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The indescribability of the order of the indescribable cardinals

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Abstract

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We prove the following consistency results about indescribable cardinals which answer a question of A. Kanamori and M. Magidor (cf. [3]).

Theorem 1.1 $(m \geqslant 2, n \geqslant 2)$. CON(ZFC + $\exists\kappa, \kappa'$ (κ *is* Π^m_n *indescribable,* κ' *is* Σ^m_n *indescribable, and* $\kappa < \kappa'$)) $\Rightarrow$ CON(ZFC + $\sigma^m_n > \pi^m_n$ + GCH).

Theorem 5.1 (ZFC). *Assuming the existence of* Σ^m_n *indescribable cardinals for all* $m < \omega$ *and* $n < \omega$ *and given a function* $\mathscr{F}$: $\{(m, n): m \geqslant 2, n \geqslant 1\} \rightarrow \{0, 1\}$, *there is a poset* $P_{\mathscr{F}} \in L[\mathscr{F}]$ *such that* GCH *holds in* $(L[\mathscr{F}])^{P_{\mathscr{F}}}$ *and*

$$\Vdash^{L[\mathscr{F}]}_{P_{\mathscr{F}}} \begin{cases} \sigma^m_n < \pi^m_n & \text{if } \mathscr{F}(m, n) = 0, \\ \sigma^m_n > \pi^m_n & \text{if } \mathscr{F}(m, n) = 1. \end{cases}$$

Theorem 1.1 extends the work begun in [2], and its proof uses an iterated forcing construction together with master condition arguments. By combining these techniques with some observations about small forcing and indescribability, one obtains the Easton-style result 5.1.

Introduction and statements of results

This paper presents a continuation of the work begun in [2]. Recall that an ordinal α is Ω indescribable if a partial reflection principle for formulas in Ω holds at the α-th level of the von-Neumann-hierarchy; i.e., for any sentence ϕ in Ω which may contain a unary predicate symbol and any subset $A \subseteq V_\alpha$

$$\langle V_\alpha, \in, A \rangle \vDash \phi \rightarrow \exists \beta < \alpha \langle V_\beta, \in, A \cap V_\beta \rangle \vDash \phi.$$

We will only be concerned with certain standardized classes of formulas. As usual Σ^m_n (Π^m_n resp.) denotes the collection of all formulas in the language of set theory

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with higher type variables and a unary predicate symbol whose prenex normal form has n alternating blocks of $(m+1)$th order quantifiers starting with $\exists$ ($\forall$ resp.) and all other quantifiers are of order $\leqslant m$.

It has been known since the early sixties (cf. [1]) that this approach leads to large cardinals, i.e., the existence of Π_n^m (or Σ_n^m) indescribable cardinals is unprovable in ZFC ($m \geqslant 1$). Moreover, larger classes of formulas yield genuinely stronger notions of indescribability: If σ_n^m (π_n^m resp.) denotes the least Σ_n^m (Π_n^m resp.) indescribable cardinal (provided it exists) then in ZFC

$$\pi_n^1 = \sigma_{n+1}^1 < \pi_{n+1}^1 = \sigma_{n+2}^1$$

and

$$\sigma_n^m,\ \pi_n^m < \sigma_{n+1}^m,\ \pi_{n+1}^m$$

for $m \geqslant 2$ and $n \geqslant 0$ (cf. [4]). It is also shown there that

$$\sigma_n^m \neq \pi_n^m$$

for $m \geqslant 2$ and $n \geqslant 1$. However, this is as far as we can go in ZFC: If $V = L$ then $\sigma_n^m < \pi_n^m$ ($m \geqslant 1$, $n \geqslant 1$, cf. [5]). On the other hand it is consistent with ZFC to have $\sigma_1^m > \pi_1^m$ for $m \geqslant 2$ (cf. [2]). In this paper we complete the picture by showing

Theorem 1.1 ($m \geqslant 2$, $n \geqslant 2$).

CON(ZFC + $\exists \kappa, \kappa'$ (*κ is Π_n^m indescribable, κ' is Σ_n^m indescribable, and* $\kappa < \kappa'$)) $\Rightarrow$ CON(ZFC + $\sigma_n^m > \pi_n^m$ + GCH). □

If one combines the techniques from the proof of Theorem 1.1 with some observations about small forcing and indescribability, one obtains the following Easton-style result which shows that we have the ultimate freedom in simultaneously arranging the relative sizes of the indescribable cardinals.

Theorem 5.1 (ZFC). *Assuming the existence of Σ_n^m indescribable cardinals for all m and n and given a function $\mathscr{F}: \{(m, n): m \geqslant 2, n \geqslant 1\} \to \{0, 1\}$ there is a poset $P_{\mathscr{F}} \in L[\mathscr{F}]$ such that* GCH *holds in $(L[\mathscr{F}])^{P_{\mathscr{F}}}$ and*

$$\Vdash_{P_{\mathscr{F}}}^{L[\mathscr{F}]} \begin{cases} \sigma_n^m < \pi_n^m & \text{if } \mathscr{F}(m, n) = 0, \\ \sigma_n^m > \pi_n^m & \text{if } \mathscr{F}(m, n) = 1. \end{cases} \qquad \square$$

These results provide an answer to a question of Kanamori and Magidor (cf. [3]). In order to prove Theorem 1.1 one defines a forcing iteration which kills off all Σ_n^m indescribable cardinals below a given Π_n^m indescribable κ. This forcing will preserve any Σ_n^m indescribable cardinal κ' above κ because it is small relative to κ'. The hard part of the proof is showing that this poset also preserves the Π_n^m indescribability of κ. For this we need a characterization of Π_n^m indescribability in terms of elementary embeddings (cf. Theorem 1.3 in [2]). A series of master condition arguments is then employed to lift these embeddings from the ground model to suitable generic extensions.

Thus the general strategy appears to resemble the one for the proof of Theorem 3.3.1 in [2]. However, there are new problems here: Recall that, working in $V[G]$ we have to lift some elementary embedding $j: M \hookrightarrow N$ (where M, N are some transitive models) to obtain an embedding $j: M[G^M] \hookrightarrow N[G^N]$. In the proof of Theorem 3.3.1 in [2] it was sufficient to make $N[G^N]$ agree with $V[G^N]$ for sets of rank less than $\kappa + m - 1$. Now we have to guarantee that in addition, $N[G^N]$ is Σ^m_{n-1} correct for κ inside $V[G^N]$, i.e. $N[G^N]$ correctly computes the Σ^m_{n-1} facts in parameters from $(N[G^N])_{\kappa+m}$ that hold in V_κ. Worse is to come: The iteration that N wants to do is of length $j(\kappa)$; at the κ-th stage we want to force a Π^m_n statement about certain objects. On the other hand, the iteration in V forces the negation of this statement. Therefore great care has to be taken in the definition of the forcing iteration in order to make the Π^m_n forcing and the Σ^m_n forcing resemble each other to a degree that allows us to carry out the above correctness argument.

Regarding our notation, the reader is referred to [2] where he will find the definitions of all nonstandard symbols that appear without explanations in this paper.

These results are the published incarnation of parts of my Caltech Ph.D. thesis. It has been both a privilege and a pleasure to work under the supervision of Prof. W. Hugh Woodin. Furthermore I would like to thank Prof. G.H. Müller (Heidelberg) for suggesting the central problem, and for his continued interest in my personal and mathematical well-being over the years.

1. The coarse structure of the iteration

Our goal is to show

Theorem 1.1 ($m \geq 2$, $n \geq 2$).

CON(ZFC + $\exists \kappa, \kappa'$ (*κ is Π^m_n indescribable, κ' is Σ^m_n indescribable, and $\kappa < \kappa'$*)) $\Rightarrow$ CON(ZFC + $\sigma^m_n > \pi^m_n$ + GCH). □

In order to prove this we can work in ZF + $V = L$ (since Π^m_n and Σ^m_n indescribability relativize down to L) and assume κ is a Π^m_n indescribable cardinal and $\kappa' > \kappa$ is a Σ^m_n indescribable cardinal. Then we define a $\kappa + 1$ stage iteration $(P_\alpha : \alpha \leq \kappa + 1)$ such that

$\Vdash_{P_{\kappa+1}}$ "there are no Σ^m_n indescribables $<\kappa$, κ is Π^m_n indescribable, κ' is Σ^m_n indescribable".

Hence we obtain

$$\Vdash_{P_{\kappa+1}} \sigma^m_n > \pi^m_n.$$

The idea behind the definition of the iteration is that we want no Σ_n^m indescribable cardinals $\leqslant \kappa$. Thus at stage $\lambda \leqslant \kappa$ of the iteration we have to force a Σ_n^m description of λ. In addition to this we also want our iteration to preserve the Σ_n^m indescribability of κ' (which is no problem) and the Π_n^m indescribability of κ. Because of the latter (cf. [2] for a discussion of this issue) it is necessary to do more than simply to force one Σ_n^m description λ.

Here is the official definition of $(P_\alpha: \alpha \leqslant \kappa + 1)$: Let P_0 be the trivial poset. For a limit ordinal $\alpha \leqslant \kappa$ let P_α be the direct or inverse limit of $(P_\zeta: \zeta < \alpha)$ depending on whether α is inaccessible or not (respectively). If P_λ has been defined for some Mahlo cardinal $\lambda \leqslant \kappa$, pick a term $Q_\lambda \in V^{P_\lambda}$ with the following properties: Q_λ is itself a $m+2$ step iteration. In the first step we add a sequence $(F_\gamma: \gamma < \lambda^+)$ of Lipschitz functions on $(2^{\lambda^{+(m-1)}})^{n-1}$, i.e., each F_γ is really a function with domain $(2^{<\lambda^{+(m-1)}})^{n-1}$ and range contained in $2^{<\lambda^{+(m-1)}}$, and we define

$$F_\gamma((X_1, \ldots, X_{n-1})) = \bigcup_{\zeta < \lambda^{+(m-1)}} F_\gamma((X_1 \cap \zeta, \ldots, X_{n-1} \cap \zeta))$$

for $X_1, \ldots, X_{n-1} \subseteq \lambda^{+(m-1)}$ provided that for $\eta < \zeta < \lambda^{+(m-1)}$, $F_\gamma((X_1 \cap \zeta, \ldots, X_{n-1} \cap \zeta))$ extends $F_\gamma((X_1 \cap \eta, \ldots, X_{n-1} \cap \eta))$. (Note: we frequently identify sets with their characteristic functions.) In the second step we force a Σ_n^m fact about F_γ where $\gamma < \lambda^+$ is even and its negation (a Π_n^m fact) about F_γ where $\gamma < \lambda^+$ is odd. The next $m-1$ steps code down each F_γ to $\tilde{S}_\gamma \subseteq \lambda$ $(\gamma < \lambda^+)$. Finally we add a sequence of club sets $C_\gamma \subseteq \lambda$ such that for each $\gamma < \lambda^+$, C_γ avoids the set of all inaccessibles μ below λ for which the above Σ_n^m fact about F_γ (or rather its code $\tilde{S}_\gamma$) reflects down to V_μ. If $\lambda < \kappa$ is not Mahlo, we let Q_λ be a term for the trivial poset. In either case define $P_{\lambda+1} = P_\lambda * Q_\lambda$. This completes the definition of the iteration.

Since for any inaccessible μ we have $\forall \alpha < \mu\ |P_\alpha| < \mu$ and since we take direct limits at inaccessibles, P_λ is λ c.c. for any Mahlo cardinal $\lambda \leqslant \kappa$. Thus such λ remain regular in V^{P_λ}. In fact their inaccessibility is preserved, since one can show by standard factoring arguments that for each $\alpha \leqslant \kappa$

$$\Vdash_{P_{\alpha+1}} \text{“}P_{\alpha+1,\kappa+1} \text{ has for each } \nu < \mu \text{ a } < \nu \text{ closed dense suborder”}$$

where $P_{\alpha+1,\kappa+1}$ denotes the tail of the iteration in $V^{P_{\alpha+1}}$ and μ is the least inaccessible cardinal $>\alpha$. This means in particular that from the viewpoint of $V^{P_{\alpha+1}}$ the tail is highly Baire. Thus once a candidate for Σ_n^m indescribability is killed off it is never resurrected later on during the iteration and we obtain

$$\Vdash_{P_{\kappa+1}} \text{“there are no } \Sigma_n^m \text{ indescribables below } \kappa\text{”}.$$

More factoring arguments together with the chain condition and closure properties of the posets in the forcing Q_λ allow us to prove by induction on α

$$\Vdash_{P_\alpha} \mathrm{GCH}.$$

It follows from $|P_{\kappa+1}| < \kappa'$ that $P_{\kappa+1}$ preserves the Σ_n^m indescribability of κ. In order to finish the proof of 1.1 we only have to show that $P_{\kappa+1}$ also preserves the

Π^m_n indescribability of κ. This is being done in Section 3 where we work out the argument for the case $m=2$. In Section 4 we briefly indicate how all this can be generalized to $m \geq 3$. Finally, in Section 5 we prove an Easton style result that shows that we can simultaneously arrange the relative sizes of the indescribables as we please.

2. The fine structure of the Σ^2_n/Π^2_n iteration

Suppose $\lambda \leq \kappa$ is a Mahlo cardinal, G_λ is generic for P_λ, and in $V[G_\lambda]$ λ is inaccessible and $\lambda^{+l} = (\lambda^{+l})^L$ for $l \geq 1$ and GCH holds from λ on. (Once the whole iteration has been defined it is easily verified that these requirements are satisfied.) The first step Q^1_λ of the four step iteration Q_λ is a λ^+ product with $<\lambda^+$ support of copies of the forcing Q_F which adds a Lipschitz function $F:(2^{\lambda^+})^{n-1} \to 2^\lambda$. Conditions in Q_F are approximations of F, i.e., conditions are functions f with

$$
\begin{aligned}
&\mathrm{dom}(f) \text{ a subtree of } (2^{<\lambda^+})^{n-1} \text{ of size } \lambda \text{ such that}\\
&\forall (s_1, \ldots, s_{n-1}) \in \mathrm{dom}(f)\ [\mathrm{lh}(s_1) = \cdots = \mathrm{lh}(s_{n-1}) \wedge \exists \alpha < \lambda^+\\
&[f((s_1, \ldots, s_{n-1})) \in 2^{\alpha+1} \wedge \alpha \geq \mathrm{lh}(s_1) \wedge f((s_1, \ldots, s_{n-1}))(\alpha) = 0\\
&\wedge \forall \zeta \leq \alpha\, [f((s_1, \ldots, s_{n-1}))(\zeta) = 1 \Rightarrow \mathrm{cf}(\zeta) = \lambda]] \text{ and}\\
&\forall (s_1, \ldots, s_{n-1}), (t_1, \ldots, t_{n-1}) \in \mathrm{dom}(f)\ [(t_1, \ldots, t_{n-1}) \text{ extends}\\
&(s_1, \ldots, s_{n-1}) \Rightarrow f((t_1, \ldots, t_{n-1})) \text{ extends } f((s_1, \ldots, s_{n-1}))]].
\end{aligned}
$$

For two conditions $f, g \in Q_F$ we let $f \leq g$ iff $f \supseteq g$. Clearly Q^1_λ is $<\lambda^+$ closed and has size λ^+. Therefore, if $(F_\gamma\colon \gamma < \lambda^+)$ is Q^1_λ generic over $V[G_\gamma]$ then in $V[G_\gamma, \vec{F}_\gamma]$ λ is still inaccessible, $\lambda^{+l} = (\lambda^{+l})^L$ for each $l \geq 1$ and GCH holds from λ on. Moreover, for each $(X_1, \ldots, X_{n-1}) \in (2^{\lambda^+})^{n-1}$ we can define

$$F_\gamma((X_1, \ldots, X_{n-1})) \stackrel{\mathrm{def}}{=} \bigcup_{\zeta < \lambda^+} F_\gamma((X_1 \cap \zeta, \ldots, X_{n-1} \cap \zeta)).$$

In the second step Q^2_λ of Q_λ we will force a Σ^2_n statement about F_γ for $\gamma < \lambda^+$ even and a Π^2_n statement about F_γ for $\gamma < \lambda^+$ odd. The Σ^2_n statement says

$$\exists X_1 \subseteq \lambda^+\, \forall X_2 \subseteq \lambda^+ \cdots \mathrm{Q} X_{n-1} \subseteq \lambda^+\, \varphi(F_\gamma((X_1, \ldots, X_{n-1})))$$

where Q is $\forall$ or $\exists$ (resp.) and φ is "$F_\gamma((X_1, \ldots, X_n))$ is a nonstationary (stationary resp.) subset of λ^+" depending on whether n is odd or even (resp.). The Π^m_n statement is just the negation of the Σ^m_n statement.

Naturally, Q^2_λ will itself be an iteration of length λ^{++}, but we prefer to think of it as a suborder of $\mathrm{Add}(\lambda^{++}, \lambda^+)$. On the outset fix a partition of λ^{++} into cofinal pieces A^0 and $A^{k,\gamma}$ where $1 \leq k \leq n-1$ and $\gamma < \lambda^+$ with $\lambda^+ \subseteq A^0$. For each $k \in \{1, \ldots, n-1\}$ and $\gamma < \lambda^+$ pick a *complete* sequence $((\tau^{1,\gamma}_\zeta, \ldots, \tau^{k,\gamma}_\zeta)\colon \zeta < \lambda^{++})$ of k-tupels of nice $\mathrm{Add}(\lambda^{++}, \lambda^+)$ names for subsets of λ^+, i.e., for each k-tupel $(\tau^1, \ldots, \tau^k)$ of nice $\mathrm{Add}(\lambda^{++}, \lambda^+)$ names for subsets of λ^+ there are confinally many $\zeta<\lambda^{++}$ with $(\tau^1,\ldots,\tau^k)=(\tau^{1,\gamma}_\zeta,\ldots,\tau^{k,\gamma}_\zeta)$. We need some notation: For $S\subseteq\lambda^{++}$ and $q\in\mathrm{Add}(\lambda^{++},\lambda^+)$ let

$$\mathrm{Add}^{\lambda^{++}}(S,\lambda^+)\stackrel{\mathrm{def}}{=}\{f\in\mathrm{Add}(\lambda^{++},\lambda^+)\colon \mathrm{supp}(f)\subseteq S\}$$

and define $q|^{\emptyset}S\in\mathrm{Add}^{\lambda^{++}}(S,\lambda^+)$ by $q|^{\emptyset}S(\zeta)=q(\zeta)$ for $\zeta\in S$ and $q|^{\emptyset}S(\zeta)=\emptyset$ for $\zeta\in\lambda^{++}\sim S$. Now we define by induction on $\alpha\leqslant\lambda^{++}$ a sequence $(Q_{\alpha}\colon \alpha\leqslant\lambda^{++})$ where each Q_α is a suborder of $\mathrm{Add}^{\lambda^{++}}(\alpha,\lambda^+)$ and $Q_{\lambda^{++}}=Q^2_\lambda$. Q_0 is the trivial order on $\{\mathbf{1}_{\mathrm{Add}(\lambda^{++},\lambda^+)}\}$. If $\alpha\leqslant\lambda^{++}$ is a limit ordinal we let

$$Q_\alpha=\{q\in\mathrm{Add}^{\lambda^{++}}(\alpha,\lambda^+)\colon \forall\zeta<\alpha\; q|^{\emptyset}\zeta\in Q_\zeta\}.$$

If $\alpha=\beta+1$ for some $\beta<\lambda^{++}$ there are two cases: For $\beta\in\lambda^{++}\sim\bigcup_{\gamma<\lambda^+}A^{n-1,\gamma}$ we simply add a subset of λ^+ at coordinate β, i.e.,

$$Q_{\beta+1}=\{q\in\mathrm{Add}^{\lambda^{++}}(\beta+1,\lambda^+)\colon q|^{\emptyset}\beta\in Q_\beta\}.$$

On the other hand suppose $\gamma<\lambda^+$ and $\beta\in A^{n-1,\gamma}$. In this case we want to add a club subset of λ^+ which is disjoint from $F_\gamma((\tau^{1,\gamma}_\beta,\ldots,\tau^{n-1,\gamma}_\beta))$ if certain '*killing conditions*' are satisfied in $(V[G_\lambda,\vec{F}_\gamma])^{Q_\beta}$. If these killing conditions are not satisfied we save $F_\gamma((\tau^{1,\gamma}_\beta,\ldots,\tau^{n-1,\gamma}_\beta))$, i.e., we force with the trivial poset at coordinate β. (Why this is called 'saving' will become clear in 2.5.) The killing conditions are essentially determined by certain agreements and disagreements in the first $k+1$ components $(0\leqslant k\leqslant n-2)$ of the tupel $(\tau^{1,\gamma}_\beta,\ldots,\tau^{n-1,\gamma}_\beta)$ at coordinate $\beta\in A^{n-1,\gamma}_\beta$ with tupels of the form $(\tau^{1,\gamma}_\zeta,\ldots,\tau^{k,\gamma}_\zeta,\Gamma^\zeta)$ where $(\tau^{1,\gamma}_\zeta,\ldots,\tau^{k,\gamma}_\zeta)$ appears at coordinate $\zeta\in A^{k,\gamma}\cap\beta$ $(1\leqslant k\leqslant n-2)$ and Γ^ζ is a canonical Q_β name for the subset of λ^+ that we add at coordinate ζ. There is a minor technical point here: In general we cannot expect any of terms appearing in tupels at coordinates up to β to be terms in the forcing language for Q_β. Therefore we have to define an operation on terms that associates with each nice $\mathrm{Add}(\lambda^{++},\lambda^+)$ name $\tau^{i,\gamma}_\zeta$ a term $\hat{\tau}^{i,\gamma}_\zeta$ in the forcing language for Q_ζ as follows:

$$\hat{\tau}^{i,\gamma}_\zeta=\{(\eta,f)\colon f\in Q_\zeta\wedge\exists g\,((\eta,g)\in\tau^{i,\gamma}_\zeta\wedge f\leqslant g)\}$$

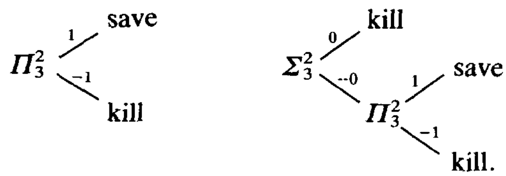
where $\leqslant$ denotes the $\leqslant$ of $\mathrm{Add}(\lambda^{++},\lambda^+)$. Note that strictly speaking this operation depends on ζ, i.e., if $\tau^{i,\gamma}_\zeta=\tau^{i',\gamma'}_{\zeta'}$ with $\zeta\neq\zeta'$ we might end up with $\hat{\tau}^{i,\gamma}_\zeta\neq\hat{\tau}^{i',\gamma'}_{\zeta'}$. Also note that if $\tau^{i,\gamma}_\zeta$ is already a Q_ζ-term then for any filter G on $\mathrm{Add}(\lambda^{++},\lambda^+)$ we have $(\hat{\tau}^{i,\gamma}_\zeta)^G=(\tau^{i,\gamma}_\zeta)^G$.

We are now ready to define the killing conditions formally. Towards this end we build by induction on $n\geqslant 2$ finite trees $T_{\Pi^2_n}$ and $T_{\Sigma^2_n}$.

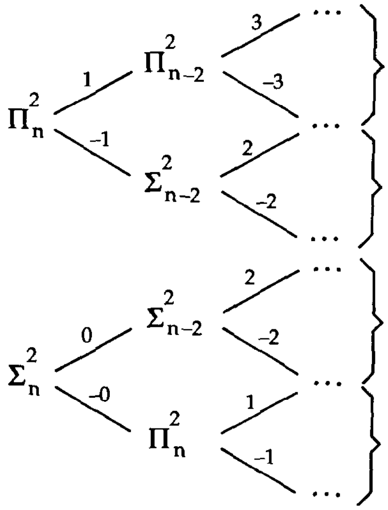
For $n=2$ these trees look very simple

$$\Pi^2_2\,(\text{kill}) \qquad\qquad \begin{tikzcd}[column sep=small, row sep=small] & \text{save} \\ \Sigma^2_2 \arrow[ur, dash, "0"] \arrow[dr, dash, "\neg 0"'] & \\ & \text{kill.} \end{tikzcd}$$

For $n = 3$ we have



For $n > 3$ we use $T_{\Pi^2_{n-2}}$ and $T_{\Sigma^2_{n-2}}$ to define $T_{\Pi^2_n}$ and $T_{\Sigma^2_{n-2}}$ and $T_{\Pi^2_n}$ to define $T_{\Sigma^2_n}$:



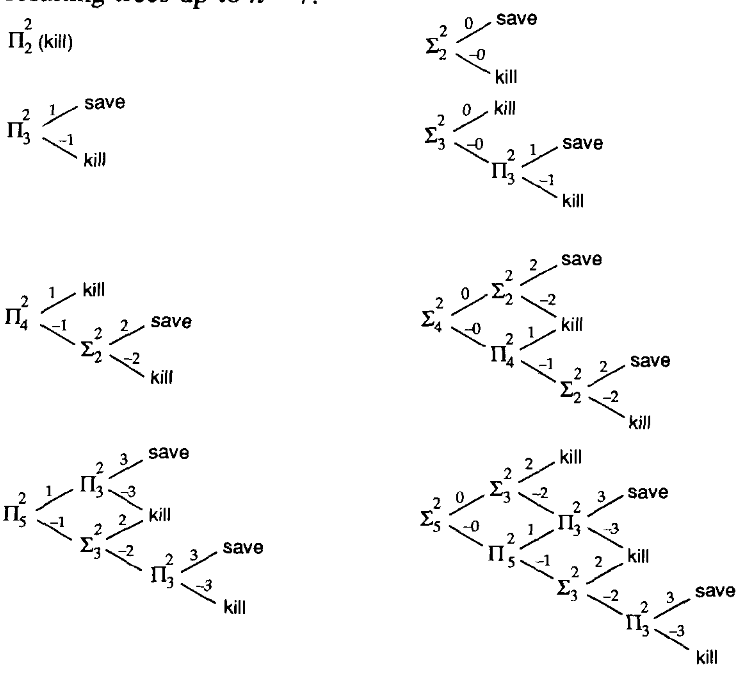
$T_{\Pi^2_{n-2}}$ where we add 2 to all positive labels and subtract 2 from all negative labels,

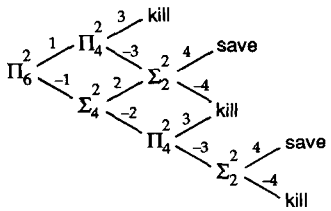
$T_{\Sigma^2_{n-2}}$ where we add 2 to all positive labels including 0 and subtract 2 from all negative labels including -0.

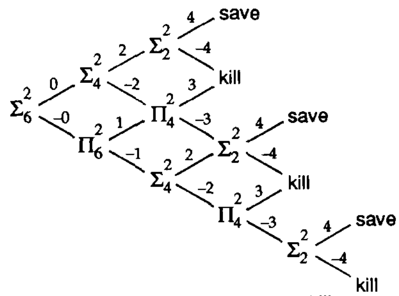
$T_{\Sigma^2_{n-2}}$ with all labels changed as above,

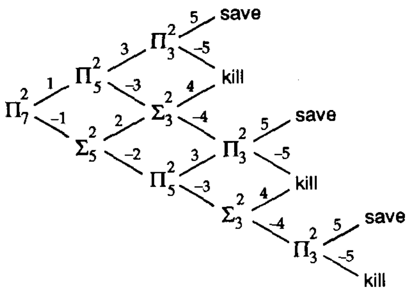
$T_{\Pi^2_n}$.

