(b \otimes c + c \otimes b)\,,$$
$$(1 + q^4)b \otimes c - 2q^2 c \otimes b - (q - q^3)(d \otimes a - a \otimes d)\,,$$

where q is once again $\sec(t) - \tan(t)$.

Note that the basic quantum skew element in $V \otimes V$ is the commutation relation for the quantum plane $\mathbf{k}_q^2$ and an easy computation shows that the quantum skew elements in $X \otimes X$ give precisely the commutation relations for $\mathbf{M}_q(2)$. Thus another way to obtain the commutation relations for these quantizations is by imposing the quantum interchanges associated to γ_q on $V \otimes V$ and $X \otimes X$ and taking the quantum skew elements relative to those interchanges. Alternatively, since the quantum skew elements coincide with the Faddeev–Reshetikhin–Takhtajan construction for the matrix Q^2, replacing Q^2 by QCQ yields the same relations for any invertible $C \in M(4)$ which commutes with $\mathscr{X} \widehat{\otimes} \mathscr{X} + (12)_X \mathscr{X} \widehat{\otimes} \mathscr{X}$. In [7] the matrix C is given such that $QCQ = R$, the standard 'quantum Yang–Baxter' matrix used to define the commutation relations for $\mathbf{M}_q(2)$.

As stated earlier, the generators $\tau_1 = \tau_V(\gamma_q) \otimes \mathrm{Id}_V$ and $\tau_2 = \mathrm{Id}_V \otimes \tau_V(\gamma_q)$ of $S_3(\gamma_q)$ do *not* satisfy the braid relation. However, the following relation does hold:

$$\tau_2 \tau_1 \tau_2 - \tau_1 \tau_2 \tau_1 = S^2(t)[\tau_1 - \tau_2]\,. \tag{1}$$

Although the following observation will not be used in this paper, we would like to mention that (1) implies that $S_n(\gamma_q) \cong H_q(n)$, the *Hecke algebra* of the symmetric group. The Hecke algebra may be viewed as the algebra having generators $\sigma_1, \ldots, \sigma_{n-1}$ with relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$, the braid relation, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, and $\sigma_i^2 = (q - 1)\sigma_i + q$, where $q \in k$. Now when q is not a root of unity, $H_q(n) \cong kS_n$, and hence $kS_n(\gamma_q) \cong kS_n$ as subgroups of $\mathrm{GL}(V^{\otimes n})$. While this can be used to prove the existence of deformations of $\mathbf{M}(2)$ and $\mathbf{k}^2$ associated with γ_q, to obtain explicit formulas for the deformed multiplication we need a precise description of the decomposition of our representations.

3. The quantum enveloping algebra

In addition to quantizing the action of the symmetric group on $V^{\otimes n}$, we also quantize the action of the universal enveloping algebra $U\mathfrak{sl}(2)$ on $V^{\otimes n}$. Using both quantizations, we can decompose $V^{\otimes n}$ into simple $S_n(\gamma_q)$-modules and then realize $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$ as the quantum symmetric elements in TX and TV, respectively.

Let $\mathfrak{sl}(2)$ be generated by H, X and Y with bracket relations

$$[H, X] = 2X\ , \quad [H, Y] = -2Y\ , \quad [X, Y] = H\ .$$

To describe the $U\mathfrak{sl}(2)$-module structure on $V^{\otimes n}$ first note that V is a Lie module for $\mathfrak{sl}(2)$ in which the action is given by the operators:

$$X = x\partial/\partial y\ , \quad Y = y\partial/\partial x\ , \quad H = x\partial/\partial x - y\partial/\partial y\ . \tag{2}$$

These then extend to derivations of the tensor algebra TV, giving the latter an $\mathfrak{sl}(2)$-module structure in which each $V^{\otimes n}$ is an $\mathfrak{sl}(2)$-submodule, and, hence, is a $U\mathfrak{sl}(2)$-module.

We view $V^{\otimes n}$ as a graded vector space where $\deg(x) = 1$, $\deg(y) = -1$ and $\deg(\alpha \otimes \beta) = \deg(\alpha) + \deg(\beta)$. Let $V(i, j) \subset V^{\otimes n}$ be the submodule spanned by all monomials of degree $i - j$ with $i + j = n$. Note that $\dim V(i, j) = \binom{n}{i} = \binom{n}{j}$. Each element of $V(i, j)$ is an eigenvector with eigenvalue $i - j$ for the operation of H. We now describe the necessary quantization of $U\mathfrak{sl}(2)$ needed to decompose $V^{\otimes n}$ into simple $S_n(\gamma_q)$-modules. This quantization is isomorphic to the one introduced by Woronowicz in [16] and will be denoted $U_q^{\mathrm{W}}\mathfrak{sl}(2)$.

