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Aiming Control: Design of Residence Probability Controllers*†

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A method for Residence Probability controllers design is developed using a two step procedure: (1) controller gains are obtained based on the cheap control approach, (2) the initial 'lock in' set is calculated using Liapunov-type considerations.

Key Words—Linear Systems; stochastic control; large deviations; approximation theory.

Abstract—Methods for design of residence probability (RP) controllers for linear stochastic systems are developed. The design of RP controllers requires the selection of a controller gain and an initial 'lock in' set. The controller gain is obtained by solving a Riccati equation and the initial set is obtained by solving a Liapunov equation. Examples illustrating the suggested design procedures are given.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

CONSIDER A SYSTEM defined by the Ito stochastic differential equation:

$$dx = (Ax + Bu)\,dt + \epsilon C\,dw, \quad x(0) = x_0, \qquad (1.1)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $0 < \epsilon \ll 1$ and w is a standard r-dimensional Brownian motion. Assume that (A, B) is controllable and (A, C) is disturbable. Let $D \subset \mathbf{R}^n$ and $D_0 \subset D$ be open bounded domains with 0 in their interiors and smooth boundaries ∂D and ∂D_0, respectively. Choose

$$u = Kx, \qquad (1.2)$$

and define the first passage time as

$$\tau_{x_0}(K) = \inf\{t \geq 0: x(t) \in \partial D \mid x_0 \in [D_0]\}, \qquad (1.3)$$

where $x(t)$ is the trajectory of the closed loop system (1.1), (1.2) originating at x_0 and $[D_0]$ is the closure of D_0. Suppose that the goal of control is defined by the *aiming process specifications*, i.e. by a pair (D, T) where D is the desired domain of operation and T is the period of operation. In Kim *et al.* (1992), the following *residence probability control problem* has been formulated.

Given (1.1), a pair (D, T), and a constant $0 < p < 1$, find a feedback law (1.2) and an open set $D_0 \subset D$ such that

$$P_{D_0}\{\tau(K) > T\} \triangleq \min_{x_0 \in [D_0]} \text{Prob}\,\{\tau_{x_0}(K) > T\} > p. \qquad (1.4)$$

Design specifications of the above form arise in many practical applications. For instance, in the problem of telescope pointing, the domain D is determined by the size of film grain, T is defined by the exposure time and p is the minimal acceptable probability of success. Laser beam pointing, gun pointing, robot arm pointing, airplane landing and missile terminal guidance are other examples of practical problems that have design specifications of the form (1.4) (see Meerkov and Runolfsson, 1988).

A controller that solves this problem is referred to as the *residence probability controller* (RP-controller).

It has been shown in Kim *et al.* (1992) that $P_{D_0}\{\tau(K) \leq T\} \triangleq 1 - P_{D_0}\{\tau(K) > T\}$ is related to $\exp\left\{-\dfrac{\varphi(D_0, K)}{\epsilon^2}\right\}$ in the following sense:

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$$\lim_{\epsilon \to 0} \epsilon^2 \ln P_{D_0}\{\tau(K) \le T\} = -\varphi(D_0, K),$$ where

$$\varphi(D_0, K) = \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2}(y - e^{(A+BK)t}x_0)^T \times X^{-1}(t, K)(y - e^{(A+BK)t}x_0), \quad (1.5)$$

$$\dot{X}(t, K) = (A + BK)X(t, K) + X(t, K) \times (A + BK)^T + CC^T \quad X(0, K) = I. \quad (1.6)$$

Thus, when ϵ is sufficiently small, problem (1.4) can be replaced by the problem of choosing K and D_0 such that

$$\varphi(D_0, K) \ge \alpha > 0, \quad (1.7)$$

where α is defined by

$$1 - \exp\left\{-\frac{\alpha}{\epsilon^2}\right\} = p. \quad (1.8)$$

The question of the existence of such K and D_0 has been studied in Kim *et al.* (1992). It has been shown that they exist for *any* $p < 1$ if and only if $\operatorname{Im} C \subseteq \operatorname{Im} B$ (the strong residence probability controllability, srp, case). If $\operatorname{Im} C \nsubseteq \operatorname{Im} B$, the maximal achievable p is bounded away from 1 (the weak residence probability controllability, wrp, case), and the estimates of this bound have been analyzed.

The present paper addresses the problem of the synthesis of RP-controllers. The design approach utilized here is based on the following considerations:

Since $\varphi(D_0, K)$ is, in general, difficult to calculate, reduce (1.5) to the inequality:

$$\varphi(D_0, K) \ge \frac{\min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2}{2\lambda_{\max}(X(T, K))},$$

where $\|\cdot\|$ is the Euclidean norm of a vector and $\lambda_{\max}(X)$ is the largest eigenvalue of X. In deriving the above inequality we have used the fact that $X(t, K)$ is a non-decreasing function of t. Further, since $\lambda_{\max}(X) \le \operatorname{Tr} X$,

$$\varphi(D_0, K) \ge \frac{\min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2}{2 \operatorname{Tr} X(T, K)} \triangleq \Phi(D_0, K). \quad (1.9)$$

Then problem (1.7) can be reformulated as follows.

Given D, T and α, find K and $D_0 \subset D$ such that

$$\Phi(D_0, K) \ge \alpha. \quad (1.10)$$

This is the problem solved in this paper. Obviously, some conservatism is introduced by replacing (1.7) by (1.10). Simulations of second order systems have shown that the degree of conservatism is of the order of 10–50%, i.e. $\left|\frac{\varphi(D_0, K) - \Phi(D_0, K)}{\varphi(D_0, K)}\right|$ is of the order of 0.1–0.5. In general, the degree of conservatism seems to be a complicated function of the system matrices and the domains D_0 and D.