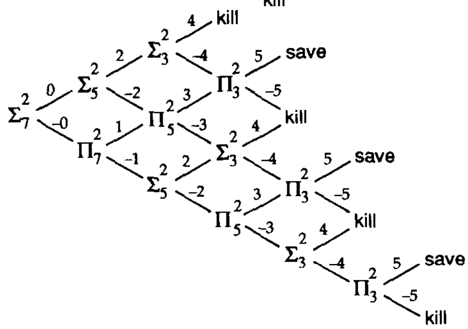
In order to get a better understanding of this definition we write out the resulting trees up to $n = 7$:











Some explanations are in order: The integer numbers that occur as labels of the segments of branches in the tree correspond to various agreements or disagreements of tupels of the kind mentioned above. Suppose $\beta \in A^{n-1,\gamma}$ and we want to decide whether to kill or to save at coordinate β. If $\gamma < \lambda^+$ is odd we consider $T_{\Pi_n^2}$ otherwise $T_{\Sigma_n^2}$. If a segment of a branch in the tree is labeled k $(1 \leq k \leq n-2)$, this corresponds to the existence of some $\zeta \in A^{k,\gamma} \cap \beta$ with $(\hat{\tau}_\zeta^{1,\gamma}, \ldots, \hat{\tau}_\zeta^{k,\gamma}, \Gamma^\zeta) = (\hat{\tau}_\beta^{1,\gamma}, \ldots, \hat{\tau}_\beta^{k+1,\gamma})$. The label $-k$ indicates that this fails for all $\zeta \in A^{\kappa,\gamma} \cap \beta$. By the label 0 ($-0$ resp.) we express that $\hat{\tau}_\beta^{1,\gamma} = \Gamma^\gamma$ ($\hat{\tau}_\beta^{1,\gamma} \neq \Gamma^\gamma$ resp.). In order to decide whether to kill or to save at coordinate β we now simply pick the unique branch through the appropriate tree that corresponds to the various agreements and disagreements of $(\hat{\tau}_\beta^{1,\gamma}, \ldots, \hat{\tau}_\beta^{n-1,\gamma})$ with tupels of the form above associated with coordinates $<\beta$. If this branch ends in '*kill*' we kill otherwise we save. Formally we define for $\beta \in A^{n-1,\gamma}$

$$Q_{\beta+1} = \{q \in \mathrm{Add}^{\lambda^{++}}(\beta+1, \lambda^+) \colon q|^\emptyset \beta \in Q_\beta \wedge$$

$$q|^\emptyset \Vdash_{Q_\beta} \begin{cases} q(\beta) \text{ is a condition for killing } F_\gamma(\hat{\tau}_\beta^{1,\gamma}, \ldots, \hat{\tau}_\beta^{n-1,\gamma}) & \text{if } \theta_{\Pi_n^2}(\Gamma^\gamma, (\hat{\tau}_\beta^{1,\gamma}, \ldots, \hat{\tau}_\beta^{n-1,\gamma}), \\ & ((\hat{\tau}_\zeta^{1,\gamma}, \ldots, \hat{\tau}_\zeta^{k,\gamma}, \Gamma^\zeta) \colon \zeta \in A^{k,\gamma} \\ & \cap \beta, k \in \{1, \ldots, n-2\})) \\ & \text{and } (\hat{\tau}_\beta^{1,\gamma}, \ldots, \hat{\tau}_\beta^{n-1,\gamma}) \in \mathrm{dom}(F_\gamma) \\ q(\beta) = \emptyset & \text{otherwise}\} \end{cases}$$

in case γ is odd (here $\theta_{\Pi_n^2}$ denotes the disjunction of the killing conditions as given by the branches of $T_{\Pi_n^2}$ ending in 'kill'). In the case γ is even we replace $\theta_{\Pi_n^2}$ by $\theta_{\Sigma_n^2}$ which is given by the branches of $T_{\Sigma_n^2}$ ending in 'kill'. This completes the definition of $(Q_\alpha \colon \alpha \leq \lambda^{++})$.

Since compatibility in Q^2_λ agrees with compatibility in $\mathrm{Add}(\lambda^{++}, \lambda^+)$ Q^2_λ is λ^{++} c.c. Moreover, it has size λ^{++}, and it is $<\lambda$ closed (because of the cofinality requirement that we included in the definition of Q_F). On the other hand Q^2_λ is not $<\lambda^+$ closed as it makes many of the sets $F_\gamma((X_1, \ldots, X_{n-1}))$ that are all stationary in $V[G_\lambda, \vec{F}_\gamma]$ nonstationary. However we will show in 2.3 below that Q^2_λ is $<\lambda^+$ Baire. The proof strategy is to define a larger model $V[G_\lambda, \vec{F}_\gamma, \vec{H}_\gamma] \supseteq V[G_\lambda, \vec{F}_\gamma]$ and to show that in this larger model Q^2_λ is $<\lambda^+$ Baire.

It is easy to modify the definition of the forcing Q^1_λ so that rather than adding Lipschitz functions $F_\gamma : (2^{\lambda^+})^{n-1} \to 2^{\lambda^+} (\gamma < \lambda^+)$ we add a sequence $((F_\gamma, H_\gamma) : \gamma < \lambda^+)$ of pairs of Lipschitz functions with $F_\gamma, H_\gamma : (2^{\lambda^+})^{n-1} \to 2^{\gamma^+}$ such that for all $X_1, \ldots, X_{n-1} \subseteq \lambda^+$, $H_\gamma((X_1, \ldots, X_{n-1}))$ is a club subset of λ^+ which is disjoint from $F_\gamma((X_1, \ldots, X_{n-1}))$. We denote this modified poset by $\bar{Q}^1_\lambda$. It is a λ^+ product with $<\lambda^+$ support of copies of a poset $Q_{F,H}$. Conditions in $Q_{F,H}$ are pairs (f, h) where

$$
\begin{aligned}
& f \in Q_F \ \wedge\ h \text{ is a function } \wedge\ \mathrm{dom}(h) = \mathrm{dom}(f) \\
& \wedge \forall (s_1, \ldots, s_{n-1}) \in \mathrm{dom}(f)\ [\exists \alpha, \beta < \lambda^+\ [\mathrm{dom}(s_1) \leq \alpha, \beta \\
& \wedge f(s_1, \ldots, s_{n-1}) \in 2^{\alpha+1} \wedge h((s_1, \ldots, s_{n-1})) \in 2^{\beta+1} \\
& \wedge h((s_1, \ldots, s_{n-1}))(\beta) = 1] \wedge \{\zeta : h((s_1, \ldots, s_{n-1}))(\zeta) = 1 \\
& = f((s_1, \ldots, s_{n-1}))(\zeta)\} = \emptyset \wedge \{\zeta : h((s_1, \ldots, s_{n-1}))(\zeta) = 1 \text{ is closed}] \\
& \wedge \forall (s_1, \ldots, s_{n-1}), (t_1, \ldots, t_{n-1}) \in \mathrm{dom}(h)\ [(s_1, \ldots, s_{n-1}) \text{ extends} \\
& (t_1, \ldots, t_{n-1}) \Rightarrow h((s_1, \ldots, s_{n-1})) \text{ extends } h((t_1, \ldots, t_{n-1}))].
\end{aligned}
$$

For two conditions (f, h) and (f', h') in $Q_{F,H}$ we let $(f, h) \leq (f', h')$ iff $f \supseteq f'$ and $h \supseteq h'$. Clearly $(f, h) \in Q_{F,H}$ implies $f \in Q_F$; conversely for any $f \in Q_F$ we can find h such that $(f, h) \in Q_{F,H}$. It follows that if D is dense in Q_F then $\{(f, h) \in Q_{F,H} : f \in D\}$ is dense in $Q_{F,H}$. Thus, if $((F_\gamma, H_\gamma) : \gamma < \lambda^+)$ is $\bar{Q}^1_\lambda$ generic over $F[G_\lambda]$ then $(F_\gamma : \gamma < \lambda^+)$ is Q^1_λ generic over $V[G_\lambda]$ and $V[G_\lambda, \vec{F}_\gamma] \subseteq V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$. Inside $V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$ we can define the following posets for $\alpha \leq \lambda^{++}$:

$$
\begin{aligned}
Q^*_\alpha \overset{\text{def}}{=} \{ & q \in Q_\alpha : \forall \gamma < \lambda^+\ \forall \beta \in A^{n-1,\gamma} [q(\beta) \neq \emptyset \Rightarrow \\
& q|^{\emptyset} \beta \Vdash^{V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]}_{Q_\beta} \sup q(\beta) \in H_\gamma((\hat{\tau}^{1,\gamma}_\beta, \ldots, \hat{\tau}^{n-1,\gamma}_\beta))]\}.
\end{aligned}
$$

Lemma 2.1. *For each $\alpha \leq \lambda^{++}$, Q^*_α is $<\lambda^+$ closed.*

Proof. Fix $\alpha \leq \lambda^{++}$ and a decreasing sequence of conditions in Q^*_α, say $(q_\eta : \eta < \lambda)$. By induction on $\zeta \leq \alpha$ we will build $q \mid \zeta$ such that $(q \mid \zeta)^\frown 1 \in Q^*_\zeta$, $\mathrm{supp}(q \mid \zeta) = \bigcup_{\eta < \lambda} \mathrm{supp}(q_\eta \mid \zeta)$ and $\forall \eta < \lambda\ q \mid \zeta \leq q_\eta \mid \zeta$. The only nontrivial case in the induction step is obtaining $q \mid (\zeta + 1)$ from $q \mid \zeta$ if ζ happens to be an element of $A^{n-1,\gamma}$ for some $\gamma < \lambda^+$ and if $q_\eta(\zeta) = \emptyset$ for some $\eta < \lambda$. In this case we define

$$q(\zeta) = \bigcup_{\eta<\lambda} q_\eta(\zeta). \cup . \{\sup_{\eta<\lambda} \sup q_\eta(\zeta)\}.$$

This works since for any $X_1, \ldots, X_{n-1} \subseteq \lambda^+$ if $H_\gamma((X_1, \ldots, X_{n-1}))$ is defined then it is closed and disjoint from $F_\gamma((X_1, \ldots, X_{n-1}))$. □

Lemma 2.2. *For each $\alpha \leqslant \lambda^{++}$, Q^*_α is dense in Q_α.*

Proof. We use induction on $\alpha \leqslant \lambda^{++}$. Note that $Q_0 = \{\emptyset\} = Q^*_0$. Suppose $\alpha = \beta + 1$ and $q \in Q_\alpha$. The only interesting case is $\beta \in A^{n-1,\gamma}$ for some $\alpha < \lambda^+$ and $q(\beta) \neq 0$. Recall that for any $X_1, \ldots, X_{n-1} \subseteq \lambda^+$ if $H_\gamma((X_1, \ldots, X_{n-1}))$ is defined then it must be unbounded in λ^+. Hence we can find some ordinal $\delta < \lambda^+$ with $\sup q(\beta) < \delta$ and a condition $q^* \in Q^*_\beta$ below $q|^\emptyset \beta$ (this uses the induction hypothesis for β) such that

$$q^* \Vdash^{V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]}_{Q_\beta} \delta \in H_\gamma((\hat{\tau}^{1,\gamma}_\beta, \ldots, \hat{\tau}^{n-1,\gamma}_\beta)).$$

Clearly $(q^* \sim \{\langle \phi, \emptyset \rangle\}) \cup \{\langle \beta, q(\beta) \cup \{\delta\} \rangle\}$ is a condition in Q^*_α below q. If α happens to be a limit ordinal there are two cases: if $\mathrm{cf}(\alpha) \geqslant \lambda^+$ then $\mathrm{supp}(q)$ is bounded below α for $q \in Q_\alpha$. Thus we can apply the induction hypothesis to find $q^* \in Q^*_\alpha$ below q. Otherwise we pick a normal sequence $(\lambda_\eta: \eta \leqslant \beta)$ where $\lambda_\beta = \alpha$ and $\beta = \mathrm{cf}(\alpha) \leqslant \lambda$. Using induction on $\eta \leqslant \beta$ we can define a decreasing sequence $(q_n: \eta \leqslant \beta)$ with $q_\eta \in Q^*_{\lambda_\eta}$ and $q_\eta \leqslant q|^\emptyset \lambda_\eta$ for all $\eta \leqslant \beta$. This works at limits $\eta \leqslant \beta$ since $Q^*_{\lambda_n}$ is $<\lambda^+$ closed by 2.1. Clearly $q_\beta \in Q^*_\alpha$ and extends q. □

Lemma 2.3. *For any $\alpha \leqslant \lambda^{++}$, Q_α is $<\lambda^+$ Baire.*

Proof. Suppose this failed for some $\alpha \leqslant \lambda^{++}$. Pick a name $\mathring{P} \in V[G_\lambda]^{Q^1_\lambda}$ for the set of parameters that we need to define Q_α and a condition $f \in Q^1_\lambda$ such that

(*) $\qquad f \Vdash^{V[G_\lambda]}_{Q^1_\lambda}$ "Q_α defined from $\mathring{P}$ is not $<\lambda^+$ Baire".

Pick some h such that $(f, h) \in \bar{Q}^1_\lambda$. Let $((F_\gamma, H_\gamma): \gamma < \lambda^+)$ be $\bar{Q}^1_\lambda$ generic over $V[G_\lambda]$ extending (f, h). In $V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$ Q_α has a $<\lambda^+$ closed dense suborder Q^*_α. So in particular Q_α is $<\lambda^+$ Baire in $V[G_\lambda, \vec{F}_\gamma]$ contradicting (*). □

As a corollary we get that for each $\beta \leqslant \lambda^{++}$

$$\Vdash^{V[G_\lambda, \vec{F}_\gamma]}_{Q_\beta} \forall \gamma < \lambda^+ \ \mathrm{dom}(F_\gamma) = (2^{\lambda^+})^{n-1}.$$

Hence in the definition of $Q_{\beta+1}$ where $\beta \in A^{n-1,\gamma}$ we can omit the clause $((\hat{\tau}^{1,\gamma}_\beta, \ldots, \hat{\tau}^{n-1,\gamma}_\beta)) \in \mathrm{dom}(F_\gamma)$.

Our next task is to verify that Q^2_λ forces the Σ^2_n (Π^2_n resp.) statement about F_γ when γ is even (odd resp.) mentioned in Section 1. We must first prove a technical fact that will be used later on.

Lemma 2.4. *For any condition $q \in Q^2_\lambda$ and any ordinal $\delta < \lambda^+$ there is a stronger condition $q' \in Q^2_\lambda$ with*

$$\forall \gamma < \lambda^+ \, \forall \alpha \in A^{n-1,\gamma} \, (q(\alpha) \neq \emptyset \to \sup q'(\alpha) > \delta).$$

Proof. The usual argument: suppose for some condition $f \in Q^1_\lambda$ and some $q \in Q^2_\lambda$ and $\delta < \lambda^+$ we have

$$f \Vdash^{V[G_\lambda]}_{Q^1_\lambda} \text{"the claim fails for } q \text{ and } \delta\text{"}.$$

Pick some h such that $(f, h) \in \bar{Q}^1_\lambda$ and let $((F_\gamma, H_\gamma)\colon \gamma < \lambda^+)$ be a $\bar{Q}^1_\lambda$ generic over $V[G_\lambda]$ that extends (f, h). In $V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$ pick an increasing enumeration $(\alpha_\eta\colon \eta < \bar{\lambda})$ of $\mathrm{supp}(q)$ with $|\bar{\lambda}| \leq \lambda$ and define by induction on η a decreasing sequence $(q_n\colon \eta < \bar{\lambda})$ such that for all $\eta < \bar{\lambda}$: $q_\eta \in Q^*_{\alpha_\eta}$, $q_\eta \leq q|^{\emptyset} \alpha_\eta$ and $\forall \gamma < \lambda^+ \, \forall \alpha \in A^{n-1,\gamma} \cap \alpha_\eta$ $[q(\alpha) \neq \emptyset \Rightarrow \sup q_\eta(\alpha) > \delta]$. From this we obtain a condition q' as in the claim. This contradicts our assumption above. $\square$

The next lemma says that after forcing with Q^2_λ any $F_\gamma((X_1, \ldots, X_{n-1}))$ is stationary unless we killed it explicitly. To be more exact:

Lemma 2.5. *Let G be Q^2_λ generic over $V[G_\lambda, \vec{F}_\lambda]$. In $V[G_\lambda, \vec{F}_\lambda, G]$, let $X_1, \ldots, X_{n-1} \subseteq \lambda^+$ and $\gamma_0 < \lambda^+$ and assume*

$$\forall \beta \in A^{n-1,\gamma} \, \forall q \in G \, [q(\beta) \neq \emptyset \Rightarrow ((\mathring{\tau}^{1,\gamma_0}_\beta)^G, \ldots, (\mathring{\tau}^{n-1,\gamma_0}_\beta)^G) \neq (X_1, \ldots, X_{n-1})].$$

Then $F_{\gamma_0}((X_1, \ldots, X_{n-1}))$ is stationary.

Proof. Pick names $\mathring{A}^{k,\gamma}$ $(0 \leq k \leq n-1,\ \gamma < \lambda^+)$, $\mathring{\tau}^{i,\gamma}_\beta$ $(\gamma < \lambda^+,\ \beta \in A^{k,\gamma},\ 1 \leq i \leq k)$ in $V[G_\lambda]^{Q^1_\lambda}$ for the parameters in the definition of Q^2_λ. Let $\sigma, \sigma^1, \ldots, \sigma^{n-1} \in V[G_\lambda]^{Q^1_\lambda * Q^2_\lambda}$ and $(\bar{f}, \bar{q}) \in \vec{F}_\gamma * G$ with

$$(\bar{f}, \bar{q}) \Vdash^{V[G_\lambda]}_{Q^1_\lambda * Q^2_\lambda} \text{"} \forall q \in \Gamma \, \forall \beta \in A^{n-1,\gamma_0} [q(\beta) \neq \emptyset \Rightarrow$$

$$((\mathring{\tau}^{1,\gamma_0}_\beta, \ldots, \mathring{\tau}^{n-1\gamma_0}_\beta) \neq (\sigma^1, \ldots, \sigma^{n-1})] \wedge \sigma \subseteq \lambda^+ \text{ is club"}.$$

The lemma is proved if we can find a condition $(f, q) \leq (\bar{f}, \bar{q})$ and $s^1, \ldots, s^{n-1} \in 2^{<\lambda^+}$ and an ordinal $\alpha < \lambda^+$ such that

$$(s^1, \ldots, s^{n-1}) \in \mathrm{dom}(f^{\gamma_0}),$$

$$f^{\gamma_0}((s^1, \ldots, s^{n-1}))(\alpha) = 1,$$

$$(f, q) \Vdash^{V[G_\lambda]}_{Q^1_\lambda * Q^2_\lambda} \text{"}(\sigma^1, \ldots, \sigma^{n-1}) \text{ extends } (s^1, \ldots, s^{n-1}) \wedge \alpha \in \sigma\text{"}$$

where f^γ denotes the γ-th component of the condition $f \in Q^1_\lambda$. In order to come up with (f, q), $(s^1, \ldots, s^{n-1})$ and α, we have to construct a decreasing sequence

$((f_\eta, q_\eta)\colon \eta < \lambda)$ of conditions below $(\bar{f}, \bar{q})$ and auxiliary sequences

$$\begin{aligned}
&(\alpha_\eta\colon \eta < \lambda),\\
&(\delta_\eta\colon \eta < \lambda),\\
&(T_\eta\colon \eta < \lambda),\\
&(b^\gamma_\eta\colon \gamma \in T_\eta)\colon \eta < \lambda),\\
&((((s^{1,\gamma}_{\beta,\eta}, \ldots, s^{n-1,\gamma}_{\beta,\eta})\colon \beta \in b^\gamma_\eta)\colon \gamma \in T_\eta)\colon \eta < \lambda),\\
&((s^1_\eta, \ldots, s^{n-1}_\eta)\colon \eta < \lambda)
\end{aligned}$$

where α_η, $\delta_\eta < \lambda^+$ and $T_\eta \subseteq \lambda^+$ and $b^\gamma_\eta \subseteq \lambda^{++}$ and $s^{i,\gamma}_{\beta,\eta}$, $s^i_\eta \subseteq \lambda^+$ and at stage η of the construction we have:

$$\alpha_\eta,\ \delta_\eta > \sup\{\operatorname{dom}(f^\gamma_{\eta'}(s_1, \ldots, s_{n-1}))\colon \eta' < \eta,\ \gamma \in \operatorname{supp}(f_{\eta'}),$$

$$(s_1, \ldots, s_{n-1}) \in \operatorname{dom}(f^\gamma_{\eta'})\} \cup \sup_{\eta'<\eta} (\alpha_{\eta'} \cup \delta_{\eta'});$$

$$f_\eta \Vdash^{V[G_\lambda]}_{Q^1_\lambda} \begin{cases} T_\eta = \left\{\gamma < \lambda^+\colon \mathring{A}^{n-1,\gamma} \cap \bigcup_{\eta'<\eta} \operatorname{supp}(q_{\eta'}) \neq \emptyset\right\} \cup \{\gamma_0\},\\ \forall \gamma \in T_\eta\ b^\gamma_\eta = \mathring{A}^{n-1,\gamma} \cap \bigcup_{\eta'<\eta} \operatorname{supp}(q_{\eta'}); \end{cases}$$

$$(f_\eta, q_\eta) \Vdash^{V[G_\lambda]}_{Q^1_\lambda * Q^2_\lambda} \begin{cases} \mathring{\hat{\tau}}^{i,\gamma}_\beta \cap \delta_\eta = s^{i,\gamma}_{\beta,\eta} \quad (i = 1, \ldots, n-1,\ \gamma \in T_\eta,\ \beta \in b^\gamma_\eta),\\ \sigma^i \cap \delta_\eta = s^i_\eta \quad (i = 1, \ldots, n-1)\\ \alpha_\eta \in \sigma; \end{cases}$$

$$\forall \beta \in b^{\gamma_0}_\eta\ (s^{1,\gamma_0}_{\beta,\eta}, \ldots, s^{n-1,\gamma_0}_{\beta,\eta}) \neq (s^1_\eta, \ldots, s^{n-1}_\eta);$$

$$\forall \gamma \in T_\eta\ \forall \beta \in b^\gamma_\eta \sup q_\eta(\beta) > \delta_\eta.$$

Note the construction of these sequences can be carried out in $V[G_\lambda]$ since $Q^1_\lambda * Q^2_\lambda$ is $<\lambda^+$ Baire.

Once the sequences have been defined we let

$$\alpha \overset{\text{def}}{=} \sup_{\eta<\lambda} \alpha_\eta \quad \text{and} \quad s^i \overset{\text{def}}{=} \bigcup_{\eta<\lambda} s^i_\eta \quad (i = 1, \ldots, n-1).$$

For $\gamma \in \bigcup_{\eta<\lambda} T_\eta$ and $\beta \in \bigcup_{\eta<\lambda, \gamma\in T_\eta} b^\gamma_\eta$ we define

$$s^{i,\gamma}_\beta \overset{\text{def}}{=} \bigcup_{\eta<\lambda} s^{i,\gamma}_{\beta,\eta} \quad (i = 1, \ldots, n-1).$$

Now we pick a condition $f \leqslant f_\eta$ $(\eta < \lambda)$ such that $(s^1, \ldots, s^{n-1}) \in \operatorname{dom}(f^{\gamma_0})$ and $(s^{1,\gamma}_\beta, \ldots, s^{n-1,\gamma}_\beta) \in \operatorname{dom}(f^\gamma)$ (for $\gamma \in \bigcup_{\eta<\lambda} T_\eta$ and $\beta \in \bigcup_{\eta<\lambda, \gamma\in T_\eta} b^\gamma_\eta$). We also want

$$f^{\gamma_0}((s^1, \ldots, s^{n-1}))(\alpha) = 1 \quad \text{and} \quad f^\gamma((s^{1,\gamma}_\beta, \ldots, s^{n-1,\gamma}_\beta))\Big(\sup_{\eta<\lambda} \sup q_\eta(\beta)\Big) = 0.$$

Note there is no conflict for $\gamma = \gamma_0$. Finally we define $q \in \mathrm{Add}(\lambda^{++}, \lambda^+)$ by

$$\mathrm{supp}(q) \stackrel{\mathrm{def}}{=} \bigcup_{\eta<\lambda} \mathrm{supp}(q_\eta)$$

and

$$q_\eta(\beta) = \begin{cases} \bigcup_{\eta<\lambda} q_\eta(\beta) \cup \left\{\sup_{\eta<\lambda} \sup q_\eta(\beta)\right\} & \text{if } \beta \in \bigcup \{b^\gamma_\eta : \eta < \lambda,\ \gamma \in T_\eta\}, \\ \bigcup_{\eta<\lambda} q_\eta(\beta) & \text{otherwise.} \end{cases}$$

Then (f, q) is a condition in $Q^1_\lambda * Q^2_\lambda$, i.e., $f \Vdash^{V[G_\lambda]}_{Q^1_\lambda} q \in Q^2_\lambda$, and clearly (f, q) and $(s^1, \ldots, s^{n-1})$ and α have the properties that we want. $\square$

Lemma 2.6. *For each odd (even resp.) $\gamma < \lambda^+$, Q^2_λ forces the Π^2_n (Σ^2_n resp.) statement about F_γ that we want.*

Proof. For the sake of the argument suppose that n is odd, i.e., $n = 2l + 1$. If $\gamma < \lambda^+$ is odd we want the following Π^2_n statement about F_γ to hold in $(V[G_\lambda, \vec{F}_\lambda])^{Q^2_\lambda}$

$$\forall X_1 \subseteq \lambda^+ \, \exists X_2 \subseteq \lambda^+ \cdots \exists X_{n-1} \subseteq \lambda^+ \, F_\gamma(X_1, \ldots, X_{n-1}) \text{ is stationary.}$$

Let G be Q^2_λ generic over $V[G_\lambda, \vec{F}_\gamma]$ and suppose in $V[G_\lambda, \vec{F}_\gamma, G]$ we are given some $X_1 \subseteq \lambda^+$. Pick some $\zeta_1 \in A^{1,\gamma}$ with $(\hat{\tau}^{1,\gamma}_{\zeta_1})^G = X_1$ and let $X_2 = G^{\zeta_1}$ (i.e., the subset of λ^+ that G adds at coordinate ζ_1). Now suppose $X_3 \subseteq \lambda^+$. Pick some $\zeta_2 \in A^{3,\gamma}$ with $(X_1, X_2, X_3) = ((\hat{\tau}^{1,\gamma}_{\zeta_2})^G, \ldots, (\hat{\tau}^{3,\gamma}_{\zeta_2})^G)$. Then let $X_4 = G^{\zeta_2}$. Continue in this fashion until $\zeta_1, \ldots, \zeta_l$ and a tupel $(X_1, \ldots, X_{n-1})$ have been defined. Now suppose that $\zeta \in A^{n-1,\gamma}$. Since for each $\beta \in \lambda^{++} \sim \bigcup_{\gamma<\lambda^+} A^{n-1,\gamma}$, $Q_{\beta+1}$ is isomorphic to $Q_\beta \otimes$ "Adding a Cohen subset of λ^+", we obtain by the product lemma that $\zeta_1 < \zeta_2 < \cdots \zeta_l$. The same argument shows that $((\hat{\tau}^{1,\gamma}_\zeta)^G, \ldots, (\hat{\tau}^{n-1,\gamma}_\zeta)^G) = (X_1, \ldots, X_{n-1})$ (with $\zeta \in A^{n-1,\gamma}$) implies $\zeta_l < \zeta$. Recall that the top branch in the tree $T_{\Pi^2_n}$ is labeled $1, 3, \ldots, n-2$ and ends in 'save'. Hence we obtain

$$\forall q \in G \, \forall \zeta \in A^{n-1,\gamma} [((\hat{\tau}^{1,\gamma}_\zeta)^G, \ldots, (\hat{\tau}^{n-1,\gamma}_\zeta)^G) = (X_1, \ldots, X_{n-1}) \Rightarrow q(\zeta) = \emptyset].$$

It follows from 2.5 that $F_\gamma(X_1, \ldots, X_{n-1})$ is stationary.

