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Some Remarkable Combinatorial Matrices

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In this paper we describe, in combinatorial terms, some matrices which arise as Laplacians connected to the three-dimensional Heisenberg Lie algebra. We pose the problem of finding the eigenvalues and eigenvectors of these matrices. We state a number of conjectures including the conjecture that all eigenvalues of these matrices are non-negative integers. We determine the eigenvalues and eigenvectors explicitly for an important subclass of these matrices.

1. INTRODUCTION

For r a non-negative integer let $\mathbf{r}_0 = \{0, 1, ..., r\}$ and let $\mathbf{r} = \{1, 2, ..., r\}$ (with $\mathbf{0} = \phi$).

DEFINITION 1.1.1. Let a, b, and k be non-negative integers with $a \leqslant k+1$ and $b \leqslant k+1$. Let $\Omega_k(a, b)$ be the set of pairs (U, V) such that U is an A-subset of $\mathbf{k}_0$ and V is a B-subset of $\mathbf{k}_0$. Define the *weight* of a pair (U, V) to be

$$\mathscr{W}(U, V) = \sum_{u \in U} u + \sum_{v \in V} v.$$

Let $\Omega_k(a, b, w)$ be the set of pairs (U, V) such that $\mathscr{W}(U, V) = w$.

EXAMPLE 1.1.2. Let $a = b = 2$ and $w = 4$. The sets $\Omega_k(a, b, w)$ are given below for every value of k:

(A) $k \geqslant 3$, $\Omega_k(a, b, w) = \{(03, 01), (12, 01), (02, 02), (01, 12), (01, 03)\}$.

(B) $k = 2$, $\Omega_k(a, b, w) = \{(12, 01), (02, 02), (01, 12)\}$.

(C) $k = 0, 1$, $\Omega_k(a, b, w) = \phi$.

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For each four-tuple (k, a, b, w) we will define a matrix $T_k(a, b, w)$ with rows and columns indexed by $\Omega_k(a, b, w)$. To define $T_k(a, b, w)$ we need a notion of when two elements of $\Omega_k(a, b, w)$ are neighbors.

DEFINITION 1.3. Let (U, V) and (X, Y) be elements of $\Omega_k(a, b, w)$, and let (u, v, z) be a triple with $u \in U$, $v \in V$, and $z \in \mathbb{Z}$. We say that (U, V) and (X, Y) are (u, v, z)-neighbors if

(1) $X = (U \setminus \{u\}) \cup \{u+z\}$

(2) $Y = (V \setminus \{v\}) \cup \{v-z\}$

(3) $u + v \leqslant k$.

In other words, (U, V) and (X, Y) are (u, v, z) neighbors if X is obtained from U by replacing u with $u+z$, Y is obtained from V by replacing v with $v-z$, and the sum $(u+v) = (u+z) + (v-z)$ of the effected elements does not exceed k.

EXAMPLE 1.4. (A) (134, 05) and (123, 25) are $(4, 0, -2)$ neighbors for $k \geqslant 4$.

(B) (123, 3) and (024, 3) are *not* neighbors since the size of $U \cap X$ is $a-2$.

(C) Let $k=3$. Then (123, 12) is a $(1, 1, 0)$, $(1, 2, 0)$, and $(2, 1, 0)$ neighbor to itself.

The relation of being (u, v, z) neighbors is not a reflexive relation. However, if (U, V) and (X, Y) are (u, v, z) neighbors then (X, Y) and (U, V) are $(u+z, v-z, -z)$ neighbors.

The last thing we need is a sign associated to each neighborhood relation.

DEFINITION 1.5. (A) Let $\mathbf{u} = (u_1, ..., u_a)$ and $\mathbf{x} = (x_1, ..., x_a)$ be sequences. Define $\varepsilon(\mathbf{u}, \mathbf{x})$ as follows:

1. $\varepsilon(\mathbf{u}, \mathbf{x}) = 0$ if either $\mathbf{u}$ or $\mathbf{x}$ has repeated elements.

2. If the elements of $\mathbf{u}$ and $\mathbf{x}$ are distinct, let σ and τ be the elements of S_a such that

$$u_{\sigma 1} < u_{\sigma 2} < \cdots < u_{\sigma a}, \qquad x_{\tau 1} < x_{\tau 2} < \cdots < x_{\tau a}.$$

Define $\varepsilon(\mathbf{u}, \mathbf{v})$ to be $\operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)$.

(B) Next let U and X be A-subsets such that $|U \cap X| \geqslant a-1$. In this case we can write $U = \{u_1, ..., u_a\}$ and $X = \{u_1, ..., u_i + z, ..., u_a\}$ for some $z \in \mathbb{Z}$. Define $\varepsilon(U, X) = \varepsilon(\mathbf{u}, \mathbf{x})$, where $\mathbf{u} = (u_1, ..., u_a)$ and $\mathbf{x} = (u_1, ..., u_i + z, ..., u_a)$.

(C) Lastly suppose that (U, V) and (X, Y) are (u, v, z)-related. Define $\varepsilon((U, V), (X, Y))$ to be

$$\varepsilon((U, V), (X, Y)) = \varepsilon(U, X)\, \varepsilon(V, Y).$$

It is important to check that the definition of $\varepsilon(U, X)$ is independent of the order $\{u_1, ..., u_a\}$ in which U is written. Also note that $\varepsilon(U, U) = 1$ for all U (here we put $z = 0$).

EXAMPLE 1.6. (a) Let $(U, V) = (156, 246)$ and $(X, Y) = (456, 126)$. Then (U, V) and (X, Y) are $(1, 4, 3)$ neighbors and this relation has sign -1. The values of α and β are 0 and 1, respectively.

(b) Let $(U, V) = (12, 12)$ and $(X, Y) = (01, 23)$. Then (U, V) and (X, Y) are $(2, 1, -2)$ neighbors and this relation has sign $+1$. In this case $\alpha = \beta = 1$.

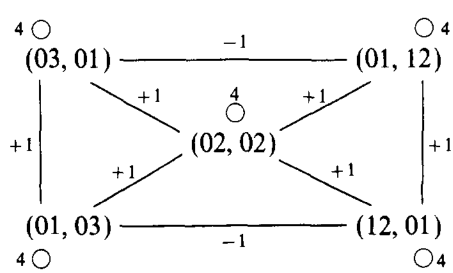
(c) Suppose (U, V) and (X, Y) are (u, v, z) neighbors and that this relation has sign ε. Then (X, Y) and (U, V) are $(u+z, v-z, -z)$ neighbors and this relation also has sign ε. The values of α and β are the same for both relations.