Definition 3.1. The *Woronowicz quantization* $U_q^{\mathrm{W}}\mathfrak{sl}(2)$ is the algebra with generators H_q, X_q, and Y_q and the relations

$$qX_qY_q - q^{-1}Y_qX_q = H_q\ ,$$
$$q^{-2}H_qX_q - q^2X_qH_q = (q + q^{-1})X_q\ ,$$
$$q^2H_qY_q - q^{-2}Y_qH_q = -(q + q^{-1})Y_q\ .$$

The operation on $V^{\otimes n}$ is defined inductively. For $n = 1$ it is given by the linear maps $V \to V$,

$$X_q(x)=0\,,\quad Y_q(x)=y\,,\quad H_q(x)=qx\,,$$
$$X_q(y)=x\,,\quad Y_q(y)=0\,,\quad H_q(y)=-q^{-1}y\,,$$

and if $\alpha \in V^{\otimes n}$ has degree r, then

$$X_q(\alpha\otimes\beta)=X_q(\alpha)\otimes\beta+q^r\alpha\otimes X_q(\beta)\,,$$
$$Y_q(\alpha\otimes\beta)=Y_q(\alpha)\otimes\beta+q^r\alpha\otimes Y_q(\beta)\,,$$
$$H_q(\alpha\otimes\beta)=H_q(\alpha)\otimes\beta+q^{2r}\alpha\otimes H_q(\beta)\,.$$

These relations then hold for all $\alpha,\beta \in TV$. (Actually, X_q, Y_q, and H_q act as *semiderivations* or *σ-derivations* of TV for appropriate automorphisms of TV; cf. [12] for a related discussion of σ-derivations and quantizations of $U\mathfrak{sl}(2)$.) The submodule $V(i, j)$ of $V^{\otimes n}$ is an eigenspace for the operator H_q. The eigenvalues involve the 'quantized' integers r_{q^2}. More generally recall the following definition:

Definition 3.2. The *q-binomial coefficient* (sometimes called the q-Gaussian binimial coefficient) is

$$\binom{n}{i}_q=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-i+1})}{(1-q^i)(1-q^{i-1})\cdots(1-q)}$$

if $n\neq 0$ and $\binom{0}{i}_q=0$.

With this we set $r_{q^2}=\binom{r}{1}_{q^2}=(1-q^{2r})/(1-q^2)$. Note that $H_q(x\otimes y)=H_q(y\otimes x)=0$ and

$$H_q(x^{\otimes r})=q(1+q^2+\cdots+q^{2r-2})x^{\otimes r}\,,$$
$$H_q(y^{\otimes r})=-q^{-1}(1+q^{-2}+\cdots+q^{-2r+2})y^{\otimes r}\,.$$

Therefore, if α has degree r, then it can be shown inductively that

$$H_q(\alpha)=q\cdot r_{q^2}\alpha\,.$$

Note that when $q=1$, $U_q^{\mathrm{W}}\mathfrak{sl}(2)$ reduces to $U\mathfrak{sl}(2)$ and its usual operation on TV. In [6], it is shown that $U_q^{\mathrm{W}}\mathfrak{sl}(2)$ is a subalgebra but not a subcoalgebra of a bialgebra isomorphic to the Drinfel'd–Jimbo quantization of $U\mathfrak{sl}(2)$ [3, 9].

4. Decomposition of tensor space

The action of $S_n(\gamma_q)$ and $U_q^{\mathrm{W}}\mathfrak{sl}(2)$ on $V^{\otimes n}$ can be used to give its decomposition into simple $S_n(\gamma_q)$-modules. The decomposition is the 'same' as the decomposition into simple S_n-modules provided that q is not a root of unity. More exactly, there is a natural correspondence between the simple $S_n(\gamma_q)$-summands and the simple S_n-summands which gives the deformations we are seeking.

It will be convenient to use the natural symmetric bilinear form $\langle -,-\rangle$ on V given by

$$\langle x, x\rangle = \langle y, y\rangle = 1\,, \qquad \langle x, y\rangle = 0\,.$$

This form induces one on $V^{\otimes n}$ where

$$\langle v_1 \otimes \cdots \otimes v_n, v_1' \otimes \cdots \otimes v_n'\rangle = \langle v_1, v_1'\rangle \cdots \langle v_n, v_n'\rangle\,.$$

With this the operations of S_n and $S_n(\gamma_q)$ on TV are given by orthogonal transformations and we have a decomposition of $V^{\otimes n}$ into orthogonal $S_n(\gamma_q)$-submodules $V(i, j)$, i.e.

$$V^{\otimes n} = \bigoplus_{i=0}^{n} V(i, j)\,.$$

To complete the decomposition we must split the $V(i, j)$ into simple $S_n(\gamma_q)$-modules. To do this we use the action of $U_q^{\mathrm{W}}\mathfrak{sl}(2)$ on $V^{\otimes n}$.

For the generators X_q, Y_q and H_q of $U_q^{\mathrm{W}}\mathfrak{sl}(2)$ we have that

$$X_q(V(i, j)) \subset V(i+1, j-1)\,,$$
$$Y_q(V(i, j)) \subset V(i-1, j+1)\,,$$
$$H_q(V(i, j)) \subset V(i, j)\,.$$

The following lemma is the analog (in this context) of Schur's classical result that kS_n and $U\mathfrak{sl}(2)$ are mutual commutants of each other when viewed as subalgebras of $\mathrm{End}(V^{\otimes n})$. There is a similar result due to Jimbo [9] that the commutant to his quantization of $U\mathfrak{sl}(2)$ is the Hecke algebra.