The remainder of this paper is structured as follows: in Section 2, preliminary considerations are presented; Section 3 is devoted to the choice of the feedback gain K; Section 4 gives methods for selecting D_0; in Section 5, design procedures for RP-controllers are formulated and examples are considered; in Section 6, the conclusions are formulated. All proofs are given in Appendices 1–3.

2. PRELIMINARY CONSIDERATIONS

Let n and d denote the numerator and the denominator of $\Phi(D_0, K)$, respectively:

$$n(D_0, T, K) \triangleq \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2, \quad (2.1)$$

$$d(T, K) \triangleq 2 \operatorname{Tr} X(T, X). \quad (2.2)$$

Introduce the deterministic system:

$$\dot{z} = (A + BK)z, \quad z(0) = z_0. \quad (2.3)$$

Lemma 2.1. Let $D = \{x \in \mathbf{R}^n : x^T x \le R^2, R > 0\}$ and $z_0 = x_0$. Then

$$n(D_0, T, K) = \begin{cases} \left(R - \max_{x_0 \in [D_0]} \|z(t, K)\|_{L^\infty_{[0,T]}}\right)^2 & \text{if } \max_{x_0 \in [D_0]} \|z(t, K)\|_{L^\infty_{[0,T]}} < R \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\|z(t, K)\|_{L^\infty_{[0,T]}} = \sup_{0 \le t \le T} [z^T(t, K) z(t, K)]^{1/2}.$$

Proof. See Appendix 1.

Lemma 2.2. Let $z_i(t, K)$ be the solution of (2.3) with $z_0 = c_i$, $i = 1, \ldots, r$, where c_i is the ith column of the noise matrix C. Then

$$d(T, K) = 2 \sum_{i=1}^{r} \|z_i(t, K)\|^2_{L^2_{[0,T]}},$$

where

$$\|z_i(t, K)\|^2_{L^2_{[0,T]}} = \int_0^T z_i^T(t, K) z_i(t, K) \, dt.$$

Proof. See Appendix 1.

As it follows from Lemmas 2.1 and 2.2,

$$\Phi(D_0, K) = \begin{cases} \dfrac{\left(R - \max\limits_{x_0 \in [D_0]} \|z(t, K)\|_{L^\infty_{[0,T]}}\right)^2}{2 \sum\limits_{i=1}^{r} \|z_i(t, K)\|^2_{L^2_{[0,T]}}} & \text{if } \|z(t, K)\|_{L^\infty_{[0,T]}} < R, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

Thus, $\Phi(D_0, K)$ is large if both the $L^\infty_{[0,T]}$-norm of all solutions $z(t, K)$ starting in D_0 is small and the $L^2_{[0,T]}$-norm of $z_i(t, K)$ starting at c_i is small. Note that due to the complexity of (2.4), a direct maximization of $\Phi(D_0, K)$ with respect to the pair (D_0, K) is prohibitively difficult. Rather than trying to carry out this maximization, the design strategies developed in Sections 3–5 are based on the following observations.

(1) The denominator $d(T, K)$ of $\Phi(D_0, K)$ is independent of D_0. Thus, select a controller $\bar{K}$ that minimizes $d(T, K)$.
(2) Find $D_0(\bar{K})$ such that (1.10) is met, i.e.

$$n(D_0(\bar{K}), T, \bar{K}) \geq d(T, \bar{K})\alpha.$$

It is shown in Section 5 that there exists a controller developed using the above strategy that maximizes $\Phi(D_0, (K), K)$ in the limit $T \to \infty$.

3. THE CHOICE OF THE FEEDBACK GAIN

Let P_ρ be the positive definite solution of the Riccati equation:

$$A^T P_\rho + P_\rho A + I - \frac{1}{\rho} P_\rho B B^T P_\rho = 0, \quad \rho > 0. \quad (3.1)$$

Choose $\bar{K} = K_\rho = -(1/\rho)B^T P_\rho$. Then the Liapunov equation

$$(A + BK_\rho)X_\infty(K_\rho) + X_\infty(K_\rho)(A + BK_\rho)^T + CC^T = 0, \quad (3.2)$$

has a unique, positive, semi-definite solution $X_\infty(K_\rho)$ such that $\mathrm{Tr}\, X_\infty(K_\rho)$ is a non-decreasing matrix-function of ρ, i.e. (see Kwakernaak and Sivan, 1972),

$$\mathrm{Tr}\, X_\infty(K_{\rho_1}) \leq \mathrm{Tr}\, X_\infty(K_{\rho_2}) \quad \text{if} \quad \rho_1 \leq \rho_2.$$

Furthermore, a simple calculation shows that

$$X(T, K_\rho) = X_\infty(K_\rho) - e^{(A+BK_\rho)^T T} \times X_\infty(K_\rho) e^{(A+BK_\rho)T}, \quad (3.3)$$

and $X(T, K_\rho)$ is a non-decreasing matrix-function of T:

$$X(T_1, K_\rho) < X(T_2, K_\rho) \quad \text{if} \quad T_1 < T_2.$$

The quality of feedback gain K_ρ can be evaluated by the comparison between $\frac{1}{2}d(T, K_\rho) = \mathrm{Tr}\, X(T, K_\rho)$ and d^* defined as

$$d^* \triangleq \inf_{\mathbf{K}} \lim_{T\to\infty} \mathrm{Tr}\, X(T, K) = \inf_{\mathbf{K}} \mathrm{Tr}\, X_\infty(K) = \lim_{\rho\to 0} \mathrm{Tr}\, X_\infty(K_\rho), \quad (3.4)$$

where $\mathbf{K}$ is the class of all stabilizing gains for (1.1). The last equality in (3.4) is due to Kwakernaak and Sivan (1972). Note that if $\mathrm{Im}\, C \subseteq \mathrm{Im}\, B$, $d^* = 0$ (see Kim *et al.*, 1992). If $\mathrm{Im}\, C \not\subseteq \mathrm{Im}\, B$, the value of d^* can be characterized as follows.

