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STRUCTURAL EVALUATION OF CONTROL CONFIGURATIONS FOR MULTIVARIABLE NONLINEAR PROCESSES

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Abstract—This paper addresses the problem of evaluation of alternative control configurations on the basis of structural characteristics of the process. Relative order is proposed as the main analysis tool for this purpose. Using tools from graph theory, it is shown that generic calculation of relative orders requires only structural information about the process. Relative order is interpreted as a structural measure of the initial sluggishness of the response, as well as a structural analog of dead time, which expresses fundamental structural limitations in the control quality. A matrix of relative orders of input/output pairs is introduced, which leads to a characterization of structural coupling among input and output process variables. On the basis of the above properties, general structural evaluation guidelines are proposed for alternative sets of manipulated inputs and alternative input/output pairs. The application of the theory is illustrated in the case of an evaporation unit, a chemical reactor and a network of heat exchangers.

INTRODUCTION

The synthesis of control configurations for multivariable processes has been recognized as an important issue and has been investigated from various points of view in recent years [see e.g. Stephanopoulos (1983)]. Mainly for methodological purposes, the synthesis of a control configuration can be viewed as consisting of the following two sub-problems:

(1) generation of all feasible control configurations,
(2) evaluation and selection of a control configuration.

The first sub-problem includes the specification of the control objectives, the identification of the available manipulated inputs and the assessment of feasibility of the resulting control configurations. Research in this area is extensive regarding linear time-invariant processes, for which the system-theoretic properties of state controllability, output controllability and output functional controllability have been used as feasibility criteria. On the other hand, research regarding nonlinear processes is still at the stage of understanding the corresponding system-theoretic properties. In analogy with linear results, right invertibility, a concept closely related to output functional controllability, is the criterion that determines the feasibility of control configurations for most practical purposes. The first attempts to study this issue for general multi-input/multi-output (MIMO) nonlinear systems have been within the framework of algorithmic procedures for the construction of inverses (Hirschorn, 1979, 1981; Singh, 1982a, b, c). In a differential–algebraic framework (Fliess, 1985, 1986), the notion of differential output rank has generalized the notion of rank of a transfer matrix in a nonlinear setting and has led to necessary and sufficient rank conditions for invertibility, analogous to the ones for linear systems. Finally, conditions for right invertibility for a particular class of nonlinear systems have also been derived in terms of the "structure at infinity" (Nijmeijer, 1986). The implications of the above theoretical results, however, in the synthesis of control configurations have not been investigated yet.

Given a number of alternative feasible control configurations, the second sub-problem consists in the evaluation of the alternative control configurations and the final selection of the one to be employed. In this direction, the majority of research effort for processes described by linear models concerns (a) dynamic resilience and (b) decentralized control studies. Dynamic resilience studies have mainly focused on identifying factors that pose limitations on the system invertibility (Morari, 1983) and consequently on the achievable control quality. Such factors include dead time (Holt and Morari, 1985a), right-half-plane zeros (Holt and Morari, 1985b), model uncertainty (Skogestad and Morari, 1987), etc. In decentralized control studies, a variety of static and dynamic interaction measures have been proposed for identifying favorable pairings of manipulated inputs and controlled outputs [for a review see Jensen *et al.* (1986)]. By far the most popular analysis tool for this purpose is the relative gain array (RGA) (Bristol, 1966) and its generalizations that take into account dynamic considerations [e.g. Tung and Edgar (1981) and Gagnepain and Seborg (1982)] or disturbance inputs [e.g. Stanley *et al.* (1985)]. Singular values have also found powerful application in this direction [e.g. Morari (1983)]. All the above approaches assume a transfer function description of the process, often obtained from experimental data, and therefore are based on linear control considerations. On the other hand, in

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nonlinear process control theory, there are essentially no results related to the problem of evaluation of control configurations except for some results concerning the calculation of nonlinear gains [e.g. Mijares *et al.* (1985) and Manousiouthakis and Nikolaou (1989)]. One possible direction is to study the effect of nonlinearities within a linear analysis (and consequently linear controller synthesis) framework. An alternative, much more meaningful, direction is to develop analytical tools and methodologies which arise from the nonlinear description of a process itself.

In the present work, we will introduce a structural perspective in the problem of evaluation of control configurations for multivariable nonlinear processes. Structural methods have already been introduced in the generation and assessment of feasibility of control configurations for linear processes (Morari and Stephanopoulos, 1980; Govind and Powers, 1982; Johnston and Barton, 1985; Johnston *et al.*, 1985; Russel and Perkins, 1987; Georgiou and Floudas, 1989). They are essentially based on graph-theoretic concepts and the notion of structural controllability (Lin, 1974; Shields and Pearson, 1976; Glover and Silverman, 1976). The major advantage of these methods is the genericity of the results and the minimum amount of process information that they require, which allows them to be efficiently used at the early stages of the design procedure. There has not been any attempt, however, to systematically introduce structural considerations in the evaluation and selection of control configurations either for linear or for nonlinear processes. On the other hand, intuitive guidelines for the selection and pairing of manipulated inputs do make implicit use of structural considerations, through the notions of "direct effect" and "physical closeness" [see e.g. the modern process control textbooks by Stephanopoulos (1984) and Seborg *et al.* (1989)]. The basic idea is that by choosing a manipulated input which is *physically close* to a controlled variable (or has a *direct effect* on it), we have good chances of obtaining favorable static and dynamic characteristics for the particular input/output pair, i.e. small time delays, small time constants as well as significant static gains. However, it is evident that, as the size and complexity of the process increase, such intuitive considerations become obscure and sometimes misleading, especially in a MIMO context. Furthermore, there is no theoretical justification for the use of such intuitive notions as evaluation criteria. The above discussion motivates the need for the development of quantitative formulations for the above intuitive notions, as well as a systematic structural evaluation framework with a sound theoretical basis. To this end, the present work deals with general nonlinear processes and its purpose is:

(1) to identify and quantify structural characteristics that pose fundamental limitations on the control quality,
(2) to develop guidelines for the structural evaluation of alternative control configurations based on control quality characteristics and structural coupling considerations.

The above guidelines will allow a systematic hierarchization of alternative control configurations at the early stages of the design procedure, based on a minimum amount of process information. Quantitative, static and dynamic, process information can be used at later stages of the design procedure to complement the results of the structural analysis. Standing assumptions throughout the paper will be that:

(1) We are dealing with the control of a single processing unit.
(2) Operational, environmental, economical, safety and production requirements have resulted in a set of control objectives (controlled outputs).
(3) The major disturbances have been identified (from physical considerations and possibly from steady-state gain information).
(4) The physical phenomena with non-negligible dynamics have been identified.

The term "alternative control configurations" will then imply alternative sets of manipulated inputs, while the term "multi-loop configuration" will be used to denote the specification of input/output pairs for a given set of manipulated inputs. In general, disturbance inputs that can be manipulated may also be considered as manipulated input candidates. Each control configuraton will correspond to a state–space process model of the form

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j + \sum_{\kappa=1}^{p} w_\kappa(x)d_\kappa$$
$$y_i = h_i(x),\ i = 1, \ldots, m \tag{1}$$

where f, g_j, w_κ are analytic vector fields on $\mathbb{R}^n$, h_i are analytic scalar fields on $\mathbb{R}^n$, $x \in \mathbb{R}^n$ denotes the state vector, and d_κ, u_j, $y_i \in \mathbb{R}$ denote the disturbance, manipulated input and output variables, respectively, expressed in deviations from some nominal values. For simplicity, we will assume equal number of manipulated inputs and control outputs.

Starting with process models of the above general form, we will first introduce our main analysis tool, the concept of relative order. Then, we will review the notion of the directed graph (digraph) representation of a process, closely related to the notion of structural models; using tools from graph theory, it will be shown that generic calculation of relative orders requires only the process digraph. Relative order will then be interpreted as a structural measure of the initial sluggishness of the response. This will lead to an alternative interpretation of relative order as a structural analog of dead time and will allow quantifying the notions of "direct effect" and "physical closeness" through the concept of relative order. In the following section, we will discuss the fundamental limitations that the structure of a process poses on the control quality, as expressed by relative orders; this will naturally lead to guidelines for the structural evaluation of control configurations on the basis of the overall servo and regulatory characteristics. Then, a matrix of relative orders will be introduced, which will allow quantifying structural coupling among input and output variables; the analysis will naturally lead to guidelines for evaluating alternative multiloop configurations, based on structural coupling considerations. Finally, chemical engineering examples will illustrate the application of the proposed generic evaluation framework.

RELATIVE ORDER: A FUNDAMENTAL STRUCTURAL CONCEPT

Definitions

In what follows, we will refer to general nonlinear processes with a model of the form of eq. (1), and we will be using the standard Lie derivative notation. In particular, the Lie derivative of the scalar field $h_i(x)$ with respect to the vector field $f(x)$ is defined as $L_f h_i(x) = \sum_{l=1}^{n} (\partial h(x)/\partial x_l) f_l(x)$, where $f_l(x)$ denotes the lth row element of $f(x)$. Note that $L_f h_i(x)$ is a scalar field itself; one can, therefore, define higher-order Lie derivatives $L_f^k h_i(x) = L_f L_f^{k-1} h_i(x)$ as well as mixed Lie derivatives $L_{g_j} L_f^{k-1} h_i(x)$ in an obvious way. We can now proceed with the definitions of the various concepts of relative order.

Definition 1: *The relative order* r_i *of the output* y_i *with respect to the manipulated input vector* u *is defined as the smallest integer for which*

$$[L_{g_1} L_f^{r_i-1} h_i(x) \cdots L_{g_m} L_f^{r_i-1} h_i(x)] \not\equiv [0 \cdots 0] \quad (2)$$

or $r_i = \infty$ *if such an integer does not exist.*

The above concept of relative order has been introduced and proved very important in a multivariable controller *synthesis* framework [e.g. Ha and Gilbert (1986) and Kravaris and Soroush (1990)], since it captures the dynamic effect of the manipulated input vector on the process outputs. In an *analysis* framework, however, it is also meaningful to consider the effect of each one of the manipulated input variables on the outputs. For this reason, we introduce here a natural generalization of the concept of relative order for single-input/single-output (SISO) systems in a MIMO context.