Lemma 4.1. *For every $f \in U_q^{\mathrm{W}}\mathfrak{sl}(2)$, the map $f : V^{\otimes n} \to V^{\otimes n}$ is an $S_n(\gamma_q)$-module morphism.*

Proof. It is sufficient to show that

$$\tau_V(\gamma_q)Y_q(\alpha) = Y_q\tau_V(\gamma_q)(\alpha) \tag{3}$$

for $\alpha \in V \otimes V$. The corresponding statement for X_q follows by switching x and y, X_q and Y_q, and replacing q by q^{-1}. The operation of any $f \in U_q^{\mathrm{W}}\mathfrak{sl}(2)$ on $V^{\otimes n}$ would then be an $S_n(\gamma_q)$-module morphism since any such f can be expressed as a polynomial in X_q and Y_q.

Writing the basis elements of $V \otimes V$ once again as a column vector and setting $q = \sec(t) - \tan(t)$, we have

$$Y_q \tau_V(\gamma_q) \begin{pmatrix} x \otimes x \\ x \otimes y \\ y \otimes x \\ y \otimes y \end{pmatrix} = Y_q \begin{pmatrix} x \otimes x \\ C(t) y \otimes x - S(t) x \otimes y \\ C(t) x \otimes y + S(t) y \otimes x \\ y \otimes y \end{pmatrix} = \begin{pmatrix} qx \otimes y + y \otimes x \\ y \otimes y \\ q^{-1} y \otimes y \\ 0 \end{pmatrix}$$

which coincides with

$$\tau_V(\gamma_q) Y_q \begin{pmatrix} x \otimes x \\ x \otimes y \\ y \otimes x \\ y \otimes y \end{pmatrix} = \tau_V(\gamma_q) \begin{pmatrix} qx \otimes y + y \otimes x \\ y \otimes y \\ q^{-1} y \otimes y \\ 0 \end{pmatrix} = \begin{pmatrix} qx \otimes y + y \otimes x \\ y \otimes y \\ q^{-1} y \otimes y \\ 0 \end{pmatrix},$$

where $q = \sec(t) - \tan(t)$ as required. □

The decompositions of the modules $V(i, j)$ and $V(j, i)$ are 'dual' to each other if the roles of x and y are switched and t is replaced by $-t$ (equivalently, q is replaced by q^{-1}).

Lemma 4.2. *Suppose $\alpha \in V(i, j)$, and that $i > j$. Then $Y_q(\alpha) \neq 0$.*

Proof. Pick an r such that $X_q^r(\alpha) = 0$ but $X_q^s(\alpha) \neq 0$ if $s < r$. Since $X_q^r(\alpha) = 0$ we have that

$$(q^{2r} X_q^r Y_q - Y_q X_q^r)\alpha = (q^{2r} X_q^r Y_q)\alpha . \tag{4}$$

Recall that from the definition of $U_q^{\mathrm{W}} \mathfrak{sl}(2)$

$$Y_q X_q = q^2 X_q Y_q - qH_q .$$

Using this we can rewrite $q^{2r} X_q^r Y_q - Y_q X_q^r$ as

$$q^{2r-1} X_q^{r-1} H_q + q^{2r-3} X_q^{r-2} H_q X_q + \cdots + qH_q X_q^{r-1} . \tag{5}$$

If d is the degree of α, then the degree of $X_q^s(\alpha)$ is $d + 2s$, so we may use (4) and (5) to obtain

$$\begin{aligned} q^{2r}(X_q^r Y_q)\alpha = \{ & q^{2r-1}(q \cdot d_{q^2}) + q^{2r-3}(q \cdot (d+2)_{q^2}) \\ & + \cdots + q(q \cdot (d + 2r - 2)_{q^2})\} X_q^{r-1}(\alpha) . \end{aligned} \tag{6}$$

This expression is not zero since $(X_q^{r-1})\alpha \neq 0$ as assumed and hence $Y_q(\alpha) \neq 0$ as claimed. □

Similarly, $X_q(\alpha) \neq 0$ for $\alpha \in V(j, i)$ when $i > j$. Note that the preceding lemma is false when q is specialized at a root of unity since $H_q(\alpha)$ can be zero even if the degree of α is nonzero.

For the remainder of this section suppose that $i \geq j$. The preceding lemma then implies that $Y_q : V(i+1, j-1) \to V(i, j)$ is a monomorphism. Let $M_q(i, j) = (Y_q(V(i+1, j-1)))^{\perp} \subset V(i, j)$ be the orthogonal complement of the image of $V(i+1, j-1)$ in $V(i, j)$ under the action of Y_q, and set $M_q(n, 0) = V(n, 0)$. Every $M_q(i, j)$ is a $S_n(\gamma_q)$-submodule of $V(i, j)$ of dimension $\binom{n}{j} - \binom{n}{j-1}$. Dually, let $M'_q(j, i) = (X_q(V(j-1, i+1)))^{\perp} \subset V(j, i)$. By the correspondence between $V(i, j)$ and $V(j, i)$ already discussed, $M_q(i, j) \cong M'_q(j, i)$ as $S_n(\gamma_q)$-modules.