If $\gamma < \lambda^+$ is even, we want the following Σ^2_n statement to hold about F_γ in $V[G_\lambda, \vec{F}_\gamma, G]$

$$\exists X_1 \subseteq \lambda^+ \, \forall X_2 \subseteq \lambda^+ \cdots \forall X_{n-1} \subseteq \lambda^+ \, F_\gamma(X_1, \ldots, X_{n-1}) \text{ is not stationary.}$$

Let $X_1 = G^0$. If $X_2 \subseteq \lambda^+$ is given, pick $\eta_1 \in A^{2,\gamma}$ with $((\hat{\tau}^{1,\gamma}_{\eta_1})^G, (\hat{\tau}^{2,\gamma}_{\eta_1})^G) = (G^0, X_2)$. Then define $X_3 = G^{\eta_1}$. For a given $X_4 \subseteq \lambda^+$ choose $\eta_2 \in A^{4,\gamma}$ such that $((\hat{\tau}^{1,\gamma}_{\eta_2})^G, \ldots, (\hat{\tau}^{4,\gamma}_{\eta_2})^G) = (X_1, \ldots, X_4)$. Then define $X_5 = G^{\eta_2}$. Continue in this fashion until $\eta_1, \ldots, \eta_l$ and a tupel $(X_1, \ldots, X_{n-1})$ have been defined. Now fix some $\eta \in A^{n-1,\gamma}$ with $((\hat{\tau}_\eta^{1,\gamma})^G, \ldots, (\hat{\tau}_\eta^{n-1,\gamma})^G) = (X_1, \ldots, X_{n-1})$. By the same argument as above we have $\eta_1 < \cdots < n_l < \eta$. Recall that the top branch in $T_{\Sigma_n^2}$ is labeled $0, 2, \ldots, n-3$ and ends in 'kill'. Thus at coordinate η we add a club set that is disjoint from $F_\gamma(X_1, \ldots, X_{n-1})$. The argument for even n is similar. $\square$

This completes our discussion of Q_λ^2 for now. Let G be Q_λ^2 generic over $V[G_\lambda, \vec{F}_\lambda]$. In the next step we want to define a forcing Q_λ^3 in $V[G_\lambda, \vec{F}_\gamma, G]$ that codes each F_γ by a subset of λ. Clearly we can think of each F_γ as a subset of λ^+: After all $2^{<\lambda^+} \subseteq V[G_\lambda]$, but $V[G_\lambda] = L[G_\lambda]$ since we started in $V = L$. Now $G_\lambda \subseteq L_\lambda$ thus $2^{<\lambda^+} \subseteq L_{\lambda^+}[G_\lambda]$, and we can use the canonical well-ordering $<_{L[G_\lambda]}$ to code $2^{<\lambda^+}$ (which has order type λ^+ under this well-ordering) by λ^+. Let $\tilde{F}_\gamma \subseteq \lambda^+$ denote the code for F_γ in this coding. Now let

$$Q_\lambda^3 \stackrel{\text{def}}{=} \prod_{\substack{\gamma<\lambda^+ \\ <\lambda \text{ support}}} Q_{\tilde{F}_\gamma}$$

where for $\gamma < \lambda^+$, $Q_{\tilde{F}_\gamma}$ codes $\tilde{F}_\gamma \subseteq \lambda^+$ by a subset $\tilde{S}_\gamma$ of λ using the $<_L$ least almost disjoint family of constructible subsets of λ of size λ^+ (cf. [2], note that we still have $\lambda^{+l} = (\lambda^{+l})^L$ for $l \geq 1$). $Q_{\tilde{F}}$ is λ centered and $<\lambda$ closed. Hence by a Δ system argument Q_λ^3 has the property λ^+ and is $<\lambda$ closed. Therefore in particular $Q_\lambda^3 \times Q_\lambda^3$ is λ^+ c.c. Hence Q_λ^3 does not add any new subsets of λ^+ all of whose initial segments are in $V[G_\lambda, \vec{F}_\lambda, G]$, i.e., $L_{\lambda^+}[G_\lambda]$. If $(S_\gamma\colon \gamma < \lambda^+)$ is Q_λ^3 generic over $V[G_\lambda, \vec{F}_\lambda, G]$ and $\tilde{S}_\gamma$ denotes the code for $\tilde{F}_\gamma$ (see [2] on how $\tilde{S}_\gamma$ is defined from S_γ) then we obtain in $V[G_\lambda, \vec{F}_\lambda, G, \vec{S}_\gamma]$:

$$\begin{aligned}&\exists \text{ good } X_1 \subseteq \lambda^+ \ \forall \text{ good } X_2 \subseteq \lambda^+ \cdots \mathrm{Q} \text{ good } X_{n-1} \subseteq \lambda^+ \ \mathrm{Q}^{-1}\mathcal{M} \\ &\quad[\mathcal{M} \text{ transitive}, \mathcal{M} \models \mathrm{ZF}^-, |\mathcal{M}| = |V_{\lambda+1}|, \mathcal{M}^{|V_\lambda|} \subseteq \mathcal{M}, X_1, \ldots, X_{n-1} \in \mathcal{M} \\ &\quad . \vec{\wedge} . \mathcal{M} \models \text{"If } F\colon (2^{\lambda^+})^{n-1} \to 2^{\lambda^+} \text{ is the Lipschitz function coded by } \tilde{S}_\gamma \\ &\qquad\quad \text{then } \varphi(F((X_1, \ldots, X_{n-1})))\text{"}]\end{aligned}$$

for even $\gamma < \lambda^+$ where φ says "$F((X_1, \ldots, X_n))$ is not stationary (stationary resp.)" for odd n (even n resp.).

In this formula "X is good" (where $X \subseteq \lambda^+$) means $\forall \alpha < \lambda^+ \ X \cap \alpha \in L_{\lambda^+}[G_\lambda]$. So this is $\Sigma_1^2(X, G_\lambda, \lambda)$ over V_λ since it is absolute for any transitive model of enough of ZF that correctly computes λ^+. Hence the whole formula is $\Sigma_n^2(\tilde{S}_\gamma, G_\lambda, \lambda)$ over V_λ. It will be abbreviated by $\phi^{\Sigma_n^2}(\tilde{S}_\gamma, G_\lambda, \lambda)$ from now on. Similarly, for odd $\gamma < \lambda^+$ we have a Π_n^2 formula $\phi^{\Pi_n^2}(\tilde{S}_\gamma, G_\lambda, \lambda)$ holding at V_λ in $V[G_\lambda, \vec{F}_\lambda, G, \vec{S}_\gamma]$ which is just the negation of $\phi^{\Sigma_n^2}$.

Finally in the last step Q_λ^4 of Q_λ we add a sequence of club sets $C_\gamma \subseteq \lambda$ $(\gamma < \lambda^+)$ such that

$$C_\gamma \cap \{\mu < \lambda\colon \mu \text{ is inaccessible} \ \wedge \ V_\mu \models \phi^{\Sigma_n^2}(\tilde{S}_\gamma \cap V_\mu, G_\lambda \cap V_\mu, \mu)\} = \emptyset.$$

Q_λ^4 is a λ^+ product with $<\lambda$ support of posets each of which is of size λ and has for each $\nu < \lambda$ a $<\nu$ closed dense suborder. Thus by a Δ system argument Q_λ^4 has property λ^+ and for each $\nu<\lambda$ a dense $<\nu$ closed suborder. In particular $Q^4_\lambda \times Q^4_\lambda$ is λ^+c.c. and hence for any Q^4_λ generic $(C_\lambda: \gamma<\lambda^+)$ we still have $V_\lambda \models \phi^{\Sigma^2_n}(\bar{S}_\gamma, G_\lambda, \lambda)$ for even $\gamma<\lambda^+$ and $V_\lambda \models \phi^{\Pi^2_n}(\bar{S}_\gamma, G_\lambda, \lambda)$ for odd $\gamma<\lambda^+$ in $V[G_\lambda, \vec{F}_\gamma, G, \vec{S}_\gamma, \vec{C}_\gamma]$. Therefore we obtain

$$\Vdash_{P_\lambda * Q_\lambda} \text{“}\lambda \text{ is } \Sigma^2_n \text{ describable”}.$$

We conclude this paragraph by proving two technical results about the iteration Q^2_λ which will be used to show that our iteration $P_{\kappa+1}$ preserves the Π^2_n indescribability of κ. As a minor technical point the reader may have wondered how we choose the parameters necessary to define Q^2_λ, i.e., we need to choose them in a uniform way in order to define Q_λ by induction on λ. This can be done by simply choosing the $<_{L[G_\lambda, \vec{F}_\gamma]}$ least family of parameters. However, as we shall see in a moment the exact way in which we choose the parameters for Q^2_λ is actually irrelevant since the outcome is always the same as long as the sequences of terms are complete. For the rest of this paragraph we work in a model, say where GCH holds from λ on and $(F_\gamma: \gamma<\lambda^+)$ denotes a sequence of Lipschitz functions $F_\gamma: (2^{\lambda^+})^{n-1} \to 2^{\lambda^+}$.

Definition 2.7. We call P a *set of parameters* if it consists of a partition of λ^{++} into cofinal pieces and of complete enumerations of tupels of terms along the coordinates in the sets in the partitions. So P will be of the form

$$\{A^0, (A^{i,\gamma}: \gamma<\lambda^+, 1\leq i\leq n-1), ((\tau^{1,\gamma}_\zeta, \ldots, \tau^{i,\gamma}_\zeta): \gamma<\lambda^+, \zeta\in A^{i,\gamma}, 1\leq i\leq n-1)\}.$$

If P is a set of parameters we denote by $(Q_\alpha(P): \alpha\leq\lambda^{++})$ the *Q-iteration* defined from P, i.e., $Q_{\lambda^{++}}(P)$ (for which we will simply write $Q(P)$) is the poset defined from P for forcing a certain Π^2_n statement about F_γ for $\gamma<\lambda^+$ odd and a certain Σ^2_n statement about F_γ for $\gamma<\lambda^+$ even. □

Definition 2.8. Let $S\subseteq\lambda^{++}$ and P be a set of parameters. We say S is a *complete set of coordinates for* P if for each $\gamma<\lambda^+$, $k\in\{1,\ldots,n-1\}$ and $i\in\{1,\ldots,k\}$

$$\forall\zeta\in A^{k,\gamma}\cap S\ \forall(\eta,f)\in\tau^{i,\gamma}_\zeta\,(\operatorname{supp}(f)\subseteq\zeta\to\operatorname{supp}(f)\subseteq S)$$

and if $\lambda^+\subseteq S$. □

Definition 2.9. Let P be a set of parameters and $S\subseteq\lambda^{++}$. For $\zeta\leq\lambda^{++}$

$$Q^S_\zeta \stackrel{\text{def}}{=} \{q\in Q_\zeta(P): \operatorname{supp}(q)\subseteq S\}$$

and $Q^S \stackrel{\text{def}}{=} Q^S_{\lambda^{++}}$. For $\gamma<\lambda^+$ and $\zeta\in A^{i,\gamma}_\zeta$ $(1\leq k\leq i\leq n-1)$

$${}^S\bar{\tau}^{k,\gamma}_\zeta = \{(\eta,f): f\in Q^S_\zeta \wedge \exists g\,((\eta,g)\in\tau^{k,\gamma}_\zeta \wedge f\leq g)\}.$$

If it is clear from the context which S we are referring to we drop the superscript S and simply write $\bar{\tau}^{k,\gamma}_\zeta$. □

The following lemma shows that for complete sets of coordinates we can thin out a given condition in $Q(P)$ and stay within $Q(P)$.

Lemma 2.10. *Suppose $S \subseteq \lambda^{++}$ is a complete set of coordinates for a set of parameters P. Then for each $\zeta \leq \lambda^{++}$*

$$\forall q \in Q_\zeta(P) \quad q|^{\emptyset} S \in Q^S_\zeta$$

and

$$Q^S_\zeta \subseteq_c Q_\zeta(P).$$

Proof. We proceed by induction on $\zeta \subseteq \lambda^{++}$. Note that the first claim clearly implies the second. The only nontrivial case in the induction step is when $\zeta = \alpha + 1$ with $\alpha \in A^{n-1,\gamma} \cap S$ for some $\gamma < \lambda^+$. W.l.o.g. let λ be odd and assume towards a contradiction that $q|^{\emptyset} S \notin Q^S_{\alpha+1}$. Hence there is a condition $q' \in Q^S_\alpha$ with $q' \leq q|^{\emptyset} S \cap \alpha$ and

$$q' \Vdash_{Q^S_\alpha} \neg\theta^*_{\Pi^2_n}((\Gamma^\eta\colon \eta \in S \cap \alpha), ((\tilde{\tau}^{1,\gamma}_\eta, \ldots, \tilde{\tau}^{k,\gamma}_\eta)\colon \eta \in A^{k,\gamma} \cap (\alpha+1) \cap S,$$
$$k \in \{1, \ldots, n-1\}), F_\gamma, \alpha, q(\alpha))$$

where $\theta^*_{\Pi^2_n}$ roughly says "If the killing conditions are satisfied then we kill at coordinate α otherwise we save". The key point is that the completeness of S implies that for any $Q_\alpha(P)$ generic H, $(\hat{\tau}^{i,\gamma}_\eta)^H = (\tilde{\tau}^{i,\gamma}_\eta)^{H \cap Q^S_\alpha}$ for all $\eta \in A^{k,\gamma} \cap (\alpha+1) \cap S$. Moreover, if for $\eta \in A^{k,\gamma} \cap \alpha$ and $k \in \{0, \ldots, n-2\}$, $((\hat{\tau}^{1,\gamma}_\alpha)^H, \ldots, (\hat{\tau}^{k+1,\gamma}_\alpha)^H) = ((\hat{\tau}^{1,\gamma}_\eta)^H, \ldots, (\hat{\tau}^{k,\gamma}_\eta)^H, H^\eta)$, then we must have $\eta \in S$ (by the induction hypothesis together with the product lemma). Now define a condition $q'' \in Q_\alpha(P)$ by

$$q''|(S \cap \alpha) = q'|(S \cap \alpha),$$
$$q''|(\alpha \sim S) = q|(\alpha \sim S).$$

Then with the above remarks

$$q'' \Vdash_{Q_\alpha(P)} \neg\theta^*_{\Pi^2_n}((\Gamma^\eta\colon \eta < \alpha), ((\tilde{\tau}^{1,\gamma}_\eta, \ldots, \tilde{\tau}^{k,\gamma}_\eta)\colon \eta \in A^{k,\gamma} \cap (\alpha+1),$$
$$k \in \{1, \ldots, n-1\}), F_\gamma, \alpha, q(\alpha))$$

which contradicts $q \in Q_{\alpha+1}$. $\square$

Now let $P = \{A^0, (A^{k,\gamma}\colon \gamma < \lambda^+, 1 \leq k \leq n-1), ((\tau^{1,\gamma}_\zeta, \ldots, \tau^{k,\gamma}_\zeta)\colon \gamma < \lambda^+, 1 \leq k \leq n-1, \zeta \in A^{k,\gamma})\}$ and $\bar{P} = (\bar{A}^0, (\bar{A}^{k,\gamma}\colon \gamma < \lambda^+, 1 \leq k \leq n-1), ((\bar{\tau}^{1,\gamma}_\zeta, \ldots, \bar{\tau}^{k,\gamma}_\zeta)\colon \gamma < \lambda^+, 1 \leq k \leq n-1, \zeta \in \bar{A}^{k,\gamma})\}$ be two sets of parameters and $Q = Q(P)$ and $\bar{Q} = \bar{Q}(\bar{P})$ the corresponding Q-iterations.

Lemma 2.11. *Q and $\bar{Q}$ are isomorphic.*

Proof. We construct an isomorphism by a back-and-forth argument. Towards this end we define a sequence $(e_\zeta\colon \zeta \leqslant \lambda^{++})$ of functions such that

$$\begin{aligned}
&\mathrm{dom}(e_\zeta),\ \mathrm{rng}(e_\zeta) \subseteq \lambda^{++},\\
&\zeta \subseteq \mathrm{dom}(e_\zeta), \qquad \zeta \subseteq \mathrm{rng}(e_\zeta),\\
&e_\zeta \text{ is } 1{:}1,\\
&|e_\zeta| < \lambda^{++},\\
&\eta < \zeta \rightarrow e_\eta \subseteq e_\zeta,\\
&e_{\lambda^+} = \mathrm{id} \restriction \lambda^+,\\
&\forall \gamma < \lambda^+\ \forall k \in \{1, \ldots, n-1\}\ \forall \eta \in \mathrm{dom}(e_\zeta)\ (\eta \in A^{k,\gamma}\ (A^0 \text{ resp.})\\
&\hspace{16em} \Leftrightarrow e_\zeta(\eta) \in \bar{A}^{k,\gamma}\ (\bar{A}^0 \text{ resp.})).
\end{aligned}$$

We begin with $e_{\lambda^+} = \mathrm{id} \restriction \lambda^+$. For a limit ordinal $\zeta \in (\lambda^+, \lambda^{++}]$ we let $e_\zeta = \bigcup_{\eta<\zeta} e_\eta$. Now suppose we have arrived at a successor ordinal $\zeta + 1 < \lambda^{++}$. If $\zeta \notin \mathrm{dom}(e_\zeta)$, we have to distinguish the following cases:

For $\zeta \in A^0$ define

$$e_{\zeta+1}(\zeta) = \min(\bar{A}^0 \sim \sup^+ \mathrm{rng}(e_\zeta)).$$

For $\zeta \in A^{k,\gamma}$ (where $1 \leqslant k \leqslant n-1$ and $\gamma < \lambda^+$) pick the minimal $\eta \in \bar{A}^{k,\gamma} \sim \sup^+ \mathrm{rng}(e_\zeta)$ such that $(\bar{\tau}^{1,\gamma}_\eta, \ldots, \bar{\tau}^{k,\gamma}_\eta) = ((\hat{\tau}^{1,\gamma}_\zeta)^{e_\zeta}, \ldots, (\hat{\tau}^{k,\gamma}_\zeta)^{e_\zeta})$ (where $(\hat{\tau}^{i,\gamma}_\zeta)^{e_\zeta} \overset{\mathrm{def}}{=} \{(\eta, q^{e_\zeta})\colon (\eta, q) \in \hat{\tau}^{i,\gamma}_\zeta\}$ and where $q^{e_\zeta} \in \mathrm{Add}(\lambda^{++}, \lambda^+)$ with $\mathrm{supp}(q^{e_\zeta}) = e_\zeta[\mathrm{supp}(q)]$ and $\forall \xi \in \mathrm{supp}(q)\, q^{e_\zeta}(e_\zeta(\xi)) = q(\xi)$) and let $e_{\zeta+1}(\zeta) = \eta$. If $\zeta \notin \mathrm{rng}(e_\zeta) \cup \{\eta\}$ again there are two cases: For $\gamma \in \bar{A}^0$ let $\xi = \min A^0 \sim \sup^+(\mathrm{dom}(e_\zeta) \cup \{\zeta\})$ and define $e_{\zeta+1}(\xi) = \zeta$. If $\zeta \in \bar{A}^{k,\gamma}$ for some $\gamma < \lambda^+$ and $k \in \{1, \ldots, n-1\}$ pick the minimal $\xi \in A^{k,\gamma} \sim \sup^+(\mathrm{dom}(e_\zeta) \cup \{\zeta\})$ such that $(\tau^{1,\gamma}_\xi, \ldots, \tau^{k,\gamma}_\xi) = ((\hat{\bar{\tau}}^{1,\gamma}_\zeta)^{e_\zeta^{-1}}, \ldots, (\hat{\bar{\tau}}^{k,\gamma}_\zeta)^{e_\zeta^{-1}})$ (again $(\hat{\bar{\tau}}^{i,\gamma}_\zeta)^{e_\zeta^{-1}}$ denotes the result of applying the shifting map induced by e_ζ^{-1} to $\hat{\bar{\tau}}^{i,\gamma}_\zeta$) and let $e_{\zeta+1}(\xi) = \zeta$.

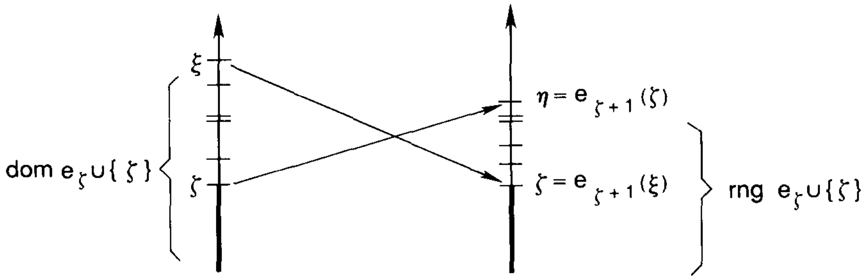
This completes the definition of $e_{\zeta+1}$. (If ζ happens to be already in the domain of e_ζ or in the range of e_ζ or the intermediate function we skip the corresponding clause in the definition.) Note that all this is possible because the sequences of tupels of terms are complete. In order to finish the proof of the lemma we have to prove the following claims:

Claim 1. *For each* $\zeta \in [\lambda^+, \lambda^{++})$, $\mathrm{dom}(e_\zeta)$ $(\mathrm{rng}(e_\zeta)$ *resp.) is a complete set of coordinates for* P ($\bar{P}$ *resp.*).

This is immediate from the way we defined the sequence $(e_\zeta\colon \zeta < \lambda^{++})$.

Claim 2. *For all* $\zeta < \lambda^{++}$, $q^{e_\zeta} \in \bar{Q}^{\mathrm{rng}(e_\zeta)}$ *for all* $q \in Q^{\mathrm{dom}(e_\zeta)}$ *and* $q^{e_\zeta^{-1}} \in Q^{\mathrm{dom}(e_\zeta)}$ *for all* $q \in \bar{Q}^{\mathrm{rng}(e_\zeta)}$.

In order to prove the first half of Claim 2 suppose that $q \in Q^{\mathrm{dom}(e_{\zeta+1})}$ for some successor $\zeta + 1 < \lambda^{++}$. We can assume $\zeta \geqslant \lambda^+$. The worst case that can happen is that we have to add first ζ to the domain of e_ζ and then ζ to the range of the intermediate function in order to get $e_{\zeta+1}$ from e_ζ:



Clearly $\mathrm{dom}(e_\zeta) \cup \{\xi\}$ is complete for P. Hence by 2.10

$$q|^{\emptyset}(\mathrm{dom}(e_\zeta) \cup \{\xi\}) \in Q^{\mathrm{dom}(e_\zeta) \cup \{\xi\}}.$$

One now argues that

$$(q|^{\emptyset}(\mathrm{dom}(e_\zeta) \cup \{\xi\}))^{e_{\zeta+1}} \in \bar{Q}^{\mathrm{rng}(e_\zeta) \cup \{\zeta\}}.$$

Then one shows $q^{e_\zeta+1} \in \bar{Q}^{\mathrm{rng}(e_{\zeta+1})}$. The second half of Claim 2 is proved similar.

