On some possible extensions of the Brodsky–Lepage–Mackenzie approach beyond the next-to-leading order

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Noting that the choice of renormalization point advocated by Brodsky, Lepage and Mackenzie (BLM) is the flavor independent prescription which removes all f-dependence from the next-to-leading order coefficients, we consider the possible generalization which requires all higher order coefficients r_i to be f-independent constants r_i^*. We point out that in QCD, setting $r_i = r_i^*$ is always possible, but leaves us with an ambiguous prescription. We consider an alternative possibility within the framework of the BLM approach and apply the corresponding prescription to the next-to-next-to-leading approximation of $\sigma_{tot}(e^+e^- \to \text{hadrons})$ in QCD. The analogous questions and the special features of the BLM and effective charge approaches in QED are also discussed.

1. Fixing the arbitrariness of the renormalization scheme (RS) is of much practical interest (both theoretically and phenomenologically) in perturbative QCD, owing to the not so small value of the coupling constant. In the next-to-leading order (NLO) one has to fix the scheme renormalization point μ or the definition of the parameter Λ. In the next-to-next-to-leading order (NNLO) one has to fix the scheme-dependent NNLO coefficient of the renormalization group (RG) β-function.

Brodsky, Lepage and Mackenzie (BLM) have proposed, for any given physical quantity R, an interesting flavor independent prescription to fix μ (or Λ) which removes all flavor f-dependence from the NLO coefficient r_1 of the perturbative series for R [1]. The motivation of this prescription is the observation that in QED the energy dependence of the invariant charge can be identified with the renormalized expression of the photon vacuum polarization graphs. Starting from this observation BLM have proposed to absorb the fermion contributions to the photon (or gluon) propagators to the scale fixing parameter μ (or Λ). Within this approach one can consider the question of applicability (or non-applicability) of the perturbative predictions for their detailed comparison with experimental data [1].

This proposal has already been discussed and used in a number of phenomenological QCD studies (see e.g. ref. [2]). However, the problem of the possibility of the extension of the BLM proposal beyond the NL level remained open (recently this problem was also investigated within the skeleton expansion in ref. [3]). In this note we discuss the generalization of the BLM proposal which requires all higher order coefficients r_i to be independent from f as well. We will show that beyond the NLO, in addition to the prescription to fix μ, one must consider the problem how to fix the higher order coefficients of the RS β-function. We will find that this problem has no unique

solution. We then consider the alternative procedure of using a fixed RS β-function, but a coupling constant dependent redefinition of μ in higher orders of perturbation theory [1]. As an example we consider the application of this prescription to deal with the scheme dependence ambiguity of the NNLO QCD approximation of $\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})$. The analogous questions in QED are also discussed.

2. Consider a physical quantity $R = R(Q)$ and its expression in an arbitrary RS with the coupling $a = \alpha_s/\pi$, $a = a(\mu)$:

$$R = a(1 + r_1 a + r_2 a^2 + r_3 a^3 + ...) . \tag{1}$$

If we restrict ourselves to RS which does not introduce the arbitrary f-dependence beyond that expected from fermion-loop insertions, r_i will be a linear function of f, and since the same is true for the one-loop coefficient β_0 of the RG QCD β-function

$$\mu^2 \frac{\partial a}{\partial \mu^2} = \beta(a)$$

$$= -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \beta_3 a^5 - ... , \tag{2}$$

one can write

$$r_1 = -\beta_0 \left(\ln \frac{Q^2}{\mu^2} + d_1^* \right) + r_1^* , \tag{3}$$

where both d_1^* and r_1^* are f-independent. BLM propose to start from the $\overline{\text{MS}}$-scheme [4] and to fix $\mu = \mu_*$ by the condition $\ln(Q^2/\mu_*^2) + d_1^* = 0$, so that $r_1 = r_1^*$. Writing $r_1 = r_{10} + r_{11} f$ and $\beta_0 = \beta_{00} + \beta_{01} f$ one has

$$r_{10} = -\beta_{00} \left(\ln \frac{Q^2}{\mu^2} + d_1^* \right) + r_1^* \tag{4}$$

and

$$r_{11} = -\beta_{01} \left(\ln \frac{Q^2}{\mu^2} + d_1^* \right) . \tag{5}$$

The BLM proposal is therefore equivalent to the condition $r_{11}^{\overline{\text{MS}}} = 0$ or $\mu_*^2 = Q^2 \exp(d_1^*)$, i.e., that r_1 is f-independent, with f, μ and Q naturally considered as independent variables. Note that the BLM pres-