Theorem 4.3. *Each $M_q(i, j)$ is a simple $S_n(\gamma_q)$-module.*

The proof of this theorem makes use of the quantum symmetric elements in the tensor algebra TX which we view as $TV \otimes TV^*$. Recall that the action of the $S_n(\gamma_q)$ on these spaces is induced by the quantum interchanges $\tau_V(\gamma_q)$, $\tau_{V^*}(\gamma_q) = (\tau_V(\gamma_q))^{-*}$, and $\tau_X(\gamma_q) = \tau_V(\gamma_q) \otimes \tau_{V^*}(\gamma_q)$ described in Section 2. The quantum symmetric elements of TX are easy to describe using the following observation:

Lemma 4.4. *If W is any finite-dimensional vector space with basis $e_1, \ldots, e_r$ and $T \in \mathrm{GL}(W)$, then the element $\alpha = \sum e_i \otimes e_i^*$ is invariant under the transformation $T \otimes T^{-*} \in \mathrm{GL}(W \otimes W^*)$.*

Proof. Note that $W \otimes W^* \cong \mathrm{End}_k(W)$ and that α corresponds to Id_W and is independent of the choice of basis. Now the dual basis to $Te_1, \ldots, Te_r$ is $T^{-*}e_1^*, \ldots, T^{-*}e_r^*$ and therefore $(T \otimes T^{-*})\alpha = \alpha$. □

Using the preceding lemma we may conclude that if M_q, and M'_q are submodules of $V^{\otimes n}$ isomorphic to some $M_q(i, j)$, then $M_q \otimes M'^*_q \subset X^{\otimes n}$ contains a quantum symmetric element.

Proof of Theorem 4.3. We begin by counting the number of quantum symmetric elements in $TV \otimes TV^* = TX$ that can be constructed using Lemma 4.4.

The modules $V(i, j)$ and $V(j, i)$ (recall $i \geq j$ and $i + j = n$) decompose into $S_n(\gamma_q)$-submodules in the following way:

$$
\begin{aligned}
V(i, j) &= \bigoplus_{r=0}^{j} Y_q^{j-r} M_q(n-r, r)\,, \\
V(j, i) &= \bigoplus_{r=0}^{j} X_q^{j-r} M'_q(r, n-r)\,.
\end{aligned}
\tag{7}
$$

From this and the fact that X_q and Y_q are $S_n(\gamma_q)$-module morphisms, the *multiplicity*, $m(i, j)$, of $M_q(i, j)$ in $V^{\otimes n}$, i.e. the number of distinct $V(r, s)$ which contain a $S_n(\gamma_q)$-submodule isomorphic to $M_q(i, j)$, is $n + 1 - 2j$. Now Lemma 4.4 guarantees that $V^{\otimes n} \otimes V^{*\otimes n}$ contains at least $\sum_{j=0}^{[n/2]} m(i, j)^2$ linearly independent quantum symmetric elements. These elements are linearly independent since they are in orthogonal submodules of $(V^{\otimes n}) \otimes (V^{*\otimes n})$. Using the multiplicities calculated above gives

$$\sum_{j=0}^{[n/2]} m(i, j)^2 = \frac{(n + 1)(n + 2)(n + 3)}{6} .$$

Thus there are at least $(n + 1)(n + 2)(n + 3)/6$ linearly independent quantum symmetric elements in each $X^{\otimes n}$. These quantum symmetric elements become linearly independent ordinary symmetric elements when $t = 0$ since they remain in the same orthogonal submodules of $X^{\otimes n}$. Recall that, since $\dim(V \otimes V^*) = 4$, $\dim(\mathrm{sym}((V \otimes V^*)^{\otimes n})) = \binom{n+3}{3} = (n + 1)(n + 2)(n + 3)/6$, so the ordinary symmetric elements found form a basis for $\mathrm{sym}((V \otimes V^*)^{\otimes n})$. Since any set of linearly independent quantum symmetric elements may be chosen to remain linearly independent when $t = 0$, it follows that $\dim(\mathrm{sym}_q(V \otimes V^*)^{\otimes n}) \leq \dim(\mathrm{sym}(V \otimes V^*)^{\otimes n})$. Therefore, the dimensions of the spaces of quantum and ordinary symmetric elements coincide for all n. Consequently, if M_q and M'_q are $S_n(\gamma_q)$-submodules of $V^{\otimes n}$ isomorphic to some $M_q(i, j)$, then $M_q \otimes M'^*_q$ contains (up to scalar multiple) a unique quantum symmetric element and contains none if $M_q \not\cong M'_q$.

The $S_n(\gamma_q)$-submodule $M_q(i, j)$ of $V(i, j)$ is therefore simple, for if it decomposed in a nontrivial way, we would be able to find at least two linearly independent quantum symmetric elements in $M_q(i, j) \otimes M^*_q(i, j)$. □

A consequence of the simplicity of $M_q(i, j)$ is that (7) gives the complete direct sum decompositions of $V(i, j)$ and $V(j, i)$ into simple $S_n(\gamma_q)$-modules. Moreover, these are orthogonal decompositions.

The correspondence of the decompositions of $V^{\otimes n}$ into simple modules for both the quantum and ordinary symmetric groups is the crucial step in realizing $\mathbf{M}_q(2)$ and $\mathbf{k}^2_q$ as deformations. In [6], the analogous result for $n > 2$ is discussed. There are some technical difficulties that do not appear for $n = 2$. For example, the analog of the modules $V(i, j)$ is $V(\mathfrak{p})$, where $\mathfrak{p}$ is a partition of n. When $n \neq 2$, the partitions are not linearly ordered making it impossible to count the multiplicities of the simple modules $M_q(\mathfrak{p})$, which are the counterparts of the $M_q(i, j)$. The modifications used in [6] do, however, establish the simplicity of $M_q(\mathfrak{p})$, thus making it possible to construct $\mathbf{M}_q(n)$ as a deformation of $\mathbf{M}(n)$.