It follows from Claim 2 that each e_ζ induces an isomorphism of $Q^{\mathrm{dom}(e_\zeta)}$ with $\bar{Q}^{\mathrm{rng}(e_\zeta)}$. Thus $\bigcup_{\zeta<\lambda^{++}} e_\zeta$ induces an isomorphism of Q and $\bar{Q}$. □

We need one more technical fact about the iteration of the form $Q(P)$ which says roughly that $Q(P)$ factors in a nice way, i.e., if we pause at some intermediate stage $\zeta < \lambda^{++}$ of $Q(P)$ then from the viewpoint of V^{Q_ζ} the rest of the iteration looks pretty much like the original iteration in V. Before we can make this precise we need to set up some notation. Suppose P is a set of parameters and $(Q_\zeta: \zeta \leqslant \lambda^{++})$ is the iteration defined from P. If $\delta < \lambda^{++}$ and H_δ is Q_δ generic then, in $V[H_\delta]$, let for $\zeta \in [\delta, \lambda^{++}]$

$$Q_{\delta,\zeta} \overset{\mathrm{def}}{=} \{s \in \mathrm{Add}([\delta, \lambda^{++}), \lambda^+): \exists q \in H_\delta\ (q \mid \delta)^\frown s \in Q_\zeta\}$$

and endow it with the ordinary $\leqslant$ on $\mathrm{Add}([\delta, \lambda^{++}), \lambda^+)$. Pick a canonical name $\mathring{Q}_{\delta,\zeta} \in V^{Q_\delta}$ for $Q_{\delta,\zeta}$. For each $q \in Q_\zeta$ $(\delta \leqslant \zeta \leqslant \lambda^{++})$ pick a term $\mathring{q}_{\delta,\zeta} \in V^{Q_\delta}$ with

$$\Vdash_{Q_\delta} (\mathring{q}_{\delta,\zeta} \in \mathring{Q}_{\delta,\zeta} \wedge (q \mid [\delta, \lambda^{++}) \in \mathring{Q}_{\delta,\zeta} \rightarrow \mathring{q}_{\delta,\zeta} = q \mid [\delta, \lambda^{++}))).$$

Lemma 2.12. *For each ζ with $\delta \leqslant \zeta \leqslant \lambda^{++}$*

$$\Phi_\delta : Q_\zeta \rightarrow Q_\delta * \mathring{Q}_{\delta,\zeta}$$
$$q \mapsto (q|^{\emptyset}\delta, \mathring{q}_{\delta,\zeta})$$

defines an isomorphism of Q_ζ with a dense suborder of $Q_\delta * \mathring{Q}_{\sigma,\zeta}$. □

Next we associate with each nice $\mathrm{Add}(\lambda^{++}, \lambda^{+})$ name τ for a subset of λ^{+} a canonical name ${}_{\delta}\tau \in V^{Q_\delta}$ such that

$$(2.13) \quad \Vdash_{Q_\delta} {}_{\delta}\tau = \{(\eta, h) \colon \eta < \lambda^{+} \wedge h \in \mathrm{Add}([\delta, \lambda^{++}), \lambda^{+}) \wedge \exists f \in \Gamma\, \exists g \in \mathrm{Add}(\delta, \lambda^{+})\, ((\eta, g^\frown h) \in \tau \wedge f \mid \delta \leqslant g)\}.$$

Note that

$$\Vdash_{Q_\delta} \text{“}{}_{\delta}\tau \text{ is a nice } \mathrm{Add}[(\delta, \lambda^{++}), \lambda^{+}) \text{ name for a subset of } \lambda^{+}\text{”}.$$

Lemma 2.14. *For any complete sequence* $(\tau_\zeta \colon \zeta < \lambda^{++})$ *of nice* $\mathrm{Add}(\lambda^{++}, \lambda^{+})$ *names for subsets of* λ^{+}

$$\Vdash_{Q_\delta} \text{“}({}_{\delta}\tau_\zeta \colon \zeta < \lambda^{++}) \text{ is a complete sequence of nice } \mathrm{Add}([\delta, \lambda^{++}), \lambda^{+}) \text{ names for subsets of } \lambda^{+}\text{”}.$$

Proof. Suppose $q \in Q_\delta$ and $\mathring{\sigma} \in V^{Q_\delta}$ such that

$$q \Vdash_{Q_\delta} \text{“}\mathring{\sigma} \text{ is a nice } \mathrm{Add}([\delta, \lambda^{++}), \lambda^{+}) \text{ name for a subset of } \lambda^{+}\text{”}.$$

Define a nice $\mathrm{Add}(\lambda^{++}, \lambda^{+})$ name τ for a subset of λ^{+} by

$$\tau = \{(\eta, h) \colon \eta < \lambda^{+} \wedge h \in \mathrm{Add}(\lambda^{++}, \lambda^{+}) \wedge h|^{\emptyset}\,\delta \in Q_\delta \wedge h|^{\emptyset}\,\delta \leqslant q \wedge h|^{\emptyset}\,\delta \Vdash_{Q_\delta} (\eta, h \mid [\delta, \lambda^{++})) \in \mathring{\sigma}\}.$$

By applying definition (2.13) we obtain

$$q \Vdash_{Q_\delta} {}_{\delta}\tau = \mathring{\sigma}.$$

By the completeness of $(\tau_\zeta \colon \zeta < \lambda^{++})$ there are arbitrarily large $\zeta < \lambda^{++}$ with $\tau = \tau_\zeta$. Obviously for each such ζ

$$q \Vdash_{Q_\delta} {}_{\delta}\tau_\zeta = \mathring{\sigma}. \qquad \square$$

We are now going to explain what we mean by *modified* δ, λ^{++} *iterations*. Suppose we have partitioned some $\delta \in (\lambda^{+}, \lambda^{++})$ into A^{0} and $\{A^{i,\gamma} \colon \gamma < \lambda^{+}, 1 \leqslant i \leqslant n-1\}$ and we have enumeration of tupels of terms $((\tau_\zeta^{1,\gamma}, \ldots, \tau_\zeta^{k,\gamma}) \colon \zeta \in A^{k,\gamma})$ for $\gamma < \lambda^{+}$ and $1 \leqslant k \leqslant n-1$, where each $\tau_\zeta^{i,\gamma}$ is a nice $\mathrm{Add}(\lambda^{++}, \lambda^{+})$ name for a subset of λ^{+}. Let $(Q_\zeta \colon \zeta \leqslant \delta)$ denote the iteration defined from these parameters. Now let H be Q_δ generic. Suppose that in $V[H]$ we choose a partition of $[\delta, \lambda^{++})$ into cofinal pieces $\bar{A}^{0}$ and $(\bar{A}^{k,\gamma} \colon \gamma < \lambda^{+}, 1 \leqslant k \leqslant n-1)$ and we have enumerations of tupels $((\bar{\tau}_\zeta^{1,\gamma}, \ldots, \bar{\tau}_\zeta^{k,\gamma}) \colon \zeta \in \bar{A}^{k,\gamma})$ of nice $\mathrm{Add}([\delta, \lambda^{++}), \lambda^{+})$ names for subsets of λ^{+} that are for each $\gamma < \lambda^{+}$ complete for $\mathrm{Add}([\delta, \lambda^{++}), \lambda^{+})$. Let $P_H = \{(H^{\gamma} \colon \gamma < \lambda^{+}, \gamma \text{ even}), (((\hat{\tau}_\zeta^{1,\gamma})^{H}, \ldots, (\hat{\tau}_\zeta^{k,\gamma})^{H}, H^{\zeta}) \colon \zeta \in A^{k,\gamma} \cap \delta, \gamma < \lambda^{+}, 1 \leqslant k \leqslant n-2)\}$ and P the set of parameters that we fixed in $V[H]$. Working in $V[H]$, we can now define the modified δ, λ^{++} iteration $(\tilde{Q}_{\delta,\alpha}(P_H, P) \colon \delta \leqslant \alpha \leqslant \lambda^{++})$ by induction on $\alpha \leqslant \lambda^{++}$ (we drop the P_H and P to avoid excessive notation): Let $\tilde{Q}_{\delta,\delta}$ be the trivial partial order on the one element set $\{\mathbf{1}_{\mathrm{Add}([\delta, \lambda^{++}), \lambda^{+})}\}$. If α is a limit then let

$$\tilde{Q}_{\delta,\alpha} \stackrel{\text{def}}{=} \{f \in \mathrm{Add}^{[\delta,\lambda^{++})}([\delta,\alpha),\lambda^+): \forall\beta \in [\delta,\alpha) f|^{\emptyset}[\delta,\beta) \in \tilde{Q}_{\delta,\beta}\},$$

where of course, for $S \subseteq [\delta, \lambda^{++})$ and $q \in \mathrm{Add}([\delta, \lambda^{++}), \lambda^+)$ we define

$$\mathrm{Add}^{[\delta,\lambda^{++})}(S,\lambda^+) \stackrel{\text{def}}{=} \{f \in \mathrm{Add}([\delta,\lambda^{++}),\lambda^+): \mathrm{supp}(f) \subseteq S\}$$

and $q|^{\emptyset} S \in \mathrm{Add}^{[\delta,\lambda^{++})}(S, \lambda^+)$ by $q|^{\emptyset} S(\zeta) = q(\zeta)$ for $\zeta \in S$. If $\alpha = \beta + 1$ for some $\beta \in [\delta, \lambda^{++})$ there are two cases: For $\beta \in [\delta, \lambda^{++}) \sim \bigcup_{\gamma<\lambda^+} \bar{A}^{n-1,\gamma}$ we simply let

$$\tilde{Q}_{\delta,\beta+1} = \{f \in \mathrm{Add}^{[\delta,\lambda^{++})}([\delta,\beta+1),\lambda^+): f|^{\emptyset}\beta \in \tilde{Q}_{\delta,\beta}\}.$$

If for some $\gamma < \lambda^+$, $\beta \in \bar{A}^{n-1,\gamma}$ and γ is odd, we let

$$\tilde{Q}_{\beta+1} \stackrel{\text{def}}{=} \{f \in \mathrm{Add}^{[\delta,\lambda^{++})}([\delta,\beta+1),\lambda^+): f|^{\emptyset}\beta \in \tilde{Q}_{\delta,\beta} \wedge f|^{\emptyset}\beta \Vdash^{V[H]}_{\tilde{Q}_{\delta,\beta}} \tilde{\theta}^*_{\Pi^2_n}\}$$

where $\tilde{\theta}^*_{\Pi^2_n}$ says:

$$\begin{aligned}&f(\beta) \text{ is a condition for killing } F_\gamma((\hat{\tilde{\tau}}^{1,\gamma}_\beta, \ldots, \hat{\tilde{\tau}}^{n-1,\gamma}_\beta)) \quad \text{if}\\ &\quad \tilde{\theta}_{\Pi^2_n}(P_H, (\hat{\tilde{\tau}}^{1,\gamma}_\beta, \ldots, \hat{\tilde{\tau}}^{n-1,\gamma}_\beta), ((\hat{\tilde{\tau}}^{1,\gamma}_\zeta, \ldots, \hat{\tilde{\tau}}^{k,\gamma}_\zeta, \Gamma^\zeta): \zeta \in \bar{A}^{k,\gamma} \cap \beta,\\ &\quad 1 \leq k \leq n-2)) \quad \text{and} \quad f(\beta) = \emptyset \quad \text{otherwise.}\end{aligned}$$

and $\tilde{\theta}_{\Pi^2_n}$ says that the killing conditions (as given by $T_{\Pi^2_n}$) are satisfied in $(V[H])^{\tilde{Q}_{\delta,\beta}}$ if we also refer to P_H. For even γ we use formulas $\tilde{\theta}^*_{\Sigma^1_n}$ and $\tilde{\theta}_{\Sigma^2_n}$ which are defined similar. The symbol $\hat{}$ denotes an operation on terms defined as follows:

$$\hat{\tilde{\tau}}^{i,\gamma}_\zeta \stackrel{\text{def}}{=} \{(\eta, f): f \in \tilde{Q}_{\delta,\zeta} \wedge \exists g\ ((\eta, g) \in \tilde{\tau}^{i,\gamma}_\zeta \wedge f \leq g)\}.$$

An analogous proof as in 2.3 shows that modified δ, λ^{++} iterations are $<\lambda^+$ Baire. Moreover, by the analogue of 2.11 once we fix H and P_H as in the above definition then, in $V[H]$ there is only one (up to isomorphism) δ, λ^{++} iteration that refers to P_H and to $(F_\gamma: \gamma \leq \lambda^+)$.

Now suppose we have a set of parameters P in V, i.e.,

$$\begin{aligned}P = \{&A^0, (A^{k,\gamma}: k \in \{1, \ldots, n-1\}, \gamma < \lambda^+),\\ &((\tau^{1,\gamma}_\zeta, \ldots, \tau^{k,\gamma}_\zeta): \gamma < \lambda^+, \zeta \in A^{k,\gamma}, k \in \{1, \ldots, n-1\})\}\end{aligned}$$

and $(Q_\zeta: \zeta \leq \lambda^+)$ denotes the iteration defined from P. Let $\delta < \lambda^{++}$ and H be Q_δ generic. Let

$$\begin{aligned}\bar{P} = \{&A^0 \cap [\delta, \lambda^{++}), (A^{k,\gamma} \cap [\delta, \lambda^{++}): 1 \leq k \leq n-1, \gamma < \lambda^+),\\ &((_\delta\tau^{1,\gamma}_\zeta, \ldots, {_\delta\tau^{k,\gamma}_\zeta}): \gamma < \lambda^+, \zeta \in A^{k,\gamma} \cap [\delta, \lambda^{++}), k \in \{1, \ldots, n-1\})\}.\end{aligned}$$

Denote by $(\tilde{Q}_\zeta: \delta \leq \zeta \leq \lambda^{++})$ the $[\delta, \lambda^{++})$ iteration defined in $V[H]$ from P_H and $\bar{P}$. Recall the poset $\mathring{Q}_{\delta,\zeta}$ from 2.12. The following lemma illustrates that $(Q_\zeta: \zeta < \lambda^{++})$ factors in a nice way.

Lemma 2.15. *For each* $\zeta \in [\delta, \lambda^{++}]$,

$$\mathring{Q}^H_{\delta,\zeta} = \tilde{Q}_{\delta,\zeta}.$$

Proof. The lemma is proved by induction on $\zeta \in [\delta, \lambda^{++}]$. For $\zeta = \delta$ the lemma is trivial. Now let ζ be a limit. If $\mathrm{cf}(\zeta) = \lambda^+$, there are no problems so suppose $\mathrm{cf}(\zeta) \leq \lambda$. $\mathring{Q}^H_{\delta,\zeta} \subseteq \tilde{Q}_{\delta,\zeta}$ follows from the induction hypothesis. Conversely, for $q \in \tilde{Q}_{\delta,\zeta}$ pick $S \subseteq [\delta, \zeta)$ cofinal in ζ with $|S| \leq \lambda$. For each $v \in S$ pick $h_v \in H$ with $(h_v \mid \delta)^\frown q|^\emptyset[\delta, v) \in Q_v$. Note that the sequence $(h_v \colon v \in S)$ is in V, thus we can pick $h \in H$ such that $h \Vdash_{Q_\delta} \forall v \in S\, h_v \in \Gamma$. One now argues that $h^\frown q \in Q_\zeta$; this proves that $q \in \mathring{Q}^H_{\delta,\zeta}$. Finally, consider a successor $\zeta + 1 < \lambda^{++}$. Here we first establish the following

Claim. *If K is $\mathring{Q}^H_{\delta,\zeta}$ generic over $V[H]$, then for each $\gamma < \lambda^+$ and $k \in \{1, \ldots, n-1\}$ and $v \in A^{k,\gamma} \cap [\delta, \zeta]$*

$$(({}_\delta\hat{\tau}^{i,\gamma}_v)^H)^K = (\hat{\tau}^{i,\gamma}_v)^{\Phi_\delta^{-1}[H*K]}$$

where Φ_δ is as in 2.12. □

The proof of the claim consists of a straightforward inspection using the definition of the $\hat{}$ operation. In order to prove the lemma for $\zeta + 1$ we can obviously restrict ourselves to considering $\zeta \in A^{n-1,\gamma}$ (for some $\gamma < \lambda^+$). Suppose

$$q \in \mathrm{Add}^{[\delta,\lambda^{++})}([\delta, \zeta+1], \lambda^+) \quad \text{and} \quad q|^\emptyset[\delta, \zeta) \in \mathring{Q}^H_{\delta,\zeta} = \tilde{Q}_{\delta,\zeta}.$$

If $q \in \mathring{Q}^H_{\delta,\zeta+1}$ pick $h \in H$ with $(h \mid \delta)^\frown q \in Q_{\zeta+1}$, i.e.,

$$(h \mid \delta)^\frown(q|^\emptyset[\delta, \zeta)) \Vdash_{Q_\zeta} \theta^*_{\Pi^2_n/\Sigma^2_n}.$$

We must show

$$q|^\emptyset[\delta, \zeta) \Vdash^{V[H]}_{\tilde{Q}_{\delta,\zeta}} \tilde{\theta}^*_{\Pi^2_n/\Sigma^2_n}.$$

Conversely, if $q \in \tilde{Q}_{\delta,\zeta+1}$ we pick $h \in H$ such that

$$h \mid \delta \Vdash_{Q_\delta} \text{“}q|^\emptyset[\delta, \zeta) \Vdash^{V[\mathring{H}]}_{\mathring{Q}_{\delta,\zeta}} \tilde{\theta}^*_{\Pi^2_n/\Sigma^2_n}\text{”}$$

(where $\mathring{H}$ and $\mathring{Q}_{\delta,\zeta}$ are canonical Q_δ names for H and $\tilde{Q}_{\delta,\zeta}$) and $(h \mid \delta)^\frown(q|^\emptyset \zeta) \in Q_\zeta$. Now we must show

$$(h \mid \delta)^\frown(q|^\emptyset[\delta, \zeta)) \Vdash_{Q_\zeta} \theta^*_{\Pi^2_n/\Sigma^2_n}.$$

However, all this is easily checked since, by the claim, the formulas $\theta^*_{\Pi^2_n/\Sigma^2_n}$ and $\tilde{\theta}^*_{\Pi^2_n/\Sigma^2_n}$ are merely restatements of each other when considered in the appropriate models. □

In the sequel we will also be using the following specialized construction: Again suppose we are working in V where $(F_\gamma \colon \gamma < \lambda^+)$ is a sequence of Lipschitz functions $F_\gamma : (2^{\lambda^+})^{n-1} \to 2^{\lambda^+}$ and GCH holds from λ on. Suppose $\delta \in (\lambda^+, \lambda^{++})$ and $P = \{A^0, (A^{k,\gamma} \colon k \in \{1, \ldots, n-1\}, \gamma < \lambda^+), ((\tau^{1,\gamma}_\zeta, \ldots, \tau^{k,\gamma}_\zeta) \colon \zeta \in A^{k,\gamma}, k \in \{1, \ldots, n-1\}, \gamma < \lambda^+)\}$ is a set of parameters and $(Q_\zeta \colon \zeta \leq \lambda^{++})$ the corresponding iteration. In addition to this suppose that for some $\lambda^* \in [\omega, \lambda^+]$ with order type of $\mathrm{Even}_{\lambda^*} = \lambda^*$

$$\forall \gamma < \lambda^+ \; \forall \zeta \in A^{k,\gamma} \cap \delta \; \mathrm{supp}(\tau_\zeta^{i,\gamma}) \cap \mathrm{Even}_{\lambda^*} = \emptyset$$

where $k \in \{1, \ldots, n-1\}$ and $1 \leq i \leq k$ and

$$\mathrm{supp}(\tau_\zeta^{i,\gamma}) = \bigcup \{\mathrm{supp}\,(q) \colon \exists \eta \; (\eta, q) \in \tau_\zeta^{i,\gamma}\}.$$

An argument from the proof of 2.10 shows that for each $q \in Q_\delta$, $q|^\emptyset(\delta \sim \mathrm{Even}_{\lambda^*}) \in Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}$ and consequently $Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}} \subseteq_c Q_\delta$.

Now let H be $Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}$ generic. In $V[H]$ define for $\zeta \leq \lambda^{++}$

$$^*Q_{\delta,\delta+\lambda^*+\zeta} \stackrel{\mathrm{def}}{=} \{q \in \mathrm{Add}^{[\delta,\lambda^{++})}([\delta, \delta + \lambda^* + \zeta) \colon \exists h \in H \; h \diamondsuit q \in Q_{\delta+\zeta}\}$$

where $h \diamondsuit q$ is defined as follows

$$h \diamondsuit q(\nu) \stackrel{\mathrm{def}}{=} q(\delta + \xi) \quad \text{if } \nu \text{ is the } \xi\text{-th ordinal} \in \mathrm{Even}_{\lambda^*},$$

$$h \diamondsuit q(\nu) \stackrel{\mathrm{def}}{=} h(\nu) \quad \text{if } \nu \in \delta \sim \mathrm{Even}_{\lambda^*},$$

$$h \diamondsuit q(\delta + \nu) \stackrel{\mathrm{def}}{=} q(\delta + \lambda^* + \nu) \quad \text{if } 0 \leq \nu < \lambda^{++}.$$

We let $^*\mathring{Q}_{\delta,\delta+\lambda^*+\zeta}$ be a canonical $Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}$ name for $^*Q_{\delta,\delta+\lambda^*+\zeta}$ and for each $q \in Q_{\delta+\zeta}$ $(\zeta < \lambda^{++})$ we pick a canonical $Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}$ name $^*\mathring{q}_{\delta,\delta+\lambda^*+\zeta}$ similar as in 2.12. Analogous to 2.12 we obtain a dense embedding

$$^*\phi_\delta \colon Q_{\delta+\zeta} \to Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}} * {}^*\mathring{Q}_{\delta,\delta+\lambda^*+\zeta}$$

$$q \mapsto (q|^\emptyset(\delta \sim \mathrm{Even}_{\lambda^*}), {}^*\mathring{q}_{\delta+\lambda^*+\zeta}).$$

Moreover, if we pick a $Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}$ generic H, then we can, working in $V[H]$, define a *special modified* δ, λ^{++} *iteration*. The idea is to choose a partition of $[\delta + \lambda^*, \lambda^{++})$ into cofinal pieces and enumerate sequences of tupels of nice $\mathrm{Add}([\delta, \lambda^{++}), \lambda^+)$ names for subsets of λ^+ along the coordinates in all but one of the pieces that are each complete for $\mathrm{Add}([\delta, \lambda^{++}), \lambda^+)$. The definition of the special modified δ, λ^{++} iteration $(^*\bar{Q}_{\delta,\zeta} \colon \zeta \in [\delta, \lambda^{++})]$ which arises from these parameters (and from the sequence of parameters for the original iteration in V up to stage δ) is entirely analogous to the definition of a modified δ, λ^{++} iteration except that at the ν-th step of the iteration (i.e., at coordinate $\delta + \nu$ where $0 \leq \nu < \lambda^*$) we add the generic witness for the Σ_n^2 statement that we want to hold about F_γ where γ is the ν-th even ordinal $< \lambda^*$. Clearly special modified δ, λ^{++} iterations are again $< \lambda^+$ Baire, and up to isomorphism there is only one special modified δ, λ^{++} iteration in $V[H]$ that refers to the parameters in V up to stage δ and to $(F_\gamma \colon \gamma < \lambda^+)$.

In an analogy with (2.13) we can associate with each nice $\mathrm{Add}(\lambda^{++}, \lambda^+)$ name τ in V a canonical term $^*_\delta\tau$ in $V^{Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}}$ such that

$$\Vdash_{Q_\delta^{\delta \sim \mathrm{Even}_{\lambda^*}}} {}^*_\delta\tau = \{(\eta, q) \colon \eta < \lambda^+, \, q \in \mathrm{Add}([\delta, \lambda^{++}), \lambda^+),$$

$$\exists h \in \Gamma \; \exists g \in \mathrm{Add}^{\lambda^{++}}(\delta \sim \mathrm{Even}_{\lambda^*}, \lambda^+) \, (h \leq g \wedge (\eta, g \diamondsuit q) \in \tau)\}.$$

For a given complete sequence $(\tau_\zeta : \zeta < \lambda^{++})$ of nice $\mathrm{Add}(\lambda^{++}, \lambda^+)$ names for subsets of λ^+ we obtain as in 2.14

$$\Vdash_{Q_\delta^{\delta\text{-Even}_{\lambda^*}}} \text{“}({}^*_\delta\tau_\zeta : \zeta < \lambda^{++}) \text{ is a complete sequence of nice } \mathrm{Add}([\delta, \lambda^{++}), \lambda^+) \text{ names for subsets of } \lambda^+.\text{”}$$

Thus, once we fix a $Q_\delta^{\delta-\mathrm{Even}_{\lambda^*}}$ generic H, our set P of parameters in V gives rise to a set ${}^*P \in V[H]$ of parameters for a special modified δ, λ^{++} iteration, i.e., if $P = \{A^0, (A^{k,\gamma}: \gamma < \lambda^+, 1 \leqslant k \leqslant n-1), ((\tau_\zeta^{1,\gamma}, \ldots, \tau_\zeta^{k,\gamma}): \zeta \in A^{k,\gamma}, 1 \leqslant k \leqslant n-1, \gamma < \lambda^+)\}$ let

$${}^*A^0 = [\delta, \delta + \lambda^*) \cup \{\delta + \lambda^* + \zeta : \delta + \zeta \in A^0\},$$
$${}^*A^{k,\gamma} = \{\delta + \lambda^* + \zeta : \delta + \zeta \in A^{k,\gamma}\} \quad (1 \leqslant k \leqslant n-1, \gamma \leqslant \lambda^+)$$

and for $\delta + \zeta \in A^{k,\gamma}$ let ${}^*\tau^{i,\gamma}_{\delta+\lambda^*+\zeta} = {}^*_\delta\tau^{i,\gamma}_{\delta+\zeta}$ and

$${}^*P = \{{}^*A^0, ({}^*A^{k,\gamma}: 1 \leqslant k \leqslant n-1, \gamma < \lambda^+), (({}^*\tau_\zeta^{1,\gamma}, \ldots, {}^*\tau_\zeta^{k,\gamma}): \zeta \in {}^*A^{k,\gamma}, k \leqslant n-1, \gamma < \lambda^+)\}$$

and

$$P_H = \{(H^\zeta : \zeta \in \mathrm{Even}_{\lambda^+} - \mathrm{Even}_{\lambda^*}), (((\bar{\tau}_\beta^{1,\gamma})^h, \ldots, (\tau_\beta^{k,\gamma})^H, H^\beta): \beta \in A^{k,\gamma} \cap \delta, 1 \leqslant k \leqslant n-2, \gamma < \lambda^+)\}$$

and denote by ${}^*\tilde{Q}_\zeta(P_H, {}^*P)$ the special modified δ, λ^{++} iteration defined from P_H and ${}^*P$. The same ideas as in the proof of 2.15 lead to the following factor lemma:

Lemma 2.16. *For all* $\zeta \leqslant \lambda^{++}$

$${}^*\mathring{Q}^H_{\delta,\delta+\lambda^*+\zeta} = {}^*\tilde{Q}_{\delta,\delta+\lambda^*+\zeta}(P_H, {}^*P). \qquad \square$$

This completes our analysis of the fine structure of the iteration for now, and we turn to the task of establishing that the iteration preserves the Π^2_n indescribability of κ.

3. Preservation of the Π^2_n indescribability of κ

Recall the following characterization of Π^m_n indescribability (cf. [2, Theorem 1.3]).

An inaccessible cardinal κ is Π^m_n indescribable ($m \geqslant 1$, $n \geqslant 1$) iff

$$\forall M\, [M \text{ trans.}, M \vDash \mathrm{ZF}^-, |M| = \kappa, M^{<\kappa} \subseteq M, \kappa \in M \Rightarrow \exists j, N\, [N \text{ trans.}, |N| = |V_{\kappa+m-1}|, N\ \Sigma^m_{n-1} \text{ correct for } \kappa, j: M \hookrightarrow N, \mathrm{crit}(j) = \kappa]].$$

Our strategy for establishing the Π^2_n indescribability of κ in $V^{P_{\kappa+1}}$ is then as follows: Suppose $\mu \in V^{P_{\kappa+1}}$ is a name for a subset of κ such that

$\Vdash_{P_{\kappa+1}}$ "the transitive collapse of the structure coded by μ is a model of ZF^-, has size κ, is closed under $<\kappa$ sequences and contains κ as an element".

Pick some δ with $\operatorname{cof}\delta > \kappa$ and $V_\delta \vDash \mathrm{ZF}^-$. By the usual arguments we can find a transitive M with $|M| = \kappa$, $M^{<\kappa} \subseteq M$, $\kappa \in M$ and an embedding $i: M \hookrightarrow V_\delta$ with $\operatorname{cpt} i > \kappa$ and $i(\mu^*) = \mu$ for some $\mu^* \in M$. Since κ is Π^2_n indescribable there is a transitive N which has size κ^+ and is Σ^2_{n-1} correct for κ and an embedding $j: M \hookrightarrow N$ with $\operatorname{crit}(j) = \kappa$. We are done if we can build a V generic G^V for $P_{\kappa+1}$, an M generic G^M for $P^M_{\kappa+1}$ and an N generic G^N for $P^N_{j(\kappa)+1}$ such that $N[G^N]$ is Σ^2_{n-1} correct for κ in $V[G^V]$ and $G^M, G^N \in V[G^V]$ and i and j lift to embeddings (called again) $i: M[G^M] \hookrightarrow V_\delta[G^V]$ and $j: M[G^M] \hookrightarrow N[G^N]$. For then, if we let M^* denote the transitive collapse of the structure coded by μ^{G^V}, $j \restriction M^*$ witnesses that κ is Π^2_n indescribable in $V[G^V]$ (note that $(\mu^*)^{G^M} = i((\mu^*)^{G^M}) = \mu^{G^V}$ since $\operatorname{crit}(i) > \kappa$).