(d) The sign of any neighborhood relation between a pair (U, V) and itself is $+1$.

In view of observations (c) and (d) in Example 1.6 the following definition makes sense.

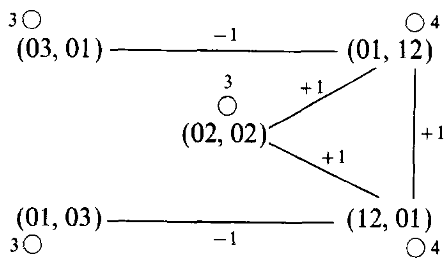
DEFINITION 1.7. Define the graph $G_k(a, b, w)$ to be the undirected signed graph which has an edge between (U, V) and (X, Y) whenever these pairs are (u, v, z) related. The sign of this edge is $\varepsilon((U, V), (X, Y))$.

EXAMPLE 1.8. (A) Let $a = b = 2$, $w = 4$ and assume k is at least 4. The set $\Omega_k(a, b, w)$ is given in Example 1.2 and the graph $G_k(2, 2, 4)$ is



Note that at each point we have consolidated the four loops labeled +1 into a single loop labeled +4.

(B) Let $a=b=2$, $w=4$ again but this time assume $k=3$. Then certain of the neighborhood relations are not allowed; hence certain of the edges in the above graph must be removed. The graph $G_3(2,2,4)$ appears below:



DEFINITION 1.9. Define the matrix $T_k(a, b, w)$ to be the adjacency matrix of the signed graph $G_k(a, b, w)$.

EXAMPLE 1.10. The matrices $T_k(2,2,4)$ $(k \geqslant 4)$ and $T_3(2,2,4)$ with respect to the ordered basis

$$\{(03, 01), (12, 01), (02, 02), (01, 12), (01, 03)\}$$

are

$$T_k(2,2,4)=\begin{pmatrix} 4 & 0 & 1 & -1 & 1 \\ 0 & 4 & 1 & 1 & -1 \\ 1 & 1 & 4 & 1 & 1 \\ -1 & 1 & 1 & 4 & 0 \\ 1 & -1 & 1 & 0 & 4 \end{pmatrix} \quad (k \geqslant 4),$$

$$T_3(2,2,4)=\begin{pmatrix} 3 & 0 & 0 & -1 & 0 \\ 0 & 4 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{pmatrix}.$$

(C) For $k=2$, the basis $\Omega_2(2,2,4)$ is reduced to

$$\{(12, 01), (02, 02), (01, 12)\}.$$

The matrix $T_2(2, 2, 4)$ is

$$T_2(2,2,4)=\begin{pmatrix}3&1&1\\1&3&1\\1&1&3\end{pmatrix}.$$

The main problem addressed in this paper is the following:

1.11. Determine the eigenvalues and corresponding eigenvectors of the matrices $T_k(a, b, w)$.

At first glance this problem seems intractable. However, the following conjecture, which is based on computational evidence and some special cases, makes the problem seem more reasonable.

Conjecture 1.12. The eigenvalues of the matrices $T_k(a, b, w)$ are non-negative integers.

EXAMPLE 1.13. In this example we will give the eigenvalues and associated eigenvectors for the matrices $T_k(2, 2, 4)$ which were given in the last example.

(A) $k \geqslant 4$. The eigenvalues are 2, 2, 4, 6, 6. Below we use independent eigenvectors associated to each of these eigenvalues:

$$\begin{matrix}2&2&4&6&6\\ \begin{pmatrix}0\\1\\-1\\0\\1\end{pmatrix}&\begin{pmatrix}1\\0\\-1\\1\\0\end{pmatrix}&\begin{pmatrix}1\\1\\0\\-1\\-1\end{pmatrix}&\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix}&\begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}\end{matrix}.$$

(B) $k = 3$. This time the eigenvalues are 2, 2, 3, 4, 6. Below we see independent eigenvectors associated to each of these eigenvalues:

$$\begin{matrix}2&2&3&4&6\\ \begin{pmatrix}0\\1\\-1\\0\\1\end{pmatrix}&\begin{pmatrix}1\\0\\-1\\1\\0\end{pmatrix}&\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix}&\begin{pmatrix}1\\1\\0\\-1\\-1\end{pmatrix}&\begin{pmatrix}1\\-3\\-2\\-3\\1\end{pmatrix}\end{matrix}.$$

Observe the interesting change in eigenspaces when we went from the $k \geqslant 4$

case to the $k=3$ case. The eigenspaces corresponding to 2 and 4 remain intact but the eigenspace of $T_4(2, 2, 4)$ corresponding to 6 splits in two with the two halves becoming the eigenspaces corresponding to 3 and 6, respectively.

(C) $k=2$. The eigenvalues are $5, 2, 2$. The corresponding eigenvectors are

$$\begin{matrix} 2 & 2 & 5 \\ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{matrix}$$

In Section 4 we have tables containing lists of eigenvalues for various values of a, b, w, and k.

If we suppress the parameter w then we can give a very elegant expression for the eigenvalues. For each a, b, w, k and each non-negative integer r let $\mu_k(a, b, w; r)$ denote the multiplicity of r as an eigenvalue of $T_k(a, b, w)$. Let

$$\mu_k(a, b; r) = \sum_{w} \mu_k(a, b, w; r).$$

Also for $i \leqslant k+1$ define $\mu_k(i, 0; 0)$ and $\mu_k(0, i; 0)$ to be 1. Lastly, let $M_k(x, y, \lambda)$ be the following generating function for the numbers $\mu_k(a, b; r)$:

$$M_k(x, y, \lambda) = \sum_{a,b,r} \mu_k(a, b; r) x^a y^b \lambda^r.$$

Based on computational evidence we conjecture that $M_k(x, y, \lambda)$ has the following simple form.

Conjecture 1.14. $M_k(x, y, \lambda) = \prod_{i=0}^{k} (1 + x + y + \lambda^{(i+1)} xy)$.