5. $\mathbf{M}_q(2)$ and $\mathbf{k}^2_q$ as deformations

With the complete decomposition of $V^{\otimes n}$ into simple modules for both $S_n(\gamma_q)$ are S_n, we can construct canonical preferred deformations of $\mathbf{M}(2)$ and $\mathbf{k}^2$ which are isomorphic to $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$. Using the explicit form of the quantum symmetric elements in TV, the deformed multiplication for the quantum plane $\mathbf{k}_q^2$ can be exhibited in a simple closed form.

Recall that for the representation of $S_n(\gamma_q)$ on TV the ideal of quantum skew elements, $\mathrm{sk}_q(TV)$, is generated $x \otimes y - qy \otimes x$, which is the defining commutation relation for the quantum plane, so there is an exact sequence

$$0 \to \mathrm{sk}_q(TV) \to TV \to \mathbf{k}_q^2 \to 0 .$$

Similarly, the commutation relations for $\mathbf{M}_q(2)$ coincide with the quantum skew elements for the representation of $S_n(\gamma_q)$ on TX so there is a corresponding sequence

$$0 \to \mathrm{sk}_q(TX) \to TX \to \mathbf{M}_q(2) \to 0 .$$

For the next lemma, W will denote either the vector space V of functions on k^2, or the vector space X of functions on $M(2)$. Recall that for TW the dimensions of the spaces of quantum and ordinary symmetric elements are the same and the actions of $S_n(\gamma_q)$ and S_n are by orthogonal transformations.

Lemma 5.1. *The ideal* $\mathrm{sk}_q(TW)$ *is the orthogonal complement*, $(\mathrm{sym}_q(TW))^{\perp}$, *to the set of quantum symmetric elements in the tensor algebra* TW.

Proof. Any element of $\mathrm{sk}_q(TW)$ can be written as a sum of elements of the form $(\mathrm{Id}_{W^{\otimes n}} - \tau_i)\alpha$ for some quantum interchange τ_i and $\alpha \in W^{\otimes n}$; conversely, all such elements are quantum skew. Each quantum interchange τ_i is self-adjoint since it has order two and acts as an orthogonal transformation on TW so

$$\langle \tau_i(u), v \rangle = \langle u, \tau_i(v) \rangle$$

for all u and v. Now if σ_q is any quantum symmetric element in TW, then

$$\langle (\mathrm{Id}_{W^{\otimes n}} - \tau_i)\alpha, \sigma_q \rangle = \langle \alpha, (\mathrm{Id}_{W^{\otimes n}} - \tau_i)\sigma_q \rangle = \langle \alpha, 0 \rangle = 0 .$$

Hence any element of $\mathrm{sk}_q(TW)$ is orthogonal to every quantum symmetric element in TW so $\mathrm{sk}_q(TW) \subset (\mathrm{sym}_q(TW))^{\perp}$. Letting $(\mathrm{sk}_q(TW))_d = (\mathrm{sk}_q(TW)) \cap W^{\otimes d}$ and $(\mathrm{sym}_q(TW))_d^{\perp} = (\mathrm{sym}_q(TW))^{\perp} \cap W^{\otimes d}$ we have that

$$\dim((\mathrm{sk}_q(TW))_d) \leq \dim((\mathrm{sym}_q(TW))_d^{\perp}) .$$

Recall that the dimensions of the spaces of quantum symmetric elements are the same as that for the symmetric elements, so their complements also have the same dimensions giving

$$\dim((\mathrm{sym}_q(TW))_d^\perp) = \dim((\mathrm{sym}(TW))_d^\perp)\,.$$

Since every element in $\mathrm{sk}(TW)$ gives rise to an element in $\mathrm{sk}_q(TW)$, we also have that

$$\dim((\mathrm{sk}(TW))_d) \le \dim((\mathrm{sk}_q(TW))_d)\,.$$

Finally, $(\mathrm{sk}(TW))_d$ and $(\mathrm{sym}(TW))_d^\perp$ coincide when the symmetric algebra is embedded into the tensor algebra so