Definition 2: *The relative order* r_{ij} *of the output* y_i *with respect to a manipulated input* u_j *is defined as the smallest integer for which*

$$L_{g_j} L_f^{r_{ij}-1} h_i(x) \not\equiv 0 \quad (3)$$

or $r_{ij} = \infty$ *if such an integer does not exist.*

Remark 1: In analogy with SISO case, it can be easily verified that $r_{ij} \leq n$, whenever r_{ij} is finite.

Remark 2: The following relation between r_i and r_{ij} is a direct consequence of definitions 1 and 2:

$$r_i = \min(r_{i1}, r_{i2}, \ldots, r_{im}).$$

Based on the above relation, the relative orders r_i can easily be calculated from the individual relative orders r_{ij}.

In analogy with definition 2, one can also define a concept of relative order between a controlled output and a disturbance input. Such a concept has been introduced in a feedforward/feedback controller synthesis setting (Daoutidis and Kravaris, 1989; Daoutidis *et al.*, 1990), leading to the solution of the nonlinear feedforward/state-feedback control problem.

Definition 3: *The relative order* $\rho_{i\kappa}$ *of the output* y_i *with respect to the disturbance input* d_κ *is defined as the smallest integer for which*

$$L_{w_\kappa} L_f^{\rho_{i\kappa}-1} h_i(x) \not\equiv 0 \quad (4)$$

or $\rho_{i\kappa} = \infty$ *if such an integer does not exist.*

In what follows, unless otherwise stated, we will refer to the concept of relative order meaning the individual relative order between an input/output pair (definitions 2 and 3).

Remark 3: For the special case of a MIMO linear system of the form

$$\dot{x} = Ax + \sum_{j=1}^{m} b_j u_j + \sum_{\kappa=1}^{p} \gamma_\kappa d_\kappa$$

$$y_i = c_i x, \quad i = 1, \ldots, m,$$

r_{ij} is the smallest integer for which

$$c_i A^{r_{ij}-1} b_j \neq 0,$$

while $\rho_{i\kappa}$ is the smallest integer for which

$$c_i A^{\rho_{i\kappa}-1} \gamma_\kappa \neq 0.$$

Furthermore, the relative order between any input/output pair is equal to the difference between the degrees of the denominator and the numerator polynomials of the corresponding transfer function.

Relative orders, graph theory and the notion of "direct effect"

The state–space model of a process described by eq. (1) can also be associated with a digraph, defined by a set of vertices (or nodes) and a set of edges as follows:

- The vertex set consists of the set of manipulated inputs $(u_1, \ldots, u_m)$, the set of disturbance inputs $(d_1, \ldots, d_p)$, the set of state variables $(x_1, \ldots, x_n)$ and the set of output variables $(y_1, \ldots, y_m)$.
- The set of edges consists of directed lines connecting two vertices according to the following rules:
  - —If $\partial f_l(x)/\partial x_k \not\equiv 0$, $k, l = 1, \ldots, n$, then there is an edge from x_k to x_l,

—If $g_{jl}(x) \not\equiv 0$, $l = 1, \ldots, n$, then there is an edge from u_j to x_l,
—If $w_{\kappa l}(x) \not\equiv 0$, $l = 1, \ldots, n$, then there is an edge from d_κ to x_l,
—If $\partial h_i(x)/\partial x_k \not\equiv 0$, $k = 1, \ldots, n$, then there is an edge from x_k to y_i,

where $f_l(x)$, $g_{jl}(x)$, $w_{\kappa l}(x)$ denote the lth element of the vector fields $f(x)$, $g_j(x)$ and $w_\kappa(x)$, respectively.

A path of a digraph is a particular directed sequence of some of its edges, such that the initial vertex of the succeeding edge is the final vertex of the preceding edge. The number of edges contained in a path is called the *length* of the path [for a detailed review of notions of graph theory see e.g. Ore (1962)].

It can be easily seen from the above rules that the digraph representation of a dynamic system contains much less information than its detailed state–space description. In particular, for nonlinear processes with a model of the form of eq. (1), their digraph representation contains no information about:

(1) the dependence of the vector fields g_j and w_κ on x
(2) the exact functional dependence of the vector field f on x
(3) the numerical values of the process parameters.

In fact, a digraph representation contains only the pattern of interdependencies among the process variables and is uniquely determined by them. This pattern of interdependencies can also be expressed through the notion of a structural model, associated with the well-known notion of structural (or structured) matrices [e.g. Shields and Pearson (1976)].

Figure 1 provides a typical illustration of a digraph corresponding to the class of dynamic systems with a structural model of the form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, x_3) + g_1(x)u \\
\dot{x}_2 &= f_2(x_1, x_2) \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + w_3(x)d \\
y &= h(x_2).
\end{aligned} \tag{5}$$

Applying definitions 2 and 3 for the calculation of the relative orders between u and y and between d and y, one easily finds that $r = 2$ and $\rho = 3$. Referring to the digraph of the above system in Fig. 1, it is also easily seen that the shortest path between u and y has a length equal to 3, while the shortest path between d and y has a length equal to 4. The above example suggests an interesting connection between relative orders and length of paths in a digraph. This connection will be rigorously established in theorem 1 that follows, which generalizes a similar result by Kasinski and Levine (1984). The proof of theorem 1 is given in Appendix A.

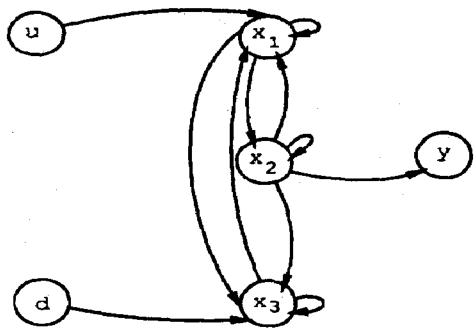

Fig. 1. A typical digraph.

Theorem 1: *Consider a nonlinear system in the form of eq.* (1) *and its corresponding digraph. Let* ℓ_{ij} *and* $\ell_{i\kappa}$ *denote the length of the shortest paths connecting* u_j *and* y_i, *and* d_κ *and* y_i, *respectively. Also, let* r_{ij} *and* $\rho_{i\kappa}$ *be the relative orders between* u_j *and* y_i, *and* d_κ *and* y_i, *respectively. Then, the following relations hold generically*: $r_{ij} = \ell_{ij} - 1$ *and* $\rho_{i\kappa} = \ell_{i\kappa} - 1$.

Remark 4: The term "generically" in the above theorem means that the result holds for all vector fields f, g_j, w_κ and all scalar fields h_i, except possibly for a "set of measure zero". Nongeneric situations in the calculation of relative orders through the digraph may arise because of the specific nonlinear dependence of the vector and scalar fields on x. A number of important observations arise from theorem 1:

- Firstly, the result of theorem 1 establishes that the generic calculation of relative orders for a process requires knowledge of its structural model only, or equivalently its digraph, i.e. the lowest level of information about the process. This fact makes the relative order a generic analysis tool, suitable for design purposes, and as such it will be used in the context of this work.
- Furthermore, it is clear from the definition of a graph that, except from the edges connecting state and output vertices, every other edge denotes the effect of one variable on another through an integration step. Therefore, the result of theorem 1 leads to a graph-theoretic interpretation of relative order as the number of integrations that an input has to go through before it affects an output, generalizing the well-known SISO result obtained through the Byrnes–Isidori normal form. In the above sense, relative order is a rigorous and meaningful measure of how direct effect an input variable has on an output variable. Theorem 2 in the next section will illustrate how this notion of direct effect manifests itself in typical response characteristics.
- Finally, the result of theorem 1 can be used to increase the efficiency of calculation of relative orders in a symbolic manipulation environment, especially for large-scale systems.

Remark 5: For linear systems, the existence of a finite relative order r_{ij} corresponds to the property of *accessibility* (Lin, 1974) of the output node y_i from the input node u_j. To denote accessibility of an output node from a disturbance node, the term *disturbability* has been used (Shah *et al.*, 1977; Morari and Stephanopoulos, 1980), which obviously corresponds to a finite relative order between a disturbance input and an output.

Relative order: a measure of sluggishness

In this section, we will provide a rigorous interpretation of relative order as a *structural measure of sluggishness* of the response of dynamic systems. The main result is summarized in theorem 2 that follows (the proof is given in Appendix B):

Theorem 2: *Consider a nonlinear system in the form of eq.* (1) *at an initial condition* $x(0) = x_0$, *where* x_0 *is the nominal steady state. Also, let* r_{ij} *be the relative order of the output* y_i *with respect to the manipulated input* u_j. *Then, the initial response of the output* y_i *under a unit-step change at the input* u_j *can be approximated, for small times* t, *by*

$$y_i(t) \cong L_{g_j} L_f^{r_{ij}-1} h_i(x_0) \frac{t^{r_{ij}}}{r_{ij}!}. \tag{6}$$

Corollary 1: *For a linear SISO time-invariant system of the form*

$$\dot{x} = Ax + bu$$
$$y = cx \tag{7}$$

with r *being the relative order of the output* y *with respect to the manipulated input* u, *the small-time response of the output under a unit-step change at the input is given by*

$$y(t) \cong (cA^{r-1}b)\frac{t^r}{r!}. \tag{8}$$

Remark 6: The result of corollary 1 is already known and proved independently in standard linear control books (the independent proof is given in Appendix C, for completeness).