cription can be restated as, first, a redefinition of μ: $\mu_{\overline{\text{MS}}}^2 \rightarrow \mu_{\text{BLM}}^2 = \mu_{\overline{\text{MS}}}^2 \exp(-d_1^*)$, or equivalently of Λ: $\Lambda_{\overline{\text{MS}}}^2 \rightarrow \Lambda_{\text{BLM}}^2 = \Lambda_{\overline{\text{MS}}}^2 \exp(-d_1^*)$, with $d_1^* = -r_{11}^{\overline{\text{MS}}}(\mu = Q)/\beta_{01}$ (this is an analogue to the transition from the MS to the $\overline{\text{MS}}$-scheme [4]). The second step is to put $\mu_{\text{BLM}}^2 = Q^2$. Note, that since $r_{11}^{\overline{\text{MS}}}$ is process dependent, the definition of μ_{BLM}^2 is process dependent too.

A straightforward extension of the BLM scheme then consists in requiring all r_i to be also f-independent. We first show that it is always possible to achieve this for any given R, by choosing arbitrary, f-independent values for the r_i. To see this it is convenient to use the RS-invariant effective charge approach [5–8] where the RS coupling a_{eff} is identified with R itself, $R = a_{\text{eff}}$ (see also ref. [9]). The corresponding RG equation for R then defines the effective charge β-function:

$$Q^2 \frac{dR}{dQ^2} = \beta_{\text{eff}}(R)$$

$$= -\beta_0 R^2 - \beta_1 R^3 - \tilde{\beta}_2 R^4 - \tilde{\beta}_3 R^5 - ... , \tag{6}$$

which is the RS-invariant object which governs the Q^2-evolution of R in QCD. In particular, the $\tilde{\beta}_i$ ($i \geqslant 2$) are the RS-invariant quantities [5–7] (simply related to the scheme-invariants ρ_i, $i \geqslant 2$, introduced in ref. [10]). Upon solving eq. (6) one introduces the effective RS-invariant scale parameter Λ_{eff}, such that

$$\beta_0 \ln \frac{Q^2}{\Lambda_{\text{eff}}^2} = \frac{1}{R} + O(\ln R) + \text{const.} + O(R) \tag{7}$$

($\beta_0 \ln(Q^2/\Lambda_{\text{eff}}^2)$ is connected with the invariant ρ_1 of ref. [10]). On the other hand, the solution of the RG equation (3), with $a(\mu^2)$ defined in the considered RS, reads

$$\beta_0 \ln \frac{\mu^2}{\Lambda^2} = \frac{1}{a} + O(\ln a) + \text{const.} + O(a) , \tag{8}$$

and introduces the RS scale parameter Λ. The effective charge RS-invariant parameters Λ_{eff} and $\tilde{\beta}_i$ are related to the RS parameter Λ and β_i by the relations [5–7,11]

$$r_1 - \beta_0 \ln \frac{\mu^2}{\Lambda^2} = -\beta_0 \ln \frac{Q^2}{\Lambda_{\text{eff}}^2} , \tag{9}$$

$$\beta_0(r_2 - r_1^2) = \beta_1 r_1 + \tilde{\beta}_2 - \beta_2 , \tag{10}$$

$\beta_0(r_3 - r_1^3)$

$$= \tfrac{5}{2}\beta_1 r_1^2 + (3\tilde{\beta}_2 - 2\beta_2)r_1 + \tfrac{1}{2}(\tilde{\beta}_3 - \beta_3), \qquad (11)$$

etc. Eq. (9) follows immediately by taking the difference between eq. (7) and eq. (8), reexpanding in powers of a and comparing with the standard expansion of eq. (1). It is clear from eqs. (10) and (11) that one can assign arbitrary values to r_1, r_2, r_3, ... at $\mu = Q$ which will determine both Λ [by $\beta_0 \ln(\Lambda^2 / \Lambda_{\text{eff}}^2) = -r_1(\mu = Q)$] and β_2, β_3, ..., i.e., the renormalization scheme parameters (this fixes the scheme). In particular, one can take the r_i to be f-independent constants. This conclusion can be generalized to all orders using the relation $\beta(a)(\partial R / \partial a) = \beta_{\text{eff}}(R)$ which determines the β_i given the r_i and the $\tilde{\beta}_i$ [6,7,11] (see ref. [12] for explicit formulae).