We would like to improve Conjecture 1.14 by giving a nice expression for the generating function

$$M_k(x, y, z, \lambda) = \sum_{a,b,w,r} \mu_k(a, b, w; r) x^a y^b z^w \lambda^r;$$

from the data available we do not see what such an expression would be. However, we can do that for the coefficient of λ^0 in $M_k(x, y, z, \lambda)$. For each a, b, w let $n_k(a, b, w)$ denote the dimension of the nullspace of $T_k(a, b, w)$ and let $N_k(x, y, z)$ be the generating function

$$N_k(x, y, z) = \sum_{a,b,w} n_k(a, b, w) x^a y^b z^w.$$

Note that $N_k(x, y, z)$ is the constant term (with respect to λ) of $M_k(x, y, z; \lambda)$. Also, if Conjecture 1.14 holds it follows that

$$N_k(x, y, 1) = (1 + x + y)^{k+1}. \tag{1.15}$$

This expression (1.15) suggests that we try to write $N_k(x, y, z)$ as a sum, over $(k+1)$-sequences σ of 0's, 1's, and 2's, of

$$x^{j_1(\sigma)} y^{j_2(\sigma)} z^{W(\sigma)},$$

where $j_i(\sigma)$ is the number of i's in σ and $W(\sigma)$ is an appropriate weight function.

DEFINITION 1.16. Let $\sigma = (\sigma_0, \sigma_1, ..., \sigma_k)$ be a sequence of 0's, 1's, and 2's. Define $W(\sigma)$ as

$$W(\sigma) = \sum_{i=0}^{k} \varepsilon_i(\sigma)$$

where

$$\varepsilon_i(\sigma) = \begin{cases} 0 & \text{if } \sigma_i = 0 \\ i & \text{if } \sigma_i = 2 \\ i + |\{j < i : \sigma_j = 2\}| & \text{if } \sigma_i = 1. \end{cases}$$

Another way to express $W(\sigma)$ is that $W(\sigma)$ is the sum of the positions in σ which have a nonzero entry plus the number of times a 2 precedes a 1 in σ.

For example, if $\sigma = 1\,0\,2\,2\,1\,0\,2$ then

$$\varepsilon_0(\sigma) = 0$$
$$\varepsilon_1(\sigma) = 0$$
$$\varepsilon_2(\sigma) = 2$$
$$\varepsilon_3 = 3$$
$$\varepsilon_4 = 6$$
$$\varepsilon_5(\sigma) = 0$$
$$\varepsilon_6(\sigma) = 6,$$

so $W(\sigma) = 17$.

Conjecture 1.16. $N_k(x, y, z) = \sum_\sigma x^{j_1(\sigma)} y^{j_2(\sigma)} z^{W(\sigma)}$.

As an example of this, let $k=1$. There are nine sequences σ given in the table below, along with the corresponding monomial $x^{j_1(\sigma)}y^{j_2(\sigma)}z^{W(\sigma)}$.

σ	$x^{j_1(\sigma)}y^{j_2(\sigma)}z^{W(\sigma)}$
00	1
10	x
20	y
01	xz
11	x^2z
21	xyz^2
02	yz
12	xyz
22	y^2z

One can check using the data in Section 4 that this is the correct polynomial $N_1(x, y, z)$. At present the author does not have a conjecture similar to Conjecture 1.16 for the whole polynomial $M_k(x, y, z; \lambda)$.

2. Stability

For certain sets (a, b, w, k) we do know the eigenvalues and eigenvectors of the $T_k(a, b, w)$. Before stating the condition which defines these sets it is worth noting that $\Omega_k(a, b, w)$ is empty unless w is at least $\binom{a}{2}+\binom{b}{2}$. Also the matrix $T_k(a, b, w)$ is similar to the matrix $T_k(b, a, w)$, so when we consider these matrices we may assume that $a \leqslant b$.

Definition 2.1. Let (a, b, w, k) be a four-tuple with $a \leqslant b$ and $w \geqslant \binom{a}{2}+\binom{b}{2}$. We say this four-tuple is *stable* if

(1) $w \leqslant \binom{a}{2}+\binom{b}{2}+a$.

(2) $k \geqslant (a-1)+(b-1)+(w-\binom{a}{2}-\binom{b}{2})$.

Proposition 2.2. *Let (a, b, w, k) be a stable four-tuple with $a \leqslant b$ and $n = w-\binom{a}{2}-\binom{b}{2}$. Then $\Omega_k(a, b, w)$ is in 1–1 correspondence with the set of pairs (λ, μ), where λ and μ are partitions of p and $n-p$ for some $p \leqslant n$.*

Proof. This follows easily from condition (1) of Definition 2.1. The exact bijection between $\Omega_k(a, b, w)$ and the set of pairs (λ, μ) is given as follows:

Let $\lambda=(\lambda_1, \lambda_2, ..., \lambda_l)$ and $\mu=(\mu_1, \mu_2, ..., \mu_m)$ be a pair of partitions of p and $(n-p)$, so that $l \leqslant p \leqslant n \leqslant a$ and $m \leqslant n-p \leqslant n \leqslant b$. Define the corresponding set (U, V) to be

$$U=\{a-1+\lambda_1, a-2+\lambda_2, ..., a-l+\lambda_l, a-l-1, ..., 0\}$$

$$V=\{b-1+\mu_1, b-2+\mu_2, ..., b-m+\mu_m, b-m-1, ..., 0\}. \quad \blacksquare$$

The main result of this section determines the eigenvalues and eigenvectors explicitly. To state this result we need some notation from the representation theory of the symmetric and hyperoctahedral groups. Let (a, b, w, k) be a stable four-tuple and let $\{x_1, ..., x_a, y_1, ..., y_b\}$ be a set of commuting indeterminates. Let $\{z_1, ..., z_{a+b}\}$ be a second set of commuting indeterminates. Symmetric functions notation follows [M].