The result of theorem 2 establishes in a rigorous way that the relative order r_{ij} is a structural measure of how sluggish the response of the output y_i is for step changes at the input u_j: the larger the relative order, the more sluggish the response is. More specifically (see Fig. 2):

- $r_{ij} = 1$ implies that the initial slope of the response will be nonzero,
- $r_{ij} = 2$ implies that the initial slope of the response will be zero, but that its rate of change will be nonzero,
- $r_{ij} > 2$ implies that the initial slope of the response as well as its rate of change will be zero, while a higher-order derivative of the slope will be nonzero if r_{ij} is finite.

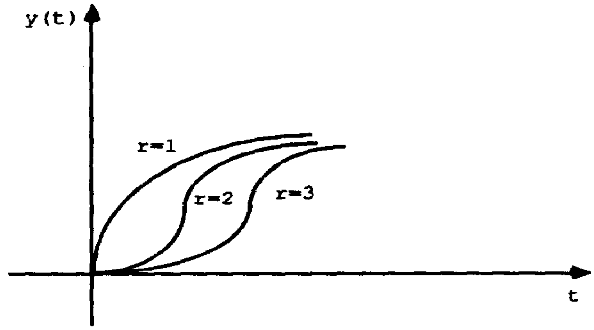

Fig. 2. Relative order as a measure of sluggishness.

Of course, the overall characteristics of the output response to an input change will also depend on:

- the time constant, which will determine how quickly the output will adjust to the input change (once it responds)
- the steady-state gain, which will determine the large-time value of the output.

As the time constant quantifies how "quick" the effect of an input variable is on an output variable and the static gain how "significant" this is, the relative order quantifies how "direct" this effect is.

Remark 7: A result similar to theorem 2 can be obtained for the relative order $\rho_{i\kappa}$, as well as for r_i. Clearly, r_i is a measure of the sluggishness of the output y_i with respect to the manipulated input vector, i.e. a measure of the maximum sluggishness of the response of the output y_i with respect to any of the manipulated inputs.

Relative order, dead time and the notion of "physical closeness"

The analysis so far has indicated that the concept of relative order quantifies how "direct" the effect of an input variable is on an output variable and has demonstrated how this property affects the small-time response characteristics. In what follows, motivated by the previous discussion, we will associate the concept of relative order with apparent dead time, which has been traditionally used to capture small-time response characteristics. Consider a typical step response of the output of a process with dynamics higher than first order (Fig. 3). Along the lines of the above treatment and assuming negligible transportation delay (which is the most common case in a single processing unit), one can obtain a clear interpretation of the sigmoidal shape of the response: it is due to the presence of a higher than one relative order between the input and the output. When such a high-order process is approximated by a first-order lag plus dead-time model, the neglected dynamics gives rise to the dead time, which is therefore an apparent but not real quantity; although it provides a useful indication of how responsive the output is, it has no physical

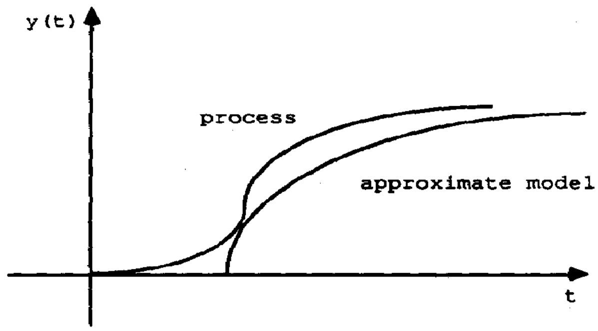

Fig. 3. Typical step response of a high-order process.

significance or rigorous justification. On the other hand, without any response data and based purely on structural information, one can rigorously assess the qualitative feature of the initial part of the response through the concept of relative order. It should be clear, therefore, that relative order represents the structural analog of apparent dead time. This analogy becomes obvious in the context of discrete linear systems, where the pole excess of the pulse transfer function (i.e. the relative order) is exactly the time delay of the process.

The above analogy between relative order and apparent dead time leads to an interpretation of relative order as a measure of "physical closeness" between an input variable and an output variable. An especially appealing illustration of this interpretation can be obtained in the case of staged processes (e.g. distillation columns, cascades of chemical reactors, etc.). Consider, for example, the cascade of two continuous stirred tank reactors (CSTR) shown in Fig. 4, where a second-order reaction A → B takes place. Under standard assumptions, the material and energy balances that describe the dynamic behavior of this process take the following form:

$$
\begin{aligned}
\frac{dc_{A1}}{dt} &= \frac{F}{V}(c_{A0} - c_{A1}) - k_0 e^{(-E/RT_1)} c_{A1}^2 \\
\frac{dc_{A2}}{dt} &= \frac{F}{V}(c_{A1} - c_{A2}) - k_0 e^{(-E/RT_2)} c_{A2}^2 \\
\frac{dT_1}{dt} &= \frac{F}{V}(T_0 - T_1) + \frac{(-\Delta H)}{\rho C_p} k_0 e^{(-E/RT_1)} c_{A1}^2 + \frac{1}{V\rho C_p} Q_1 \\
\frac{dT_2}{dt} &= \frac{F}{V}(T_1 - T_2) + \frac{(-\Delta H)}{\rho C_p} k_0 e^{(-E/RT_2)} c_{A2}^2 + \frac{1}{V\rho C_p} Q_2
\end{aligned}
\tag{9}
$$

where

C_p = heat capacity
E = activation energy
F = volumetric flow rate
Q_1, Q_2 = heat inputs to tanks 1 and 2
T_1, T_2 = temperatures in tanks 1 and 2
T_0 = inlet temperature
V = volume
$-\Delta H$ = heat of reaction
c_{A1}, c_{A2} = molar concentration of A in tanks 1 and 2
c_{A0} = inlet molar concentration of A
k_0 = Arrhenius frequency factor
ρ = density.

From the dynamic model of eq. (9), one can easily obtain the digraph of the process, which is shown in Fig. 5. Suppose that we wish to control the concentration at the exit of the second reactor, c_{A2}, and available manipulated inputs are the heat inputs to the reactors, Q_1 and Q_2. For notational consistency, set $y_1 = c_{A2} - c_{A2s}$ and $u_1 = Q_1 - Q_{1s}$, $u_2 = Q_2 - Q_{2s}$ for the alternative manipulated inputs, where the subscript s denotes a nominal steady-state value. Based

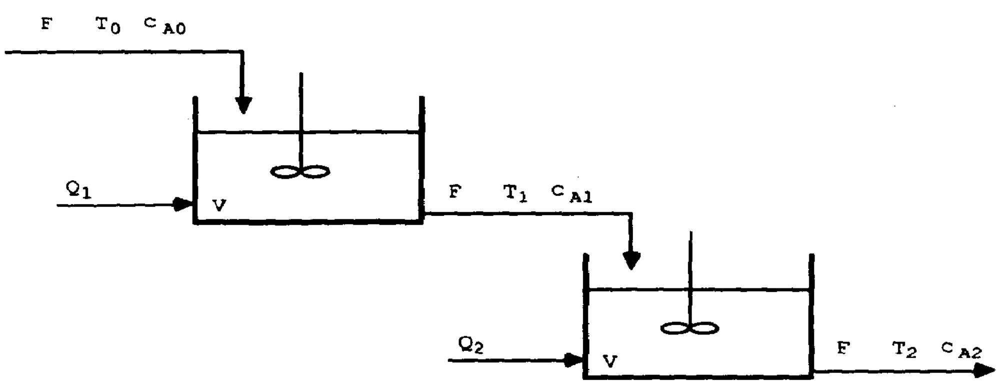

Fig. 4. A cascade of two continuous stirred tank reactors.

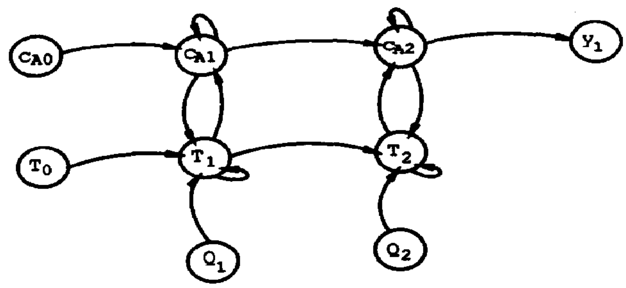

Fig. 5. The digraph of the reactor cascade.

on the result of theorem 1 and the digraph of Fig. 5, we can easily calculate the corresponding relative orders which take the values: $r_{11} = 3$ and $r_{12} = 2$. Clearly, the smallest relative order corresponds to the heat input Q_2 which is "physically closer" to the controlled output and has a more "direct effect" on it than the heat input Q_1. Furthermore, if we had to choose the manipulated input for this process between Q_1 and Q_2, the intuitively obvious choice would be Q_2. In other words, we would choose the manipulated input with the smallest relative order with respect to the controlled output, expecting improved control quality characteristics. The above intuitive argument will be rigorously established and expanded in the next section, where fundamental structural limitations in the control quality, as well as the role of relative order in the evaluation of control configurations, will be investigated.

STRUCTURAL LIMITATIONS IN THE CONTROL QUALITY AND OVERALL EVALUATION OF CONTROL CONFIGURATIONS

At a first level of evaluation of alternative control configurations (i.e. alternative sets of manipulated inputs), one would like to identify inherent limitations in the control quality imposed by the structure of the process itself. Since the whole treatment is based on structural considerations, issues like non-minimum-phase behavior, open-loop instability or constraints on the manipulated inputs are beyond consideration at this point, since their assessment requires more quantitative information. Instead, we are concerned with the general tracking and regulatory characteristics of the control configurations and the way that they are affected by structural constraints. The above issues will be investigated in the light of results on nonlinear inversion and nonlinear feedforward/state-feedback control. In both cases, relative order will be shown to encode fundamental structural limitations in the control quality. The analysis will lead to a set of guidelines for the structural evaluation of the overall servo and regulatory characteristics of alternative control configurations.