We next investigate what the constraints are on the RS β-functions which follow from the requirement that the r_i are f-independent. We first show that this condition cannot be achieved by the mere BLM choice of μ, but imply additional restrictions on the RS dependent β-function coefficients. We shall only consider the schemes where the general form of f-dependence is the same as expected if induced solely by fermion-loop insertions, namely that r_i are polynomial in f of maximal degree i. Thus we assume

$$r_1 = r_{10} + r_{11}f, \quad r_2 = r_{20} + r_{21}f + r_{22}f^2,$$

$$r_3 = r_{30} + r_{31}f + r_{32}f^2 + r_{33}f^3, \qquad (12)$$

etc., and also

$$\beta_0 = \beta_{00} + \beta_{01}f, \quad \beta_1 = \beta_{10} + \beta_{11}f,$$

$$\beta_2 = \beta_{20} + \beta_{21}f + \beta_{22}f^2 + \beta_{23}f^3,$$

$$\tilde{\beta}_2 = \tilde{\beta}_{20} + \tilde{\beta}_{21}f + \tilde{\beta}_{22}f^2 + \tilde{\beta}_{23}f^3. \qquad (13)$$

Note that the $\tilde{\beta}_2$-coefficient can have f^3-terms as has been observed [13] to occur on the basis of the recent NNLO $O(a^3)$ calculations [14,13,15] (the QCD results [15] are in agreement with the ones presented in ref. [16], for a critical discussion see ref. [17]), although they are absent in the MS-scheme [18] and MOM scheme [19]. We generalize the foundations of ref. [13] and stress that once the f^3-terms appeared in the $\tilde{\beta}_2$-coefficients in the effective scheme, they can also exist in another arbitrary scheme, and that in general $\beta_{23} \neq 0$.

Note also that some special contributions to the f-

dependence like light-by-light scattering diagrams in QED, which form a class of RG-invariant diagrams, may and probably should be treated separately.

Let us now consider the condition for r_2 to be f-independent. Inserting eqs. (12) and (13) into eq. (10) and taking $r_1 = r_1^* = r_{10}^*$ f-independent, one gets a set of relations which determine the corresponding coefficient $r_2 = r_{20} + r_{21}f + r_{22}f^2$ at $\mu = \mu_*$, namely $\bar{r}_2 = \bar{r}_{20} + \bar{r}_{21}f + \bar{r}_{22}f^2$:

$$\bar{r}_{22} = \frac{\tilde{\beta}_{23} - \beta_{23}}{\beta_{01}}, \qquad (14)$$

$$\bar{r}_{21} = \frac{\tilde{\beta}_{22} - \beta_{22}}{\beta_{01}} - \frac{\beta_{00}}{\beta_{01}^2}(\tilde{\beta}_{23} - \beta_{23}), \qquad (15)$$

$$\bar{r}_{20} - r_1^{*2} = \frac{\beta_{11}}{\beta_{01}}r_1^* - \frac{\beta_{00}}{\beta_{01}}\bar{r}_{21} + \frac{\tilde{\beta}_{21} - \beta_{21}}{\beta_{01}}, \qquad (16)$$

$$\bar{r}_{20} - r_1^{*2} = \frac{\beta_{10}}{\beta_{00}}r_1^* + \frac{\tilde{\beta}_{20} - \beta_{20}}{\beta_{00}}. \qquad (17)$$

Eqs. (16) and (17) allow to determine r_1^* and $\bar{r}_{20}$ in terms of the β-function coefficients [after eliminating $\bar{r}_{21}$ through eq. (15)]. One finds

$$r_1^* = \beta_{01}\frac{\tilde{\beta}_{20} - \beta_{20} - (\beta_{00}/\beta_{01})(\tilde{\beta}_{21} - \beta_{21})}{\beta_{11}\beta_{00} - \beta_{01}\beta_{10}}$$

$$+ \frac{\beta_{00}^2}{\beta_{01}}\frac{\tilde{\beta}_{22} - \beta_{22} - (\beta_{00}/\beta_{01})(\tilde{\beta}_{23} - \beta_{23})}{\beta_{11}\beta_{00} - \beta_{01}\beta_{10}}, \qquad (18)$$

with $\bar{r}_{20}$ following from eq. (17). Eq. (18) is simply the relation derived in ref. [20]:

$$r_1^* = -\frac{\tilde{\beta}_2 - \beta_2}{\beta_1}\Bigg|_{\beta_0 = 0} = -\frac{\tilde{\beta}_2 - \beta_2}{\beta_1}\Bigg|_{f = -\beta_{00}/\beta_{01}}, \qquad (19)$$

which follows more generally from taking the limit $\beta_0 \to 0$ ($f = -\beta_{00}/\beta_{01}$) on both sides of eq. (10) and assuming regularity of r_1 and r_2 in this limit. Eqs. (14)–(17) show that both r_1^* and $\bar{r}_2$ are determined in terms of β- and β_{eff}-function coefficients. Note, that requiring $r_1 = r_1^*$ does not imply in general that $\bar{r}_2$ is f-independent. This can be gained by putting $\bar{r}_{21} = 0$, $\bar{r}_{22} = 0$ in eqs. (14)–(17). This in turn is equivalent to the unique definition from eqs. (14) and (15) of the part of the NNLO coefficient β_2^* of the β-function in the prescription considered, namely to

$$\beta_{22}^* = \tilde{\beta}_{22}, \quad \beta_{23}^* = \tilde{\beta}_{23}. \qquad (20)$$

We are then left with *two* equations [eqs. (16) and (17) with $\bar{r}_{20} = r_2^*$, $\bar{r}_{21} = 0$, $\beta_{21} = \beta_{21}^*$ and $\beta_{20} = \beta_{20}^*$] to determine *four* unknowns $r_1^*, r_2^*, \beta_{21}^*$ and β_{20}^*. Even if one adapts the BLM choice of μ in the $\overline{\text{MS}}$ scheme $\mu^2 = \mu_*^2 = Q^2 \exp[-r_{11}^{\overline{\text{MS}}}(\mu = Q)/\beta_{01}]$, which is equivalent [20] to

$$r_1^* = r_{11}^{\overline{\text{MS}}}\bigg|_{f = -\beta_{00}/\beta_{01}}, \qquad (21)$$

one does not have enough equations to determine all unknowns. This simply means that, according to eq. (19), one should also (arbitrarily) fix β_2^* ($f = -\beta_{00}/\beta_{01}) = \beta_2^{\overline{\text{MS}}}$ ($f = -\beta_{00}/\beta_{01}$), which thus fixes one linear combination of β_{20}^* and β_{21}^* to be equal to its $\overline{\text{MS}}$ value. Note, that eq. (19) shows explicitly [20] that r_1^* is RS dependent through the β_2 ($f = -\beta_{00}/\beta_{01}$) term. This fact can also be seen from eq. (21), if one considers schemes where r_1 differs from $r_1^{\overline{\text{MS}}}$ by an f-independent constant.

One can then further require that both β_{20}^* and β_{21}^* be equal to their $\overline{\text{MS}}$ values, however, this looks rather artificial since

$$\beta_{22}^* = \tilde{\beta}_{22} \neq \beta_{22}^{\overline{\text{MS}}}, \quad \beta_{23}^* = \tilde{\beta}_{23} \neq \beta_{23}^{\overline{\text{MS}}} \equiv 0,$$

i.e., one would set a part of β_2^* to be $\beta_2^{\overline{\text{MS}}}$, and a part of it to be $\tilde{\beta}_2$! Further one we will not fix the value of r_1^* through eq. (21) but consider it as a free quantity.

Similar constraints on the BLM scheme β-function arise in higher orders. For instance, assuming the arbitrary f-independent values r_1^*, r_2^* and r_3^* and the existence in the β_3 β-functions coefficients of the f^4-terms, eq. (11) yields

$$\tilde{\beta}_{34} - \beta_{34}^* = 0, \quad \tfrac{1}{2}(\tilde{\beta}_{33} - \beta_{33}^*) + \tilde{\beta}_{23} r_1^* = 0,$$

$$\tfrac{1}{2}(\tilde{\beta}_{32} - \beta_{32}^*) + \tilde{\beta}_{22} r_1^* = 0, \qquad (22)$$

and

$$\beta_{01}(r_3^* - r_1^{*3})$$

$$= \tfrac{5}{2}\beta_{11} r_1^{*2} + (3\tilde{\beta}_{21} - 2\beta_{21}^*) r_1^* + \tfrac{1}{2}(\tilde{\beta}_{31} - \beta_{31}^*), \quad (23)$$

$$\beta_{00}(r_3^* - r_1^{*3})$$

$$= \tfrac{5}{2}\beta_{10} r_1^{*2} + (3\tilde{\beta}_{20} - 2\beta_{20}^*) r_1^* + \tfrac{1}{2}(\tilde{\beta}_{30} - \beta_{30}^*). \quad (24)$$

Since β_2^* is already fixed in the previous step in terms of r_1^*, r_2^*, eq. (22) and eqs. (23), (24) clearly determine as expected β_3^* for given r_1^*, r_2^* and r_3^*.