For each $r \in \mathbb{N}$ let $p_r^+(\mathbf{x}, \mathbf{y})$ and $p_r^-(\mathbf{x}, \mathbf{y})$ denote

$$p_r(\mathbf{x}+\mathbf{y}) = \sum_{i=1}^{a} x_i^r + \sum_{j=1}^{b} y_j^r$$

$$p_r(\mathbf{x}-\mathbf{y}) = \sum_{i=1}^{a} x_i^r - \sum_{j=1}^{b} y_j^r.$$

For $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$, a partition, define $p_\lambda(\mathbf{x}+\mathbf{y})$ by

$$p_\lambda(\mathbf{x}+\mathbf{y}) = \prod_{s=1}^{l} p_{\lambda_s}(\mathbf{x}+\mathbf{y})$$

$$p_\lambda(\mathbf{x}-\mathbf{y}) = \prod_{s=1}^{l} p_{\lambda_s}(\mathbf{x}-\mathbf{y}).$$

Lastly, if μ is a partition of n define $s_\mu(\mathbf{x}+\mathbf{y})$ as follows: first write the Schur function $s_\mu(z_1, ..., z_{a+b})$ in the form

$$s_\mu(\mathbf{z}) = \sum_{\lambda \vdash f} c_\lambda^\mu p_\lambda(\mathbf{z}).$$

Then

$$s_\mu(\mathbf{x}+\mathbf{y}) = \sum_\lambda c_\lambda^\mu p_\lambda(\mathbf{x}+\mathbf{y})$$

and

$$s_\mu(\mathbf{x}-\mathbf{y}) = \sum_\lambda c_\lambda^\mu p_\lambda(\mathbf{x}-\mathbf{y}).$$

The signed Schur functions $s_\mu(\mathbf{x} \pm \mathbf{y})$ and the signed power sums $p_\lambda(\mathbf{x} \pm \mathbf{y})$ originally came up in the representation theory of the hyperoctahedral group. The two sets

$$\{s_\lambda(\mathbf{x}+\mathbf{y})\, s_\mu(\mathbf{x}-\mathbf{y}) \colon |\lambda| + |\mu| = n\} = \mathscr{B}_1$$

and

$$\{p_\lambda(\mathbf{x}+\mathbf{y})\, p_\mu(\mathbf{x}-\mathbf{y}) \colon |\lambda| + |\mu| = n\} = \mathscr{B}_2$$

each form a $\mathbb{Q}$-basis for the homogeneous polynomials of degree n that are symmetric in the x's and symmetric in the y's. The transition matrix between these two bases is the character table of the hyperoctahedral group H_n.

There is a more natural basis for the same set of homogeneous polynomials, that being

$$\{s_\lambda(\mathbf{x})\, s_\mu(\mathbf{y}) \colon |\lambda| + |\mu| = n\} = \mathscr{B}_3.$$

We will make use of the transition matrix between this basis $\mathscr{B}_3$ and the basis $\mathscr{B}_1$ above.

DEFINITION 2.3. Define $v^{(\alpha,\beta)}_{(\lambda,\mu)}$ for all (α, β), (λ, μ) with $|\alpha| + |\beta| = |\lambda| + |\mu| = n$ by the equation

$$s_\alpha(\mathbf{x}+\mathbf{y})\, s_\beta(\mathbf{x}-\mathbf{y}) = \sum_{(\lambda,\mu)} v^{(\alpha,\beta)}_{(\lambda,\mu)} s_\lambda(\mathbf{x})\, s_\mu(\mathbf{y}).$$

EXAMPLE 2.4. (A) Suppose $\beta = \phi$ so α is a partition of n. Then $s_\alpha(\mathbf{x}+\mathbf{y})\, s_\beta(\mathbf{x}-\mathbf{y}) = s_\alpha(\mathbf{x}+\mathbf{y}) = s_\alpha(x_1, ..., x_a, y_1, ..., y_b)$. By standard symmetric function arguments we have

$$s_\alpha(\mathbf{x}, \mathbf{y}) = \sum_{\lambda,\mu} g_{\lambda\mu\alpha} s_\lambda(\mathbf{x})\, s_\mu(\mathbf{y}),$$

where $g_{\lambda\mu\alpha}$ is the number of Littlewood–Richardson fillings of α/λ with content μ. So we have

$$v^{(\alpha,\phi)}_{(\lambda,\mu)} = g_{\lambda\mu\alpha}.$$

(B) Suppose $\alpha = \phi$ so β is a partition of n. Then $s_\alpha(\mathbf{x}+\mathbf{y})\, s_\beta(\mathbf{x}-\mathbf{y}) = s_\beta(\mathbf{x}-\mathbf{y})$. It is not hard to apply the arguments from (A) above to show that

$$v^{(\phi,\beta)}_{(\lambda,\mu)} = (-1)^{|\mu|}\, g_{\lambda\mu'\beta}.$$

(C) Let $n = 2$. Cases (A) and (B) above determine $v^{(\alpha,\beta)}_{(\lambda,\mu)}$ for $(\alpha, \beta) = (2, \phi)$, $(1^2, \phi)$, $(\phi, 1^2)$, and $(\phi, 2)$. To compute the remaining values $v^{(1,1)}_{(\alpha,\beta)}$ note that:

$$\begin{aligned} s_1(\mathbf{x}+\mathbf{y})\, s_1(\mathbf{x}-\mathbf{y}) &= \left(\sum x_i + \sum y_j\right)\left(\sum x_i - \sum y_j\right) \\ &= p_1(x)^2 - p_1(y)^2 \\ &= (s_2(\mathbf{x}) + s_{1^2}(\mathbf{x})) - (s_2(\mathbf{y}) + s_{1^2}(\mathbf{y})). \end{aligned}$$

So the values of $v^{(1,1)}_{(\lambda,\mu)}$ are $v^{(1,1)}_{(2,\phi)} = v^{(1,1)}_{(1^2,\phi)} = 1$, $v^{(1,1)}_{(1,1)} = 0$, and $v^{(1,1)}_{(\phi,1^2)} = v^{(1,1)}_{(\phi,2)} = -1$. A complete listing of the $v^{(\alpha,\beta)}_{(\lambda,\mu)}$ for $n=2$ appears in the table below:

$(\lambda,\mu)\backslash(\alpha,\beta)$	$(2,\phi)$	$(1^2,\phi)$	$(1,1)$	$(\phi,1^2)$	$(\phi,2)$
$(2,\phi)$	1	0	1	0	1
(1^2)	0	1	1	1	0
$(1,1)$	1	1	0	-1	-1
$(\phi,1^2)$	0	1	-1	0	1
$(\phi,2)$	1	0	-1	1	0

The reader should compare the columns of the above table to the eigenvectors for $T_k(2,2,4)$ given in Example 1.13 (A).