For the nonlinear system described by eq. (1), the characteristic matrix is defined by

$$C(x) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{r_m - 1} h_m(x) \end{bmatrix}. \tag{10}$$

For the rest of the paper we are going to focus on control configurations that result in a nonsingular characteristic matrix. This will guarantee the feasibility of the control configuration, since the nonsingularity of the characteristic matrix is a sufficient condition for invertibility of a nonlinear system (e.g. Daoutidis and Kravaris, 1991). Following Daoutidis and Kravaris (1991), for a nonlinear system described by eq. (1) with a nonsingular characteristic matrix $C(x)$, the dynamic system

$$\dot{\xi} = f(\xi) + g(\xi) C(\xi)^{-1} \times \left(\begin{bmatrix} \dfrac{d^{r_1} y_1}{dt^{r_1}} \\ \vdots \\ \dfrac{d^{r_m} y_m}{dt^{r_m}} \end{bmatrix} - \begin{bmatrix} L_f^{r_1} h_1(\xi) \\ \vdots \\ L_f^{r_m} h_m(\xi) \end{bmatrix} \right)$$

$$u = C(\xi)^{-1} \left(\begin{bmatrix} \dfrac{d^{r_1} y_1}{dt^{r_1}} \\ \vdots \\ \dfrac{d^{r_m} y_m}{dt^{r_m}} \end{bmatrix} - \begin{bmatrix} L_f^{r_1} h_1(\xi) \\ \vdots \\ L_f^{r_m} h_m(\xi) \end{bmatrix} \right) \tag{11}$$

represents one realization of the inverse of the input/outut map. Note that the order of the output derivatives required in eq. (11) is determined by the relative orders $r_1, r_2, \ldots, r_m$, which therefore represent a measure of the "improperness" of the inverse system. Therefore, in any explicit inversion-based control structure like IMC, Inferential Control, etc. (Economou *et al.*, 1986; Parrish and Brosilow, 1988), the relative orders $r_1, r_2, \ldots, r_m$ will determine the order of the filter required in order to make the control action finite and consequently the order of the closed-loop response. In the above sense, the relative orders r_i play a fundamental role in "shaping" the closed-loop response.

The above considerations become even more transparent in a feedforward/state-feedback control framework (Daoutidis *et al.*, 1990). For a minimum-phase nonlinear system described by eq. (1), consider the following partition of the set of disturbance inputs to the classes $\mathscr{A}_i$, $\mathscr{B}_i$ and $\mathscr{C}_i$, associated with the output y_i:

$$d_\kappa \in \mathscr{A}_i \Leftrightarrow \rho_{i\kappa} > r_i, \qquad d_\kappa \in \mathscr{B}_i \Leftrightarrow \rho_{i\kappa} = r_i,$$
$$d_\kappa \in \mathscr{C}_i \Leftrightarrow \rho_{i\kappa} < r_i.$$

Then, the control law

$$u = \left[\sum_{i=1}^{m} \beta_{ir_i}, L_{g_1} L_f^{r_i - 1} h_i(x) \cdots \sum_{i=1}^{m} \beta_{ir_i}, L_{g_m} L_f^{r_i - 1} h_i(x) \right]^{-1}$$
$$\times \left\{ v - \sum_{i=1}^{m} \sum_{k=0}^{r_i} \beta_{ik} L_f^k h_i(x) - \sum_{i=1}^{m} \sum_{d_\kappa \in \mathscr{B}_i} \beta_{ir_i} L_{w_\kappa} L_f^{r_i - 1} h_i(x) d_\kappa \right.$$
$$\left. - \sum_{i=1}^{m} \sum_{d_\kappa \in \mathscr{C}_i} \sum_{l=0}^{r_i - \rho_{i\kappa}} \sum_{k=\rho_{i\kappa}+l}^{r_i} \beta_{ik} \frac{\mathrm{d}^l}{\mathrm{d}t^l} (L_{w_\kappa} L_f^{k-l-1} h_i(x) d_\kappa) \right\} \quad (12)$$

- completely eliminates the effect of the disturbances on the process outputs
- induces the linear input/output response

$$\sum_{i=1}^{m} \sum_{k=0}^{r_i} \beta_{ik} \frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = v \quad (13)$$

where $\beta_{ik} = [\beta_{ik}^1 \, \beta_{ik}^2 \ldots \beta_{ik}^m]^T \in \mathbb{R}^m$ are vectors of adjustable constant parameters and $v = [v_1 v_2 \ldots v_m]^T \in \mathbb{R}^m$ is a vector of reference inputs.

Under the above control law, the overall order of the closed-loop response is exactly $(r_1 + r_2 + \cdots + r_m)$. This should not be surprising since such an input/output linearizing control law can also be interpreted as an implicit and finite approximation of an inverse-based controller. Furthermore, considering the relative orders of the outputs y_i with respect to the external input vector v, it is clear that they are exactly equal to r_i. This implies that the order of the closed-loop response for the individual outputs y_i is exactly equal to r_i. It also implies, in loose terms, that the relative orders r_i are preserved in closed loop and the outputs cannot be made more responsive than they were in open loop. Similar characteristics have been attributed to dead time within the framework of linear control (Holt and Morari, 1985a), which is consistent with the connection of the relative order with apparent dead time established in the previous section. In the above feedforward/feedback framework, the role of the relative orders $\rho_{i\kappa}$ is also significant. In particular, the extent to which the condition $r_i \leqslant \rho_{i\kappa}$ is satisfied determines the extent to which measurements of the disturbances and derivatives of the disturbances are required for complete disturbance rejection on the output y_i; moreover, the difference $(r_i - \rho_{i\kappa})$ represents the order of finite approximation required for the derivatives of the disturbances in the control law.

The above considerations allow the structural evaluation of alternative control configurations, on the basis of their overall servo and regulatory characteristics. In particular, the following criteria naturally arise as the basis of such an evaluation:

(1) low order response characteristics for the individual outputs (min r_i)
(2) low order overall response characteristics $[\min(r_1 + \cdots + r_m)]$
(3) more direct effect of the manipulated inputs than the disturbance inputs on the controlled outputs $(r_i < \rho_{i\kappa})$.

The intuitive basis of the above criteria lies exactly on the notions of "direct effect" and "physical closeness" (see e.g. the reactor cascade example), for which they provide a quantitative expression. Obviously, the most favorable control configuration would be the one for which $r_i = 1$ and $\rho_{i\kappa} > 1$ for all outputs y_i and disturbances d_κ. When such a configuration does not exist, one must carefully hierarchize the alternative control configurations depending on the nature and the specific control needs of the process under consideration. A ranking of the outputs according to their importance may then be helpful in order to identify the most favorable control configurations. The above procedure will also allow identifying disturbances for which feedforward compensation may be required.

Remark 8: It is clear from the above discussion that the relative orders r_i (instead of the individual relative orders r_{ij}) capture the overall control quality characteristics. This is a consequence of the fact that we have used multivariable control considerations as the basis of the discussion. In the next section, multi-loop configurations will also be discussed and the individual relative orders r_{ij} will naturally arise.

STRUCTURAL COUPLING AND EVALUATION OF MULTI-LOOP CONFIGURATIONS

At a second level of evaluation, one would like to identify control configurations with favorable input/output coupling characteristics. This is especially important when one is faced with the possibility of employing a multi-loop control configuration (i.e. a partially or completely decentralized control configuration). Steady-state gain and time constant considerations, encoded in appropriate interaction measures, have been traditionally used in the linear control literature to identify favorable input/output pairs and evaluate the resulting configurations.

The graph-theoretic representation of a process introduced earlier in the paper lends itself naturally to a notion of *structural coupling* (or structural interaction), i.e. coupling in the sense of structural interdependencies among the process variables. In the light of theorem 1, relative order arises then as a natural measure of structural coupling between input and output variables. Based on the above, in what follows, we will introduce a matrix of relative orders and use it to systematically formulate intuitive guidelines for the synthesis and evaluation of multi-loop configurations based on structural coupling considerations.

Definition 4: *For a nonlinear process with a model of the form of eq.* (1), *we define the relative order matrix*:

$$M_r = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & & \vdots \\ r_{m1} & \cdots & r_{mm} \end{bmatrix} \quad (14)$$

whose elements are the individual relative orders r_{ij} *between the manipulated input and output variables.*

Clearly, the relative-order matrix of eq. (14) captures the overall picture of structural coupling among manipulated input and output variables in the process under consideration. Before we proceed any further, we now review the well-known notion of a structural matrix and its generic rank (e.g. Shields and Pearson, 1976; Glover and Silverman, 1976):

Definition 5: *A structural matrix is a matrix having fixed zeros in certain locations and arbitrary entries in the remaining locations. For a given matrix, its equivalent structural matrix is the one which has zeros and arbitrary entries in exactly the same locations as the zeros and the nonzero entries of the original matrix.*

Definition 6: *The generic rank of a structural matrix is the maximal rank that the matrix achieves as a function of its arbitrary nonzero elements.*

We can now proceed with theorem 3, which will facilitate the synthesis and evaluation of multiloop configurations based on structural coupling considerations (the proof is given in Appendix D):

Theorem 3: *Consider a nonlinear system in the form of eq.* (1) *and its characteristic matrix* $C(x)$. *Then, the generic rank of the structural matrix which is equivalent to* $C(x)$ *will be equal to* m, *if and only if the outputs can be rearranged so that the minimum relative order in each row of the relative order matrix appears in the major diagonal position, i.e.* M_r *takes the form*:

$$M_r = \begin{bmatrix} r_1 & r_{12} & \cdots & r_{1m} \\ r_{21} & r_2 & \cdots & r_{2m} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_m \end{bmatrix}. \tag{15}$$

Remark 9: If the matrix $C(x)$ itself is nonsingular (i.e. has full numerical rank), its equivalent structural matrix will also have full generic rank, and the output rearrangement will therefore be possible. The converse, however, is not necessarily true.