$$\dim((\mathrm{sk}(TW))_d) = \dim((\mathrm{sym}(TW))_d^\perp)\,,$$

which, along with the previous equations, yields

$$\dim((\mathrm{sk}_q(TW))_d) = \dim((\mathrm{sym}_q(TW))_d^\perp)$$

for each d and therefore $\mathrm{sk}_q(TW) = (\mathrm{sym}_q(TW))^\perp$. □

We can now describe the appropriate transformations of TX and TV which give $\mathbf{M}_q(2)$ and $\mathbf{k}_q^2$. To do this we use the decomposition of TV into simple $S_n(\gamma_q)$ and simple S_n-modules obtained in Section 4. Let $T_V^n : V^{\otimes n} \to V^{\otimes n}$ be an orthogonal transformation which takes the simple $S_n(\gamma_q)$-summands of $V^{\otimes n}$ to the corresponding simple S_n-summands. Combining all the T_V^n gives an orthogonal transformation $T_V : TV \to TV$ in which $T_V(\mathrm{sk}_q(TV)) = \mathrm{sk}(TV)$ and $T_V(\mathrm{sym}_q(TV)) = \mathrm{sym}(TV)$. Set $T_{V^*} = (T_V)^{-*}$ and let $T_X = T_V \otimes T_{V^*} : TX \to TX$ where TX is again viewed as the tensor product $TV \otimes TV^*$. Recall that every quantum symmetric element of TX is contained in a module of the form $M_q \otimes M_q'^*$, where $M_q, M_q' \cong M_q(i, j)$ for some (i, j). It follows that $T_X(M_q \otimes M_q'^*) = M \otimes M'^*$, where $M, M' \cong M(i, j)$ and $M(i, j)$ is the simple S_n-module of $V^{\otimes n}$ corresponding to $M_q(i, j)$. Thus $T_x(\mathrm{sym}_q(TX)) = \mathrm{sym}(TX)$ and, since T_X is orthogonal, $T_X(\mathrm{sk}_q(TX)) = \mathrm{sk}(TX)$.

The transformation T_X can be used to give a 'trivial' deformation of the tensor algebra TX. The deformed tensor multiplication, denoted $\circledast$, is obtained by setting

$$\alpha \circledast \beta = T_X(T_X^{-1}(\alpha) \otimes T_X^{-1}(\beta))$$

for $\alpha, \beta \in TX$. In particular,

$$x_1 \circledast \cdots \circledast x_n = T_X(x_1 \otimes \cdots \otimes x_n)\,.$$

Associativity of $\circledast$ is guaranteed since it is the transport of the associative tensor multiplication. The transformation T_X may therefore be viewed as an algebra isomorphism from $(TX, \otimes)$, the k-module TX with multiplication $\otimes$, to $(TX, \circledast)$, the k-module TX with multiplication $\circledast$. Note that it takes the ideal $\mathrm{sk}_q(TX)$ in $(TX, \otimes)$ to the subspace $\mathrm{sk}(TX)$ in $(TX, \circledast)$ which then must be an ideal for the $\circledast$-multiplication. We thus have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{sk}_q(TX) & \longrightarrow & (TX, \otimes) & \longrightarrow & \mathbf{M}_q(2) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle T_X} & & \Big\downarrow{\scriptstyle T_X} & & \Big\downarrow{\scriptstyle \bar{T}_X} & & \\ 0 & \longrightarrow & \mathrm{sk}(TX) & \longrightarrow & (TX, \circledast) & \xrightarrow{\mu} & (\mathbf{M}(2), *) & \longrightarrow & 0 \end{array} \tag{8}$$

where $*$ is the multiplication in $\mathbf{M}(2)$ induced from $\circledast$.

The next theorem guarantees that the deformed tensor algebra $(TX, \circledast)$ is also a preferred deformation of the tensor *bialgebra* TX. In other words, the *original* comultiplication on TX is compatible with the deformed tensor multiplication, $\circledast$, on all elements.

Theorem 5.2. *The transformation $T_X : TX \to TX$ is a coalgebra automorphism. Thus, $(TX, \circledast, \Delta)$ is a bialgebra and $\bar{T}_X$ is a bialgebra isomorphism from the quantum matrix bialgebra $\mathbf{M}_q(2)$ to the preferred deformation, $(\mathbf{M}(2), *)$, of $\mathbf{M}(2)$ in which the comultiplication is unchanged on all elements of $\mathbf{M}(2)$.*

Proof. A basis for $X^{\otimes n}$ consists of all elements of the form $v_I \otimes v_J^*$, where the subscripts I and J run through the multi-indices $i_1, \ldots, i_n$ of length n and $v_I = v_{i_1} \otimes \cdots \otimes v_{i_n}$ with all $v_{i_j} \in V$; v_J^* is defined similarly. With this, the comultiplication has the form

$$\Delta(v_I \otimes v_J^*) = \sum_K (v_I \otimes v_K) \otimes (v_K^* \otimes v_J^*) .$$

The operation of T_X is given by

$$T_X(v_I \otimes v_J^*) = T_V v_I \otimes T_V^{-*} v_J^* ,$$

so we have

$$\begin{aligned} \Delta T_X(v_I \otimes v_J^*) &= \sum_K (T_V v_I \otimes v_K) \otimes (v_K^* \otimes T_V^{-*} v_J^*) \\ &= \sum_K (T_V v_I \otimes T_V v_K) \otimes (T_V^{-*} v_K^* \otimes T_V^{-*} v_J^*) . \end{aligned} \tag{9}$$

The last equality follows from Lemma 4.4 since the set of all $\{v_K\}$ forms a basis for $V^{\otimes n}$ and $T_X = T_V \otimes T_{V^*}$. Now

$$(T_X \otimes T_X)\Delta(v_I \otimes v_J) = (T_X \otimes T_X) \sum_K (v_I \otimes v_K) \otimes (v_K^* \otimes v_J^*)$$
$$= \sum_K (T_V v_I \otimes T_V v_K) \otimes (T_V^{-*} v_K^* \otimes T_V^{-*} v_J^*) . \quad (10)$$