Theorem 3.1. Assume without loss of generality that B has the form:

$$B = \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (3.5)$$

Then

$$d^* = \mathrm{Tr}\left\{C^T \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix} C\right\}, \quad (3.6)$$

where P_{22} is the positive definite solution of

$$A_{22}^T P_{22} + P_{22} A_{22} + I - P_{22} A_{21} A_{21}^T P_{22} = 0, \quad (3.7)$$

and A_{21}, A_{22} are defined by

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} I \\ 0 \end{bmatrix} u + C\dot{w}.$$

Proof. Follows from the proof of Theorem 4.2 of Kim *et al.* (1992).

Thus, for all $d_0 > d^*$, a ρ can be found according to (3.1) so that $d(T, K_\rho) < 2d_0$. An illustrative example is given below.

Consider a missile guidance problem introduced in Hotz and Skelton (1986):

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \varphi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \epsilon \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{w}, \quad (3.8)$$

where δ is the aileron deflection, ω is the roll angular velocity, φ is the roll angle, u is the control, and $\dot{w}$ is the white noise. Using Theorem 3.1, calculate d^* to be

$$d^* = 0.1.$$

If $\rho = 0.001$, from (3.1) to (3.3),

$$K_\rho = -[40.4 \quad 31.6 \quad 31.6], \quad (3.9)$$

$$\mathrm{Tr}\,X_\infty(K_\rho) = 0.1154,$$

$$\mathrm{Tr}\,X(T, K_\rho) = \begin{cases} 0.0792 & \text{for} \quad T = 0.1 \\ 0.1120 & \text{for} \quad T = 1 \\ 0.1154 & \text{for} \quad T = 4. \end{cases} \tag{3.10}$$

Thus, for $\rho = 0.001$, $X_\infty(K_\rho)$ is in a 15% neighborhood of its lower bound d^* and $\mathrm{Tr}\,X(T, K_\rho)$ is in a 20% neighborhood of its steady state value $\mathrm{Tr}\,X_\infty(K_\rho)$ if $T \geq 0.1$.

4. THE CHOICE OF THE INITIAL SET

It is well known that when $\|K\| \to \infty$, the $L^\infty_{[0,T]}$-norm of the solution $z(t, K)$ of (2.3) may diverge for any $T > 0$. This happens due to the so-called peaking phenomenon (Francis and Glover, 1978; Izmailov, 1987). Obviously, in the case of infinite peaking, $\varphi(D_0, K) = 0$ for all $D_0 \subset D$. Fortunately, however, under the feedback K_ρ specified in the previous section, the infinite peaking does not occur (Francis and Glover, 1978):

$$\lim_{\rho \to 0} \max_{x_0 \in [D_0]} \|z(t, K_\rho)\|^2_{L^\infty_{[0,T]}} < \infty.$$

Therefore, for any K_ρ, a $D_0 \subset D$ can be found so that $n(D_0, T, K_\rho)$ is positive. To accomplish this, consider the Liapunov equation

$$(A + BK_\rho)^T M_\rho + M_\rho(A + BK_\rho) = -N, \tag{4.1}$$

where $N > 0$ is a symmetric matrix. Using the positive definite solution M_ρ of this equation, define

$$D_0 = \{x \in \mathbf{R}^n : x^T M_\rho x < R_0^2\}, \tag{4.2}$$

where R_0 is chosen so that

$$\Gamma_\rho \triangleq \min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M_\rho)}} = R - \frac{R_0}{\sqrt{\lambda_{\min}(M_\rho)}} > 0. \tag{4.3}$$

Note that D_0 defined by (4.2) is an invariant set of (2.3) and

$$\max_{x_0 \in [D_0]} \|z(t, K_\rho)\|_{L^\infty_{[0,T]}} \leq \frac{R_0}{\sqrt{\lambda_{\min}(M_\rho)}}.$$

Thus, using Lemma 2.1,

$$n(D_0, T, K_\rho) \geq \Gamma_\rho^2.$$

Note that equality holds if $D = \{x \in \mathbf{R}^n : x^T x < R^2\}$. To illustrate this procedure, consider again example (3.8) and use $N = I$ and $D = \{x \in \mathbf{R}^n : x^T x < R^2\}$ in equation (4.1). Then

$$n(D_0, T, K_\rho) = \begin{cases} \left(\min\limits_{y \in \partial D} R - 3.9R_0\right)^2 & \text{if} \quad \rho = 0.1, \\ \left(\min\limits_{y \in \partial D} R - 5.4R_0\right)^2 & \text{if} \quad \rho = 0.01, \\ \left(\min\limits_{y \in \partial D} R - 8.5R_0\right)^2 & \text{if} \quad \rho = 0.001. \end{cases} \tag{4.4}$$

The negative feature of this procedure is that, due to the dependence of Γ_ρ on ρ, R_0 is decreasing as $\rho \to 0$; for instance, as it follows from (4.4), the "radius" of D_0 is about 26% of the 'radius' of D if $\rho = 0.1$, and is about 12% if $\rho = 0.001$. This means, in particular, that as $\rho \to 0$ the matrix M_ρ may be converging to a singular matrix and, consequently, the initial set D_0 may be converging to a very 'thin' set. In order to eliminate this problem, another procedure, based on a precompensator, is suggested below.