Remark 10: The output rearrangement contained in theorem 3 is similar to the output rearrangement suggested by Holt and Morari (1985a) in studying the effect of dead time in dynamic resilience and by Jerome and Ray (1986) in the context of dead-time compensation for MIMO linear systems. This is consistent with the connection between apparent dead time and relative order established earlier.

Given a process model with a characteristic matrix whose equivalent structural matrix has full generic rank, the result of theorem 3 is important in two ways:

- The suggested output rearrangement indicates the input/output pairings u_i/y_i with the dominant structural coupling.
- After the output rearrangement, the off-diagonal relative orders allow the evaluation of structural coupling between a specific input/output pair and the remaining input and output variables.

In particular, off-diagonal relative orders in a row indicate the coupling between a specific output and the other inputs, and they will necessarily (due to the rearrangement) be larger or equal to the diagonal relative order. On the other hand, off-diagonal relative orders in a column indicate the coupling between a specific input and the other outputs, and there is no guarantee that they will be larger or equal to the diagonal relative order. The differences between off-diagonal and diagonal relative orders (a) in a column of the relative-order matrix: $(r_{li} - r_i)$, and (b) in a row of the relative-order matrix: $(r_{ij} - r_i)$, provide then a measure of the overall structural coupling in the system, for the particular input/output assignment. The larger these differences are, the weaker the structural coupling is in the system, and the more favorable the employment of a multi-loop configuration is from a structural point of view. In the above spirit, it is also possible to identify groups of inputs and outputs such that structural coupling among members of different groups is weak, providing thus favorable candidates for partially decentralized control structures.

Remark 11: In the special case of an input/output decoupled system, M_r becomes:

$$M_r = \begin{bmatrix} r_1 & \infty & \cdots & \infty \\ \infty & r_2 & \cdots & \infty \\ \vdots & \vdots & & \vdots \\ \infty & \infty & \cdots & r_m \end{bmatrix}.$$

The linear analog of this case would be a diagonal transfer function matrix.

CONCLUDING REMARKS

The analysis so far has established that the relative order is a fundamental structural concept, which quantifies the notions of "direct effect" and "physical closeness", expresses fundamental structural limitations in the control quality and allows the evaluation of structural coupling among input and output variables in a process. The above properties allowed us to develop general guidelines for the structural evaluation of alternative control configurations. In summary, for a particular process and after we identify the alternative control configurations,

- we calculate the relative orders r_{ij} and $\rho_{i\kappa}$ for all i, j, κ,
- we form the relative order matrix M_r.

Then, after checking the nonsingularity of the characteristic matrix $C(x)$ (or its equivalent structural matrix), we proceed with an evaluation of the overall servo and regulatory characteristics of the alternative configurations and the evaluation of structural coupling. Clearly, the above evaluation framework is a generic one; it allows quantifying structural differences of control configurations, if there are any, and allows *a hierarchization of alternative control configurations*, often based on the specific control needs of the process under consideration. At the early stages of the design procedure, with a minimum amount of information available, this is clearly the best we can hope for. In later stages of the design procedure, when more quantitative infomation becomes available, additional analytical tools have to be employed in order to check the modeling assumptions and make sure that the structurally favorable control configurations are statically and dynamically well defined and well behaved.

ILLUSTRATIVE EXAMPLES

In this section, we will apply the structural evaluation guidelines developed previously in three typical chemical engineering processes. In the first two examples and without loss of generality, the analysis will be based on detailed state–space models in order to better illustrate the procedure. In the third example, the analysis will be based on purely structural information.

A single-effect evaporator

In this example, we consider the single-effect evaporator shown in Fig. 6. A solution stream at solute molar concentration x_F enters the evaporator at a molar flow rate F. Heat provided by steam is used to vaporize the water, producing a vapor stream D and a liquid effluent B at a solute concentration x_B. For the purpose of the example, the following simplifying assumptions are made:

(1) The liquid is perfectly mixed.
(2) The solute concentration in the vapor stream is negligible compared with that of the liquid stream ($x_D = 0$).
(3) The vapor holdup is insignificant.
(4) The feed and bottom stream have a constant molar density c.
(5) The vapor and liquid are in thermal equilibrium at all times.
(6) All the heat input to the evaporator is used for vaporization.
(7) The heat capacities of the stream chests, tube walls etc. are negligible.

Under the above assumptions, the following equations describe the dynamic behavior of the process:

Total material balance

$$Ac\frac{dh}{dt} = F - B - D \tag{16}$$

Solute balance

$$Ac\frac{d(hx_B)}{dt} = Fx_F - Bx_B \tag{17}$$

where

A = cross-sectional area
F, B, D = molar flow rates
c = molar density of feed and bottom streams
h = liquid level in the evaporator
x_F, x_B = solute concentration at the feed and bottom stream, respectively (in mole fractions).

Assumption (3) implies that the flow rate D is equal to the rate of evaporation, and together with assumption (6), implies that

$$D = \frac{Q}{\Delta H_v} \tag{18}$$

where ΔH_v is the latent heat of vaporization and Q is the heat input to the evaporator. The above equation can then be substituted to the total material balance.

Clearly, the variables to be controlled are the liquid level in the evaporator, h, and the concentration of the effluent stream, x_B. Available manipulated variables are the flow rate B and the heat input Q, while x_F is the major disturbance. Thus, setting

$$x_1 = h - h_s,\ x_2 = x_B - x_{Bs}$$

and also

$$u_1 = B - B_s,\ u_2 = Q - Q_s$$
$$d_1 = x_F - x_{Fs}$$
$$y_1 = x_1,\ y_2 = x_2$$

where the subscript s denotes a nominal steady-state value, the dynamic equations assume the following state–space form:

$$\dot{x} = \begin{bmatrix} 0 \\ -\dfrac{B_s x_2}{Ac(x_1 + h_s)} \end{bmatrix} + \begin{bmatrix} -\dfrac{1}{Ac} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -\dfrac{1}{\Delta H_v Ac} \\ \dfrac{(x_2 + x_{Bs})}{\Delta H_v Ac(x_1 + h_s)} \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ \dfrac{F}{Ac(x_1 + h_s)} \end{bmatrix} d_1 \tag{19}$$

$$y_1 = x_1$$
$$y_2 = x_2.$$

The vector fields $f(x)$, $g_1(x)$, $g_2(x)$, $w_1(x)$ and the scalar fields $h_1(x)$, $h_2(x)$ can be easily identifed from the above equations. A straightforward calculation of the relative orders r_{ij} and the relative-order matrix M_r yields

$$M_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

while the characteristic matrix of the above system is

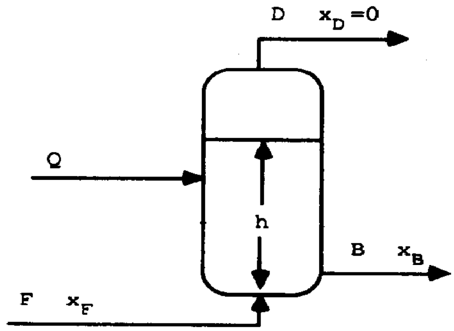

Fig. 6. A single-effect evaporator.

found to be equal to

$$C(x)=\begin{bmatrix} L_{g_1}h_1(x) & L_{g_2}h_1(x) \\ 0 & L_{g_2}h_2(x) \end{bmatrix}$$

$$=\begin{bmatrix} -\left(\dfrac{1}{Ac}\right) & -\left(\dfrac{1}{\Delta H_v Ac}\right) \\ 0 & \dfrac{(x_2+x_{Bs})}{\Delta H_v Ac(x_1+h_s)} \end{bmatrix}$$

and is nonsingular, which guarantees the feasibility of the control configuration and allows the application of theorem 3.

Clearly,

$$r_1 = 1, \qquad r_2 = 1$$

and the overall servo characteristics of the configuration are the best possible from a structural point of view. Moreover, the relative orders of the two outputs with respect to the disturbance input take the values

$$\rho_{11} = \infty,\ \rho_{21} = 1$$

which indicate that the output y_1 is not affected by the disturbance d_1, while y_2 is affected in a direct way, and moreover $\rho_{21} = r_2$. This implies that feedforward compensation will be required for the disturbance d_1 in order to completely eliminate its effect on y_2.

Proceeding with the evaluation of structural coupling for the given control configuraton, note that the relative-order matrix is in a form such that all the r_i are in the major diagonal. This automatically suggests an input/output pairing of the form

$$(u_1/y_1),\quad (u_2/y_2)$$

i.e.

$$(B/h),\quad (Q/x_B)$$

as the most favorable input/output pairing from a structural point of view, while the off-diagonal relative orders in the relative-order matrix indicate a one-way structural coupling. The above conclusion clearly agrees with intuitive considerations based on the criteria of direct effect or physical closeness.

A continuous stirred tank reactor

Consider the CSTR shown in Fig. 7. Two solution streams consisting of species A and B, at volumetric flow rates F_A and F_B, temperatures T_A and T_B and concentrations c_{A0} and c_{B0}, respectively, enter the reactor, where the elementary reaction $A + B \rightarrow C + D$ takes place. The effluent stream leaves the reactor at a flow rate F, concentrations c_A, c_B, c_C, c_D and temperature T. Heat may be added to or removed from the system at a rate Q, using an appropriate heating/cooling system. Assuming constant density ρ and constant heat capacity C_p for the liquid streams and neglecting heat of solution effects, the material and energy balances that describe the dynamic behavior of the process take the following form:

$$\frac{dV}{dt} = F_A + F_B - F$$

$$\frac{dc_A}{dt} = \frac{F_A}{V}(c_{A0} - c_A) - c_A\frac{F_B}{V} - kc_Ac_Be^{(-E/RT)}$$

$$\frac{dc_B}{dt} = \frac{F_B}{V}(c_{B0} - c_B) - c_B\frac{F_A}{V} - kc_Ac_Be^{(-E/RT)} \quad (20)$$

$$\frac{dc_C}{dt} = -c_C\frac{F_A+F_B}{V} + kc_Ac_Be^{(-E/RT)}$$

$$\frac{dT}{dt} = \frac{F_A}{V}(T_A - T) + \frac{F_B}{V}(T_B - T) + \frac{(-\Delta H)}{\rho C_p}kc_Ac_Be^{(-E/RT)} + \frac{1}{V\rho C_p}Q$$

where

C_p = heat capacity
E = activation energy
F_A, F_B, F = volumetric flow rates
Q = heat input to the tank
T_A, T_B, T = temperatures
V = volume
$-\Delta H$ = heat of reaction
c_i = molar concentrations of species i
ρ = density.