Note, that as well as in the NNLO we have more unknowns $(\beta_{30}^*, \beta_{31}^*, \beta_{32}^*, \beta_{33}^*, \beta_{34}^*, r_3^*)$ then available equations (*five*).

A new feature which appears first at this order is that when r_1^* is determined in terms of β-functions by eq. (18) and eqs. (23), (24) this implies one relation between β_{30}^* and β_{31}^* (whatever the value of r_3^*) which cannot thus be taken as free parameters (contrary to the situation with β_{20}^* and β_{21}^*). In fact, taking the limit $\beta_0 \to 0$ ($f \to -\beta_{00}/\beta_{01}$) on both sides of eq. (11), and assuming only regularity of r_1, r_2 and r_3 in this limit, one easily derives [using eq. (19)] the general relation

$$(\beta_1 \beta_3 - \beta_2^2)\bigg|_{\beta_0 = 0} = (\tilde{\beta}_1 \tilde{\beta}_3 - \tilde{\beta}_2^2)\bigg|_{\beta_0 = 0}, \qquad (25)$$

i.e., $\beta_1 \beta_3 - \beta_2^2$ is RS invariant when $\beta_0 = 0$ or $f = -\beta_{00}/\beta_{01}$.

Eq. (25) implies the above mentioned relation between β_{30}^* and β_{31}^*. A more systematic choice of free parameters to all orders, and one which has an analogue in the QED case (see below), might then consist in taking r_1^* (i.e., β_2^* for $f = -\beta_{00}/\beta_{01}$) and $\partial \beta_i^*/\partial f$ ($i \geq 2$) for $f = -\beta_{00}/\beta_{01}$ as the free quantities (of course one might also use another set, e.g. r_1^* and β_{i1}^*). However, these choices do not help to find a natural physical way to fix them either.

In summary, this first proposed extension of the BLM approach is ambiguous, since it does not allow to fix in a unique way the f-independent coefficients r_i^* for any physical quantity R. Given this situation, the simplest and in our opinion most sensible way out remains to take $r_i^* = 0$ (hence $\beta_i^* = \tilde{\beta}_i$) and thus use the effective scheme prescription (note that this is not proposed to be achieved by a mere choice of a renormalization point μ in a given RS as was incorrectly stated at the end of ref. [1]). The motivations for the effective charge scheme as the RS-invariant method in QCD was argued from more general viewpoints in refs. [5–9].

3. In the above proposed extension of the BLM scheme we tried to adapt the RS to each observable $R(Q)$ in order to obtain f-independent coefficients. This required to fix both the scale μ and the β-function coefficients β_i, $i \geq 2$ in accordance with a given quantity R. Now we will consider a different possi-

bility, namely to stick to a *fixed* definition of the coupling (i.e., work with the given β_i, the same for all R), but to use different scales at each order of perturbation theory, to achieve f-independence. We will show that this alternative does not work in general. To see this, let us introduce a set of scales μ_i, such that μ_i is fixing the scale of the coupling in the $(i+1)$st order, and put $\mu_0 = \lambda_i \mu_i$, $i \geqslant 1$.

Let us now use the standard RG relation $a(\mu_0) = a(\lambda_1 \mu_1) = a(\mu_1)[1 - \beta_0 \ln \lambda_1^2 a(\mu_1) + ...]$ and present the expression for R in the following way:

$$R(Q) = a(\mu_0)[1 + r_1 a(\mu_0) + r_2 a^2(\mu_0) + ...]$$
$$= a(\mu_0)[1 + r_1 a(\mu_1) + (r_2 - r_1 \beta_0 \ln \lambda_1^2) a^2(\mu_1) + ...]. \tag{26}$$