THEOREM 2.5. *Let (a,b,w,k) be a stable four-tuple and let $n = w - \binom{a}{2} - \binom{b}{2}$. Let $\mathscr{P}_n$ be the set of pairs of partitions (λ,μ) such that $|\lambda| + |\mu| = n$. Assume $\mathscr{P}_n$ is ordered in some way and that $T_k(a,b,w)$ is constructed according to that ordering (here we have identified $\mathscr{P}_n$ with $\Omega_k(a,b,w)$). Then*

(a) *for each $(\alpha,\beta) \in \mathscr{P}_n$ the vector $(v^{(\alpha,\beta)}_{(\lambda,\mu)})_{(\lambda,\mu)\in\mathscr{P}_n}$ is an eigenvector of $T_k(a,b,w)$ with corresponding eigenvalue $|\alpha| - |\beta| + ab$.*

(b) *The eigenvectors constructed in part* (a) *are linearly independent; hence they give a complete set of eigenvectors.*

Proof. Condition (2) in the definition of stability implies that if (U,V) is a pair in $\Omega_k(a,b,w)$ and $u \in U$, $v \in V$ then $u + v \leqslant k$. This means that k is large enough so that we never need to consider condition (3) of Definition 1.3 when we determine the entries of $T_k(a,b,w)$. In particular the diagonal entries of $T_k(a,b,w)$ are all equal to ab. Let M be the matrix

$$M = T_k(a,b,w) - (ab)I.$$

We will determine the eigenvalues and eigenvectors of M acting on the basis $\{s_\alpha(\mathbf{x})\, s_\beta(\mathbf{y})\}$.

Let $\Lambda(\mathbf{x})$ be the ring of symmetric functions in $x_1, ..., x_a$. For each r, multiplication by the power sum symmetric function $p_r(\mathbf{x})$ constitutes a linear transformation of $\Lambda(\mathbf{x})$. To avoid confusion we denote this linear transformation by $P_r(\mathbf{x})$. Our immediate goal is to express M in terms of these transformations $P_r(\mathbf{x})$.

Recall the following notation from the theory of symmetric functions. If $\gamma = (\gamma_1, ..., \gamma_a)$ is any non-negative integral sequence then $s_\gamma(\mathbf{x})$ denotes

$$s_\gamma(\mathbf{x}) = \frac{\det(x_j^{\gamma_i + a - i})}{\det(x_j^{a-i})} = \varepsilon(\boldsymbol{\gamma} + \boldsymbol{\rho})\, s_{\sigma(\gamma+\rho)-\rho}(\mathbf{x}),$$

where $\rho = (a-1, a-2, ..., 0)$ and σ is the permutation in S_a which puts the entries of $\boldsymbol{\gamma} + \boldsymbol{\rho}$ in decreasing order.

Lemma 2.6 [M, p. 32]. *Let λ be a partition of length less than or equal to a. Write $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_a)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_a \geqslant 0$. Let r be a positive integer. Then*

$$P_r(\mathbf{x})(s_\lambda(\mathbf{x})) = \sum_{i=1}^{a} s_{\boldsymbol{\lambda} + re_i}(\mathbf{x}),$$

where $e_1, ..., e_a$ are the unit coordinate vectors in $\mathbb{Z}^a$.

Let $\langle\ ,\ \rangle$ be the usual inner product on the ring $\Lambda(\mathbf{x})$. So we have that the Schur functions are an orthonormal basis of $\Lambda(\mathbf{x})$ with respect to $\langle\ ,\ \rangle$. The power sum symmetric functions satisfy

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x})\rangle = z_\lambda \delta_{\lambda\mu},$$

where z_λ is the order of the centralizer in $S_{|\lambda|}$ of a permutation with cycle type λ. Let $P_r^*(\mathbf{x})$ denote the adjoint of $P_r(\mathbf{x})$ with respect to $\langle\ ,\ \rangle$. So $P_r^*(\mathbf{x})$ is defined by

$$\langle P_r^*(x)\, u(\mathbf{x}), v(\mathbf{x})\rangle = \langle u(\mathbf{x}), P_r(\mathbf{x})\, v(\mathbf{x})\rangle.$$

From Lemma 2.6 and the fact that $\{s_\lambda(\mathbf{x})\}$ is an orthonormal basis we have

$$P_r^*(x)\, s_\mu(\mathbf{x}) = \sum_i s_{\mu - re_i}(\mathbf{x}). \tag{2.7}$$

On the right-hand side of (2.7) the sum is over those i such that $\mu_i - re_i > 0$.

We will need to rewrite Lemma 2.6 and (2.7) in slightly different forms. Let $\lambda = (\lambda_1, ..., \lambda_l)$ be a partition of f where $f \leqslant a$. Let U be the **a**-set $\{\lambda_1 + a - 1, \lambda_2 + a - 2, ..., \lambda_a\} = \{u_1, ..., u_a\}$ and let r be a positive integer. Fix i, let $X^{(+)}$ denote the set $\{u_1, ..., u_i + r, ..., u_a\}$ and let $X^{(-)}$ denote the set $\{u_1, ..., u_i - r, ..., u_a\}$. Let $\mu^{(+)}$ and $\mu^{(-)}$ denote the partitions of $f + r$ and $f - r$ corresponding to $X^{(+)}$ and $X^{(-)}$, respectively. Then

$$\begin{aligned}&\text{the coefficient of } s_{\mu^{(+)}}(\mathbf{x}) \text{ in } P_r(\mathbf{x})\, s_\lambda(x) \text{ equals } \varepsilon(U, X^{(+)})\\ &\text{the coefficient of } s_{\mu^{(-)}}(\mathbf{x}) \text{ in } P_r^*(x)\, s_\lambda(x) \text{ equals } \varepsilon(U, X^{(-)}).\end{aligned} \tag{2.8}$$

From (2.8) it follows that M represents the linear transformation

$$M = \sum_{r=1}^{\infty} P_r(\mathbf{x})\, P_r^*(\mathbf{y}) + P_r^*(\mathbf{x})\, P_r(\mathbf{y})$$

on $V_n = (\Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{y}))^{(n)}$, where the subscript (n) denotes the subspace of $\Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{y})$ of polynomials which are homogeneous of degree n. Interestingly enough, the same linear transformation M comes up in the work of Stanley on r-differential posets (see [St]).