For the above process, we wish to control the volume of the liquid in the tank, V, the concentrations of the effluent stream, c_A, c_C, and the temperature of

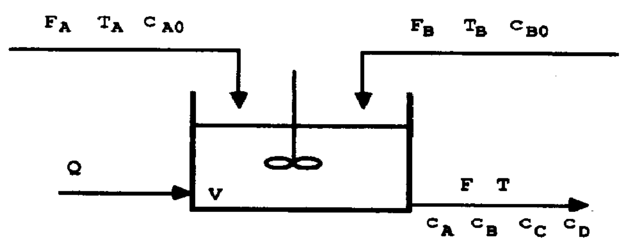

Fig. 7. A continuous stirred tank reactor.

the effluent stream, T. Available manipulated variables are the flow rates F_A, F_B, F and the heat input Q. Thus, setting

$$x_1 = V - V_s,\ x_2 = c_A - c_{As},\ x_3 = c_B - c_{Bs},$$
$$x_4 = c_C - c_{Cs},\ x_5 = T - T_s$$

and also

$$u_1 = F_A - F_{As},\ u_2 = F_B - F_{Bs},\ u_3 = F - F_s,$$
$$u_4 = Q - Q_s$$
$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_4,\ y_4 = x_5$$

where the subscript s denotes a nominal steady-state value, the dynamic equations can be put in the standard state–space form of eq. (1). Then, the calculation of the relative orders and the relative-order matrix is straightforward and yields:

$$M_r = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \infty \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

The characteristic matrix is given by

$$C(x) = \begin{bmatrix} L_{g1}h_1(x) & L_{g2}h_1(x) & L_{g3}h_1(x) & 0 \\ L_{g1}h_2(x) & L_{g2}h_2(x) & 0 & 0 \\ L_{g1}h_3(x) & L_{g2}h_3(x) & 0 & 0 \\ L_{g1}h_4(x) & L_{g2}h_4(x) & 0 & L_{g4}h_4(x) \end{bmatrix}$$

and its equivalent structural matrix has full generic rank.

The overall servo characteristics of the control configuration are clearly the best possible from a structural point of view, since all r_i are equal to 1.

Following theorem 3, we interchange the first and the third rows of M_r, obtaining the following form of the relative order matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & \infty \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

with the relative orders r_i in the major diagonal. Further rearrangement of the first and second rows is possible, without affecting the form of the relative-order matrix. Consequently, the input/output pairs with the dominant structural coupling are:

$$(u_1/y_2),\quad (u_2/y_3),\quad (u_3/y_1),\quad (u_4/y_4)$$

i.e.

$$(F_A/c_A),\quad (F_B/c_C),\quad (F/V),\quad (Q/T)$$

or

$$(u_1/y_3),\quad (u_2/y_2),\quad (u_3/y_1),\quad (u_4/y_4)$$

i.e.

$$(F_A/c_C),\quad (F_B/c_A),\quad (F/V),\quad (Q/T).$$

On the other hand, the off-diagonal relative orders indicate a significant overall structural coupling, induced mainly by F_A, F_B.

Note that as in the previous example, the results conform with intuitive considerations about the process.

A heat exchanger network

Consider the network of heat exchangers shown in Fig. 8 (Georgiou and Floudas, 1989). The energy balances that describe the dynamic behavior of the process have the following structural form

$$\begin{aligned} \frac{dT_1}{dt} &= \phi_1(T_1, T_2, T_{10}, F_1) \\ \frac{dT_2}{dt} &= \phi_2(T_1, T_2, T_{20}, F_2) \\ \frac{dT_3}{dt} &= \phi_3(T_3, T_4, T_{30}, F_3) \\ \frac{dT_4}{dt} &= \phi_4(T_3, T_4, T_{40}, F_4) \\ \frac{dT_5}{dt} &= \phi_5(T_5, T_6, T_{50}, F_5) \\ \frac{dT_6}{dt} &= \phi_6(T_5, T_6, T_{60}, F_6) \end{aligned} \qquad (21)$$

where

F_i = flow rate of stream i

T_i = exit temperature of stream i

T_{i0} = entrance temperature of stream i

and $\phi_i(.)$ denotes a functional dependence.

Assuming steady-state conditions at the mixing junction, the following algebraic equations also hold:

$$F_6 = \phi_7(F_2, F_4) \qquad (22)$$

$$T_{60} = \phi_8(T_2, T_4, F_2, F_4). \qquad (23)$$

Consequently, the last of eqs (21) can be more appropriately represented as

$$\frac{dT_6}{dt} = \phi_9(T_2, T_4, T_5, T_6, F_2, F_4). \qquad (24)$$

The control objective, determined by the operational needs of the plant under consideration, is to keep the temperatures T_1 and T_6 at some desired values. The major disturbances are considered to be the temperatures T_{30}, T_{50}. For notational consistency, let

$$d_1 = T_{30},\quad d_2 = T_{50}$$
$$y_1 = T_1,\quad y_2 = T_6.$$

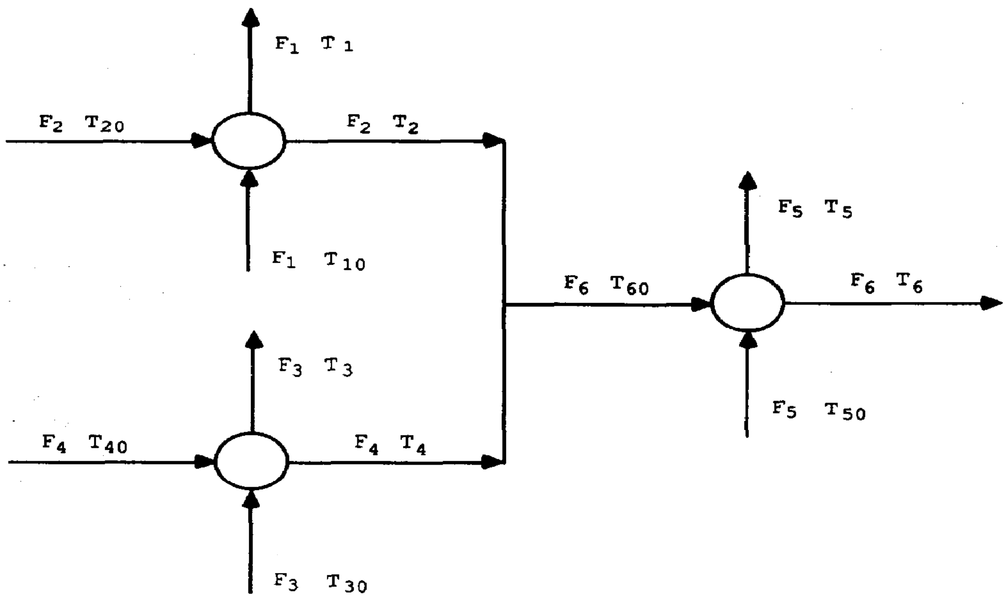

Fig. 8. A heat exchanger network.

Available manipulated inputs are the flow rates F_1, F_2 and F_4. Therefore, three alternative control configurations are possible, corresponding to the pairs of manipulated inputs (F_1, F_2), (F_2, F_4), and (F_1, F_4).

The structural dynamic model of the above process corresponds to the digraph representation shown in Fig. 9 (where only the input nodes that correspond to the possible manipulated inputs and the disturbances are shown, for simplicity). For the three alternative control configurations under consideration, the calculation of the various relative orders can be based on the result of theorem 1 and can be readily performed from the digraph representation of the process. More specifically:

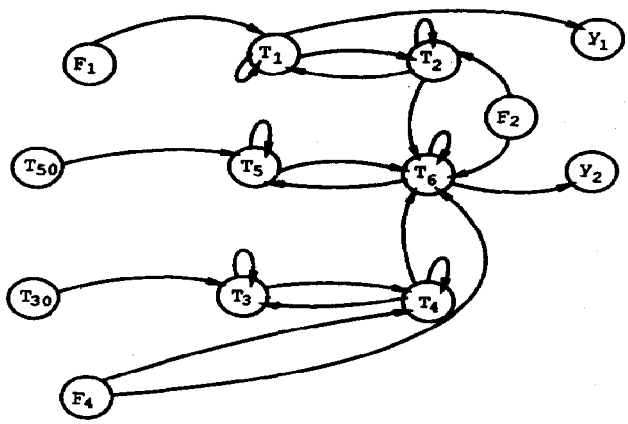

Fig. 9. The digraph of the heat exchanger network.

Configuration 1: $u_1 = F_1, u_2 = F_2$

$$M_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

and the characteristic matrix has the form

$$C(x) = \begin{bmatrix} L_{g_1}h_1(x) & 0 \\ 0 & L_{g_2}h_2(x) \end{bmatrix}$$

which guarantees full generic rank of its equivalent structural matrix.

Configuration 2: $u_1 = F_2, u_2 = F_4$

$$M_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 2 & \infty \\ 1 & 1 \end{bmatrix}$$

and the characteristic matrix has the form

$$C(x) = \begin{bmatrix} L_{g_1}L_f h_1(x) & 0 \\ L_{g_1}h_2(x) & L_{g_2}h_2(x) \end{bmatrix}$$

which also guarantees full generic rank of its equivalent structural matrix.