Choosing now $\mu_0 = \mu_*$ (the BLM prescription), hence $r_1 = r_1^*$, it can be shown that this expression and the requirement that $\tilde{r}_2 = r_2^* - r_1^* \beta_0 \ln \lambda_1^2$ be f-independent give the following set of conditions:

$$\tilde{\beta}_{23} - \beta_{23} = 0 \tag{27}$$

and

$$\tilde{\beta}_{22} - \beta_{22} = \beta_{01}^2 \ln \lambda_1^2 r_1^* ,$$

$$\beta_{01}(\tilde{r}_2^* - r_1^{*2}) = (\beta_{11} - 2\beta_{00}\beta_{01} \ln \lambda_1^2) r_1^* + \tilde{\beta}_{21} - \beta_{21} ,$$

$$\beta_{00}(\tilde{r}_2^* - r_1^{*2}) = (\beta_{10} - \beta_{00}^2 \ln \lambda_1^2) r_1^* + \tilde{\beta}_{20} - \beta_{20} . \tag{28}$$

From eq. (27) one sees that one cannot use a fixed scheme since $\tilde{\beta}_{23} \neq \beta_{23}$ in general. We conclude this alternative extension of the BLM scheme does not work owing to the eventual presence of $O(f^3)$-terms in $\tilde{\beta}_2$.

4. We will now show that using the definition of the scale parameter, mentioned in the original BLM paper [1], one can get rid of the f-dependent terms in r_2 starting from the MS-like schemes and thus resolve the ambiguities found in sections 2,3. Indeed, let us define μ^2 in accordance with ref. [1] as

$$\mu^2 = \mu_0^2[1 + \gamma_1(f)a(\mu_0) + ...] , \tag{29}$$

where $\gamma_1(f)$ are f-dependent functions and μ_0^2 is the usual scale parameter [*1]. Using the explicit dependence of r_1 upon Q^2/μ^2, namely

[*1] This possibility has been pointed out to us by Brodsky and Hung Jung Lu. For discussions of the general questions, complementary to our analysis, within the framework of the dressed skeleton expansion, see ref. [3].

$$r_1 = r_1(Q^2 = \mu^2) - \beta_0 \ln \frac{Q^2}{\mu^2} , \tag{30}$$

one can show that the shift $\mu_0^2 \rightarrow \mu^2$ of eq. (29) is equivalent to the following redefinition of the coefficients r_2:

$$r_2 = r_2 + \beta_0 \gamma_1(f) . \tag{31}$$

Assuming now the polynomial dependence of $\gamma_1(f)$,

$$\gamma_1(f) = \gamma_{10} + \gamma_{11}f , \tag{32}$$

we obtain from eq. (10) the following system of linear equations [analogous to those of eqs. (14)–(16) and of eqs. (27)–(28)] which comes from the requirement that r_2^* must be f-independent:

$$\tilde{\beta}_{23} - \beta_{23} = \beta_{01}^2 \gamma_{11} , \tag{33}$$

$$\tilde{\beta}_{22} - \beta_{22} = \beta_{01}^2 \gamma_{10} + 2\beta_{00}\beta_{01}\gamma_{11} , \tag{34}$$

$$\tilde{\beta}_{21} - \beta_{21}$$
$$= \beta_{01}(r_2^* - r_1^{*2}) - \beta_{11}r_1^* + 2\beta_{00}\beta_{01}\gamma_{10} + \beta_{00}^2 \gamma_{11} , \tag{35}$$

$$\tilde{\beta}_{20} - \beta_{20} = \beta_{00}(r_2^* - r_1^{*2}) - \beta_{10}r_1^* + \beta_{00}^2\gamma_{10} . \tag{36}$$

It can be shown that this system is compatible with the relation obtained in eqs. (18), (19) for the NLO coefficient r_1^* in the BLM prescription.

One can now see that using the definition of the scale parameter of eq. (29) we can absorb the f^3-coefficient $\tilde{\beta}_{23}$ of the $\beta_{\rm eff}$-function even within the MS-like schemes (contrarily to the cases discussed in sections 2,3). Indeed, putting $\beta_{23} = \beta_{23}^{\overline{\rm MS}} = 0$, $\beta_{2i} = \beta_{2i}^{\overline{\rm MS}}$ ($0 \leqslant i \leqslant 2$), we have *four* equations [eqs. (33)–(36)] to determine *four* unknowns ($\gamma_{11}, \gamma_{10}, r_1^*, r_2^*$). Therefore, the ambiguities found can be fixed at the NNLO.