There is a second basis for V_n, namely the set $\{s_\alpha(\mathbf{x}+\mathbf{y})\, s_\beta(\mathbf{x}-\mathbf{y}): |\alpha| + |\beta| = n\}$. Let $V_n^{(r,s)}$ denote the span of all $s_\alpha(\mathbf{x}+\mathbf{y})\, s_\beta(\mathbf{x}-\mathbf{y})$, where $|\alpha| = r$ and $|\beta| = s$. We will show:

1. M preserves the space $V_n^{(r,s)}$.
2. M restricted to $V_n^{(r,s)}$ is just the scalar matrix $r - s$. (2.9)

To check (2.9) we can apply M to any basis of $V_n^{(r,s)}$ that we like. Let $\mathscr{B}_n^{(r,s)}$ denote the basis $\{p_\alpha(\mathbf{x}+\mathbf{y})\, p_\beta(\mathbf{x}-\mathbf{y}): |\alpha| = r,\ |\beta| = s\}$. Note that $P_r(x)\, P_r^*(y)$ is a derivation so M is a derivation as well. It follows that

$$M \cdot p_\alpha(\mathbf{x}+\mathbf{y})\, p_\beta(\mathbf{x}-\mathbf{y}) = \{M p_\alpha(\mathbf{x}+\mathbf{y})\}\, p_\beta(\mathbf{x}-\mathbf{y}) + p_\alpha(\mathbf{x}+\mathbf{y}) \{M p_\beta(\mathbf{x}-\mathbf{y})\}.$$

It remains to show that $M p_\alpha(\mathbf{x}+\mathbf{y}) = |\alpha|\, p_\alpha(\mathbf{x}+\mathbf{y})$ and $M p_\beta(\mathbf{x}-\mathbf{y}) = -|\beta|\, p_\beta(\mathbf{x}-\mathbf{y})$. Again using that M is a derivation, we have

$$M p_\alpha(\mathbf{x}+\mathbf{y}) = \sum_{i=1}^{l} \{M p_{\alpha_i}(\mathbf{x}+\mathbf{y})\} \left\{\prod_{j \neq i} p_{\alpha_j}(\mathbf{x}+\mathbf{y})\right\}. \tag{2.10}$$

We must determine $M p_{\alpha_i}(\mathbf{x}+\mathbf{y}) = M \cdot (p_{\alpha_i}(\mathbf{x}) + p_{\alpha_i}(\mathbf{y}))$. Note that

$$\begin{aligned}
&\langle P_r^*(y)\, p_s(\mathbf{y}), p_\mu(\mathbf{y}) \rangle \\
&= \langle p_s(\mathbf{y}), p_r(\mathbf{y})\, p_\mu(\mathbf{y}) \rangle \\
&= \begin{cases} 0 & \text{if } r \neq s \\ \langle p_r(\mathbf{y}), p_r(\mathbf{y})\, p_\mu(\mathbf{y}) \rangle & \text{if } r = s \end{cases} \\
&= \begin{cases} 0 & \text{unless } r = s,\ \mu = \phi \\ \langle p_r(\mathbf{y}), p_r(\mathbf{y}) \rangle & \text{if } r = s,\ \mu = \phi \end{cases} \\
&= \begin{cases} r & \text{if } r = s,\ \mu = \phi \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.11}$$

So $M \cdot (p_s(\mathbf{x}) + p_s(\mathbf{y})) = P_s(\mathbf{y})\, P_s^*(\mathbf{x})\, p_s(\mathbf{x}) + P_s(\mathbf{x})\, P_s^*(\mathbf{y})\, p_s(\mathbf{y}) = s(p_s(\mathbf{y}) + p_s(\mathbf{x}))$. By (2.11) we have $M \cdot p_\alpha(\mathbf{x}+\mathbf{y}) = |\alpha|\, p_\alpha(\mathbf{x}+\mathbf{y})$.

Similarly, $M \cdot (p_s(\mathbf{x}) - p_s(\mathbf{y})) = sp_s(\mathbf{y}) - sp_s(\mathbf{x}) = (-s)(p_s(\mathbf{x}) - p_s(\mathbf{y}))$. So

$$M \cdot p_\beta(\mathbf{x} - \mathbf{y}) = -|\beta| \, p_\beta(\mathbf{x} - \mathbf{y}).$$

This proves (2.9) which completes the proof of Theorem 2.5. ∎

Theorem 2.5 tells us what happens when a and b are large with respect to w, and k is large with respect to a and b. Example 1.13 shows how the eigenvalues and eigenvectors can change when this ideal situation destabilizes by k becoming small. We now do one example where destabilization occurs by a and b becoming small. Begin with the stable parameters $a = b = 2$, $w = 4$, $k \geqslant 4$. As a and b become small we will change w to preserve $n = w - \binom{a}{2} - \binom{b}{2}$.

Example 2.12. (A) Let $a = 1$, $b = 2$, $w = 3$, and $k = 3$. We have $\Omega_3(1, 2, 3) = \{(2, 01), (1, 02), (0, 03), (0, 12)\}$. The matrix $T_3(1, 2, 3)$ with respect to that ordered basis is

$$\begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 2 \end{pmatrix}.$$

The eigenvalues and corresponding eigenvectors for this matrix are

eigenvalue λ	0	1	3	4
eigenvector	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(B) Keeping $a = 1$, $b = 2$, let $k = 2$. The ordered basis $\Omega_2(1, 2, 3)$ is $\{(2, 01), (1, 02), (0, 12)\}$. The matrix $T_2(1, 2, 3)$ is

$$T_2(1, 2, 3) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

λ	0	1	3
$\mathbf{v}$	$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

(C) Lastly, let $a=b=1$, $w=2$, and $k=2$. Then $\Omega_2(1,1,2)=\{(2,0),(1,1),(0,2)\}$ and

$$T_2(1,1,2)=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}.$$

The eigenvalue and corresponding eigenvectors are

λ	0	0	3
v	$\begin{pmatrix}1\\-1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$

By considering extreme values of a, b, and k it is possible to make a number of specialized conjectures about how this destabilization occurs. But the general procedure remains quite mysterious.

3. Laplacians of Nilpotent Upper Summands

The matrices $L_k(a,b,w)$ arise in an algebraic setting. In this section we will briefly describe their algebraic interpretation and we will state a conjecture about Laplacians of nilpotent upper summands which generalizes Conjecture 1.14.