3.1. Construction of G^M and G^V

Let G_κ be V generic for P_κ. Since $M^{<\kappa} \subseteq M$, $P^M_\kappa = P_\kappa$. Clearly G_κ is M generic and i lifts to $i: M[G_\kappa] \hookrightarrow V_\delta[G_\kappa]$ because $i(p) = p$ for all $p \in P_\kappa$. Next we consider $Q^1_\kappa * Q^2_\kappa$. Let $(D_\alpha : \alpha < \kappa) \in V_\delta[G_\kappa]$ be an enumeration of all the dense sets of $(Q^1_\kappa * Q^2_\kappa)^{M[G_\kappa]}$ that belong to $M[G_\kappa]$. By induction on $\eta < \kappa$ we now build a decreasing sequence $(({}^Mf_\eta, {}^Mq_\eta): \eta < \kappa)$ such that $({}^Mf_\eta, {}^Mq_\eta) \in D_\eta$ (for $\eta < \kappa$) and there is a condition $(f, q) \in Q^1_\kappa * Q^2_\kappa$ which extends $(i({}^Mf_\eta), (i({}^Mq_\eta))$ for all $\eta < \kappa$. The construction of this sequence takes place in $V[G_\kappa]$, but any initial piece of it is an element of $M[G_\kappa]$ which is $<\kappa$ closed in $V[G_\kappa]$. At stage η of this construction we pick ${}^MT_\eta \subseteq (\kappa^+)^M$ and for each $\gamma \in {}^MT_\eta$ we pick ${}^Mb^\gamma_\eta \subseteq (\kappa^{++})^M$ such that

$$
{}^Mf_\eta \Vdash^{M[G_\kappa]}_{(Q^1_\kappa)^{M[G_\kappa]}} \begin{cases} {}^MT_\eta = \left\{\gamma < \kappa^+ : {}^M\mathring{A}^{n-1,\gamma} \cap \bigcup_{\eta' < \eta} \operatorname{supp}({}^Mq_{\eta'}) \neq \emptyset\right\}, \\ \forall \gamma \in {}^MT_\eta\, {}^Mb^\gamma_\eta = {}^M\mathring{A}^{n-1,\gamma} \cap \bigcup_{\eta' < \eta} \operatorname{supp}({}^Mq_{\eta'}). \end{cases}
$$

In addition to this we pick an ordinal ${}^M\delta_\eta < (\kappa^+)^M$ and sets ${}^Ms^{1,\gamma}_{\beta,\eta}, \ldots, {}^Ms^{n-1,\gamma}_{\beta,\eta} \subseteq (\kappa^+)^M$ for each $\gamma \in {}^MT_\eta$ and $\beta \in {}^Mb^\gamma_\eta$ with

$$
{}^M\delta_\eta > \sup\{\operatorname{dom}({}^Mf^\gamma_{\eta'}((s_1, \ldots, s_{n-1}))) : \eta' < \eta,\ \gamma \in \operatorname{supp}({}^Mf_{\eta'}),
$$

$$
(s_1, \ldots, s_{n-1}) \in \operatorname{dom}(f^\gamma_{\eta'})\} \cup \sup_{\eta' < \eta} {}^M\delta_{\eta'}
$$

$$
({}^Mf_\eta, {}^Mq_\eta) \Vdash^{M[G_\kappa]}_{(Q^1_\kappa * Q^2_\kappa)^{M[G_\kappa]}} \forall \gamma \in {}^MT_\eta\ \forall \beta \in {}^Mb^\gamma_\eta
$$

$$
{}^Ms^{i,\gamma}_{\beta,\eta} = {}^M\hat{\mathring{t}}^{i,\gamma}_\beta \cap {}^M\delta_\eta \quad (\text{for } 1 \leqslant i \leqslant n-1);
$$

and

$$\forall \gamma \in {}^{M}T_{\eta}\ \forall \beta \in {}^{M}b_{\eta}^{\gamma} \sup {}^{M}q_{\eta}(\beta) > {}^{M}\delta_{\eta}.$$

This completes the definition of the sequence $(({}^{M}f_{\eta}, {}^{M}q_{\eta})\colon \eta < \kappa)$. Now let $f_{\eta} = i({}^{M}f_{\eta})$ and $q_{\eta} = i({}^{M}q_{\eta})$. By the elementarity of $i : M[G_{\kappa}] \hookrightarrow V_{\delta}[G_{\kappa}]$ applied at each stage $\eta < \kappa$ of the above construction it follows that there is a condition $(f, q) \in Q^1_{\kappa} * Q^2_{\kappa}$ below all (f_{η}, q_{η}). The argument for this was given in a more complicated context in the proof of 2.5. Pick some $V[G_{\kappa}]$ generic $\vec{F}_{\gamma} * G$ for $Q^1_{\kappa} * Q^2_{\kappa}$ that extends (f, q) and let ${}^{M}\vec{F}_{\gamma} * {}^{M}G$ denote the filter generated by $(({}^{M}f_{\eta}, {}^{M}q_{\eta})\colon \eta < \kappa)$. ${}^{M}\vec{F}_{\gamma} * {}^{M}G$ is $M[G_{\kappa}]$ generic for $(Q^1_{\kappa} * Q^2_{\kappa})^{M[G_{\kappa}]}$ and i lifts, i.e., $i\colon M[G_{\kappa}, {}^{M}\vec{F}_{\gamma}, {}^{M}G] \hookrightarrow V_{\delta}[G_{\kappa}, \vec{F}_{\gamma}, G]$.

For the last two steps of stage κ of $P_{\kappa+1}$ recall that $Q^3_{\kappa} * Q^4_{\kappa}$ has the κ^+ c.c., and $\operatorname{crit}(i) = (\kappa^+)^M$. Thus after we pick a $V[G_{\kappa}, \vec{F}_{\gamma}, G]$ generic $\vec{S}_{\gamma} * \vec{C}_{\gamma}$ for $Q^3_{\kappa} * Q^4_{\kappa}$ we simply let ${}^{M}\vec{S}_{\gamma}$ and ${}^{M}\vec{C}_{\gamma}$ be the pointwise preimages of $\vec{S}_{\gamma}$ and $\vec{C}_{\gamma}$ respectively. Then with

$$G^M = G_{\kappa} * {}^{M}\vec{F}_{\gamma} * {}^{M}G * {}^{M}\vec{S}_{\gamma} * {}^{M}\vec{C}_{\gamma} \quad \text{and} \quad G^V = G_{\kappa} * \vec{F}_{\gamma} * G * \vec{S}_{\gamma} * \vec{C}_{\gamma}$$

i will lift, i.e.,

$$\begin{tikzcd} M[G^M] \arrow[r, "i"] \arrow[d, dash, "P^M_{\kappa+1}"] & V_{\delta}[G^V] \arrow[d, dash, "P_{\kappa+1}"] \\ M \arrow[r, "i"] & V_{\delta}. \end{tikzcd}$$

Clearly $M[G^M]$ is still closed under $<\kappa$ sequences in $V[G^V]$.

3.2. Construction of G^N

Note that $P^N_{\kappa} = P_{\kappa}$ and $j \mid P_{\kappa} = \mathrm{id} \mid P_{\kappa}$. Thus with *any* H that is $N[G_{\kappa}]$ generic for the tail $P^N_{\kappa, j(\kappa)}$, $j : M \hookrightarrow N$ will lift to $j : M[G_{\kappa}] \hookrightarrow N[G_{\kappa} * H]$. However, we also want to pick H such that $N[G_{\kappa} * H]$ is still Σ^2_{n-1} correct for κ in $V[G^V]$. Towards this end let π denote the $<_L$ least permutation of κ^+ such that

$$\pi\colon \begin{cases} \mathrm{Even}_{\kappa^+} \xrightarrow[\text{onto}]{1:1} \mathrm{Even}_{\kappa^+} \sim \mathrm{Even}_{(\kappa^+)^M}, \\ \\ \mathrm{Odd}_{\kappa^+} \xrightarrow[\text{onto}]{1:1} \mathrm{Odd}_{\kappa^+} \cup \mathrm{Even}_{(\kappa^+)^M}. \end{cases}$$

Note that $N[G_{\kappa}]^{\kappa} \subseteq N[G_{\kappa}]$, in $V[G_{\kappa}]$. Thus the forcing that $N[G_{\kappa}]$ wants to do at the first step of stage κ of $P^N_{j(\kappa)}$ equals Q^1_{κ}. Now define ${}^{N}F_{\gamma} \overset{\text{def}}{=} F_{\pi(\gamma)}$ for $\gamma < \lambda^+$. Since π is in the ground model ${}^{N}\vec{F}_{\gamma}$ is Q^1_{κ} generic over $N[G_{\kappa}]$.

Lemma 3.2.1. *$N[G_{\kappa}, {}^{N}\vec{F}_{\gamma}]$ is Σ^2_{n-1} correct for κ in $V[G_{\kappa}, \vec{F}_{\gamma}]$.*

Proof. Note that $N[G_\kappa, {}^N\vec{F}_\gamma] = N[G_\kappa, \vec{F}_\gamma]$. Now the proof of the lemma is routine since $|Q^1_\kappa| = |V_{\kappa+1}|$. (Cf. [2, Lemma 1.2].) $\square$

Next we have to find a generic for the second step of stage κ of $P^N_{j(\kappa)}$. Recall that in $N[G_\kappa, {}^N\vec{F}_\gamma]$ we have a partition of κ^{++} into cofinal pieces ${}^NA^0$ and ${}^NA^{k,\gamma}$ ($1 \leq k \leq n-1$, $\gamma < \kappa^+$) and complete sequences $(({}^N\tau^{1,\gamma}_\beta, \ldots, {}^N\tau^{k,\gamma}_\beta): \beta \in {}^NA^{k,\gamma})$ from which we define an iteration $({}^NQ_\zeta: \zeta < (\kappa^{++})^N)$ that refers to ${}^N\vec{F}_\gamma$. Now let

$$\pi^* : \kappa^{++} \xrightarrow[\text{onto}]{1:1} \kappa^{++} \sim \mathrm{Even}_{(\kappa^+)^M}$$

be defined as

$$\pi^*(\zeta) = \pi(\zeta) \quad \text{if } \zeta \in \mathrm{Even}_{\kappa^+},$$
$$\pi^*(\zeta) = \zeta \qquad \text{if } \zeta \in \kappa^{++} \sim \mathrm{Even}_{\kappa^+}.$$

π^* induces an isomorphism of $\mathrm{Add}(\kappa^{++}, \kappa^+)$ with $\mathrm{Add}^{\kappa^{++}}(\kappa^{++} \sim \mathrm{Even}_{(\kappa^+)^M}, \kappa^+)$. For $q \in \mathrm{Add}(\kappa^{++}, \kappa^+)$ we denote by q^{π^*} the image of q under this isomorphism. We also use this isomorphism to associate with each nice $\mathrm{Add}(\kappa^{++}, \kappa^+)$ name τ for a subset of κ^+ a nice $\mathrm{Add}^{\kappa^{++}}(\kappa^{++} - \mathrm{Even}_{(\kappa^+)^M}, \kappa^+)$ name τ^{π^*} in the usual way. Now in $V[G_\kappa, \vec{F}_\gamma]$ let P^* denote the following set of parameters: Partition κ^{++} into cofinal pieces ${}^*A^0$, ${}^*A^{k,\gamma}$ ($1 \leq k \leq n-1$, $\gamma < \kappa^+$) and choose complete sequences $(({}^*\tau^{1,\gamma}_\beta, \ldots, {}^*\tau^{k,\gamma}_\beta): \beta \in A^{k,\gamma})$ such that ${}^*A^0 \cap (\kappa^{++})^N = {}^NA^0$ and ${}^*A^{k,\pi(\gamma)} \cap (\kappa^{++})^N = {}^NA^{k,\gamma}$ and ${}^*\tau^{i,\pi(\gamma)}_\beta = ({}^N\hat{\tau}^{i,\gamma}_\beta)^{\pi^*}$ for $\beta \in {}^NA^{k,\gamma}$, $\gamma < \kappa^+$ and $1 \leq k \leq n-1$. Let $({}^*Q_\zeta: \zeta \leq \kappa^{++})$ denote the iteration defined from P^* and $\vec{F}_\gamma$. Note that for $\gamma < \kappa^+$, $k \in \{1, \ldots, n-1\}$ and $\beta \in {}^*A^{k,\gamma} \cap (\kappa^{++})^N$

$$\mathrm{supp}({}^*\tau^{i,\gamma}_\beta) \cap \mathrm{Even}_{(\kappa^+)^M} = \emptyset \quad \text{for } 1 \leq i \leq k.$$

Thus for $\zeta \leq (\kappa^{++})^N$ and $q \in {}^*Q_\zeta$

$$q|^\emptyset(\zeta \sim \mathrm{Even}_{(\kappa^+)^M}) \in {}^*Q_\zeta \quad \text{and} \quad {}^*Q_\zeta^{\zeta \sim \mathrm{Even}_{(\kappa^+)^M}} \subseteq_c {}^*Q_\zeta.$$

Moreover, π^* induces an isomorphism of ${}^NQ_{(\kappa^{++})^N}$ with ${}^*Q^{(\kappa^{++})^N \sim \mathrm{Even}_{(\kappa^+)^M}}_{(\kappa^{++})^N}$. Recall that for noncritical $\gamma < \kappa^+$ (i.e., γ and $\pi(\gamma)$ are both odd or both even) ${}^*Q$ and ${}^NQ$ agree on whether to make a Σ^2_n or a Π^2_n statement true about ${}^NF_\gamma = F_{\pi(\gamma)}$. However, for critical $\gamma < \kappa^+$ (i.e., γ odd and $\pi(\gamma) \in \mathrm{Even}_{(\kappa^+)^M}$, ${}^*Q$ wants to force a Σ^2_n statement about $F_{\pi(\gamma)}$, and ${}^NQ$ wants to force a Π^2_n statement about ${}^NF_\gamma = F_{\pi(\gamma)}$. Note that $\mathrm{range}(\pi^*) \cap \mathrm{Even}_{(\kappa^+)^M} = \emptyset$; hence no term that appears in ${}^*Q_{(\kappa^{++})^N}$ can possibly 'see' the witness for the Σ^2_n statement about $F_{\pi(\gamma)}$ (for critical γ) that ${}^*Q$ adds at coordinate $\pi(\gamma) \in \mathrm{Even}_{(\kappa^+)^M}$. Now the key point is that any branch in the tree for Σ^2_n which is labeled -0 leads into a subtree which is identical with the tree for Π^2_n.

Denote by G the generic (coming from G^V) for Q, the second step of stage κ of $P_{\kappa+1}$. We know from 2.10 that Q and ${}^*Q$ are isomorphic. Let G^* be the pullback of G to ${}^*Q$ via the isomorphism constructed in 2.10 and $g^* = G^* \cap {}^*Q^{(\kappa^{++})^N \sim \mathrm{Even}_{(\kappa^+)^M}}_{(\kappa^{++})^N}$. Finally let g be the pullback of g^* to ${}^NQ$ via the isomorphism of

${}^NQ$ and ${}^*Q^{(\kappa^{++})^N \sim \mathrm{Even}_{(\kappa^+)^M}}_{(\kappa^{++})^N}$ induced by π^*. Clearly g is $N[G_\kappa, {}^N\vec{F}_\gamma]$ generic for ${}^NQ$. The argument that establishes that $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ is Σ^2_{n-1} correct for κ in $V[G_\kappa, \vec{F}_\gamma, G]$ is quite complicated. Therefore we defer its proof and show how to finish the construction of G^N from here on.

In the third step of stage κ of $P^N_{j(\kappa)}$ the poset ${}^NQ^3_\kappa$ codes each ${}^NF_\gamma$ by a subset of κ. Since ${}^NF_\gamma = F_{\pi(\gamma)}$ we can take ${}^NS_\gamma \overset{\mathrm{def}}{=} S_{\pi(\gamma)}$ (where $(S_\gamma\colon \gamma < \kappa^+)$ is the $V[G_\kappa, \vec{F}_\gamma, G]$ generic for Q^3_κ), and $({}^NS_\gamma\colon \gamma < \kappa^+)$ will be $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ generic for ${}^NQ^3_\kappa$. Moreover, if $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ is Σ^2_{n-1} correct in $V[G_\kappa, \vec{F}_\gamma, G]$ for κ then $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma]$ is Σ^2_{n-1} correct for κ in $V[G_\kappa, \vec{F}_\gamma, G, \vec{S}_\gamma]$ because $|Q^3_\kappa| = \kappa^+$ (cf. Lemma 1.2 of [2]).

For the fourth step ${}^NQ^4_\kappa$ of stage κ of ${}^NP_{j(\kappa)}$ we simply take ${}^NC_\gamma \overset{\mathrm{def}}{=} C_{\pi(\gamma)}$ (where $(C_\gamma\colon \gamma < \kappa^+)$ is the $V[G_\kappa, \vec{F}_\gamma, g, \vec{S}_\gamma]$ generic for Q^4_κ). Again $({}^NC_\gamma\colon \gamma < \kappa^+)$ is ${}^NQ^4_\kappa$ generic over $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma]$ and $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]$ is still Σ^2_{n-1} correct for κ in $V[G^V]$.

Next we have to deal with the tail $P^N_{\kappa+1,j(\kappa)}$. This is no problem as it has (from the view point of $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]$) for each $\nu <$ the least inaccessible of N above κ a $<\nu$ closed dense suborder. In the usual way we can construct an $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]$ generic H for $P^N_{\kappa+1,j(\kappa)}$ since we only have to meet few dense sets. With $j(G_\kappa) = G_\kappa * {}^N\vec{F}_\gamma * g * {}^N\vec{S}_\gamma * {}^N\vec{C}_\gamma * H$, j lifts, i.e.,

$$\begin{tikzcd}
 & N[j(G_\kappa)] \arrow[d, dash, "P^N_{j(\kappa)}"] \\
M[G_\kappa] \arrow[ur, "j"] \arrow[d, dash] & \\
M \arrow[r, "j"'] & N
\end{tikzcd}$$

and $N[j(G_\kappa)]$ is still Σ^2_{n-1} correct for κ in $V[G^V]$ since the tail is highly Baire.

Finally we have to consider stage $j(\kappa)$ of $P^N_{j(\kappa)+1}$. In the first 3 steps we use standard master condition arguments to lift j, i.e.,

$$\begin{tikzcd}
 & N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^Mg), j({}^M\vec{S}_\gamma)] \arrow[d, dash] \\
M[G_\kappa, {}^M\vec{F}_\gamma, {}^Mg, {}^M\vec{S}_\gamma] \arrow[ur, "j"] \arrow[d, dash] & \\
M \arrow[r, "j"'] & N
\end{tikzcd}$$

where ${}^M\vec{F}_\gamma$, ${}^Mg$, ${}^M\vec{S}_\gamma$ denote the generics for the first 3 steps of stage κ of $P^M_{\kappa+1}$ coming from G^M. Clearly $N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^Mg), j({}^M\vec{S}_\gamma)]$ is still Σ^2_{n-1} correct in $V[G^V]$ because of the closure of these posets. Now let ${}^M\vec{C}_\gamma$ denote the generic for the fourth step of stage κ of $P^M_{\kappa+1}$ coming from G^M. There is only one candidate for a master condition in the forcing that $N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^Mg), j({}^M\vec{S}_\gamma)]$ wants to do in the last step of stage $j(\kappa)$ of $P^N_{j(\kappa)+1}$: define c^* by

$$\mathrm{dom}(c^*) = j((\kappa^+)^M)$$

and

$$c^*(j(\gamma)) = {}^M C_\gamma \cup \{\kappa\} \quad \text{for } \gamma < (\kappa^+)^M,$$
$$c^*(\zeta) = \emptyset \quad \text{for } \zeta \in j((\kappa^+)^M) \sim j[(\kappa^+)^M].$$

This c^* works provided we can show that c^* is a condition. For this it suffices to argue that in $N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^M g), j({}^M\vec{S}_\gamma)]$ for $\gamma < (\kappa^+)^M$

(3.2.2) $$c^*(j(\gamma)) \cap \{\mu < j(\kappa) \colon \mu \text{ is inaccessible} \wedge V_\mu \models \phi^{\Sigma^2_n}(j({}^M\tilde{S})_{j(\gamma)} \cap V_\mu, j(G_\kappa) \cap V_\mu, j(\kappa) \cap \mu)\} = \emptyset.$$

Here $\phi^{\Sigma^2_n}$ is the Σ^2_n statement from above and $({}^M\tilde{S}_\gamma \colon \gamma < (\kappa^+)^M)$ is the sequence of codes that one obtains from the generic $({}^M S_\gamma \colon \gamma < (\kappa^+)^M)$. Since $(N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^M g), j({}^M\vec{S}_\gamma)])_\kappa = (M[G_\kappa, {}^M\vec{F}_\gamma, {}^M g, {}^M\vec{S}_\gamma])_\kappa$ we only have to worry about $\mu = \kappa$. Fix $\gamma < (\kappa^+)^M$. Note that $j(\kappa) \cap \kappa = \kappa$, $j(G_\kappa) \cap V_\kappa = G_\kappa$ and $j({}^M\tilde{S})_{j(\gamma)} \cap V_\kappa = j({}^M\tilde{S}_\gamma) \cap V_\kappa = {}^M\tilde{S}_\gamma = i({}^M\tilde{S}_\gamma) = \tilde{S}_\gamma$ (which is the γ-th code that we add in the third step of stage κ of $P_{\kappa+1}$). Recall that $\tilde{S}_\gamma = {}^N\tilde{S}_{\pi^{-1}(\gamma)}$ and $\pi^{-1}[(\kappa^+)^M] \subseteq \mathrm{Odd}_{\kappa^+}$. Thus in $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ the Π^2_n statement $\phi^{\Pi^2_n}(\tilde{S}_\gamma, G_\kappa, \kappa)$ holds at V_κ. In the last two steps of stage κ of $P^N_{j(\kappa)}$ we do not add any new subsets of κ^+ all of whose initial segments are in $N[G_\kappa, {}^N\vec{F}_\gamma, g]$, and the rest of the forcing up to $N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^M g), j({}^M\vec{S}_\gamma)]$ is highly Baire. Thus $V_\kappa \models \phi^{\Pi^2_n}(\tilde{S}_\gamma, G_\kappa, \kappa)$ is still true in $N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^M g), j({}^M\vec{S}_\gamma)]$ and (3.2.2) is proved.

Now build an $N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^M g), j({}^M\vec{S}_\gamma)]$ generic that extends c^* in the usual way, then j lifts, i.e.

$$\begin{tikzcd}
 & N[G^N] \arrow[dd, dash] \\
M[G^M] \arrow[ur, "j"] \arrow[d, dash] & \\
M \arrow[r, "j"'] & N
\end{tikzcd}$$

where $G^N = j(G_\kappa) * j({}^M\vec{F}_\gamma) * j({}^M g) * j({}^M\vec{S}_\gamma) * j({}^M\vec{C}_\gamma)$. Moreover, $N[G^N]$ is still Σ^2_{n-1} correct for κ in $V[G^V]$ since the forcing in the last step of stage $j(\kappa)$ of $P^N_{j(\kappa)+1}$ is $<j(\kappa)$ Baire.

3.3. Σ^2_{n-1} correctness of $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ for κ in $V[G_\kappa, \vec{F}_\gamma, G]$

We begin with a general lemma

Lemma 3.3.1 ($r \geq 0$). *Suppose $\mathcal{N} \models \mathrm{ZF}^-$ is transitive and Σ^2_r correct for κ in V. Suppose, in V that for some $S \subseteq \kappa$, $2^{<\kappa^+} \subseteq L[S]$ and $(\kappa^+)^L = \kappa^+$. Let $(H_\gamma \colon \gamma < \kappa^+) \in \mathcal{N}$ be a sequence of Lipschitz functions $H_\gamma : (2^{\kappa^+})^{n-1} \to 2^{\kappa^+}$; and in $\mathcal{N}$, fix a set of parameters consisting of ${}^{\mathcal{N}}A^0$, ${}^{\mathcal{N}}A^{k,\gamma}$ and $(({}^{\mathcal{N}}\tau^{1,\gamma}_\beta, \ldots, {}^{\mathcal{N}}\tau^{k,\gamma}_\beta) \colon \beta \in A^{k,\gamma})$ for $\gamma < \kappa^+$, $1 \leq k \leq n-1$. In V pick parameters $\bar{A}^0$, $\bar{A}^{k,\gamma}$ and $((\bar{\tau}^{1,\gamma}_\beta, \ldots, \bar{\tau}^{k,\gamma}_\beta) \colon \beta \in A^{k,\gamma})$ for $\gamma < \kappa^+$, $1 \leq k \leq n-1$ such that $\bar{A}^0 \cap (\kappa^{++})^{\mathcal{N}} = {}^{\mathcal{N}}A^0$, $\bar{A}^{k,\gamma} \cap (\kappa^{++})^{\mathcal{N}} =$*

${}^{\mathcal{N}}A^{k,\gamma}$ and $\bar{\tau}^{i,\gamma}_{\beta} = {}^{\mathcal{N}}\tau^{i,\gamma}_{\beta}$ for $\beta < (\kappa^{++})^{\mathcal{N}}$. Let ${}^{\mathcal{N}}Q$ and $\bar{Q}$ be the corresponding iterations defined from the sets of parameters and from $(H_\gamma : \gamma < \lambda^+)$. If $\bar{G}$ is V generic for $\bar{Q}$ then $\mathcal{N}[\bar{G} \cap {}^{\mathcal{N}}Q]$ is Σ^2_r correct for κ in $V[\bar{G}]$.