5. Let us apply our theoretical considerations to the analysis of the NNLO perturbative QCD approximation of the quantity $R(s) = \sigma_{\rm tot}({\rm e^+e^-} \rightarrow {\rm hadrons})/\sigma({\rm e^+e^-} \rightarrow \mu^+\mu^-)$. In the $\overline{\rm MS}$ scheme the result recently obtained reads [15] (see also ref. [16])

$$R(s) = 3 \sum Q_f^2 [1 + a + (1.986 - 0.115f)a^2$$
$$+ (-6.637 - 1.200f - 0.005f^2)a^3 + ...]$$
$$- \left(\sum Q_f\right)^2 \cdot 1.239 a^3 , \tag{37}$$

where Q_f are the quark charges and $a=a(\mu_{\overline{MS}}^2=s)$. Using now the results of calculations of the coefficients of the β-function in the MS-like schemes at the three-loop level [18], reminded now in the numerical form

$$\beta_0 = 2.75-0.167f, \quad \beta_1 = 6.375-0.792f,$$

$$\beta_2 = 22.320-4.370f+0.094f^2, \qquad (38)$$

we obtain the numerical expression of the scheme-invariant $\tilde{\beta}_2$ without taking into account proportional to $(\Sigma Q_f)^2$ light-by-light-type contribution to $R(s)$:

$$\tilde{\beta}_2 = -19.422-2.334f+0.076f^2+0.003f^3 \qquad (39)$$

(compare with the similar expression obtained in ref. [15]). Fixing now the NNLO ambiguities of the BLM approach in accordance with the discussions of section 4, we get the following numerical values of the scale fixing parameters introduced in eqs. (30), (31):

$$\mu_0^2 = \mu_{BLM}^2 = \mu_{\overline{MS}}^2 \exp(0.69),$$

$$\gamma_{11} = 0.11, \quad \gamma_{10} \approx 3, \qquad (40)$$

and the NNLO approximation of $R(s)$ in the considered generalization of the BLM approach:

$$R(s) = 3 \sum Q_f^2(1+a_*+0.08a_*^2-23.3a_*^3)$$

$$-\left(\sum Q_f\right)^2 \cdot 1.24a_*^3, \qquad (41)$$

where $a_* = a\{\mu_0^2[1+\gamma_1(f)a(\mu_0^2)]=s\}$.

6. Let us now discuss the QED case. The corresponding results can be obtained from the QCD ones after putting $C_A=0$, $C_F=1$ and $Tf=N$ where C_A and C_F are the corresponding Casimir operators, $T=\frac{1}{2}$ and N is the number of lepton types. As a result, in QED $\beta_{i0} \equiv 0$, which represents the main difference for the present discussion with respect to the QCD case. Since $\beta_{i0} \equiv 0$, there is no analogue in QED of the relation of eq. (19), so that r_i^* is not fixed in terms of β-function coefficients. Otherwise, the main conclusions remain similar: one can find RS with n-independent coefficients r_i^* for a given $R(Q)$ either by assigning arbitrary values to the r_i^* themselves, or, alternatively consider r_i^* and the coefficients β_{i1}^* of the terms linear in N in the β-function coefficients as the free parameters, as is clear from eqs. (14)–(16) with $\beta_{00}=0$.

We note that in general the β_{i1}-coefficient $(i \geqslant 0)$

are not RS invariant, even in QED: the known fact that in QED $\beta_{i1}^{MS} = \beta_{i1}^{MOM} = \beta_{i1}^{OS}$ for the MS-like, the momentum (MOM) and the on-shell (OS) subtraction schemes is a consequence of the absence of an $O(N^0)$-term in the expression coefficients r_i for the standard QED invariant charge $\alpha_{inv}(Q^2) = \alpha_{MOM}(Q^2) = \alpha/(1+\alpha\Pi) = \alpha(1+r_1\alpha+ ...)$, where α corresponds to the renormalized coupling constant of the MS-like or OS schemes, so that $\lim r_i(N \to 0)=0$, and $\alpha_{inv} \equiv \alpha_{MOM}$ coincides with $\alpha_{\overline{MS}}$ (or α_{MS}) and α_{OS} in the limit $N \to 0$. This statement follows immediately from the QED analogue of eq. (10) [or eq. (35)], which gives in the limit $N \to 0$ (or $\beta_{00}=0$)