Assume for simplicity that u is a scalar and define

$$\phi(s) \triangleq \det(sI - A), \quad H(s) \triangleq (sI - A)^{-1}B. \tag{4.5}$$

Assume also that A is Hurwitz and that $n(s) = \phi(-s)\phi(s)H^T(-s)H(s)$ has no zeros on the imaginary axis. Let $Z_1, \ldots, Z_{n-1}$ be the zeros of $n(s)$ with $\mathrm{Re}\, Z_i < 0$. Define $Q > 0$ by

$$A^T Q + QA + I = 0. \tag{4.6}$$

Theorem 4.1. Assume $Z_1, \ldots, Z_{n-1}$ are distinct and are contained in the spectrum of A. Then the following inequality holds:

$$\left(A - \frac{1}{\rho} BB^T P_\rho\right)^T Q + Q\left(A - \frac{1}{\rho} BB^T P_\rho\right) < 0, \quad \forall \rho > 0. \tag{4.7}$$

Proof. See Appendix 2.

This implies that, under the conditions stated, the set $D_0(R_0) = \{x \in \mathbf{R}^n : x^T Q x \leq R_0^2\}$ is an invariant set of (2.3) with $K = K_\rho = -(1/\rho)B^T P_\rho$ for all $\rho > 0$. This property can be exploited as follows:

Choose $D_0(R_0)$ as above. Then, as it follows from (4.7),

$$z^T(t, K_\rho) Q z(t, K_\rho) \leq R_0^2, \quad \forall \rho > 0,$$

where $z(t, K_\rho)$ is the solution of (2.3) with $K = K_\rho$ and $z(0, K_\rho) \in D_0(R_0)$. Therefore, if R_0 is chosen so that $D_0(R_0) \subset D$, by Lemma 2.1,

$$n(D_0(R_0), T, K_\rho) \geq [\mathrm{dist}(\partial D, \partial D_0(R_0))]^2, \quad \forall \rho > 0, \tag{4.8}$$

where $\mathrm{dist}(\partial A, \partial B)$ is the distance between domain A and B.

It is well known (Kwakernaak, 1976) that, as $\rho \to 0$, $n - 1$ of the eigenvalues of $A + BK_\rho$

converge to the zeros $Z_1, \ldots, Z_{n-1}$. Under the assumptions of Theorem 4.1, $n-1$ of the eigenvalues of A are already fixed at the terminal locations. Furthermore, these eigenvalues are not affected by the feedback K_ρ. Thus, roughly speaking, as a function of ρ the closed loop system behaves like a first order stable system whose eigenvalue converges to infinity as $\rho \to 0$. Furthermore, for a stable first order system any open set containing the origin is an invariant set. That explains why it is possible to construct the invariant set $D_0(R_0)$ independently of ρ.

To insure that Z_is are contained in the spectrum of A, choose a precompensator,

$$u = Lx, \tag{4.9}$$

that places the eigenvalues of $(A + BL)$ at the desired positions. That is why this choice of D_0 is referred to as the precompensator based design. As it follows from (4.8), D_0 can be chosen here so that $n(D_0(R_0), T, K_\rho)$ takes any desired value from the open interval $\left(0, \min_{y \in \partial D} \|y\|^2\right)$, independent of the choice of K_ρ.

Note that in order to design the precompensator L the exact locations of the Z_is are required. At the present time the sensitivity to errors in the values of the Z_is is unknown. The analysis of the robustness properties of the design procedures developed in this paper are the subject of a future research.

The precompensator based design can be generalized for systems with multiple inputs as well.

5. DESIGN PROCEDURES

The goal of the design procedures is to choose K and D_0, based on the specifications D, T and α, so that (1.10) is satisfied. Using the results of Sections 3 and 4, two such procedures are formulated below. They are similar as far as the choice of K is concerned and differ only in the choice of D_0. In the first one, D_0 is chosen directly for the original A; in the second a precompensator is used.

5.1. *Design procedure*

(1) Calculate d^* according to (3.6).

(2) Check whether the inequality

$$\alpha^* \triangleq \frac{\min_{y \in \partial D} \|y\|^2}{2d^*} > \alpha, \tag{5.1}$$

is satisfied. If it is, proceed to step (3); if it is not, this procedure may result in no controller satisfying the specifications. (Note that in the srp-controllability case, 5.1 is always met.)

(3) Choose $\rho > 0$ and solve (3.1)–(3.3) to find K_ρ and $\operatorname{Tr} X(T, K_\rho)$.

(4) Select $N > 0$ and find $M_\rho > 0$ from (4.1).