Proof. Under the hypotheses it is clear that ${}^{\mathcal{N}}Q$ is an initial segment of $\bar{Q}$, i.e., ${}^{\mathcal{N}}Q = \bar{Q}_{(\kappa^{++})^{\mathcal{N}}}$. Hence $\bar{G} \cap {}^{\mathcal{N}}Q$ generic over $\mathcal{N}$. Now we proceed by induction on r: The case $r = 0$ is clear since $\bar{Q}$ is $<\kappa^+$ Baire. If we have arrived at $r+1$ it suffices to consider $\phi(A)$ in Π^2_{r+1} and $A \in (\mathcal{N}[\bar{G} \cap {}^{\mathcal{N}}Q])_{\kappa+2}$ with $\mathcal{N}[\bar{G} \cap {}^{\mathcal{N}}Q] \models$ "$V_\kappa \models \phi(A)$" and to argue that $V[G] \models$ "$V_\kappa \models \phi(A)$". Fix $\delta < (\kappa^{++})^{\mathcal{N}}$ and a nice name $\mathring{A} \in \mathcal{N}^{\bar{Q}_\delta}$ for A and a condition $q \in \bar{Q}_\delta \cap \bar{G}$ with

$$q \Vdash^{\mathcal{N}}_{\mathcal{N}_Q} \text{“} V_\kappa \models \phi(\mathring{A}) \text{”}.$$

It follows from the factor Lemma 2.15 and from the analogue of 2.10 for modified δ, $(\kappa^{++})^N$ iterations that

$$\begin{aligned}\mathcal{N}[\bar{G} \cap \bar{Q}_\delta]) \models \text{“for all sets of parameters } P \text{ and all modified}\\ \delta, \kappa^{++} \text{ iterations } Q(P_{\bar{G}\cap\bar{Q}_\delta}, P) \text{ that refer to}\\ (H_\gamma : \gamma < \kappa^+),\ \Vdash_{Q(P_{\bar{G}\cap\bar{Q}_\delta}, P)} [V_\kappa \models \phi(A)]\text{”}\end{aligned}$$

where

$$\begin{aligned}P_{\bar{G}\cap\bar{Q}_\delta} = \{(\bar{G}^\gamma : \gamma \in \mathrm{Even}_{\kappa^+}), (((\mathring{\tau}^{1,\gamma}_\zeta)^{\bar{G}}, \ldots, (\mathring{\tau}^{k,\gamma}_\zeta)^{\bar{G}}, \bar{G}^\zeta) :\\ \zeta \in \bar{A}^{k,\gamma} \cap \delta,\ \gamma < \kappa^+,\ 1 \leq k \leq n-2)\}.\end{aligned}$$

The induction hypothesis applied within $\mathcal{N}[\bar{G} \cap \bar{Q}_\delta]$ yields in $\mathcal{N}[\bar{G} \cap \bar{Q}_\delta]$

$$\begin{aligned}(3.3.2)\quad & \forall \mathcal{M}[\mathcal{M} \text{ trans.}, \mathcal{M} \models \mathrm{ZF}^-, |\mathcal{M}| = |V_{\kappa+1}|, \mathcal{M}\ \Sigma^2_r \text{ correct for } \kappa,\ P_{\bar{G}\cap\bar{Q}_\delta} \in \mathcal{M},\\ & A \in \mathcal{M}, (H_\gamma : \gamma < \kappa^+) \in \mathcal{M}, \mathcal{M} \models \delta < \kappa^{++} \cdot\!\!\Rightarrow\!\!\cdot\\ & \mathcal{M} \models \text{“for all sets of parameters } P \text{ and all modified}\\ & \quad \delta, \kappa^{++} \text{ iterations } Q(P_{\bar{G}\cap\bar{Q}_\delta}, P) \text{ that refer to}\\ & \quad (H_\gamma : \gamma < \kappa^+), \Vdash_{Q(P_{\bar{G}\cap\bar{Q}_\delta}, P)} [V_\kappa \models \phi(A)]\text{”}].\end{aligned}$$

Since $P_{\bar{G}\cap\bar{Q}_\delta}$ consists of κ^+ many subsets of κ^+, we can code it by one subset of $V_{\kappa+1}$ (using $(\kappa^+)^L = \kappa^+$). By our assumptions for each $\gamma < \kappa^+$, $H_\gamma \subseteq L[S]$ with $S \subseteq \kappa$, thus we can use the canonical well-ordering of $L[S]$ to code each H_γ by a subset of κ^+ and then code these κ^+ many subsets of κ^+ by one subset of $V_{\kappa+1}$. Finally $\mathcal{M} \models \delta < \kappa^{++}$ can be expressed by choosing a well-ordering R of $V_{\kappa+1}$ of order type δ and then requiring that $R \in \mathcal{M}$. Since $R \subseteq V_{\kappa+1}$ this is $\Sigma^2_1(R)$. Therefore the formula (3.3.2) is Π^2_{r+1} in a parameter $\in (\mathcal{N}[\bar{G} \cap {}^{\mathcal{N}}Q_\delta])_{\kappa+2}$. Since $|Q_\delta| \leq \kappa^+$, $\mathcal{N}[\bar{G} \cap {}^{\mathcal{N}}Q_\delta]$ is Π^2_{r+1} correct for κ in $V[\bar{G} \cap \bar{Q}_\delta]$ (cf. [2, Lemma 1.2]). It follows that (3.3.2) holds in $V[\bar{G} \cap \bar{Q}_\delta]$. Therefore, by reflection in $V[\bar{G} \cap \bar{Q}_\delta]$ together with another application of 2.15 we obtain

$$V[\bar{G}] \models \text{“} V_\kappa \models \phi(A) \text{”}. \qquad \square$$

Now let $A \in (N[G_\kappa, {}^N\vec{F}_\gamma, g])_{\kappa+2}$ and $\phi(A)$ be a formula in $\Sigma^2_{n-1} \cup \Pi^2_{n-1}$ with

$$N[G_\kappa, {}^N\vec{F}_\gamma, g] \models \text{“} V_\kappa \models \phi(A) \text{”}.$$

Lemma 3.3.1 together with the factor Lemma 2.15 and the analogue of 2.10 for modified $(\kappa^{++})^N$, κ^{++} iterations imply

(3.3.3) $V[G_\kappa, {}^N\vec{F}_\gamma, g] \models$ [for all sets of parameters P and all modified $(\kappa^{++})^N$, κ^{++} iterations $Q(P_g, P)$ that refer to $({}^NF_\gamma: \gamma < \kappa^+)$, $\Vdash_{Q(P_g, P)}$ "$V_\kappa \models \phi(A)$"]

where

$$P_g = \{(g^\gamma: \gamma \in \mathrm{Even}_{\kappa^+}), (({}^N\hat{\tau}_\beta^{1,\gamma})^g, \ldots, ({}^N\hat{\tau}_\beta^{k,\gamma})^g, g^\beta): \beta \in {}^NA^{k,\gamma}, \gamma < \kappa^+, 1 \leqslant k \leqslant n-2)\}$$

Recall that $V[G_\kappa, {}^N\vec{F}_\gamma, g] = V[G_\kappa, \vec{F}_\gamma, g^*]$ which we call V^* from here on. Moreover, by 2.16 in V^* the 'tail' ${}^*Q/{}^*Q^{(\kappa^{++})^N \sim \mathrm{Even}_{(\kappa^+)^M}}_{(\kappa^{++})^N}$ is isomorphic to a special modified $(\kappa^{++})^N$, κ^{++} iteration that refers to

$$P_{g^*} = \{(g^{*,\gamma}: \gamma \in \mathrm{Even}_{(\kappa^+)} \sim \mathrm{Even}_{(\kappa^+)^M}), ((({}^*\tilde{\tau}_\beta^{1,\gamma})^{g^*}, \ldots, ({}^*\tilde{\tau}_\beta^{k,\gamma})^{g^*}, g^{*,\beta}): \beta \in {}^*A^{k,\gamma} \cap (\kappa^{++})^N, 1 \leqslant k \leqslant n-2, \gamma < \kappa^+)\}$$

and uses $(F_\gamma: \gamma < \kappa^+)$. In order to finish the correctness argument we have to show that $V[G_\kappa, \vec{F}_\gamma, G] \models$ "$V_\kappa \models \phi(A)$". This will follow from

(3.3.4) $V^* \models$ [for all sets of parameters P and all special modified $(\kappa^{++})^N$, κ^{++} iterations ${}^*Q(P_{g^*}, P)$ that refer to $(F_\gamma: \gamma < \kappa^+)$, $\Vdash_{{}^*Q(P_{g^*}, P)}$ "$V_\kappa \models \phi(A)$"].

Therefore we have to argue that (3.3.3) implies (3.3.4). We will be able to do so because of a special feature of Σ^2_n/Π^2_n iterations: the $(n-1)$ *Back-and-Forth Property for* Σ^2_n/Π^2_n for $n \geqslant 2$. Before we explain this, we remark that from here on all modified iterations refer to P_g and ${}^N\vec{F}_\gamma$ and all special modified iterations refer to P_{g^*} and $\vec{F}_\gamma$.

Suppose now that $Q_{(\kappa^{++})^N, \delta_1}$ (where $\delta_1 < \kappa^{++}$) is an initial piece of a modified $(\kappa^{++})^N$, κ^{++} iteration. The $(n-1)$ *Back and Forth Property* claims that there is an initial piece ${}^*Q_{(\kappa^{++})^N, \delta_1'}$ ($\delta_1' < \kappa^{++}$) of a special modified $(\kappa^{++})^N$, κ^{++} iteration and a complete embedding $Q_{(\kappa^{++})^N, \delta_1} \hookrightarrow {}^*Q_{(\kappa^{++})^N, \delta_1'}$ such that if $n \geqslant 3$ and ${}^*Q_{(\kappa^{++})^N, \delta_1'+\delta_2}$ (where $\delta_2 < \kappa^{++}$) is an initial piece of a special modified $(\kappa^{++})^N$, κ^{++} iteration extending ${}^*Q_{(\kappa^{++})^N, \delta_1'}$ then we can find an initial piece of a modified $(\kappa^{++})^N$, κ^{++} iteration $Q_{(\kappa^{++})^N, \delta_1+\delta_2'}$ (where $\delta_2' < \kappa^{++}$) extending $Q_{(\kappa^{++})^N, \delta_1}$ and a complete embedding $i_2: {}^*Q_{(\kappa^{++})^N, \delta_1'+\delta_2} \hookrightarrow Q_{(\kappa^{++})^N, \delta_1+\delta_2'}$ such that $i_2 \circ i_1 = \mathrm{id}_{Q_{(\kappa^{++})^N, \delta_1}}$. Repeating this procedure $n-1$ times we can define a sequence of complete embeddings $i_1, i_2, \ldots, i_{n-1}$ with $i_{k+1} \circ i_k = \mathrm{id} \mid \mathrm{dom}(i_k)$ $(1 \leqslant k \leqslant n-2)$ and $i_{k+2} \mid \mathrm{dom}(i_k) = i_k$ $(1 \leqslant k \leqslant n-3)$. Moreover, a sequence of embeddings $i_1, i_2, \ldots, i_{n-1}$ with analogous properties can be obtained if one starts with an initial piece $Q_{(\kappa^{++})^N, \delta_1}$ (where $\delta_1 < \kappa^{++}$) of a special modified $(\kappa^{++})^N$, κ^{++} iteration. The proof of the $(n-1)$ *Back-and-Forth Property* for Σ^2_n/Π^2_n is somewhat lengthy and will be done in gory detail below. Until then assume this property. The following lemma is the heart of the correctness argument.

Lemma 3.3.5. *Suppose $Q_{(\kappa^{++})^N,\kappa^{++}}$ is a modified $(\kappa^{++})^N,\kappa^{++}$ iteration, $\mathring{A}$ a nice $Q_{(\kappa^{++})^N,\kappa^{++}}$ name for a subset of $V_{\kappa+1}$ and $\phi(\mathring{A})$ a formula in Σ^2_{n-1} such that for some condition*

$$q_1 \Vdash^{V^*}_{Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \phi(\mathring{A})\text{”}.$$

Pick $\delta_1 < \kappa^{++}$ large enough so that $q_1 \in Q_{(\kappa^{++})^N,\delta_1}$ and $\mathring{A} \in (V^*)^{Q_{(\kappa^{++})^N,\delta_1}}$ and there is a witness in $(V^*)^{Q_{(\kappa^{++})^N,\delta_1}}$ for the Σ^2_{n-1} statement ϕ. Now pick $\delta_1' < \kappa^{++}$ and define a special modified κ^{++}, δ_1' iteration${}^*Q_{(\kappa^{++})^N,\delta_1'}$ such that there is a complete embedding i_1: $Q_{(\kappa^{++})^N,\delta_1} \hookrightarrow {}^*Q_{(\kappa^{++})^N,\delta_1'}$ and such that this procedure can be continued as required in the $(n-1)$ Back and Forth Property. Then we have for ***any*** *special modified $(\kappa^{++})^N$, κ^{++} iteration ${}^*Q_{(\kappa^{++})^N,\kappa^{++}}$ that extends ${}^*Q_{(\kappa^{++})^N,\delta_1'}$*

$$i_1(q_1) \Vdash^{V^*}_{{}^*Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \phi(\mathring{A}^{i_1})\text{”}$$

(where $\mathring{A}^{i_1}$ is obtained from $\mathring{A}$ by replacing all conditions q in $\mathring{A}$ by $i_1(q)$ as usual).

A similar fact holds when one starts out with an initial piece of a special modified iteration.

Proof. The proof proceeds by induction on $n \geq 2$ and we only present the argument for the first half of the lemma (the argument for the second half is totally analogous).

Case $n = 2$. Suppose $\phi(\mathring{A}) = \exists X\, \varphi(X, \mathring{A})$ where X ranges over $V_{\kappa+2}$ and φ is Σ^2_0. Now pick δ_1 as above and let $\mathring{X} \in (V^*)^{Q_{(\kappa^{++})^N,\delta_1}}$ such that

$$q_1 \Vdash^{V^*}_{Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \varphi(\mathring{X}, \mathring{A})\text{”}.$$

If ${}^*Q_{(\kappa^{++})^N,\kappa^{++}}$ is any special modified iteration extending ${}^*Q_{(\kappa^{++})^N,\delta_1'}$ where δ_1' is as above then

$$i_1(q_1) \Vdash^{V^*}_{{}^*Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \varphi(\mathring{X}^{i_1}, \mathring{A}^{i_1})\text{”}$$

since φ is Σ^2_0 and all models involved have the same $V_{\kappa+1}$.

Case $n \geq 3$. Suppose $\phi(\mathring{A}) \equiv \exists X\, \forall Y\, \varphi(X, Y, \mathring{A})$ where X, Y range over $V_{\kappa+2}$ and φ is Σ^2_{n-3}. Pick δ_1 as above and let $\mathring{X} \in (V^*)^{Q_{(\kappa^{++})^N,\delta_1}}$ such that

$$(3.3.6) \quad q_1 \Vdash^{V^*}_{Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \forall Y\, \varphi(\mathring{X}, Y, \mathring{A})\text{”}.$$

Assume towards a contradiction that there is some special modified iteration ${}^*Q_{(\kappa^{++})^N,\kappa^{++}}$ extending ${}^*Q_{(\kappa^{++})^N,\delta_1'}$ where δ_1' is as above such that

$$(3.3.7) \quad \neg\, i_1(q_1) \Vdash^{V^*}_{{}^*Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \forall Y\, \varphi(\mathring{X}^{i_1}, Y, \mathring{A}^{i_1})\text{”}.$$

Pick $\delta_2 < \kappa^{++}$ and a condition $q_2 \leq i_1(q_1)$ in ${}^*Q_{(\kappa^{++})^N,\delta_1'+\delta_2}$ and a name

$\mathring{Y} \in (V^*)^{*Q_{(\kappa^{++})^N,\delta_1'+\delta_2}}$ such that

(3.3.8) $\quad q_2 \Vdash^{V^*}_{*Q_{(\kappa^{++})^N,\kappa^{++}}}$ "$V_\kappa \models \neg\, \varphi(\mathring{X}^{i_1}, \mathring{Y}, \mathring{A}^{i_1})$".

Let $\delta_2' < \kappa^{++}$ and $\bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'}$ be a modified iteration extending $Q_{(\kappa^{++})^N,\delta_1}$ such that there is a complete embedding

$$i_2\colon {}^*Q_{(\kappa^{++})^N,\delta_1'+\delta_2} \hookrightarrow \bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'} \quad \text{with } i_2 \circ i_1 = \mathrm{id} \restriction Q_{(\kappa^{++})^N,\delta_1}$$

and such that this procedure can be continued as in the $(n-1)$ Back and Forth Property.

Claim. $i_2(q_2) \Vdash^{V^*}_{\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}}$ "$V_\kappa \models \neg\varphi(\mathring{X}, \mathring{Y}^{i_2}, \mathring{A})$" *for any modified iteration* $\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}$ *extending* $\bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'}$.

Proof of the Claim. For $n=3$ note that all models involved have the same $V_{\kappa+1}$ and φ is Σ^2_0. Furthermore $(\mathring{X}^{i_1})^{i_2} = \mathring{X}$ and $(\mathring{A}^{i_1})^{i_2} = \mathring{A}$. If $n \geq 4$ assume towards a contradiction that there is some modified iteration $\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}$ extending $\bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'}$ such that for some condition $q_3 \in \bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'+\delta_3}$ with $\delta_3 < \kappa^{++}$ and $q_3 \leq i_2(q_2)$

$$q_3 \Vdash^{V^*}_{\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \varphi(\mathring{X}, Y^{i_2}, \mathring{A})\text{”}$$

and there is a witness $\in (V^*)^{\bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'+\delta_3}}$ for the Σ^2_{n-3} statement φ. Fix an ordinal $\delta_3' < \kappa^{++}$ and define a special modified iteration ${}^*\bar{Q}_{(\kappa^{++})^N,\delta_1'+\delta_2+\delta_3'}$ extending ${}^*Q_{(\kappa^{++})^N,\delta_1'+\delta_2}$ such that there is a complete embedding

$$i_3\colon \bar{Q}_{(\kappa^{++})^N,\delta_1+\delta_2'+\delta_3} \hookrightarrow {}^*\bar{Q}_{(\kappa^{++})^N,\,\delta_1'+\delta_2+\delta_3'} \text{ with the properties that}$$

$$i_3 \circ i_2 = \mathrm{id} \restriction {}^*Q_{(\kappa^{++})^N,\delta_1'+\delta_2} \quad\text{and}\quad i_3 \restriction Q_{(\kappa^{++})^N,\delta_1} = i_1$$

and that this process can be continued as required in the Back and Forth Property. Since φ is Σ^2_{n-3} it follows from the induction hypothesis that

(3.3.9) $\quad i_3(q_3) \Vdash^{V^*}_{*\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}}$ "$V_\kappa \models \varphi(\mathring{X}^{i_1}, \mathring{Y}, \mathring{A}^{i_1})$"

for any special modified iteration ${}^*\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}$ extending ${}^*\bar{Q}_{(\kappa^{++})^N,\delta_1'+\delta_2+\delta_3'}$ (note that $\mathring{X}^{i_3} = \mathring{X}^{i_1}$, $(\mathring{Y}^{i_2})^{i_3} = \mathring{Y}$ and $\mathring{A}^{i_3} = \mathring{A}^{i_1}$).

Recall that for any such ${}^*\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}$ there is an isomorphism with ${}^*Q_{(\kappa^{++})^N,\kappa^{++}}$ that is the identity on ${}^*Q_{(\kappa^{++})^N,\delta_1'+\delta_2}$. Thus $i_3(q_3) \leq q_2$ together with (3.3.8) and (3.3.9) yield a contradiction and the claim is proved. □ Claim

Now fix a modified iteration $\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}$ extending $\bar{Q}_{(\kappa^{++})^N,\delta_1'+\delta_2}$. By the claim,

(3.3.10) $\quad i_2(q_2) \Vdash^{V^*}_{\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}}$ "$V_\kappa \models \neg\phi(\mathring{X}, \mathring{Y}^{i_2}, \mathring{A})$".

Recall that there is an isomorphism of $Q_{(\kappa^{++})^N,\kappa^{++}}$ with $\bar{Q}_{(\kappa^{++})^N,\kappa^{++}}$ that is the identity on $Q_{(\kappa^{++})^N,\delta_1}$. Thus $i_2(q_2) \leq q_1$ together with (3.3.10) and (3.3.6) yield a contradiction. Hence our assumption (3.3.7) was false and the lemma is proved. □ 3.3.5

We use this lemma to show that (3.3.3) implies (3.3.4) from which the Σ^2_{n-1} correctness of $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ for κ inside $V[G_\kappa, \vec{F}_\gamma, G]$ follows as remarked earlier. Assume that (3.3.3) holds. If $\phi(A)$ is Σ^2_{n-1} then by the first half of the lemma we obtain (3.3.4). If $\phi(A)$ is Π^2_{n-1} assume towards a contradiction that for some special modified iteration ${}^*Q_{(\kappa^{++})^N,\kappa^{++}}$ and a condition q

$$q \Vdash^{V^*}_{{}^*Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_\kappa \models \neg\phi(\check{A})\text{”}.$$

It follows from the second half of the lemma that there is a modified iteration $Q_{(\kappa^{++})^N,\kappa^{++}}$ and a condition q' with

$$q' \Vdash^{V^*}_{Q_{(\kappa^{++})^N,\kappa^{++}}} \text{“}V_k \Vdash \neg\phi(\check{A})\text{”}.$$

contradicting (3.3.3).

In order to complete the proof of Theorem 1.1, we establish the $(n-1)$ Back and Forth Property in the next section.

Proof of the $(n-1)$ Back and Forth Property ($(n-1)BFP$) for Σ^2_n/Π^2_n

We begin with some remarks in order to avoid excessive notation which would only blur the important ideas. Recall that $F^N_\gamma = F_{\pi(\gamma)}$ $(\gamma < \kappa^+)$ and that for ‘many’ γ modified and special modified iterations either both want to force a Σ^2_n or both want to force a Π^2_n fact about F^N_γ. For critical γ (i.e., $\gamma < \kappa^+$ odd and $\pi(\gamma) < (\kappa^+)^M$ even), however, modified iterations force a Π^2_n statement about F^N_γ and special modified iterations force a Σ^2_n statement about $F_{\pi(\gamma)}$. In what is going to follow we can therefore safely ignore the noncritical $\gamma < \kappa$. Moreover, it makes no difference how many critical γ we have to consider. Finally, the fact that we have to deal with tails of Σ^2_n and Π^2_n iterations rather than Σ^2_n and Π^2_n iterations themselves does not have any bearing on the method that we are going to use. For these reasons we choose to work in the following context: Assume we are living in some model which we call V for the remainder of this section and $F:(2^{\kappa^+})^{n-1} \to 2^{\kappa^+}$ Is a Lipschitz function. Now, in V define Σ^2_n and Π^2_n iterations which make a Σ^2_n and Π^2_n statement (respectively) true about this single Lipschitz function F in the usual way. We want to establish the $(n-1)$BFP for Σ^2_n and Π^2_n iterations in this context.

Towards this end suppose first that n is odd, i.e., $n = 2r + 1$ and suppose ${}^\Sigma Q_{\delta_1}$ (where $\delta_1 < \kappa^{++}$) is an initial piece of a Σ^2_n iteration. Denote $[0, \delta_1)$ by $I_{1,\Sigma}$ and $[0, 1 + \delta_1)$ by $I_{1,\Pi}$ and define $i_1 : I_{1,\Sigma} \to I_{1,\Pi}$ by $i_1(\zeta) = 1 + \zeta$ for $\zeta \in I_{1,\Sigma}$. Note that $0 \notin \operatorname{rng}(i_1)$, and for reasons that will become apparent below we call 0 the *new coordinate in* $I_{1,\Pi}$. Now define an initial piece of a Π^2_n iteration ${}^\Pi Q_{1+\delta_1}$ where we choose as the underlying partition the partition of $1 + \delta_1$ induced by i_1 and the partition of δ_1 for ${}^\Sigma Q_{\delta_1}$ and where we associate no terms with coordinate 0. If $\zeta \in I_{1,\Sigma}$ and a k-tuple of terms, say $(\tau^\zeta_1, \ldots, \tau^\zeta_k)$, appears at ζ and $k > 1$, then with coordinate $1 + \zeta \in I_{1,\Pi}$ we associate the k-tuple $((\hat{\tau}^1_\zeta)^{i_1}, \ldots, (\hat{\tau}^k_\zeta)^{i_1})$. Here $\hat{}$ refers to the iteration ${}^{\Sigma}Q_{\delta_1}$ and as usual for some τ in $\mathrm{Add}^{\kappa^{++}}(\delta_1, \kappa^+)$, τ^{i_1} is the term in $\mathrm{Add}^{\kappa^{++}}(1+\delta_1, \kappa^+)$ that is obtained by replacing all conditions q in τ by their image under the map induced by i_1. From here on we call τ^{i_1} the *shifted image of τ under* i_1. If a single term τ_ζ appears at coordinate $\zeta \in I_{1,\Sigma}$ then, assuming inductively that ${}^{\Pi}Q_{1+\zeta}$ has already been defined, we pick a canonical term $\tau^* \in V^{{}^{\Pi}Q_{1+\zeta}}$ such that, in $V^{{}^{\Pi}Q_{1+\zeta}}$

τ^* = the set that we add at the new coordinate in $I_{1,\Pi}$ (i.e., at 0) if certain *changing conditions* are satisfied in $V^{{}^{\Sigma}Q_\zeta}$ about $\hat{\tau}_\zeta$ and
$\tau^* = (\hat{\tau}_\zeta)^{i_1}$ otherwise.

We will explain below what these *changing conditions* are, and we will show that in fact for each $\zeta \in I_{1,\Sigma}$

i_1 induces an isomorphism of ${}^{\Sigma}Q_\zeta$ with ${}^{\Pi}Q_{1+\zeta}^{1+\zeta-\{0\}}$
which is a complete suborder of ${}^{\Pi}Q_{1+\zeta}$.

This shows in particular that $V^{{}^{\Sigma}Q_\zeta}$ can be regarded as contained in $V^{{}^{\Pi}Q_{1+\zeta}}$ and the clauses in the definition of τ^* make sense.

Now suppose ${}^{\Pi}Q_{1+\delta_1}$ has been extended to a Π_n^2 iteration ${}^{\Pi}Q_{1+\delta_1+\delta_2}$ (with $\delta_2 < \kappa^{++}$). Let $I_{2,\Pi} = [1+\delta_1, 1+\delta_1+\delta_2)$ and $I_{2,\Sigma} \overset{\text{def}}{=} [\delta_1, \delta_1+2+\delta_2)$ and define $i_2 : I_{1,\Pi} \cup I_{2,\Pi} \to I_{1,\Sigma} \cup I_{2,\Sigma}$ by

$$i_2(0) = \delta_1,$$
$$i_2(1+\zeta) = \zeta \quad (0 \le \zeta < \delta_1),$$
$$i_2(1+\delta_1+\zeta) = \delta_1 + 2 + \zeta \quad (0 \le \zeta < \delta_2).$$

Note that $i_2 \circ i_1 = \mathrm{id}_{I_{1,\Sigma}}$ and $\delta_1 + 1 \notin \mathrm{rng}(i_2)$. We call $\delta_1 + 1$ the *new coordinate* in $I_{2,\Sigma}$.

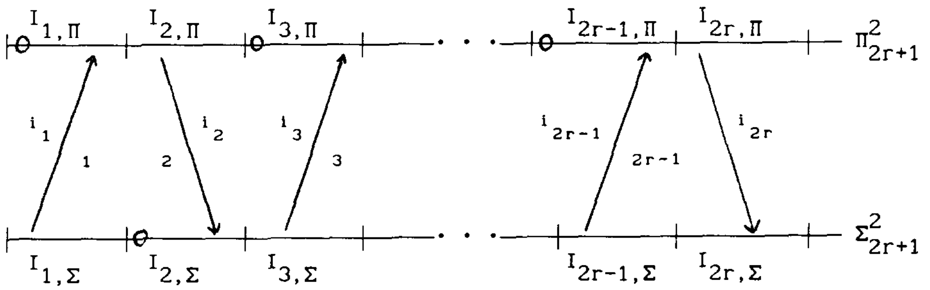
Now we define a Σ_n^2 iteration ${}^{\Sigma}Q_{\delta_1+2+\delta_2}$ extending ${}^{\Sigma}Q_{\delta_1}$ by fixing the partition of $I_{2,\Sigma}$ that is induced by i_2. We do not assign terms to coordinates $\delta_1 \in I_{2,\Sigma}$ and no terms are assigned to the new coordinate $\delta_1 + 1$ in $I_{2,\Sigma}$. If a k-tuple $(\tau_\zeta^1, \ldots, \tau_\zeta^k)$ appears at coordinate $\zeta \in I_{2,\Pi}$ then the tuple $((\hat{\tau}_\zeta^1)^{i_2}, \ldots, (\hat{\tau}_\zeta^k)^{i_2})$ consisting of the i_2-shifts of the terms in the tuple at coordinate $\zeta \in I_{2,\Pi}$ is associated with coordinate $i_2(\zeta)$ for $k \ne 2$. If a pair $(\tau_\zeta^1, \tau_\zeta^2)$ appears at coordinate $\zeta \in I_{2,\Pi}$ we associate with $i_2(\zeta)$ the pair $(\tau^*, (\hat{\tau}_\zeta^2)^{i_2})$ where $\tau^* \in V^{{}^{\Sigma}Q_{i_2(\zeta)}}$ is a canonical term such that in $V^{{}^{\Sigma}Q_{i_2(\zeta)}}$

τ^* = the set that gets added at coordinate $\delta_1 + 1$ if certain changing conditions are satisfied in $V^{{}^{\Pi}Q_\zeta}$ about $\hat{\tau}_\zeta^1$ and
$\tau^* = (\hat{\tau}_\zeta^1)^{i_2}$ otherwise.

Again we have to check that for each $\zeta \in I_{2,\Pi}$, i_2 induces an isomorphism of ${}^{\Pi}Q_\zeta$ with ${}^{\Sigma}Q_{i_2(\zeta)}^{i_2(\zeta)-\{\delta_1+1\}}$ which is a complete suborder of ${}^{\Sigma}Q_{i_2(\zeta)}$.