Let $\mathscr{G}$ be a semisimple complex Lie algebra and let H be a Cartan subalgebra. Let R be the root system of $\mathscr{G}$ and let

$$\mathscr{G}=H\oplus\bigoplus_{\alpha\in R}\mathscr{G}_\alpha$$

be the root space decomposition of $\mathscr{G}$.

For S a set of simple roots, let R_S be the root system of R generated by S. Let $\mathscr{G}_S$ be the semisimple subalgebra of $\mathscr{G}$ given by

$$\mathscr{G}_S=H\oplus\bigoplus_{\alpha\in R_S}\mathscr{G}_\alpha.$$

Definition 3.1. For S a set of simple roots let $N^{(S)}$ be the nilpotent Lie algebra given by

$$N^{(S)}=\bigoplus_{\alpha\in R^+\setminus R_S^+}\mathscr{G}_\alpha.$$

We call $N^{(S)}$ a *nilpotent upper summand* of $\mathscr{G}$.

EXAMPLE 3.2. Let $\mathscr{H}$ be the three-dimensional Heisenberg Lie algebra. So $\mathscr{H}$ has $\mathbb{C}$-basis $\{e, f, x\}$ with non-zero brackets

$$[e, f] = -[f, e] = x.$$

Then $\mathscr{H}$ is the nilpotent upper summand of $\mathscr{G} = sl_3(\mathbb{C})$ obtained by taking $S = \phi$. The isomorphism is given by sending z_{12} to e, z_{23} to f, and z_{13} to x (where z_{ij} is the matrix in $sl_3(\mathbb{C})$ which has a 1 in the i, j entry and 0's elsewhere).

DEFINITION 3.3. Let N be a complex Lie algebra and let k be a non-negative integer. Define N_k to be the Lie algebra

$$N_k = N \otimes \{\mathbb{C}[t]/(t^{k+1})\}.$$

The bracket in N_k is given by

$$[x \otimes a(t), y \otimes b(t)] = [x, y] \otimes (a(t)\, b(t)).$$

Let M be any complex Lie algebra. For each r, define $\partial_r \colon \Lambda^r M \to \Lambda^{r-1} M$ by

$$\partial^r(m_1 \wedge \cdots \wedge m_r) = \sum_{i<j} (-1)^{i+j-1} [m_i, m_j] \wedge m_1 \wedge \cdots \wedge \widehat{m_i} \wedge \cdots \wedge \widehat{m_j} \wedge \cdots \wedge m_r.$$

It is straightforward to check that $\partial_{r-1} \circ \partial_r = 0$. Define $H_r(M)$ by

$$H_r(M) = \ker \partial_r / im\, \partial_{r+1}.$$

$H_*(M)$ is called the *homology of M*.

Suppose, in addition, that M is a graded Lie algebra, i.e., $M = \bigoplus_{s=0}^{\infty} M_s$, where $[M_u, M_v] \subseteq M_{u+v}$. In this case we introduce a second grading on ΛM (which we call *weight*) as follows: if $m_i \in M_{s_i}$ then the weight of $m_1 \wedge \cdots \wedge m_r$ is $(s_1 + \cdots + s_r)$. Let $\Lambda^{r,w} M$ denote the span of all $m_1 \wedge \cdots \wedge m_r$ which have weight w.

It is easy to see that ∂_* preserves weight, i.e.,

$$\partial_r \colon \Lambda^{r,w} M \to \Lambda^{r-1,w} M.$$

This implies that the notion of weight passes to the homology of M and we obtain a subsequent bigrading of $H(M)$.

Let $\langle\ ,\ \rangle$ be a Hermitian form on M with the property that

$\langle M_u, M_v \rangle = 0$ if $u \neq v$. Then $\langle \ , \ \rangle$ gives rise to a Hermitian form on ΛM (also denoted $\langle \ , \ \rangle$) by

$$\langle m_1 \wedge \cdots \wedge m_r, p_1 \wedge \cdots \wedge p_s \rangle = \begin{cases} 0 & \text{if } r \neq s \\ \det(\langle m_i, p_j \rangle) & \text{if } r = s. \end{cases}$$

It is easy to see that

$$\langle \Lambda^{r_1, w_1} M, \Lambda^{r_2, w_2} M \rangle = 0 \qquad \text{unless} \quad r_1 = r_2 \text{ and } w_1 = w_2.$$

DEFINITION 3.4. Define ∂'_* to be the adjoint of ∂_* with respect to $\langle \ , \ \rangle$ and define L to be

$$L = \partial \partial' + \partial' \partial.$$

We call L the *Laplacian* of M.

Note that ∂' and L both depend on $\langle \ , \ \rangle$. In practice the form $\langle \ , \ \rangle$ will be understood and so we will not refer to $\langle \ , \ \rangle$ when we talk about ∂' and L. It is easy to see that

$$\partial' \colon \Lambda^{r, w} M \to \Lambda^{r+1, w} M.$$

So $L\colon \Lambda^{r, w} M \to \Lambda^{r, w} M$.

It is straightforward to check that $\partial'_{r+1} \partial'_r = 0$. Define $H^*(M)$ by

$$H^*(M) = \ker \partial'_r / im\, \partial'_{r-1}.$$

$H^*(M)$ is called the *cohomology* of M. As the following result indicates, the kernel of the Laplacian is of particular interest.

THEOREM 3.5 (see [K]). *Any basis for the kernel of L simultaneously gives a complete set of homology representatives and cohomology representatives.*

For the rest of this section we will look at a particular graded Lie algebra M. Let $\mathscr{H} = \langle e, f, x \rangle$ denote the three-dimensional Heisenberg–Lie algebra and let $M = \mathscr{H}_k$. Grade M by

$$M_u = \begin{cases} \mathscr{H} \oplus t^u & \text{if } u \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience we let e_u, f_u, and x_u denote $e \oplus t^u$, $f \otimes t^u$, and $x \otimes t^u$, respectively.

Let $\langle\ ,\ \rangle$ be the Hermitian form whose Gram matrix is the identity with respect to the basis $\mathscr{B}$,

$$\mathscr{B}=\{e_u : u=0, 1, ..., k\} \cup \{f_u : u=0, 1, ..., k\} \cup \{x_u : u=0, 1, ..., k\}.$$

We henceforth assume that ∂' and L are defined with respect to the form $\langle\ ,\ \rangle$.