So combining (9) and (10) gives

$$\Delta T_X = (T_X \otimes T_X)\Delta ,$$

which says that T_X is a coalgebra automorphism and, since T_X defines the deformed tensor multiplication ⊛, also guarantees that ⊛ is compatible with Δ, making $(TX, ⊛, \Delta)$ a bialgebra. There is then an induced bialgebra isomorphism $\bar{T}_X : \mathbf{M}_q(2) \to (\mathbf{M}(2), *)$. □

The quantum special linear group $\mathbf{SL}_q(2)$ is defined to be the bialgebra obtained as the quotient $\mathbf{M}_q(2)/\det_q(\mathscr{X}) - 1)$, where $\det_q(\mathscr{X}) = a \otimes d - qb \otimes c$ is the *quantum determinant*. In [4], it is shown that $\det_q(\mathscr{X})$ is a group-like element of $\mathbf{M}_q(2)$, that is $\Delta(\det_q(\mathscr{X})) = (\det_q(\mathscr{X})) \otimes (\det_q(\mathscr{X}))$. Now it is well known that in $\mathbf{M}(2)$, $\det(\mathscr{X}) = a \otimes d - b \otimes c$ is the unique group-like element of $X \otimes X$. Consequently, since $\bar{T}_X$ is, in particular, an isomorphism of coalgebras $\mathbf{M}_q(2) \cong \mathbf{M}(2)$, it must take $\det_q(\mathscr{X})$ to $\det(\mathscr{X})$. Thus, in the preferred presentation of $\mathbf{M}_q(2)$, the quantum determinant is just the ordinary determinant and $\mathbf{SL}_q(2) \cong (\mathbf{M}(2), *)/(\det(\mathscr{X}) - 1)$.

Similarly, there is a preferred presentation, $(\mathbf{k}^2, *)$, for the quantum plane as a $(\mathbf{M}(2), *)$-comodule algebra. This means that $\mathbf{k}_q^2 \cong (\mathbf{k}^2, *)$ and for any u and v in $\mathbf{k}^2$ with coactions $\lambda(u) = \sum w_{(1)} \otimes u_{(2)}$ and $\lambda(v) = \sum w'_{(1)} \otimes v_{(2)}$, where $w_{(1)}, w'_{(1)} \in \mathbf{M}(2)$ and $u_{(2)}, v_{(2)} \in \mathbf{k}^2$, we have $\lambda(u * v) = \sum (w_{(1)} * w'_{(1)}) \otimes (u_{(2)} * v_{(2)})$.

The 'star' multiplication for $(\mathbf{k}^2, *)$ is obtainable using the orthogonal transformation T_V since the explicit form of the quantum symmetric elements in TV is easy to compute. The module $V(i, j)$ is spanned by all monomials α such that $\mu\alpha = x^i y^j$ (these are the monomials of degree $i - j$ and μ is the multiplication) so $\sum \alpha$ is, up to a multiple, the only symmetric element of $V(i, j)$. Since $\dim(V(i, j)) = \binom{n}{i}$, the unique symmetric element of norm one is

$$\alpha(i, j) = \binom{n}{i}^{-1/2} \sum \alpha .$$

In each $V(i, j)$ there is also, up to a multiple, only one quantum symmetric element; for $V(1, 1)$ we have seen that this element is $qx \otimes y + y \otimes x$. From this it is easy to intuit that the quantum symmetric element in the module $V(i, j)$ is $\sum q^{l(\alpha)}\alpha$, where $l(\alpha)$ is the number of *inversions* present in the monomial α.

Definition 5.3. For a monomial $\alpha \in V(i, j)$, let $\sigma \in S_n$ be any permutation which satisfies $\sigma(y^{\otimes i}x^{\otimes n-i}) = \alpha$. The number $l(\alpha)$ of *inversions* in α is the length of the shortest expression of σ as a product of transpositions.

It can easily be shown that $l(\alpha)$ is well defined. To compute the quantum symmetric elements of norm one we need to know the number of monomials in $V(i, j)$ having exactly d inversions. The generating function for this is shown in [10] to be the q-binomial coefficient $\binom{n}{i}_q$. In other words

$$\binom{n}{i}_q = \sum_\alpha q^{l(\alpha)}, \tag{11}$$

where the sum ranges over all monomials α in $V(i, j)$. The quantum symmetric element of norm one, $\alpha_q(i, j)$, is thus

$$\alpha_q(i, j) = \binom{n}{i}_{q^2}^{-1/2} \sum q^{l(\alpha)}\alpha, \tag{12}$$

where the sum ranges over all monomials $\alpha \in V(i, j)$. Some examples of the various $\alpha_q(i\ j)$ are

$$\alpha_q(n, 0) = x^{\otimes n},$$
$$\alpha_q(1, 1) = (1 + q^2)^{-1/2}(qx \otimes y + y \otimes x),$$
$$\alpha_q(2, 1) = (1 + q^2 + q^4)^{-1/2}(q^2 x \otimes x \otimes y + qx \otimes y \otimes x + y \otimes x \otimes x).$$

The products for the quantum plane are deduced from the following analog of (8):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{sk}_q(TV) & \longrightarrow & (TV, \otimes) & \longrightarrow & \mathbf{k}_q^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_V} & & \downarrow{\scriptstyle T_V} & & \downarrow{\scriptstyle \bar{T}_V} & & \\ 0 & \longrightarrow & \mathrm{sk}(TV) & \longrightarrow & (TV, \circledast) & \xrightarrow{\mu} & (\mathbf{k}^2, *) & \longrightarrow & 0 \end{array} \tag{13}$$