We continue in this fashion until we have defined $I_{1,\Sigma}, \ldots, I_{n-1,\Sigma}$ and $I_{1,\Pi}, \ldots, I_{n-1,\Pi}$ and embeddings $i_1, \ldots, i_{n-1}$. In the last step of the construction (i.e., in step $n = 2r$) we do not introduce a new coordinate to define $I_{2r,\Sigma}$ from $I_{2r,\Pi}$ and we shift all terms at coordinates in $I_{2r,\Pi}$ to get the terms for the corresponding coordinates in $I_{2r,\Sigma}$.

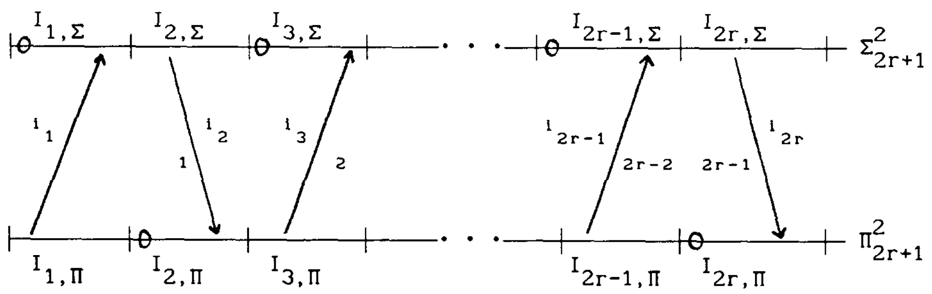
The schematic picture for this construction looks like this:



The numbers below the arrows stand for the arity of the tuple of terms whose first terms get changed at this stage of the construction. The symbol 0 indicates that we have a new coordinate in the interval where it occurs.

Here is the definition of the changing conditions: Suppose we are at stage $k \leqslant 2r - 1$ of the construction above and k is odd (even resp.) and the k-tuple $(\tau^1_\zeta, \ldots, \tau^k_\zeta)$ is assigned to coordinate $\zeta \in I_{k,\Sigma}$ ($\zeta \in I_{k,\Pi}$ resp.). Consider the tree $T_{\Sigma^2_{k+2}}$ ($T_{\Pi^2_{k+2}}$ resp.). Among the branches that end in 'kill' consider the *positive killing branches*, i.e., those killing branches whose last label is a nonnegative integer. Each such branch determines a certain combination of agreements and disagreements of $(\hat{\tau}^1_\zeta, \ldots, \hat{\tau}^k_\zeta)$ with earlier tuples. The changing conditions at stage k are satisfied if at least one of these combinations of agreements and disagreements is valid in $V^{^\Sigma Q_\zeta}$ ($V^{^\Pi Q_\zeta}$ resp.).

Now let me describe the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ where we start with an initial piece ${}^\Pi Q_{\delta_1}$ of a Π^2_{2r+1} iteration ($\delta_1 < \kappa^{++}$). Let $L_{1,\Pi} = [0, \delta_1)$ and $I_{1,\Sigma} = [0, 1 + \delta_1)$ and as before $i_1 \colon I_{1,\Pi} \hookrightarrow I_{1,\Sigma}$ be defined by $i_1(\zeta) = 1 + \zeta$ for $\zeta < \delta_1$. Now define a Σ^2_{2r+1} iteration ${}^\Sigma Q_{1+\delta_1}$ by choosing as the underlying partition of $I_{1,\Sigma}$ the one which is induced by i_1 and assign no terms to coordinate 0. If $\zeta < \delta_1$ and some tuple of terms is associated with coordinate $\zeta \in I_{1,\Pi}$ then with coordinate $i_1(\zeta)$ associate the tuple consisting of the i_1 shifts of terms in that tuple. Clearly i_1 induces an isomorphism ${}^\Pi Q_{\delta_1}$ with ${}^\Sigma Q^{\delta_1 - \{0\}}_{1+\delta_1}$ which is a complete suborder of ${}^\Sigma Q_{1+\delta_1}$. This follows from the fact that no term appearing at any coordinates in $I_{1,\Sigma}$ can 'see' the generic that we add at the new coordinate $0 \in I_{1,\Sigma}$. Recall also that in $T_{\Sigma^2_{2r+1}}$ the edge labeled -0 leads into a subtree which is identical with $T_{\Pi^2_{2r+1}}$. (This argument has already been used in the construction of the generic g from G above.) In the remaining steps of the construction we proceed similarly as in the first $n - 2$ steps of the construction for Σ^2_n/Π^2_n starting with an initial piece of a Σ^2_n iteration which we described above. Thus the schematic picture looks like this:



Again the numbers below the arrows indicate the arity of the tuples whose first terms gets changed if the changing conditions are satisfied where now the changing conditions for even (odd resp.) stage $k \in \{2, \dots, 2r\}$ are given by the positive killing branches in $T_{\Sigma^2_{k+1}}$ ($T_{\Pi^2_{k+1}}$ resp.). As before the symbol 0 indicates that we have a new coordinate in the interval where it occurs.

The constructions for Σ^2_n/Π^2_n where n is even are totally analogous to the ones presented above. Now we have to argue that this really works. Let us examine first the Π^2_n/Σ^2_n construction starting with an initial piece of a Π^2_n iteration and assume that $n = 2r+1$. We check at one stage after another that the construction works. For the sake of the argument suppose that we are at stage $k \leqslant n-1$ with k being odd and $\zeta \in I_{k,\Pi}$. We need to argue that

(3.3.11) $\quad {}^{\Sigma}Q_{i_k(\zeta)}^{I_{1,\Sigma} \cup \cdots \cup I_{k,\Sigma} \sim \{\text{new coordinate} \in I_{k,\Sigma}\}} \subseteq_c {}^{\Sigma}Q_{i_k(\zeta)}$

(3.3.12) $\quad i_k$ induces an isomorphism ${}^{\Pi}Q_{\zeta}$ with ${}^{\Sigma}Q_{i_k(\zeta)}^{I_{1,\Sigma} \cup \cdots \cup I_{k,\Sigma} \sim \{\text{new coordinate} \in I_{k,\Sigma}\}}$

where ${}^{\Sigma}Q_{i_k(\zeta)}$ (${}^{\Pi}Q_{\zeta}$ resp.) denotes the Σ^2_n (Π^2_n resp.) iteration up to coordinate $i_k(\zeta) \in I_{k,\Sigma}$ ($\zeta \in I_{k,\Pi}$ resp.).

We show this by induction on $\zeta \in I_{k,\Pi}$. The nontrivial case is that ζ is a successor, say, $\eta + 1$, where at coordinate $\eta \in I_{k,\Pi}$ we add a set that kills $F((\hat{\tau}^1_\eta, \dots, \hat{\tau}^{2r}_\eta))$ if the killing conditions are satisfied. First we handle (3.3.12). By induction hypothesis ${}^{\Pi}Q_\eta$ is isomorphic to a complete suborder ${}^{\Sigma}Q_{i_k(\eta)}$, therefore we can think of $V^{{}^{\Pi}Q_\eta}$ being contained in $V^{{}^{\Sigma}Q_{i_k(\eta)}}$. In order to prove (3.3.12) it will therefore suffice to show

(3.3.13) It cannot happen that we are on a killing branch of $T_{\Pi^2_n}$ in $V^{{}^{\Pi}Q_\eta}$ and on a saving branch of $T_{\Sigma^2_n}$ in $V^{{}^{\Sigma}Q_{i_k(\eta)}}$.

(3.3.14) It cannot happen that we are on a killing branch of $T_{\Sigma^2_n}$ in $V^{{}^{\Sigma}Q_{i_k(\eta)}}$ and on a saving branch of $T_{\Pi^2_n}$ in $V^{{}^{\Pi}Q_\eta}$.

Once this has been shown we can discard of (3.3.11): Let $q \in {}^{\Sigma}Q_{i_k(\zeta)}$ and $q' = q|^{\emptyset}(i_k(\zeta) \sim \{\text{the new coordinate in } I_{k,\Sigma}\})$ and $q'' \in \mathrm{Add}^{\kappa^{++}}(\zeta, \kappa^+)$ such that $(q'')^{i_k} = q'$. It follows from (3.3.14) that $q'' \in {}^{\Pi}Q_\zeta$. Thus $q' = (q'')^{i_k} \in {}^{\Sigma}Q_{i_k(\zeta)}$ by (3.3.13) and we have shown (3.3.11).

Similarly we can use analogues of (3.3.13) and (3.3.14) to argue that in the first $n-2$ steps of the construction for Σ^2_n/Π^2_n (n odd) starting with an initial piece of a Σ^2_n iteration things work out. Recall that in the last step of this construction we have to show that $i_{n-1}: {}^{\Pi}Q_\zeta \hookrightarrow {}^{\Sigma}Q_{i_{n-1}(\zeta)}$ is for each $\zeta \in I_{n-1}$ a complete embedding. In order to argue that $i_{n-1}(q) \in {}^{\Sigma}Q_{i_{n-1}(\zeta)}$ for $q \in {}^{\Pi}Q_\zeta$ we use the analogue of (3.3.13). In order to find for each $q \in {}^{\Sigma}Q_{i_{n-1}(\zeta)}$ an i_{n-1}-reduction in ${}^{\Pi}Q_\zeta$ we proceed by induction on $\zeta \in I_{n-1,\Pi}$. For the case where ζ is a limit ordinal with cofinality κ we use the method that was developed for the proof of 2.3.

Finally we can formulate analogues of (3.3.13) and (3.3.14) which guarantee that things work out for the Σ^2_n/Π^2_n constructions when n is even.

In order to establish the BFP for Σ^2_n/Π^2_n we thus have to prove (3.3.13) and (3.3.14).

Proof of (3.3.13) *and* (3.3.14) *for the Σ^2_n/Π^2_n construction*

The proof proceeds by induction on n. We shall see how the rather complicated definitions of the killing and changing conditions together with certain patterns in the structure of the associated trees allow us to go back and forth sufficiently many times between Σ and Π iterations.

We begin with the two cases Σ^2_2/Π^2_2 and Σ^2_3/Π^2_3. If we are given an initial piece of a Π^2_2 iteration ${}^{\Pi}Q_{\delta_1}$, then we can define an initial piece of a Σ^2_2 iteration ${}^{\Sigma}Q_{1+\delta_1}$ by shifting all the terms in ${}^{\Pi}Q_{\delta_1}$ so that they cannot see the Σ^2_2 witness that we add a coordinate 0 in ${}^{\Sigma}Q_{1+\delta_1}$. Clearly the two central facts hold in this case. If we start with an initial piece of a Σ^2_2 iteration ${}^{\Sigma}Q_{\delta_1}$ and define an initial piece of a Π^2_2 iteration ${}^{\Pi}Q_{\delta_1}$ by using the same parameters, then ${}^{\Sigma}Q_{\delta_1} \subseteq {}^{\Pi}Q_{\delta_1}$ (since the graph for Π^2_2 has no saving paths at all) and in fact this inclusion is complete.

Now we examine the case where we start with an initial segment of a Π^2_3 iteration ${}^{\Pi}Q_{\delta_1}$. In the first step we define an initial piece of a Σ^2_3 iteration ${}^{\Sigma}Q_{1+\delta_1}$ by shifting all the terms in ${}^{\Pi}Q_{\delta_1}$ so they cannot see the Σ^2_3 witness at coordinate 0 in ${}^{\Sigma}Q_{1+\delta_1}$. (3.3.13) and (3.3.14) clearly hold at this step. In the second step suppose we are at coordinate $\zeta \in I_{2,\Sigma}$ and on a killing path for Σ^2_3 in $V^{{}^{\Sigma}Q_\zeta}$ so that we have either 0 or -0, -1. If we have 0 in $V^{{}^{\Sigma}Q_\zeta}$ then there cannot be a 1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$ with a term that appears at a coordinate $\in I_{1,\Pi}$ since none of these terms can see the Σ^2_3 witness that we add at the new coordinate $0 \in I_{1,\Sigma}$. Furthermore there cannot be a 1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$ with a term that appears at a coordinate $\in I_{2,\Pi}$ since when going from $I_{2,\Sigma}$ to $I_{2,\Pi}$ we change all 1's in $I_{2,\Sigma}$ because we assumed we have 0 in $V^{{}^{\Sigma}Q_\zeta}$. Thus we have -1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$; i.e., we kill in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$. If we have -0, -1 in $V^{{}^{\Sigma}Q_\zeta}$ then we clearly must have -1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$; i.e., again we kill in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$. On the other hand suppose we kill at coordinate $i_2(\zeta)$ of $I_{2,\Pi}$, i.e., we have -1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$. In this case -0 in $V^{{}^{\Sigma}Q_\zeta}$ clearly implies that we also must have -1 in $V^{{}^{\Sigma}Q_\zeta}$, because any 1 would survive when going from $I_{2,\Sigma}$ to $I_{2,\Pi}$. Thus we kill in $V^{{}^{\Sigma}Q_\zeta}$.

The arguments for establishing (3.13) and (3.14) when one starts with an initial piece of a Σ^2_3 iteration are similar.

Now suppose $n \geq 4$ and we have already proved (3.13) and (3.14) for all constructions $\Sigma^2_{n'}/\Pi^2_{n'}$ with $n' < n$. We will restrict ourselves to looking at odd $n = 2r + 1$ ($r \geq 2$) and consider the case of $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ where we start with an initial piece of a Π^2_{2r+1} iteration. The arguments for the other cases are similar to the argument that we present here in detail.

First we want to argue that (3.3.13) and (3.3.14) are satisfied through the first $2r-2$ stages of the construction. For this we make the following observation: The subtree of $T_{\Sigma^2_{2r+1}}$ ($T_{\Pi^2_{2r+1}}$ resp.) which consists of all edges that are labeled by an integer of absolute value $\leq 2r-3$ is identical with $T_{\Sigma^2_{2r-1}}$ ($T_{\Pi^2_{2r-1}}$ resp.) except that all nodes in $T_{\Sigma^2_{2r-1}}$ ($T_{\Pi^2_{2r-1}}$ resp.) of the form Q^2_k ($\mathrm{Q} \in \{\forall, \exists\}$) have to be changed to Q^2_{k+2} and we must replace each save node by $T_{\Pi^2_3}$ where the labels 1, -1 get replaced by $2r-1$ and $-(2r-1)$ resp. and each kill node must be replaced by $T_{\Sigma^2_3}$ where the labels 0, -0, 1, -1 get replaced by $2r-2$, $-(2r-2)$, $2r-1$, $-(2r-1)$ resp. If we now apply the induction hypothesis about $\Sigma^2_{2r-1}/\Pi^2_{2r-1}$ together with this observation then we see that throughout the first $2r-2$ stages of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ we have the following:

> Any time we are on a branch in the subtree of $T_{\Pi^2_{2r+1}}$ mentioned above that ends in 'Σ^2_3' we cannot be on a branch in the subtree of $T_{\Sigma^2_{2r+1}}$ that ends in 'Π^2_3' and similarly if we interchange Π^2_{2r+1} and Σ^2_{2r+1}.

Then note that throughout the first $2r-2$ stages of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ all the terms in $2r-2$ and $2r-1$ tuples merely get shifted. Moreover, a Σ^2_3 iteration clearly does 'more killing' than a Π^2_3 iteration. Hence the two central facts hold throughout the first $2r-2$ stages of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$.

Now we consider the last two stages of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ where we start with an initial piece of a Π^2_{2r+1} iteration. First we show that if we kill on the Π^2_{2r+1} side, we cannot save on the Σ^2_{2r+1} side. Inspection of $T_{\Pi^2_{2r+1}}$ tells us that there are two cases for killing branches:

negative killing; i.e., the last edge in the branch is labeled $-(2r-1)$ and

positive killing; i.e., the last edge in the branch is labeled $2r-2$.

On the other hand there are two ways of saving in $T_{\Sigma^2_{2r+1}}$:

negative saving; i.e., the last two edges of the branch are labeled $-(2r-2)$, $2r-1$ and

positive saving; i.e., the last two edges of the path are labeled $2r-3$, $2r-1$.

It can never happen that we are on a negative killing branch for Π^2_{2r+1} (i.e., $-(2r-1)$) and on a saving path for Σ^2_{2r+1} (i.e., $2r-1$) since at stage $2r$ of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ starting with an initial piece of a Π^2_{2r+1} iteration, a $2r-1$ tuple gets altered only when we are on a positive killing branch of $T_{\Sigma^2_{2r+1}}$.

It cannot happen that we are on a positive killing branch for Π^2_{2r+1} (i.e., ends in $2r-2$) and on a negative saving branch for Σ^2_{2r+1} (i.e., the next-to last edge is labeled $-(2r-2)$). This is so because at stage $2r-1$ of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ (starting with an initial piece of a Π^2_{2r+1} iteration) a $2r-2$ tuple gets altered only if we are on positive killing branches of $T_{\Pi^2_{2r}}$. However, the positive killing branches in $T_{\Pi^2_{2r}}$ can be extended to save branches or negative killing branches in $T_{\Pi^2_{2r+1}}$.

Finally we consider positive killing in Π^2_{2r+1} and positive saving in Σ^2_{2r+1}. We observe that the positive killing branches in $T_{\Pi^2_{2r+1}}$ are just all the killing branches in $T_{\Pi^2_{2r-1}}$ extended by one edge which is labeled $2r-2$ and the positive saving branches $T_{\Sigma^2_{2r+1}}$ are just all saving branches in $T_{\Sigma^2_{2r-1}}$ extended by one edge labeled $2r-1$. Now we can apply our induction hypothesis about the $\Sigma^2_{2r-1}/\Pi^2_{2r-1}$ construction which tells us that this constellation can never arise.

Next we show that in the last two stages of the construction $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ (starting with an initial piece of Π^2_{2r+1} iteration) we cannot kill on the Σ^2_{2r+1} side and save on the Π^2_{2r+1} side.

We have to check three cases here:

Negative killing branch for Σ^2_{2r+1} (i.e., ending in $-(2r-1)$) versus *saving branch for* Π^2_{2r+1} (i.e., ending in $2r-1$) cannot occur since the first term in a $(2r-1)$-tuple that gets altered at stage $2r$ of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ (starting with an initial piece of a Π^2_{2r+1} iteration) will then denote the set at the new coordinate $\in I_{2r,\Pi}$. But no term at a coordinate $\in I_{1,\Pi} \cup \cdots \cup I_{2r,\Pi}$ can 'see' this set.

Next we consider a *positive killing branch for* Σ^2_{2r+1} (i.e., ending in $2r-2$) versus a *negative saving path in* Π^2_{2r+1} (i.e., ending in $-(2r-2)$, $2r-1$). In this situation the $2r-2$ agreement on the Σ^2_{2r+1} side had to occur with a $(2r-2)$-tuple whose first term denotes the set that we add at the new coordinate $\in I_{2r-1,\Sigma}$. Therefore the $2r-1$ agreement in Π^2_{2r+1} had to occur at a coordinate $\in I_{2r,\Pi}$. Then this must come from a $2r-1$ agreement in Σ^2_{2r+1} at a coordinate $\in I_{2r,\Sigma}$. However, we assumed we were on a positive killing branch in Σ^2_{2r+1}. Thus any such $2r-1$ agreement would get destroyed at stage $2r$ of the construction for $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ when going from $I_{2r,\Sigma}$ to $I_{2r,\Pi}$—a contradiction.

Finally we consider the case of a *positive killing branch in* Σ^2_{2r+1} (ending in $2r-2$) versus a *positive saving branch in* Π^2_{2r+1} (i.e., ending in $2r-3$, $2r-1$). We prove the following:

Claim. *If we are on a positive killing branch for* Σ^2_{2r+1} *and a positive saving branch for* Π^2_{2r+1}*, then there cannot be a* $2r-1$ *agreement on the* Π^2_{2r+1} *side with a* $2r-1$ *tuple that appears at a coordinate* $\in I_{1,\Pi} \cup \cdots \cup I_{2r,\Pi}$.

It follows from the claim together with the fact that the positive saving branches in $T_{\Pi^2_{2r+1}}$ end with $2r-1$ that the case of positive killing in Σ^2_{2r+1} versus positive saving in Π^2_{2r+1} cannot arise during the $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ construction (starting with an initial piece of a Π^2_{2r+1} iteration).

Proof of the Claim. First we note that the $2r-2$ agreement on the Σ^2_{2r+1} side cannot occur with a $2r-2$ tuple that appears at a coordinate $\in I_{2r,\Sigma}$ because in that case any $2r-1$ agreement on the Σ^2_{2r+1} side had to occur at a coordinate $\in I_{2r,\Sigma}$. Since we are on a positive killing branch for Σ^2_{2r+1}, the first term in any such $2r-1$ tuple will be altered when going from $I_{2r,\Sigma}$ to $I_{2r,\Pi}$. This will result in $-(2r-1)$ on the Π^2_{2r+1} side—a contradiction.

Now there are 2 possibilities for a positive killing branch in Σ^2_{2r+1}. If its next-to-the-last edge is labeled $-(2r-3)$, then the first term in any $2r-3$ agreement on the Π^2_{2r+1} side has to agree with the set that we add at the new coordinate $\in I_{2r-2,\Pi}$. From this it follows that any $2r-2$ agreement on the Σ^2_{2r+1} has to occur at a coordinate $\in I_{2r-1,\Sigma} \cup I_{2r,\Sigma}$ because none of the terms appearing at coordinates $\in I_{1,\Sigma} \cup \cdots \cup I_{2r-2,\Sigma}$ can see the set that we add at the new coordinate $I_{2r-2,\Pi}$. By the remark at the beginning of the proof of the claim, the $2r-2$ agreement on the Σ^2_{2r+1} side must therefore occur at a coordinate $\in I_{2r-1,\Sigma}$. Now we observe that the positive saving branches in Π^2_{2r+1} are exactly the extensions of the positive killing branches in Π^2_{2r} by one edge that is labeled $2r-1$. Therefore the first term in any $2r-2$ tuple that gives a $2r-2$ agreement on the Σ^2_{2r+1} side has to agree with the set that we add at the new coordinate $\in I_{2r-1,\Sigma}$. This implies that any $2r-1$ agreement on the Σ^2_{2r+1} side has to occur at a coordinate $\in I_{2r,\Sigma}$. However, this is impossible by the remark above.

So we have shown that the positive killing branch in Σ^2_{2r+1} cannot end with $-(2r-3)$, $2r-2$. Thus it must end in $2r-4$, $2r-2$. Inspection of the trees for Σ^2_{2r+1} and Σ^2_{2r} shows that we are on a positive saving branch for Σ^2_{2r} in this case.

Inspection of the trees for Π^2_{2r+1} and Π^2_{2r} shows that the positive saving branches in Π^2_{2r+1} are obtained from the positive killing branches in Π^2_{2r} by extending them with an edge labeled $2r-1$. We can assume by induction that:

> If we are on a positive killing branch for Π^2_{2r} and a positive saving branch for Σ^2_{2r}, then there cannot be a $2r-2$ agreement on the Σ^2_{2r} side with a $2r-2$ tuple that appears at a coordinate $\in I_{1,\Sigma} \cup \cdots \cup I_{2r-1,\Sigma}$ in the construction for Σ^2_{2r}/Π^2_{2r} (starting with an initial piece of a Π^2_{2r} iteration).

It follows that the $2r-2$ agreement on the Σ^2_{2r+1} side in the $\Sigma^2_{2r+1}/\Pi^2_{2r+1}$ construction (starting with an initial piece of a Π^2_{2r+1} iteration) has to occur at a coordinate in $I_{2r,\Sigma}$. But we already know this is impossible. □ Claim

This finishes the proof of (3.3.13) and (3.3.14), and the proof of the $(n-1)$ BFP for Σ^2_n/Π^2_n is complete.

4. Σ^m_n/Π^m_n $(m \geqslant 3, n \geqslant 2)$

The main ideas for establishing the consistency of $\sigma^m_n > \pi^m_n$ $(m \geqslant 3, n \geqslant 2)$ have already been developed in the Σ^2_n/Π^2_n case. Let us describe the $m+2$ step iteration that we use at stage λ (where λ is Mahlo) in order to make λ Σ^m_n describable in $V^{P_{\lambda+1}}$.

Suppose that G_λ is P_λ generic over $V=L$ and in $V[G_\lambda]$ λ is inaccessible and $\lambda^{+l}=(\lambda^{+l})^L$ for $l\geq 1$ and $\mathrm{GCH}^{\geq\lambda}$ holds. In the first step we add a sequence $(F_\gamma\colon \gamma<\lambda^+)$ where each F_γ is a Lipschitz function $(2^{\lambda^{+(m-1)}})^{n-1}\to 2^{\lambda^{+(m-1)}}$. Thus the forcing Q^1_λ is a λ^+ product (with full support) of copies of the forcing notion P_F where conditions in P_F are functions f such that

$$\begin{aligned}&\mathrm{dom}(f) \text{ is a subtree of } (2^{\lambda^{+(m-1)}})^{n-1} \text{ of size } <\lambda^{+(m-1)}\\ &\wedge \forall(s_1,\dots,s_{n-1})\in\mathrm{dom}(f)\ [\exists\alpha<\lambda^{+(m-1)}[\alpha\geq\mathrm{dom}(s_1)\\ &\wedge f((s_1,\dots,s_n))\in 2^{\alpha+1}\wedge f((s_1,\dots,s_{n-1}))(\alpha)=0]\\ &\wedge \forall\zeta\,[f((s_1,\dots,s_{n-1}))(\zeta)=1\Rightarrow \mathrm{cf}(\zeta)=\lambda^{+(m-2)}]\\ &\wedge \forall(t_1,\dots,t_{n-1})\in\mathrm{dom}(f)\ [(t_1,\dots,t_{n-1}) \text{ extends } (s_1,\dots,s_{n-1})\\ &\Rightarrow f((t_1,\dots,t_{n-1})) \text{ extends } f((s_1,\dots,s_{n-1}))]]\end{aligned}$$

and for $f,g\in P_F$ we let $f\leq g$ iff $f\supseteq g$. Clearly $|Q^1_\lambda|=\lambda^{+(m-1)}$ and Q^1_λ is $<\lambda^{+(m-2)}$ closed. Hence if $(F_\gamma\colon\gamma<\lambda^+)$ is Q^1_λ generic in $V[G_\lambda,\vec{F}_\gamma]$ we still have that λ is inaccessible, $\lambda^{+l}=(\lambda^{+l})^L$ for $l\geq 1$ and $\mathrm{GCH}^{\geq\lambda}$ holds.

In the second step we will do an iteration Q^2_λ which will make a Σ^m_n fact true about F_γ for γ even and its negation for γ odd. Q^2_λ will be a certain suborder of $\mathrm{Add}(\lambda^{+m},\lambda^{+(m-1)})$. We partition λ^{+m} into cofinal pieces $(A^{k,\gamma}\colon 1\leq k\leq n-1,\ \gamma<\lambda^+)$ and A^0 with $\lambda^+\subseteq A^0$. For each $k\in\{1,\dots,n-1\}$ and each $\gamma<\lambda^+$ we fix a complete enumeration $((\tau^{1,\gamma}_\zeta,\dots,\tau^{k,\gamma}_\zeta)\colon\zeta\in A^{k,\gamma})$ of k-tupels of nice $\mathrm{Add}(\lambda^{+m},\lambda^{+(m-1)})$ names for subsets of $\lambda^{+(m-1)}$. (Note that this is possible since $\mathrm{Add}(\lambda^{+m},\lambda^{+(m-1)})$ is λ^{+m} c.c. and has size λ^{+m}.) The poset Q^2_λ will add a subset of $\lambda^{+(m-1)}$ at each coordinate in $\lambda^{+m}\sim\bigcup_{\gamma<\lambda^+}A^{n-1,\gamma}$. At a coordinate $\alpha\in A^{n-1,\gamma}$ (for some $\gamma<\lambda^+$) Q^2_λ will add a club set $\subseteq\lambda^{+(m-1)}$ that is disjoint from $F_\gamma((\hat\tau^{1,\gamma}_\alpha,\dots,\hat\tau^{n-1,\gamma}_\alpha))$ if certain killing conditions are met. If these killing conditions are not satisfied we just force with the trivial poset, i.e., we save $F_\gamma((\hat\tau^{1,\gamma}_\alpha,\dots,\hat\tau^{n-1,\gamma}_\alpha))$.