(5) Calculate

$$c^* = \min_{y \in \partial D} \|y\| - \sqrt{2\alpha d^*}, \tag{5.2}$$

choose $c < c^*$ and determine

$$R_0(\rho) = c\sqrt{\lambda_{\min}(M_\rho)}. \tag{5.3}$$

(6) Check whether (1.10), i.e.

$$\frac{\left[\min_{y \in \partial D} \|y\| - c\right]^2}{2 \operatorname{Tr} X(T, K_\rho)} > \alpha, \tag{5.4}$$

is satisfied. If it is, the design of RP-controller is accomplished with

$$\begin{aligned} K^{5.1} &= K_\rho \\ D_0^{5.1} &= \{x \in \mathbf{R}^n: x^T M_\rho x \le R_0^2(\rho)\}. \end{aligned} \tag{5.5}$$

If it is not, go to step (3) and choose a smaller ρ.

Note that under condition (5.1) this procedure always converges to a RP controller satisfying the specifications. To illustrate its application, consider again the missile guidance problem (3.8) and assume that the residence probability control specifications are given as follows:

$$\begin{aligned} &D = \{(\delta, \omega, \varphi): \delta^2 + \omega^2 + \varphi^2 = 2\}, \\ &T = 3, \\ &\alpha = 0.75. \end{aligned} \tag{5.6}$$

Then, knowing from Section 3 that $d^* = 0.1$, we calculate α^* to be 10 and therefore (5.1) is met. Choosing $\rho = 0.00063$, $N = I$, $c = 1 < c^*$, and following steps (3)–(6) we obtain the RP-controller:

$$\begin{aligned} K^{5.1} &= K_\rho = [-48.8 \quad -39.8 \quad -39.8], \\ D_0^{5.1} &= \{(\delta, \omega, \varphi): 0.0128\delta^2 + 0.1123\omega^2 \\ &\quad + 1.1123\varphi^2 + 0.025\delta\omega + 0.025\delta\varphi \\ &\quad + 0.2246\omega\varphi \le 0.01123\}, \end{aligned} \tag{5.7}$$

which ensures $\Phi(D_0^{5.1}, K_0^{5.1}) = 0.763$.

5.2. *Design procedure*

Repeat steps (1) and (2) of 5.1.

(3) Find the zeros Z_i, $i = 1, \ldots, n-1$ of $\phi(-s)\phi(s)H^T(-s)H(s)$ with negative real parts. If they are multiple, this procedure is not applicable. If they are simple, go to the next step.

(4) Find a precompensator L such that Z_is are in the spectrum of $(A + BL)$.

(5) Choose $\rho > 0$ and solve (3.1)–(3.3), with A

replaced by $(A+BL)$, to find K_ρ and $\operatorname{Tr} X(T, K_\rho)$.
(6) Find $Q>0$ satisfying (4.7) with A replaced by $(A+BL)$.
(7) Find $R_0>0$ such that

$$D_0^{5.2} \triangleq \{x \in \mathbf{R}^n : x^T Q x \leq R_0^2\} \subset D,$$

and (1.10) is satisfied, i.e.

$$\frac{[\operatorname{dist}(\partial D, \partial D_0^{5.2})]^2}{2 \operatorname{Tr} X(T, K_\rho)} > \alpha.$$

If such R_0 exists, the design of RP-controller is accomplished with

$$K^{5.2} = K_\rho + L,$$

and $D_0^{5.2}$ defined above. If it does not, go to step (5) and choose smaller ρ.

Illustrating this procedure, consider again (3.8) with specifications (5.6). The calculation of Z_is gives:

$$Z_1 = -1; \quad Z_2 = -10.$$

Therefore, (5.2) is applicable and the precompensator L can be chosen as

$$L = [-12 \quad -2 \quad -2].$$

Choosing $\rho = 0.00063$ and following steps (5)–(7), we arrive at the following RP-controller:

$$\begin{aligned} K^{5.2} &= K_\rho + L \\ &= [-49.8911 \quad -39.8911 \quad -39.8911], \\ D_0^{5.2} &= \{(\delta, \omega, \varphi) : 0.25\delta^2 + 0.35\omega^2 + 1.35\varphi^2 \\ &\quad + 0.5\delta\omega + 0.5\delta\varphi + 0.7\omega\varphi \leq 0.0441\}, \end{aligned} \tag{5.8}$$

which again assures $\Phi(D_0^{5.2}, K^{5.2}) \simeq 0.763$.

Next we compare controllers (5.7) and (5.8): both are defined by the same ρ and ensure identical residence probability. Furthermore, the control effort, as measured by $\|K\|$, is also similar for the two designs, i.e. $\|K^{5.1}\| \simeq 74.5$, $\|K^{5.2}\| \simeq 75.3$. Therefore, the only remaining ground for comparison is the size, i.e. the volume, of the initial domain D_0. These volumes are as follows

$$\operatorname{vol}(D_0^{5.1}) = 4.27 \times 10^{-5}\pi,$$
$$\operatorname{vol}(D_0^{5.2}) = 146.4 \times 10^{-5}\pi.$$

Thus, 5.2 results in a considerably larger initial, "lock in", set.

To conclude this section, we formulate two theorems describing general properties of RP-controllers.

Theorem 5.1. When $\rho \to 0$, the closed loop poles of system (1.1) with feedback gains $K^{5.1}$ and $K^{5.2}$ are identical, i.e.

$$\lim_{\rho \to 0} \Lambda(A + BK_\rho) = \lim_{\rho \to 0} \Lambda(A + BK_\rho + BL), \quad \forall L,$$

where $\Lambda(N)$ is the spectrum of N.

Proof. See Appendix 3.