Suppose $x^k y^l, x^r y^s \in \mathbf{k}^2$ are given with $k + l = n$ and $r + s = m$. The elements $\binom{n}{l}^{-1/2}\alpha(k, l)$ and $\binom{m}{s}^{-1/2}\alpha(r, s)$ are then the canonical symmetric elements in TV corresponding to $x^k y^l$ and $x^r y^s$ (remember that $\|\alpha(k, l)\| = \|\alpha(r, s)\| = 1$). According to (13),

$$\begin{aligned} x^k y^l * x^r y^s &= \left(\binom{n}{l}\binom{m}{s}\right)^{-1/2} \mu(T_V(T_V^{-1}(\alpha(k, l)) \otimes T_V^{-1}(\alpha(r, s)))) \\ &= \left(\binom{n}{l}\binom{m}{s}\right)^{-1/2} \mu(T_V(\alpha_q(k, l) \otimes \alpha_q(r, s))) \\ &= \left(\binom{n}{l}\binom{m}{s}\binom{n}{l}_{q^2}\binom{m}{s}_{q^2}\right)^{-1/2} \mu \sum_{\alpha, \alpha'} q^{l(\alpha)+l(\alpha')} T_V(\alpha \otimes \alpha'), \end{aligned} \tag{14}$$

where the sum ranges through all monomials α and α' in $V(k, l)$ and $V(r, s)$, respectively.

Now, if $\zeta \in V(i', j')$, then

$$T_V(\zeta) = c\alpha(i', j') + \beta ,$$

where $\mu(\beta) = 0$, hence to find $\mu(T_V(\zeta))$ it suffices to find the constant c since $\mu(\alpha(i', j')) = \binom{n'}{i'}^{1/2} x^{i'} y^{j'}$. Since $\alpha(i', j')$ may be viewed as an element of an orthonormal basis for $V(i', j')$, we have that $c = \langle T_V(\zeta), \alpha(i', j') \rangle$. Now because T_V is orthogonal and $T_V(\alpha_q(n', i')) = \alpha(n', i')$ it follows that

$$c = \langle T_V(\zeta), \alpha(i', j') \rangle = \langle \zeta, \alpha_q(i', j') \rangle .$$

For monomials $\alpha \in V(k, l)$ and $\alpha' \in V(r, s)$,

$$\langle \alpha \otimes \alpha', \alpha_q(k + r, l + s) \rangle = \left(\binom{n + m}{l + s}_{q^2} \right)^{-1/2} q^{l(\alpha) + l(\alpha') + ks} ,$$

where $k + l = n$ and $r + s = m$.

Returning to (14) we now have that

$$\begin{aligned} x^k y^l * x^r y^s &= \left(\binom{n}{l} \binom{m}{s} \binom{n}{l}_{q^2} \binom{m}{s}_{q^2} \binom{n + m}{l + s}_{q^2} \right)^{-1/2} \\ &\quad \times q^{ks} \sum_{\alpha, \alpha'} q^{2(l(\alpha) + l(\alpha'))} \mu(\alpha(k + r, l + s)) \\ &= \left(\binom{n}{l} \binom{m}{s} \binom{n}{l}_{q^2} \binom{m}{s}_{q^2} \binom{n + m}{l + s}_{q^2} \right)^{-1/2} \\ &\quad \times q^{ks} \binom{n}{l}_{q^2} \binom{m}{s}_{q^2} \binom{n + m}{l + s}^{1/2} x^{k + r} y^{l + s} \\ &= \left(\frac{\binom{n}{l}_{q^2} \binom{m}{s}_{q^2} \binom{n + m}{l + s}}{\binom{n}{l} \binom{m}{s} \binom{n + m}{l + s}_{q^2}} \right)^{1/2} q^{ks} x^{k + r} y^{l + s} . \end{aligned}$$

Setting $\nu_{i,j} = (\binom{i+j}{i} / \binom{i+j}{i}_{q^2})^{1/2}$ to be the 'norm factor', we have the following theorem:

Theorem 5.4. *The products*

$$x^k y^l * x^r y^s = \left(\frac{\nu_{k+r, l+s}}{\nu_{k,l} \nu_{r,s}} \right) q^{ks} x^{k + r} y^{l + s}$$

define a preferred deformation from $\mathbf{k}^2$ *to* $(\mathbf{k}^2, *)$ *which, by construction, is isomorphic as an* $(\mathbf{M}(2), *)$*-comodule algebra to the quantum plane* $\mathbf{k}_q^2$. □

For an example, in $V(1,1)$ we have the following:

$$x * y = q\left(\frac{2}{1+q^2}\right)^{1/2} xy\,, \qquad y * x = \left(\frac{2}{1+q^2}\right)^{1/2} xy\,,$$

so $x * y = qy * x$, which is the standard defining commutation relation for the quantum plane. This construction is only valid when q is not a root of unity since otherwise some of the quantum symmetric elements, $\alpha_q(i, j)$ become isotropic which makes the norm factors $\nu(i, j)$ infinite. Thus a preferred presentation for this quantum plane need not exist when q is specialized at a root of unity, a value where quantum groups are known to have different properties (cf. [11]).

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