Configuration 3: $u_1 = F_1, u_2 = F_4$

$$M_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & \infty \\ 3 & 1 \end{bmatrix}$$

and the characteristic matrix has the form

$$C(x) = \begin{bmatrix} L_{g_1}h_1(x) & 0 \\ 0 & L_{g_2}h_2(x) \end{bmatrix}$$

which also guarantees full generic rank of its equivalent structural matrix.

Also, the relative orders with respect to the disturbance inputs are given by

$$\rho_{11} = \infty, \quad \rho_{12} = \infty$$
$$\rho_{21} = 3, \quad \rho_{22} = 2.$$

Clearly, the relative orders with respect to the manipulated input vectors take the following values:

Configuration 1: $r_1 = 1, r_2 = 1$

Configuration 2: $r_1 = 2, r_2 = 1$

Configuration 3: $r_1 = 1, r_2 = 1.$

Since $r_i < \rho_{i\kappa}$ for all i, κ, all three configurations have very favorable regulatory characteristics from a structural point of view. Configurations 1 and 3 have better overall structural characteristics, since $r_1 = r_2 = 1$ for both, while configuration 2 has less favorable structural characteristics since $r_1 = 2$.

We can now proceed evaluating the structural coupling in the three configurations. The relative-order matrices do not require any rearrangement and they immediately indicate the most favorable input/output pairings for each configuration. A close inspection of the off-diagonal elements indicates that configuration 2 has an unfavorable structural coupling, since the off-diagonal relative order in the first column of M_r is smaller than the diagonal. Comparing the structural coupling in configurations 2 and 3, it is clear that configuration 3 is the most favorable one, since it is characterized by the weakest structural coupling. In the case of a multi-loop configuration, the most structurally favorable input/output pairing would then be

$$(F_1/T_1), \quad (F_4/T_6).$$

CONCLUSIONS

In this work, we introduced a structural perspective on the issue of evaluation of control configurations for multivariable nonlinear processes, using appropriate formulations of the concept of relative order. A number of attractive properties of the relative order were rigorously established: its generic calculation requires only structural information for the process, it provides a measure of sluggishness and structural coupling, and it expresses fundamental structural limitations in the control quality. Relative order was also interpreted as the structural analog of apparent dead time and it was shown that it quantifies the "direct effect" and "physical closeness" criteria for the selection and pairing of manipulated inputs. On the basis of these properties, general guidelines were developed for a hierarchization of alternative control configurations. The proposed approach has a purely structural character and does not substitute static gain or dominant time-constant considerations, neither does it address limitations arising from open-loop instability and non-minimum-phase characteristics. It is applicable to both linear and nonlinear systems at the preliminary stages of the design procedure, allowing the designer to systematically evaluate alternative control configurations on the basis of their structural characteristics.

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NOTATION

$C(x)$	characteristic matrix
M_r	relative-order matrix
d_κ	disturbance input
f, g_j, w_κ	vector fields
h_i	output scalar field
ℓ_{ij}	the length of the shortest path connecting u_j and y_i
$\ell_{i\kappa}$	the length of the shortest path connecting d_κ and y_i
r_i	relative order of the output y_i with respect to the manipulated input vector
r_{ij}	relative order of the output y_i with respect to the manipulated input u_j
t	time
u_j	manipulated input
v	external input vector
x	vector of state variables
y_i	output to be controlled
$\mathscr{A}_i, \mathscr{B}_i, \mathscr{C}_i$	partition of disturbances for the output y_i

Greek letters

β^j_{ik}	parameters of the feedforward/state-feedback law
$\rho_{i\kappa}$	relative order of the output y_i with respect to the disturbance d_κ

Mathematical symbols

$\in$	belongs to
$\cup$	union
$\exists$	there exists
$\mathbb{R}^n$	n-dimensional Euclidean space

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APPENDIX A: PROOF OF THEOREM 1

We will only prove the part of the theorem concerning the relative order r_{ij}. The same arguments will hold for ρ_{ik}. The procedure follows closely the one by Kasinski and Levine (1984). For the purpose of the proof, define v_{ij} as the smallest integer such that there exist integers $k_1, k_2, \ldots, k_{v_{ij}} \in \{1, \ldots, n\}$ for which

$$g_j k_{v_{ij}}(x) \frac{\partial f_{k_{v_{ij}-1}}(x)}{\partial f_{k_{v_{ij}}}} \cdots \frac{\partial f_{k_1}(x)}{\partial x_{k_2}} \frac{\partial h_i(x)}{\partial x_{k_1}} \not\equiv 0.$$

The proof of the theorem will then go through the following steps:

Step 1: We will show that $v_{ij} = l_{ij} - 1$.
Step 2: We will show that $v_{ij} \leqslant r_{ij}$.
Step 3: We will show that generically $v_{ij} = r_{ij}$.

Step 1: $v_{ij} = l_{ij} - 1$

From the definition of v_{ij}, we have that

$$g_j k_{v_{ij}}(x), \quad \frac{\partial f_{k_{v_{ij}-1}}(x)}{\partial x_{k_{v_{ij}}}}, \ldots, \frac{\partial f_{k_1}(x)}{\partial x_{k_2}}, \quad \frac{\partial h_i(x)}{\partial x_{k_1}} \not\equiv 0.$$

Consequently, according to the definition of the graph, the sequence $(u_j, x_{k_{v_{ij}}}, \ldots, x_{k_1}, y_i)$ corresponds to a directed path connecting u_j and y_i, of length $(v_{ij} + 1)$. By its definition, ℓ_{ij} is the length of the shortest path connecting u_j and y_i. We will therefore have that $\ell_{ij} \leqslant (v_{ij} + 1)$. Suppose now that ℓ_{ij} is strictly less than $(v_{ij} + 1)$. Then, by the definition of the graph and ℓ_{ij}, there exist integers $k_1, k_2, \ldots, k_{l_{ij}-1} \in \{1, \ldots, n\}$, such that

$$g_j k_{\ell_{ij}-1}(x) \frac{\partial f_{k_{\ell_{ij}-2}}(x)}{\partial x_{k_{\ell_{ij}-1}}} \cdots \frac{\partial f_{k_1}(x)}{\partial x_{k_2}} \frac{\partial h_i(x)}{\partial x_{k_1}} \not\equiv 0$$

with $l_{ij} - 1 < v_{ij}$. But this leads to contradiction, since v_{ij} is by its definition the smallest integer for which such a sequence of integers exists. Consequently, the strict inequality does not hold and $\ell_{ij} = (v_{ij} + 1)$.

Step 2: $v_{ij} \leqslant r_{ij}$

In order to proceed with the proof, we need to define some auxiliary notation. In particular, we define the subsets Γ_i^j and

$\tilde{\Gamma}_i^j$ of $\{1, \ldots, n\}$, with $j \geq 1$, by induction, as follows:

$$\Gamma_i^1 = \left\{ k_1 \in \{1, \ldots, n\} : \frac{\partial h_i(x)}{\partial x_{k_1}} \not\equiv 0 \right\} = \tilde{\Gamma}_i^1$$

$$\Gamma_i^j = \left\{ k_j \in \{1, \ldots, n\} : \exists k_{j-1} \in \tilde{\Gamma}_i^{j-1}, \ldots, k_1 \in \tilde{\Gamma}_i^1 : \frac{\partial f_{k_{j-1}}(x)}{\partial x_{k_j}} \cdots \frac{\partial f_{k_1}(x)}{\partial x_{k_2}} \frac{\partial h_i(x)}{\partial x_{k_1}} \not\equiv 0 \right\} - \tilde{\Gamma}_i^{j-1}$$

$$\tilde{\Gamma}_i^j = \Gamma_i^j \cup \tilde{\Gamma}_i^{j-1}.$$

Also, we define the following analytic functions:

$$\pi_i(k_1, \ldots, k_j) = \frac{\partial f_{k_{j-1}}(x)}{\partial x_{k_j}} \cdots \frac{\partial f_{k_1}(x)}{\partial x_{k_2}} \frac{\partial h_i(x)}{\partial x_{k_1}}.$$

We have suppressed the dependence of π_i on x mainly for notational convenience. Taking into account the previous definitions of the sets Γ_i^j and $\tilde{\Gamma}_i^j$, it can be deduced that $\pi_i(k_1, \ldots, k_j)$ is a function of the state variables x_{k_j}, with $k_j \in \tilde{\Gamma}_i^j$. We will finally need the following lemma, for the proof of which we refer the reader to Kasinski and Levine (1984):

Lemma:

$$L_f^l h_i = \sum_{k_1 \in \Gamma_i^1, \ldots, k_l \in \Gamma_i^l} f_{k_l} \pi_i(k_1, \ldots, k_l) + \Phi_i(\tilde{\Gamma}_i^{l-1})$$

where $\Phi_i(\tilde{\Gamma}_i^{l-1})$ is a linear combination with analytic coefficients of all the terms of the form $\pi_i(k_1, \ldots, k_{l-1})$ and $\partial^{l-s-1}\pi_i(k_1, \ldots, k_s)/\partial x_{k_{l-1}} \ldots \partial x_{k_{s+1}}$, for every $s < l - 1$ and every $k_1, \ldots, k_{l-1} \in \tilde{\Gamma}_i^{l-1}$.

In the above lemma, we have also suppressed the exact dependence on x. While $\pi_i(k_1, \ldots, k_l)$ is a function of x_{k_l}, with $k_l \in \tilde{\Gamma}_i^l$, it can also be deduced that f_{k_l} is a function of $x_{k_{l+1}}$, with $k_{l+1} \in \tilde{\Gamma}_i^{l+1}$.