The killing conditions for $\alpha\in A^{n-1,\gamma}$ were γ is even (i.e., the killing conditions for Σ^m_n) are given by $T_{\Sigma^m_n}$. Similarly $T_{\Pi^m_n}$ tells us whether we kill at some $\alpha\in A^{n-1,\gamma}$ where γ is odd. Now the tree for Σ^m_n (Π^m_n resp.) looks exactly like the tree for Σ^2_n (Π^2_n resp.) except that the nodes are labeled Σ^m_k and Π^m_k instead of Σ^2_k and Π^2_k. Clearly Q^2_λ is $<\lambda^{+(m-2)}$ closed (because of the cofinality restriction in the definition of conditions in Q^1_λ) and λ^{+m} c.c. (since compatibility in Q^2_λ agrees with compatibility in $\mathrm{Add}(\lambda^{+m},\lambda^{+(m-1)})$).

An analogous proof as in the Σ^2_n/Π^2_n case shows that Q^2_λ is $<\lambda^{+(m-1)}$ Baire. In particular this implies that for each $\gamma<\lambda^+$

$$\Vdash^{V[G_\lambda,\vec{F}_\gamma]}_{Q}\mathrm{dom}(F_\gamma)=(2^{\lambda^{+(m-1)}})^{n-1}.$$

Moreover the analogue of 2.5 can be proved for Q^2_λ; hence after forcing with Q^2_λ, $F_\gamma((X_1,\dots,X_{n-1}))$ (for $\gamma<\lambda^+$, $X_1,\dots,X_n\subseteq\lambda^{+(m-1)}$) will be stationary unless Q^2_λ explicitly killed it. Therefore, if G is Q^2_λ generic over $V[G_\lambda,\vec{F}_\gamma]$ we have in $V[G_\lambda,\vec{F}_\gamma,G]$ for odd $\gamma<\kappa^+$:

$$\forall X_1\subseteq\lambda^{+(m-1)}\,\exists X_2\subseteq\lambda^{+(m-1)}\cdots\mathrm{Q}X_{n-1}\subseteq\lambda^{+(m-1)}\ \psi(F_\gamma((X_1,\dots,X_{n-1})))$$

where $Q = \exists$ ($Q = \forall$ resp.) and ψ says $F_\gamma((X_1, \ldots, X_{n-1}))$ is stationary (nonstationary resp.) in $\lambda^{+(m-1)}$ for odd n (even n resp.). Clearly this is $\Pi^m_n(F_\gamma)$. For even $\gamma < \kappa^+$ the negation of this statement will hold about F_γ, i.e., a $\Sigma^m_n(F_\gamma)$ fact.

For each $\gamma < \lambda^+$ we can find a code $\bar{F}_\gamma \subseteq \lambda^{+(m-1)}$ for F_γ in $V[G_\lambda, \vec{F}_\gamma, G]$. This uses the fact that

$$(2^{<\lambda^{+(m-1)}})^{V[G_\lambda, \vec{F}_\gamma, G]} = (2^{<\lambda^{+(m-1)}})^{V[G_\lambda]}$$

and $V[G_\lambda] = L[G_\lambda]$ where $G_\lambda \subseteq P_\lambda \subseteq L_\lambda$. Hence we can use the canonical well-ordering $<_{L[G_\lambda]}$ on $2^{<\lambda^{+(m-1)}}$ to do this coding.

The posets $Q^3_\lambda, \ldots, Q^{3+m-2}_\lambda$ will generically code each $\bar{F}_\gamma \subseteq \lambda^{+(m-1)}$ down to a subset $S_\gamma \subseteq \lambda$. This is done in exactly the same way as in the σ^m_1/π^m_1 proof (cf. [2]). Finally in the last step we add a sequence $(C_\gamma\colon \gamma < \lambda^+)$ where each $C_\gamma \subseteq \lambda$ is club and

$$C_\gamma \cap \{\mu < \lambda\colon \mu \text{ inaccessible } \wedge\ V_\mu \Vdash \phi^{\Sigma^m_n}(S_\gamma \cap V_\mu, G_\lambda \cap V_\mu, \lambda \cap \mu)\} = \emptyset.$$

Here $\phi^{\Sigma^m_n}$ is the analogue of $\phi^{\Sigma^2_n}$ from Section 2 for Σ^m_n. Now we can proceed as outlined in Section 1 and prove

$$\Vdash_{P_{\kappa+1}} \text{"there are no } \Sigma^m_n \text{ indescribables} \leqslant \kappa, \quad \kappa \text{ is } \Pi^m_n \text{ indescribable, } \kappa' \text{ is } \Sigma^m_n \text{ indescribable".}$$

The hard part of the proof of $\Vdash_{P_{\kappa+1}}$ "κ is Π^m_n indescribable" is again to show (in the notation of Section 3) that

$$N[G_\kappa, {}^N\vec{F}_\gamma, g] \text{ is } \Sigma^m_{n-1} \text{ correct for } \kappa \text{ in } V[G_\kappa, \vec{F}_\gamma, G].$$

The strategy for this is the same as in the Σ^2_n/Π^2_n case; i.e., the key point is that Σ^m_n/Π^m_n has the $(n-1)$ Back and Forth Property which is proved by the same arguments as in the Σ^2_n/Π^2_n case.

5. Oracles—the final word on indescribability

In order to state the final theorem we introduce the notion of an oracle. An oracle is simply a subset of ω that codes a function with domain $\{(m, n)\colon m \geqslant 2, n \geqslant 1\}$ that takes values in $\{0, 1\}$.

The final theorem is

Theorem 5.1 (ZFC). *Assuming the existence of Σ^m_n indescribables for all m and n and given any oracle $\mathscr{F}$, there is a poset $P_{\mathscr{F}} \in L[\mathscr{F}]$ such that* GCH *holds in $(L[\mathscr{F}])^{P_{\mathscr{F}}}$ and*

$$\Vdash^{L[\mathscr{F}]}_{P_{\mathscr{F}}} \begin{cases} \sigma^m_n < \pi^m_n & \text{if } \mathscr{F}(m, n) = 0, \\ \sigma^m_n > \pi^m_n & \text{if } \mathscr{F}(m, n) = 1. \end{cases} \tag{5.2}$$ □

Before defining $P_{\mathscr{F}}$ (for a given oracle function $\mathscr{F}$) we make some observations about small forcing and indescribability. In [2] it was shown that a forcing of size $<\kappa$ cannot destroy the Σ_n^m indescribability of κ. The same statement is true about a Π_n^m indescribable cardinal κ and the proof is analogous to the Σ_n^m case. However, the characterization of Π indescribability in [2] makes it possible to give an even easier proof.

To complete the picture we show (cf. Corollary 5.8) that no poset can create new Σ_n^m or Π_n^m indescribable cardinals (for any $m, n \geq 1$) that are larger than the cardinality of the forcing. First we prove:

Lemma 5.3 (ZFC). *Suppose that κ is inaccessible and P is a notion of forcing with $|P| < \kappa$. Let G be a P generic. Then, in $V[G]$ for any $X \in V_{\kappa+1}$,*

(5.4) $\quad X \in V \;\Leftrightarrow\; X \subseteq V \wedge \forall s\,(s \in V \Rightarrow X \cap s \in V)$,

and for any $\mathscr{X} \in V_{\kappa+m}$ $(m \geq 2)$

(5.5) $\quad \mathscr{X} \in V \;\Leftrightarrow\; \mathscr{X} \subseteq V \wedge \forall \mathscr{T}\,[\mathscr{T} \in V \wedge |\mathscr{T}| \leq \kappa \cdot\!\Rightarrow\!\cdot\, \mathscr{X} \cap \mathscr{T} \in V]$.

Here s ranges over V_κ and $\mathscr{T}$ over $V_{\kappa+m}$ (of $V[G]$).

Proof. To prove the nontrivial direction of (5.4) assume towards a contradiction that for some condition $p^* \in G$ and some $\mathring{X} \in V^P$ we have

$$p^* \Vdash_P \text{“}\mathring{X} \subseteq (V)_\kappa \wedge \forall s\,[s \in V \Rightarrow \mathring{X} \cap s \in V] \wedge \mathring{X} \notin V\text{”}.$$

In V, pick a well-ordering of V_κ of order type κ and let seg_α denote the segment of the first α-many elements ($\alpha < \kappa$). We can (in V) for each $\alpha < \kappa$ pick $p_\alpha \leq p^*$ and $x_\alpha \in (V)_\kappa$ with

$$p_\alpha \Vdash \mathring{X} \cap \mathrm{seg}_\alpha = x_\alpha.$$

$|P| < \kappa$ implies that there is some $p \in P$ with $p = p_\alpha$ for cofinally many α. Then let

$$\check{X} = \bigcup_{p_\alpha = p} x_\alpha.$$

Clearly $p \Vdash \check{X} = \mathring{X}$, contradicting $p \Vdash \mathring{X} \notin V$.

To prove the nontrivial direction in (5.5) we assume (without loss of generality) $m = 2$ and suppose towards a contradiction that for some $p^* \in G$ and $\mathring{\mathscr{X}}$ in V^P

$$p^* \Vdash \text{“}\mathring{\mathscr{X}} \subseteq (V)_{\kappa+1} \wedge \forall \mathscr{T}\,[\mathscr{T} \in V \wedge |\mathscr{T}| \leq \kappa \cdot\!\Rightarrow\!\cdot\, \mathring{\mathscr{X}} \cap \mathscr{T} \in V] \wedge \mathring{\mathscr{X}} \notin V\text{”}.$$

Now we

Claim. $p^* \Vdash \exists \mathscr{Y} \subseteq \mathring{\mathscr{X}}\,[|\mathscr{Y}| = \kappa \wedge \forall \mathscr{Z}\,[\mathscr{Y} \subseteq \mathscr{Z} \subseteq \mathring{\mathscr{X}} \Rightarrow \mathscr{Z} \notin V]]$.

Proof of the Claim. Let $B \overset{\text{def}}{=} \text{r.o.}(P)$ and H be B generic over V with $p^* \in H$. Since $|B| < \kappa$ we can find $\mathring{\mathscr{Y}} \in V^B$ such that in $V[H]$, $|\mathring{\mathscr{Y}}^H| = \kappa$ and $\mathring{\mathscr{Y}}^H \subseteq \mathring{\mathscr{X}}^H$ and

$$\{\|\check{X} \in \mathring{\mathscr{X}}\|^B : X \in \mathring{\mathscr{X}}^H \cap (V)_{\kappa+1}\} = \{\|\check{X} \in \mathring{\mathscr{X}}\|^B : X \in \mathring{\mathscr{Y}}^H \cap (V)_{\kappa+1}\}.$$

Then $\mathring{\mathscr{X}}^H \notin V$ implies

$$\text{(5.6)} \qquad \neg \exists b \in H\, \forall X \in \mathring{\mathscr{Y}}^H \cap (V)_{\kappa+1}\; b \leqslant \|\check{X} \in \mathring{\mathscr{X}}\|^B.$$

Now suppose $\mathring{\mathscr{Z}} \in V^B$ with $\mathring{\mathscr{Y}}^H \subseteq \mathring{\mathscr{Z}}^H \subseteq \mathring{\mathscr{X}}^H$; it follows that $\mathring{\mathscr{Z}}^H \notin V$. Otherwise we can pick $\mathscr{Z} \in V$ with $\|\check{\mathscr{Z}} = \mathring{\mathscr{Z}}\|^B \in H$ and we get for all $X \in \mathring{\mathscr{Y}}^H \cap (V)_{\kappa+1}$

$$\begin{aligned} \|\check{X} \in \mathring{\mathscr{X}}\|^B &\geqslant \|\check{X} \in \mathring{\mathscr{Z}}\|^B \cdot \|\mathring{\mathscr{Z}} \subseteq \mathring{\mathscr{X}}\|^B \\ &\geqslant \|\check{X} \in \check{\mathscr{Z}}\|^B \cdot \|\check{\mathscr{Z}} = \mathring{\mathscr{Z}}\|^B \cdot \|\mathring{\mathscr{Z}} \subseteq \mathring{\mathscr{X}}\|^B \\ &= \|\check{\mathscr{Z}} = \mathring{\mathscr{Z}}\|^B \cdot \|\mathring{\mathscr{Z}} \subseteq \mathring{\mathscr{X}}\|^B \in H, \end{aligned}$$

since for $X \in \mathring{\mathscr{Y}}^H \cap (V)_{\kappa+1}$ we clearly have $\|\check{X} \in \check{\mathscr{Z}}\| = 1$ because $\mathring{\mathscr{Y}}^H \subseteq \mathring{\mathscr{Z}}^H = \mathscr{Z}$. But this contradicts (5.6). Hence $\mathring{\mathscr{Y}}^H$ works and the claim is proved. □ Claim

By the claim we can fix $\mathring{\mathscr{Y}} \in V^P$ with

$$p^* \Vdash \text{“}\mathring{\mathscr{Y}} \subseteq \mathring{\mathscr{X}} \wedge |\mathring{\mathscr{Y}}| = \kappa \wedge \forall \mathscr{Z}\, [\mathring{\mathscr{Y}} \subseteq \mathscr{Z} \subseteq \mathring{\mathscr{X}} \Rightarrow \mathscr{Z} \neq V]\text{”}.$$

Let $\mathring{f} \in V^P$ such that $p^* \Vdash \mathring{f} : \kappa \xrightarrow[\text{onto}]{1:1} \mathring{\mathscr{Y}}$. Then define (in V):

$$\mathscr{Y} \overset{\text{def}}{=} \{X \in (V)_{\kappa+1} : \exists p \leqslant p^*\, \exists \alpha < \kappa\, p \Vdash X = \mathring{f}(\alpha)\}.$$

Clearly $|\mathscr{Y}| = \kappa$ and $p^* \Vdash \mathring{\mathscr{Y}} \subseteq \check{\mathscr{Y}}$. Now (in V) well-order $\mathscr{Y}$ in order type κ and for $\alpha < \kappa$ denote by seg_α the segment of the first α elements. Note that for $\alpha < \kappa$

$$p^* \Vdash \mathring{\mathscr{X}} \cap \text{seg}_\alpha \in V.$$

Hence (in V) we can find for each $\alpha < \kappa$ a condition $p_\alpha \leqslant p^*$ and $\mathscr{Y}_\alpha$ with

$$p_\alpha \Vdash \mathring{\mathscr{X}} \cap \text{seg}_\alpha = \mathscr{Y}_\alpha.$$

Since $|P| < \kappa$ there must be some $p \leqslant p^*$ with $p = p_\alpha$ for cofinally many α. Then

$$p \Vdash \text{“}\mathring{\mathscr{X}} \cap \mathscr{Y} = \bigcup_{p_\alpha = p} \mathscr{Y}_\alpha \in V\text{”}$$

contradicting

$$p^* \Vdash \mathring{\mathscr{Y}} \subseteq \mathring{\mathscr{X}} \cap \mathscr{Y} \subseteq \mathring{\mathscr{X}}. \qquad \square$$

Lemma 5.7 (ZFC). *If κ, P, G are as in* 5.3 *then in $V[G]$ for any $\mathscr{X} \in V_{\kappa+m}$ (where $m \geqslant 1$) the formula “$\mathscr{X} \in V$” is $\Sigma_0^m(\mathscr{X}, (V)_\kappa)$ over V_κ.*

Proof. We use induction on $m \geqslant 1$. For $m = 1$, (5.4) implies (s ranges over V_κ)

$$\mathscr{X} \in V \quad \text{iff} \quad \mathscr{X} \subseteq (V)_\kappa \wedge \forall s\, [s \in (V)_\kappa \Rightarrow \mathscr{X} \cap s \in (V)_\kappa].$$

Clearly this is $\Sigma_0^1(\mathscr{X}, (V)_\kappa)$. Now suppose $m \geq 1$ and $\mathscr{X} \in V_{\kappa+m+1}$; then by (5.5) (where $\mathscr{T}$ ranges over $V_{\kappa+m+1}$)

$$\mathscr{X} \in V \quad \text{iff} \quad \mathscr{X} \subseteq (V)_{\kappa+m} \wedge \forall \mathscr{T}\, [\mathscr{T} \in V, |\mathscr{T}| \leq \kappa \Rightarrow \mathscr{X} \cap \mathscr{T} \in V].$$

By the induction hypothesis and because any $\mathscr{T} \subseteq V_{\kappa+m}$ of cardinality $\leq\kappa$ can be coded by some element of $V_{\kappa+m}$, the whole formula is $\Sigma_0^{m+1}(\mathscr{X}, (V)_\kappa)$. □

Corollary 5.8 (ZFC). *If κ is inaccessible and P a poset of size $<\kappa$ and G is P generic, then for $m, n \geq 1$*

$$(\kappa \textit{ is } \Sigma_n^m \ (\Pi_n^m \textit{ resp.}) \textit{ indescribable})^{V[G]}$$

implies

$$(\kappa \textit{ is } \Sigma_n^m \ (\Pi_n^m \textit{ resp.}) \textit{ indescribable})^{V}.$$

Proof. If ϕ is Σ_n^m (Π_n^m resp.) then in $V[G]$ for $A \in (V)_{\kappa+1}$, $(\phi(A))^V$ is $\Sigma_n^m(A, (V)_\kappa)$ ($\Pi_n^m(A, (V)_\kappa)$ resp.) over V_κ uniformly for all inaccessible $\kappa > |P|$. Note that we are allowed to use $(V)_\kappa$ as a parameter in $V[G]$ since $(V)_\kappa \in (V[G])_{\kappa+1}$. □

We are now turning to the proof of 5.1. Suppose $\mathscr{F}$ is an oracle and we have Σ_n^m indescribables for all m, n. We know that in $L[\mathscr{F}]$ the following picture holds for $m \geq 2$, $n \geq 1$: (cf. [5])

$$(5.9) \qquad \cdots < {}^{L[\mathscr{F}]}\sigma_n^m < {}^{L[\mathscr{F}]}\pi_n^m < {}^{L[\mathscr{F}]}\sigma_{n+1}^m < {}^{L[\mathscr{F}]}\pi_{n+1}^m < \cdots.$$

For the sake of completeness we give a proof of this fact.

Proof of (5.9). Fix $m \geq 2$ and $n \geq 1$. We work in $L[\mathscr{F}]$. Let κ be the least Π_n^m indescribable. The proof strategy is to find a Π_n^m statement $\Phi(A, \mathscr{F}, \kappa)$ with $A \subseteq V_\kappa$ such that $V_\kappa \Vdash \Phi(A, \mathscr{F}, \kappa)$ and any inaccessible λ to which Φ reflects is Σ_n^m indescribable. Φ can be found as follows: We know that κ being the least Π_n^m indescribable is Σ_n^m describable. We fix some $A \subseteq V_\kappa$ and a Σ_n^m formula $\Psi(A)$ such that $V_\kappa \models \Psi(A)$ and $\Psi(A)$ does not reflect to any inaccessible $\lambda < \kappa$ and such that the witness in the Σ_n^m formula Ψ is least in the canonical well-ordering $<_{L[\mathscr{F}]}$ with the property that it is a witness for a Σ_n^m formula Ψ' in a parameter A' as above. We pick a sufficiently large finite fragment T of ZF + $V = L[\mathscr{F}]$ such that any transitive model M of T with $\mathscr{F} \in M$ is of the form $L_\alpha[\mathscr{F}]$ for some α. Then we take $\Phi(A, \mathscr{F}, \kappa)$ to be the formula

$$\begin{aligned}\forall \mathscr{M}\, [&\mathscr{M} \text{ trans.}, \mathscr{M} \models T, |\mathscr{M}| = |V_{\kappa+m-1}|, \mathscr{M}\ \Sigma_{n-1}^m \text{ correct for } \kappa,\\ &\mathscr{M} \models \text{“}\kappa \text{ is not } \Sigma_n^m \text{ indescribable”} \cdot\!\!\Rightarrow\!\!\cdot\ \mathscr{M} \models \text{“}V_\kappa \models \Psi(A)\text{”}].\end{aligned}$$

Φ is Π_n^m over V_κ, and by the choice of T we get $V_\kappa \models \Phi(A, \mathscr{F}, \kappa)$. If $\lambda < \kappa$ is inaccessible and $V_\lambda \models \Phi(A \cap V_\lambda, \mathscr{F}, \lambda)$, then λ must be Σ_n^m indescribable because we cannot have $V_\lambda \models \Psi(A \cap V_\lambda)$ by our choice of $\Psi(A)$. Thus Φ has the properties that we want. $\square$ (5.9)

Actually the proof that we just gave works for a large class of inner models. The key point is that the inner model under consideration (or at least its truncation up to the first measurable) must have a certain 'good' well-ordering.

We now resume the proof of 5.1. Working in $L[\mathcal{F}]$ we define for $m \geq 2$ and $n \geq 1$ the poset $P_{\mathcal{F}}^{m,n}$ to be the trivial poset if $\mathcal{F}(m, n) = 0$. If $\mathcal{F}(m, n) = 1$ then we use the exact same definition that we used in the Σ_n^m/Π_n^m case (with $\kappa = {}^{L[\mathcal{F}]}\pi_n^m$ and κ' = the least Σ_n^m indescribable cardinal in the sense of $L[\mathcal{F}]$ above κ) except that we replace L by $L[\mathcal{F}]$ and we do something only at Mahlo stages $\geq {}^{L[\mathcal{F}]}\sigma_n^m$. Then we let

$$P_{\mathcal{F}} \stackrel{\text{def}}{=} \prod_{m\geq 2, n\geq 1} P_{\mathcal{F}}^{m,n}.$$

We must show that (5.2) holds. So fix $m' \geq 2$ and $n' \geq 1$. Note that $P_{\mathcal{F}} \approx P_1 \times P_2 \times P_3$ where

$$P_1 \stackrel{\text{def}}{=} \prod_{\substack{m<m' \text{ or} \\ (m=m' \wedge n<n')}} P_{\mathcal{F}}^{m,n}, \qquad P_2 \stackrel{\text{def}}{=} P_{\mathcal{F}}^{m',n'}, \qquad P_3 = \prod_{\substack{m>m' \text{ or} \\ (m=m' \wedge n>n')}} P_{\mathcal{F}}^{m,n}.$$

First assume that $\mathcal{F}(m', n') = 1$. We know from Section 2 that for each $\alpha < {}^{L[\mathcal{F}]}\sigma_{n'+1}^{m'}$, P_3 has a $<\alpha$ closed, dense suborder. Hence P_3 is $< {}^{L[\mathcal{F}]}\sigma_{n'+1}^{m'}$ Baire. Thus if G_3 is P_3 generic over $L[\mathcal{F}]$,

$$(L[\mathcal{F}, G_3])_{{}^{L[\mathcal{F}]}\sigma_{n'+1}^{m'}} = (L[\mathcal{F}])_{{}^{L[\mathcal{F}]}\sigma_{n'+1}^{m'}}.$$

This implies that in $L[\mathcal{F}, G_3]$, ${}^{L[\mathcal{F}]}\pi_{n'}^{m'}$ is still $\Pi_{n'}^{m'}$ indescribable and that there are many $\Sigma_{n'}^{m'}$ indescribables above ${}^{L[\mathcal{F}]}\pi_{n'}^{m'}$. It implies also that $L[\mathcal{F}, G_3]$'s version of $P_{\mathcal{F}}^{m',n'}$ agrees with the $P_{\mathcal{F}}^{m',n'}$ of $L[\mathcal{F}]$. Thus if we denote by κ' at least $\Sigma_{n'}^{m'}$ indescribable $> {}^{L[\mathcal{F}]}\pi_{n'}^{m'}$ in $L[\mathcal{F}]$ then for any G_2 that is P_2 generic over $L[\mathcal{F}, G_3]$: In $L[\mathcal{F}, G_3, G_2]$, ${}^{L[\mathcal{F}]}\pi_{n'}^{m'}$ is $\Pi_{n'}^{m'}$ indescribable, κ' is $\Sigma_{n'}^{m'}$ indescribable and there are no $\Sigma_{n'}^{m'}$ indescribables below ${}^{L[\mathcal{F}]}\pi_{n'}^{m'}$.

Clearly $|P_1| < {}^{L[\mathcal{F}]}\sigma_{n'}^{m'}$. Hence by 5.8 for any G_1 that is P_1 generic over $L[\mathcal{F}, G_3, G_2]$ we obtain that in $L[\mathcal{F}, G_3, G_2, G_1]$ there are no $\Sigma_{n'}^{m'}$ indescribables $\in [{}^{L[\mathcal{F}]}\sigma_{n'}^{m'}, {}^{L[\mathcal{F}]}\pi_{n'}^{m'}]$ and clearly we cannot have any $\Sigma_{n'}^{m'}$ indescribables below ${}^{L[\mathcal{F}]}\sigma_{n'}^{m'}$. Also by $|P_1| < {}^{L[\mathcal{F}]}\sigma_{n'}^{m'}$ we get that ${}^{L[\mathcal{F}]}\pi_{n'}^{m'}$ is still $\Pi_{n'}^{m'}$ indescribable and κ' is still $\Sigma_{n'}^{m'}$ indescribable in $L[\mathcal{F}, G_3, G_2, G_1]$. This shows that for $\mathcal{F}(m', n') = 1$ we have

$$\Vdash_{P_{\mathcal{F}}}^{L[\mathcal{F}]} \sigma_{n'}^{m'} > \pi_{n'}^{m'}.$$

Now we assume that $\mathcal{F}(m', n') = 0$. Then $P_{\mathcal{F}} \approx P_1 \times P_3$. The $<\sigma_{n'+1}^{m'}$ Baireness of P_3 implies

$$\Vdash_{P_3}^{L[\mathcal{F}]} \sigma_{n'}^{m'} = {}^{L[\mathcal{F}]}\sigma_{n'}^{m'} < {}^{L[\mathcal{F}]}\pi_{n'}^{m'} = \pi_{n'}^{m'}.$$

$|P_1| < {}^{L[\mathcal{F}]}\sigma_{n'}^{m'}$ together with the observation that no generic extension of $L[\mathcal{F}]$ can have any $\Pi_{n'}^{m'}$ indescribables $< {}^{L[\mathcal{F}]}\sigma_{n'}^{m'}$ yield that

$$\Vdash_{P_{\mathcal{F}}}^{L[\mathcal{F}]} \sigma_{n'}^{m'} < \pi_{n'}^{m'}.$$

Another factoring argument shows that for any cardinal μ, $\Vdash_{P_{\mathcal{F}}}^{L[\mathcal{F}]} 2^{\mu} = \mu^{+}$. Hence we get

$$\Vdash_{P_{\mathcal{F}}}^{L[\mathcal{F}]} \mathrm{GCH}.$$

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