$$r_{20} = r_{10}^2 + \frac{\beta_{11}}{\beta_{01}} r_{10} + \frac{\tilde{\beta}_{21}-\beta_{21}}{\beta_{01}}, \qquad (42)$$

so that $r_{10}=r_{20}=0$ clearly implies $\tilde{\beta}_{21}=\beta_{21}$. In general, however, r_{10} and r_{20} do not vanish (note that r_{10} is just the BLM coefficient r_1^* in QED!) and consequently $\tilde{\beta}_{21} \neq \beta_{21}$. This has been explicitly checked to be the case for the QED part of the NNLO QCD result [15] for $R(s)$ (see also ref. [16]). The result $r_{10}=r_1$ $(N=0) \neq 0$ also holds in the case of the $(g-2)_\mu$ quantity, calculated in the OS scheme at the four-loop level [22] and discussed in ref. [1].

Taking now the free parameters to be r_{10} and the β_{10}^*, one can choose $\beta_{i1}^* = \beta_{i1}^{MS} \equiv \beta_{i1}^{OS}$. Then, only r_{10} remains free and sensible choices are $r_{10}=r_1^*$ (which will insure the corresponding RS coupling α_* to coincide with $\alpha_{\overline{MS}}$, α_{OS} and α_{inv} as $N \to 0$) or even $r_1^* = 0$. Note that $r_1^*=0$ does not in general imply $r_i^*=0$ $(i \geqslant 2)$ if $\beta_{i1}^* = \beta_{i1}^{MS} \neq \tilde{\beta}_{i1}$, i.e., we will not recover in this case the effective charge prescription (which is equivalent, under the assumption of N-independent r_i, to $r_i^*=0$ and $\beta_{i1}^* = \tilde{\beta}_{i1}$). However, given the other relations like $\beta_{23}^* = \tilde{\beta}_{23}$ and $\beta_{22}^* = \tilde{\beta}_{22}$ (which are necessary for N-independence within the framework of considerations of section 2) the most natural choice for this purpose in this case remains the effective charges prescription.

However, as in QCD one can consider the alternative higher order extension of the BLM scheme which uses a coupling constant dependent definition of the scale via eq. (29) (see ref. [1]). In this case it is possible to achieve N-independence at the NNLO by fixing $\gamma_1(f)$-function coefficients through eqs. (33), (34) with $\beta_{00}=0$, $\beta_{23}=0$ [it should be stressed that, as well as in the QCD case, the corresponding coefficients of the $\beta_{eff}(R)$-functions, namely $\tilde{\beta}_2$ contain the

additional N^3 terms $\tilde{\beta}_{23}$] and $\beta_{22} = \beta_{22}^{\overline{MS}}$ or $\beta_{22} = \beta_{22}^{OS}$. Therefore, as well as in QCD, one can realize the BLM ideas starting from the initial $\overline{MS}$ or OS schemes. The concrete QED examples will be considered elsewhere.

7. Finally, we remark that the question of reliability of perturbative QCD is a priori distinct from the problem of fixing the ambiguities in the BLM approach in higher orders. We emphasize that in the framework of the effective charges scheme the applicability of perturbative QCD to a given physical quantity requires only the applicability of perturbation theory to the RS invariant equations (6), (7). This in turn depends solely on the value of the ratio Q^2/Λ_{eff}^2 and on the perturbative behaviour of the $\beta_{eff}(R)$-function, which has no straightforward relation to the value of r_1^* (except in the QCD case at $f = -\beta_{00}/\beta_{01}$ through the relation of eq. (20) derived in ref. [20]). If the expansion of the $\beta_{eff}(R)$-function for some possible observed values of R happens to converge poorly (this sometimes happens for not so large values of α_s, see in particular the discussions of the characteristics of the NNLO approximations for $H^0 \rightarrow$ hadrons [14,13] and $e^+e^- \rightarrow$ hadrons [15]) the transformation of the results from the effective charges scheme to the BLM-type scheme where $r_i^* \neq 0$ will simply transfer the bad convergence properties in the lower region of energies from the $\beta_{eff}(R)$-series to the $R(a^*)$-series (compare eq. (39) and eq. (41), see also the independent discussions of similar topics in ref. [8] and ref. [13]). Moreover, even if the NLO BLM condition indicates large values of r_1^* one can use a variant of the effective charge scheme, proposed in ref. [23].

and in the foundation of the prescription discussed in sections 4, 5 to extend the BLM approach to the NNLO of perturbative QCD.

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