The relative order r_{ij} is defined as the smallest integer for which $L_{g_j} L_f^{r_{ij}-1} h_i(x) \not\equiv 0$. Applying the above lemma to the case of the scalar field $L_f^{r_{ij}-1} h_i(x)$, we obtain the following expression:

$$L_f^{r_{ij}-1} h_i(x) = \sum_{k_1 \in \Gamma_i^1, \ldots, k_{r_{ij}-1} \in \Gamma_i^{r_{ij}-1}} f_{k_{r_{ij}-1}} \times \pi_i(k_1, \ldots, k_{r_{ij}-1}) + \Phi_{r_{ij}-1}(\tilde{\Gamma}_i^{r_{ij}-2})$$

and

$$L_{g_j} L_f^{r_{ij}-1} h_i(x) = \sum_{k_{r_{ij}} \in \tilde{\Gamma}_i^{r_{ij}}} g_{jk_{r_{ij}}} \frac{\partial L_f^{r_{ij}-1} h_i(x)}{\partial x_{k_{r_{ij}}}}$$

$$= \sum_{k_1 \in \Gamma_i^1, \ldots, k_{r_{ij}} \in \Gamma_i^{r_{ij}}} g_{jk_{r_{ij}}} \pi_i(k_1, \ldots, k_{r_{ij}})$$
$$+ \sum_{k_1, \ldots, k_{r_{ij}} \in \tilde{\Gamma}_i^{r_{ij}-1}} g_{jk_{r_{ij}}} \pi_i(k_1, \ldots, k_{r_{ij}-1}) \frac{\partial f_{k_{r_{ij}-1}}}{\partial x_{k_{r_{ij}}}}$$
$$+ \sum_{k_1, \ldots, k_{r_{ij}} \in \tilde{\Gamma}_i^{r_{ij}-1}} g_{jk_{r_{ij}}} \frac{\partial \pi_i(k_1, \ldots, k_{r_{ij}-1})}{\partial x_{k_{r_{ij}}}} f_{k_{r_{ij}-1}}$$
$$+ \sum_{k_{r_{ij}} \in \tilde{\Gamma}_i^{r_{ij}-1}} g_{jk_{r_{ij}}} \frac{\partial \Phi_{r_{ij}-1}}{\partial x_{k_{r_{ij}}}}.$$

In order that the above expression is not identically equal to zero, at least one term should be nonzero. If the first term is nonzero, then at least one product $g_j k_{r_{ij}} \pi_i(k_1, \ldots, k_{r_{ij}})$ should be nonzero. Since, by definition v_{ij} is the smallest integer such that there exist integers for which this is true, we must have $v_{ij} \leq r_{ij}$. If the first term is equal to zero, but the second is nonzero, there must be a product $g_j k_{r_{ij}} \pi_i(k_1, \ldots, k_{r_{ij}})$, with $k_1, \ldots, k_{r_{ij}} \in \tilde{\Gamma}_i^{r_{ij}-1}$ which is nonzero. In this case, we should also have $v_{ij} \leq r_{ij}$. Similar arguments can be used for the other terms, proving that $v_{ij} \leq r_{ij}$.

Step 3: generically, $v_{ij} = r_{ij}$

Given the result of step 2, suppose that v_{ij} is strictly less than r_{ij}. Then, the following system of equations will hold:

$$L_{g_j} L_f^{v_{ij}-1} h_i(x) = 0$$
$$\vdots$$
$$L_{g_j} L_f^{r_{ij}-2} h_i(x) = 0.$$

This is a system of $(r_{ij} - v_{ij})$ nontrivial partial differential equations in f, g_j, h_i and their partial derivatives. The set of solutions of the above system will be a closed subset with empty interior of the space of analytic vector-valued functions on $\mathbb{R}^n$. Consequently, $v_{ij} = r_{ij}$ generically, i.e. for almost all functions f, g_j, h_i.

APPENDIX B: PROOF OF THEOREM 2

Under the assumptions of the theorem, in a neighborhood of x_0 and for sufficiently small times, the output y_i of the system assumes a unique Volterra series expansion of the form (Fliess, 1980)

$$y_i(t) = k_i^0(t) + \int_0^t k_i^1(t, \tau_1) u_j(\tau_1) \, d\tau_1 + \int_0^t \int_0^{\tau_2} k_i^2(t, \tau_2, \tau_1) u_j(\tau_2) u_j(\tau_1) \, d\tau_2 \, d\tau_1 + \cdots$$

where $k_i^j(t, \tau_j, \ldots, \tau_1)$ are the Volterra kernels associated with the output y_i, which assume a Taylor series expansion of the form:

$$k_i^0(t) = \sum_{j_1=0}^{\infty} L_f^{j_1} h_i(x_0) \frac{t^{j_1}}{j_1!}$$

$$k_i^1(t, \tau_1) = \sum_{j_2=0}^{\infty} \sum_{j_1=0}^{\infty} L_f^{j_2} L_{g_j} L_f^{j_1} h_i(x_0) \frac{(t - \tau_1)^{j_1} \tau_1^{j_2}}{j_1! j_2!}$$

$$k_i^2(t, \tau_1, \tau_2) = \sum_{j_3=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_1=0}^{\infty} L_f^{j_3} L_{g_j} L_f^{j_2} L_{g_j} L_f^{j_1} h_i(x_0) \times \frac{(t - \tau_1)^{j_1} (\tau_2 - \tau_1)^{j_2} \tau_2^{j_3}}{j_1! j_2! j_3!}.$$

The first term of the expansion, $k_i^0(t)$, which corresponds to the part of the response that depends only on the initial conditions, will vanish at the given initial condition x_0, since the output is in deviation variable form. Then, we obtain in a straightforward way the following form for the response under a unit-step change at the input:

$$y_i(t) = [L_{g_j} h_i(x_0)]t + [L_{g_j} L_f h_i(x_0) + L_f L_{g_j} h_i(x_0) + L_{g_j}^2 h_i(x_0)] \frac{t^2}{2} + [L_{g_j} L_f^2 h_i(x_0) + L_f L_{g_j} L_f h_i(x_0) + L_f^2 L_{g_j} h_i(x_0) + 2L_{g_j}^2 L_f h_i(x_0) + L_{g_j} L_f L_{g_j} h_i(x_0) + 2L_f L_{g_j}^2 h_i(x_0)] \frac{t^3}{6} + \text{higher-order terms.}$$

One can then easily verify that

- if $r_{ij} = 1$, $y_i(t) \cong L_{g_j} h_i(x_0) t$ as $t \to 0$
- if $r_{ij} = 2$, $y_i(t) \cong L_{g_j} L_f h_i(x_0) t^2/2$ as $t \to 0$
- if $r_{ij} = 3$, $y_i(t) \cong L_{g_j} L_f^2 h_i(x_0) t^3/6$ as $t \to 0$

and, by induction, $y_i(t) \cong L_{g_j} L_f^{r_{ij}-1} h_i(x_0) t^{r_{ij}}/r_{ij}!$ as $t \to 0$.

APPENDIX C: PROOF OF COROLLARY 1

A simple proof of corollary 1, independent of the result of theorem 2, goes as follows: Consider the transfer function between u and y, $G(s) = c(sI - A)^{-1}b$ and its expansion in terms of the Markov parameters [see e.g. Kailath (1980)]:

$$G(s) = \frac{cb}{s} + \frac{cAb}{s^2} + \frac{cA^2b}{s^3} + \cdots.$$

Then, calculating the response of the output under a unit-step change at the input, we obtain

$$y(t) = (cb)t + (cAb)\frac{t^2}{2} + (cA^2b)\frac{t^3}{6} + \text{higher-order terms}$$

and the result of corollary 1 follows immediately.

APPENDIX D: PROOF OF THEOREM 3

First, we prove the "only if part" of the theorem. Suppose that given that the structural matrix equivalent to $C(x)$ has generic rank equal to m, the output rearrangement is not possible. This implies that there is at least one input u_{j^*} for which one of the following two is true:

(1) There is no output y_i with the minimum relative order at the j^*th column of the relative order matrix M_r, i.e. there is no output y_i such that $r_i = r_{ij^*}$.
(2) There are two or more outputs, e.g. y_{i_1} and y_{i_2}, whose minimum relative order appears at the j^*th column of the relative-order matrix M_r and nowhere else, i.e. $r_{i_1} = r_{i_1j^*}$, $r_{i_2} = r_{i_2j^*}$ and $r_{i_1j} > r_{i_1j^*}$, $r_{i_2j} > r_{i_2j^*}$ for $j \neq j^*$.

In the first case, we would have

$$L_{g_{j^*}} L_f^{r_i - 1} h_i(x) = 0$$

for every i, and therefore the j^*th column of the characteristic matrix (and its structural equivalent) would be zero. In the second case, we would have

$$L_{g_{j^*}} L_f^{r_{i_1}-1} h_{i_1}(x) \not\equiv 0, \; L_{g_{j^*}} L_f^{r_{i_2}-1} h_{i_2}(x) \not\equiv 0$$

and

$$L_{g_j} L_f^{r_{i_1}-1} h_{i_1}(x) = 0, \; L_{g_j} L_f^{r_{i_2}-1} h_{i_2}(x) = 0$$

for every $j \neq j^*$. But then, the corresponding to the outputs y_{i_1} and y_{i_2} rows of the characteristic matrix would have only one nonzero element, at the same position (the j^*th). In both cases, a rank deficiency would result, contrary to our assumption. Therefore, by contradiction, the suggested output rearrangement is always possible.

Now, we prove the "if part" of the theorem. Suppose that the suggested output rearrangement is possible, but the structural matrix equivalent to $C(x)$ has rank deficiency. This implies either of the following for this matrix:

(1) At least one row or column has zeros in all positions.
(2) There are $k (k \geqslant 2)$ columns or rows that cause the rank deficiency in a nontrivial way.

In the first case, we would have the case where all relative orders in a row or column are equal to infinity. In the second case, in order that the rank deficiency may exist, we must have at least $m - (k - 1)$ zeros at the same positions in all k columns or rows. This leaves $(k - 1)$ or less nonzero elements at the same $(k - 1)$ positions of all k rows or columns. However, because of the rearrangement, there should be k nonzero elements in the diagonal positions of these k rows or columns, i.e. in k distinct positions. In both cases, the contradiction is clear, and the theorem is proved.