Theorem 5.2. Define,

$$\Phi_\infty(D_0, K) = \frac{\min\limits_{x_0 \in [D_0]} \min\limits_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \|y - e^{(A+BK)t} x_0\|^2}{2 \operatorname{Tr} X_\infty(K)}.$$

Then

$$\sup_{\mathbf{K}} \Phi_\infty(D_0^{5.2}, K) = \lim_{\rho \to 0} \Phi_\infty(D_0^{5.2}, K^{5.2}),$$

where $\mathbf{K}$ is the class of all stabilizing gains for A.

Proof. See Appendix 3.

To interpret this theorem, note that

$$\begin{aligned} \frac{\Phi(D_0, K) - \Phi_\infty(D_0, K)}{\Phi(D_0, K)} &= \frac{\operatorname{Tr} X_\infty(K) - \operatorname{Tr} X(T, K)}{\operatorname{Tr} X_\infty(K)} \\ &= \frac{1}{\operatorname{Tr} X_\infty(K)} \int_T^\infty e^{(A+BK)t} C \\ &\quad \times C^T e^{(A+BK)^T} \, dt. \end{aligned} \tag{5.9}$$

For any stabilizing K the right-hand side of (5.9) converges to zero exponentially fast as $T \to \infty$. Thus, $\Phi_\infty(D_0, K)$ is a good approximation of $\Phi(D_0, K)$ for T sufficiently large. Theorem 5.2 states that controller $K^{5.2}$ maximizes $\Phi_\infty(D_0^{5.2}, K)$ over all stabilizing gains. Thus, for T sufficiently large, gain $K^{5.2}$ is approximately optimal for $\Phi(D_0^{5.2}, K)$ as well. Note, however, that the initial domain $D_0^{5.2}$ has not been shown to be optimal.

6. CONCLUSIONS

Residence probability controllers constitute a new class of controllers for linear stochastic systems. They are advantageous in applications defined by aiming control specifications of the form: keep the system in a desired domain, D during a given time interval, T, with a given probability, p. The design of gains for RP-controllers is not more complex than that in the LQG method: it involves only solving a Riccati equation. However, unlike the LQG, RP-controllers require the selection of the initial set D_0 as well; this is accomplished by solving a Liapunov equation.

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APPENDIX 1: PROOFS FOR SECTION 2

Proof of Lemma 2.1. If $e^{(A+BK)t}x_0 \in D$ for all t and $x_0 \in D$,

$$\begin{aligned}\min_{x_0\in[D_0]} \min_{\substack{y\in\partial D\\ 0\le t\le T}} \|y - e^{(A+BK)t}x_0\|^2 &= \min_{x_0\in[D_0]}\left(R - \sup_{0\le t\le T}\|e^{(A+BK)t}x_0\|\right)^2\\ &= \min_{x_0\in[D_0]}(R - \|z(t,K)\|_{L^\infty_{[0,T]}})^2\\ &= \left(R - \max_{x_0\in[D_0]}\|z(t,K)\|_{L^\infty_{[0,T]}}\right)^2.\end{aligned}$$

If there exist $x^* \in [D_0]$ and $t^* \in [0, T]$ such that $e^{(A+BK)t^*}x_0^* \in \partial D$, i.e. $\max_{x_0\in[D_0]}\|z(t,K)\|_{L^\infty_{[0,T]}} \ge R$, then $n(D_0, T, K) = 0$.

Q.E.D.

Proof of Lemma 2.2. From (2.2) and (1.6),

$$\begin{aligned}d(T,K) &= 2\,\mathrm{Tr}\int_0^T e^{(A+BK)t}CC^Te^{(A+BK)^Tt}\,dt\\ &= 2\,\mathrm{Tr}\int_0^T C^Te^{(A+BK)^Tt}e^{(A+BK)t}C\,dt\\ &= 2\sum_{i=1}^{r}\int_0^T c_i^Te^{(A+BK)^Tt}e^{(A+BK)t}c_i\,dt\\ &= 2\sum_{i=1}^{r}\|z_i(t,K)\|^2_{L^2_{[0,T]}}.\end{aligned}$$

Q.E.D.

APPENDIX 2: PROOFS FOR SECTION 4

To prove Theorem 4.1, we need the following.

Lemma A2.1. Under the assumptions of Theorem 4.1, for any $\rho > 0$ there exists $\alpha(\rho) > 0$ such that

$$BB^TP_\rho = \alpha(\rho)BB^TQ,$$

where P_ρ and Q are defined by (3.1) and (4.6), respectively.

Proof. As it follows from Kwakernaak (1976), the eigenvalues of $(A - (1/\rho)BB^TP_\rho)$ are the left half plane roots of

$$\phi(-s)\phi(s)\left[1 + \frac{1}{\rho}H^T(-s)H(s)\right] = 0, \quad \forall\rho > 0, \quad \text{(A2.1)}$$

where $\phi(s) = \det(sI - A)$ and $H(s) = (sI - A)^{-1}B$. Let $n(s)$ denote $\phi(-s)\phi(s)H^T(-s)H(s)$, then (A2.1) can be rewritten as

$$\rho\phi(-s)\phi(s) + n(s) = 0. \quad \text{(A2.2)}$$

By definition, $n(Z_i) = 0$, and, by assumption, $\phi(Z_i) = 0$. Therefore, (A2.2) is satisfied for $s = Z_i$, $i = 1, \ldots, n-1$, and, hence,

$$\det\left(Z_iI - A + \frac{1}{\rho}BB^TP_\rho\right) = 0, \quad \forall\rho > 0, i = 1, \ldots, n-1. \quad \text{(A2.3)}$$

Since (A, B) is controllable, there exists a nonsingular M such that

$$A_c = M^{-1}AM, \quad B_c = M^{-1}B,$$

and (A_c, B_c) is in the controller canonical form. Then

$$\det\left(sI - A + \frac{1}{\rho}BB^TP_\rho\right) = \det\left(sI - A_c + \frac{1}{\rho}B_cB_c^TP_{\rho c}\right), \quad \text{(A2.4)}$$

where $P_{\rho c}$ is the positive definite solution of

$$A_c^TP_{\rho c} + P_{\rho c}A_c + M^TM - \frac{1}{\rho}P_{\rho c}B_cB_c^TP_{\rho c} = 0. \quad \text{(A2.5)}$$

Let $-a_i$ and $-P_{\rho c}(i)$ be the ith elements of the nth row of matrices A_c and $P_{\rho c}$, respectively. Then from (A2.4),

$$\det\left(sI - A + \frac{1}{\rho}BB^TP_\rho\right) = s^n + \left(a_n - \frac{1}{\rho}P_{\rho c}(n)\right)s^{n-1} + \cdots + \left(a_2 - \frac{1}{\rho}P_{\rho c}(2)\right)s + \left(a_1 - \frac{1}{\rho}P_{\rho c}(1)\right). \quad \text{(A2.6)}$$

Since Z_i belong to the spectrum of A,

$$Z_i^n + a_nZ_i^{n-1} + \cdots + a_2Z_i + a_1 = 0. \quad \text{(A2.7)}$$

Hence, due to (A2.3) and (A2.7), from (A2.6)

$$P_{\rho c}(n)Z_i^{n-1} + \cdots + P_{\rho c}(2)Z_i + P_{\rho c}(1) = 0, \quad i = 1, \ldots, n-1.$$

This implies that vector $[P_{\rho c}(1)\cdots P_{\rho c}(n)]^T$ is in the null space of the $(n-1)\times n$ matrix

$$\bar{Z} = \begin{bmatrix}1 & Z_1 & \cdots & Z_1^{n-1}\\ \hdashline 1 & Z_{n-1} & \cdots & Z_{n-1}^{n-1}\end{bmatrix},$$

which, due to the assumption that $Z_i \ne Z_j$, $\forall i, j = 1, \ldots, n-1$, is of the rank $n-1$. Moreover, since Z_is are independent of ρ, $[P_{\rho c}(1)\cdots P_{\rho c}(n)]^T$ is in the one-dimensional null space of $\bar{Z}$ for all $\rho > 0$. Therefore, for any $\rho_1 > 0$ and $\rho_2 > 0$ there exists β such that

$$\begin{bmatrix}P_{\rho_1c}(1)\\ \vdots\\ P_{\rho_1c}(n)\end{bmatrix} = \beta\begin{bmatrix}P_{\rho_2c}(1)\\ \vdots\\ P_{\rho_2c}(n)\end{bmatrix},$$

and, since $P_{\rho c}(n) > 0$, for all $\rho > 0$, β is a positive number.

Consider now (A2.5) in the limit as $\rho\to\infty$, i.e.

$$A_c^TQ_c + Q_cA_c + M^TM = 0.$$

It is known (Kwakernaak and Sivan, 1972) that

$$Q_c \geqslant P_{\rho c} \quad \text{and} \quad \lim_{\rho\to\infty}P_{\rho c} = Q_c \quad \forall\rho > 0.$$

Hence, $[Q_c(1)\cdots Q_c(n)]^T$ is also in the null space of $\bar{Z}$, where $Q_c(i)$ is the ith element of the nth row of Q_c. This implies that there exists $\alpha(\rho) > 0$ such that

$$[P_{\rho c}(1) \cdots P_{\rho c}(n)]^T = \alpha(\rho)[Q_c(1) \cdots Q_c(n)]^T. \quad \text{(A2.8)}$$

Taking into account that

$$B_c B_c^T P_{\rho c} = \begin{bmatrix} 0 \\ P_{\rho c}(1) \cdots P_{\rho c}(n) \end{bmatrix},$$

$$B_c B_c^T Q_c = \begin{bmatrix} 0 \\ Q_c(1) \cdots Q_c(n) \end{bmatrix},$$

and

$$P_{\rho c} = M^T P_\rho M,$$
$$Q_c = M^T Q M,$$

where Q is the positive definite solution of (4.6), from (A2.8) we obtain

$$\begin{aligned} B_c B_c^T P_{\rho c} &= M^{-1} B B^T P_\rho M \\ &= \alpha(\rho) B_c B_c^T Q_c \\ &= \alpha(\rho) M^{-1} B B^T Q M, \end{aligned}$$

i.e.

$$B B^T P_\rho = \alpha(\rho) B B^T Q.$$

Q.E.D.

Proof of Theorem 4.1. Taking into account Lemma A2.1,

$$\frac{1}{\rho}(BB^T P_\rho)^T Q + \frac{1}{\rho} Q(BB^T P_\rho) = \frac{2\alpha(\rho)}{\rho} QBB^T Q \geq 0. \quad \text{(A2.9)}$$

Therefore, using (4.6) and (A2.9),

$$\left(A - \frac{1}{\rho} BB^T P_\rho\right)^T Q + Q\left(A - \frac{1}{\rho} BB^T P_\rho\right) = -I - \frac{2\alpha(\rho)}{\rho} QBB^T Q < 0.$$

Q.E.D.

APPENDIX 3: PROOFS FOR SECTION 5

Proof of Theorem 5.1. We prove below that

$$\lim_{\rho \to 0} \Lambda(A + BL + BK_\rho) = \lim_{\rho \to 0} \Lambda(A + BK_\rho), \quad \forall L \in R^{1 \times n},$$

where $\Lambda(A)$ is the spectrum of A. First, we observe that, as it is shown in Kwakernaak (1976), $n-1$ eigenvalues of $(A + BL + BK_\rho)$ converge, as $\rho \to 0$, to the left half plane roots of $\phi^L(-s)\phi^L(s)[H^{L^T}(-s)H^L(s)]$, where $\phi^L(s) = \det(sI - A - BL)$ and $H^L(s) = (sI - A - BL)^{-1}B$. Next we show that these roots are independent of L, and therefore $n-1$ finite eigenvalues of $(A + BL + BK_\rho)$ coincide, in the limit as $\rho \to 0$, with $n-1$ finite eigenvalues of $(A + BK_\rho)$. Indeed, define

$$H_i^L(s) \triangleq c_i(sI - A - BL)^{-1}B,$$

where $c_i = [0 \cdots 010 \cdots 0]$. Then, since zeros are not affected by feedback,

$$\phi^L(s)c_i(sI - A - BL)^{-1}B = \phi(s)c_i(sI - A)^{-1}B, \quad i = 1, \ldots, n,$$

where

$$\phi(s) = \det(sI - A),$$

or equivalently,

$$\phi^L(s)H^L(s) = \phi(s)H(s), \quad \text{(A3.1)}$$

where

$$H(s) = (sI - A)^{-1}B.$$

Hence from (A3.1),

$$\begin{aligned} &\phi^L(-s)\phi^L(s)[H^{L^T}(-s)H^L(s)] \\ &= \frac{\phi^L(-s)\phi(s)}{\phi^L(-s)\phi^L(s)}[\phi^L(-s)H^{L^T}(-s)\phi^L(s)H^L(s)] \\ &= \phi(-s)H^T(-s)\phi(s)H(s), \end{aligned}$$

which implies that $\phi^L(-s)\phi^L(s)H^{L^T}(-s)H^L(s)$ is independent of L.

As far as the nth eigenvalue of $(A + BL + BK_\rho)$ is concerned, it converges to $-\infty$ (Kwakernaak, 1976) as does the nth eigenvalue of $(A + BK_\rho)$.

Q.E.D.

Proof of Theorem 5.2. Due to Kwakernaak and Sivan (1972) and Theorem 5.1,

$$\inf_{K \in \mathbf{K}} X_\infty(K) = \lim_{\rho \to 0} X_\infty(K_\rho) = \lim_{\rho \to 0} X_\infty(K^{5.2}), \quad \text{(A3.2)}$$

where $K_\rho = -(1/\rho)B^T P_\rho$ and P_ρ is the positive definite solution of (3.1). Also, by the definition of $D_0^{5.2}$,

$$\begin{aligned} &\sup_{K \in \mathbf{K}} \min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2 \\ &\le \sup_{K \in \mathbf{K}} \min_{x_0 \in [D_0^{5.2}]} \min_{y \in \partial D} \|y - x_0\|^2 \\ &= [\mathrm{dist}(\partial D, \partial D_0^{5.2})]^2. \end{aligned}$$

Conversely, from Theorem 4.1 and (4.8),

$$\begin{aligned} &\sup_{K \in \mathbf{K}} \min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2 \\ &\ge \lim_{\rho \to 0} \min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK^{5.2})t}x_0\|^2 \\ &= [\mathrm{dist}(\partial D, \partial D_0^{5.2})]^2. \end{aligned}$$

Hence,

$$\sup_{K \in \mathbf{K}} \min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2 = [\mathrm{dist}(\partial D, \partial D_0^{5.2})]^2. \quad \text{(A3.3)}$$

From (A3.2) and (A3.3),

$$\begin{aligned} \sup_{K \in \mathbf{K}} \Phi_\infty(D_0^{5.2}, K) &= \sup_{K \in \mathbf{K}} \frac{\min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2}{2\,\mathrm{Tr}\, X_\infty(K)} \\ &\le \frac{\sup_{K \in \mathbf{K}} \min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK)t}x_0\|^2}{2 \inf_{K \in \mathbf{K}} \mathrm{Tr}\, X_\infty(K)} \\ &= \frac{[\mathrm{dist}(\partial D, \partial D_0^{5.2})]^2}{2 \lim_{\rho \to 0} \mathrm{Tr}\, X_\infty(K^{5.2})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup_{K \in \mathbf{K}} \Phi_\infty(D_0^{5.2}, K) &\ge \frac{\lim_{\rho \to 0} \min_{x_0 \in [D_0^{5.2}]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{(A+BK^{5.2})t}x_0\|^2}{2 \lim_{\rho \to 0} \mathrm{Tr}\, X_\infty(K^{5.2})} \\ &= \frac{[\mathrm{dist}(\partial D, \partial D_0^{5.2})]^2}{2 \lim_{\rho \to 0} \mathrm{Tr}\, X_\infty(K^{5.2})}. \end{aligned}$$

Hence,

$$\sup_{K \in \mathbf{K}} \Phi_\infty(D_0^{5.2}, K) = \lim_{\rho \to 0} \Phi_\infty(D_0^{5.2}, K^{5.2}).$$

Q.E.D.