Let E, F, and X be the subspaces of M spanned by the e_u, f_u, and x_u, respectively. We have $M=E\oplus F\oplus X$ so

$$\Lambda M=(\Lambda E)\otimes(\Lambda F)\otimes(\Lambda X). \tag{3.6}$$

We need an observation concerning the way ∂ behaves with respect to the decomposition (3.6). The only non-zero brackets in M are of the form

$$[e_u, f_v]=x_{u+v}.$$

So,

$$\partial\colon (\Lambda^a E)\otimes(\Lambda^b F)\otimes(\Lambda^c X)\to(\Lambda^{a-1}E)\otimes(\Lambda^{b-1}F)\otimes(\Lambda^{c+1}X). \tag{3.7}$$

Using (3.7) it is easy to see that

$$\partial'\colon (\Lambda^a E)\otimes(\Lambda^b F)\otimes(\Lambda^c X)\to(\Lambda^{a+1}E)\otimes(\Lambda^{b+1}F)\otimes(\Lambda^{c-1}X). \tag{3.8}$$

Combining (3.7) and (3.8) we have that

$$L\colon (\Lambda^a E)\otimes(\Lambda^b F)\otimes(\Lambda^c X)\to(\Lambda^a E)\otimes(\Lambda^b F)\otimes(\Lambda^c X).$$

Let $V_k(a, b, c; w)$ denote the subspace of ΛM given by

$$V_k(a, b, c; w)=(\Lambda^{*,w}(M))\cap\{(\Lambda^a E)\otimes(\Lambda^b F)\otimes(\Lambda^c X)\}. \tag{3.9}$$

From (3.9) and previous observation that L preserves weight, we have

$$L\colon V_k(a, b, c; w)\to V_k(a, b, c; w). \tag{3.10}$$

We have the following theorem which concerns the map (3.10) just in the case $c=0$.

THEOREM 3.11. (A) *We can choose a basis for $V_k(a, b, 0; w)$ which is in natural bijective correspondence with $\Omega_k(a, b, w)$.*

(B) *The matrix for L with respect to that basis is $L_k(a, b, w)$.*

The proof of Theorem 3.11 is not difficult and we leave it to the reader. One could ask if the restriction of L to $V_k(a, b, c; w)$ has a combinatorial description for any c. The answer is yes but the description is significantly more complicated when c is nonzero.

As mentioned earlier the three-dimensional Heisenberg–Lie algebra is one example of a nilpotent upper summand. At least one conjecture from Section 1 seems to have an analogue in the more general situation. Let $N = N^{(S)}$ be a nilpotent upper summand of a semisimple Lie algebra $\mathscr{G}$. Kostant [K] showed that there exists a positive definite Hermitian form $\{\ ,\ \}$ on N with the property that the Laplacian L of N with respect to $\{\ ,\ \}$ has non-negative integer eigenvalues. For each pair of non-negative integers r, n let $m(N; r, n)$ denote the multiplicity of n as an eigenvalue of the restriction of L to $\Lambda^r N$. Define the generating function $E(N; z, \lambda)$ to be the generating function for the numbers $m(N; r, n)$, i.e.,

$$E(N; z, \lambda) = \sum_{r,n} m(N_k; r, n) z^r \lambda^n.$$

Define a positive definite Hermitian form $\langle\ ,\ \rangle$ on N_k by

$$\langle x \otimes t^i, y \otimes t^j \rangle = \begin{cases} \{x, y\} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let L_k denote the Laplacian of N_k with respect to $\langle\ ,\ \rangle$ and define $m(N_k; r, n)$ as above to be the multiplicity of n as an eigenvalue of the restriction of L_k to $\Lambda^r N_k$. Also define $E(N_k; z, \lambda)$ by

$$E(N_k; z, \lambda) = \sum_{r,n} m(N; r, n) z^r \lambda^n.$$

We end with the following conjecture which generalizes Conjecture 1.14 from the three-dimensional Heisenberg to arbitrary nilpotent upper summands.

Conjecture 3.12. With notation as above we have

$$E(N_k; z, \lambda) = \prod_{i=1}^{k+1} E(N_k; z, \lambda^i).$$

Conjecture 3.12 is one of a series of conjectures made by the author concerning the homology of N_k for various Lie algebras N. The reader can find a thorough discussion of these conjectures in [H].

4. Computational Results

Below we see lists of the eigenvalues of the matrices $T_k(a, b, w)$ for some small values of k. Since $T_k(a, b, w) = T_k(b, a, w)$ we only list those triples with $a \leqslant b$. Also if $a=0$ then the matrix $T_k(0, b, w)$ is the $p \times p$ matrix of 0's, where p is the numbers of partitions of w with length b. So we also omit the case $a=0$ on our list.

$k=1$	a	b	w	Eigenvalues of $T_1(a, b, w)$
	1	2	0	1
			1	0, 2
			2	0
	1	2	1	2
			2	1
	2	2	2	3

$k=2$	a	b	w	Eigenvalues of $T_2(a, b, w)$
	1	1	0	1
			1	0, 2
			2	0, 0, 3
			3	0, 0
			4	0
	1	2	1	2
			2	1, 3
			3	0, 1, 3
			4	0, 2
			5	0
	1	3	3	3
			4	2
			5	1
	2	2	2	4
			3	3, 3
			4	2, 2, 5
			5	1, 3
			6	1
	2	3	4	5
			5	4
			6	3
	3	3	6	6

$k=3$	a	b	w	Eigenvalues of $T_3(a, b, w)$
	1	1	0	1
			1	0, 2
			2	0, 0, 3
			3	0, 0, 0, 4
			4	0, 0, 0
			5	0, 0
			6	0
	1	2	1	2
			2	1, 3
			3	0, 1, 3, 4
			4	0, 0, 1, 2, 4

k = 3—*Cont.*	a	b	w	Eigenvalues of $T_3(a, b, w)$
			5	0, 0, 0, 2, 4
			6	0, 0, 0, 3
			7	0, 0
			8	0
	1	3	3	3
			4	2, 4
			5	1, 2, 4
			6	0, 1, 3, 4
			7	0, 1, 3
			8	0, 2
			9	0
	1	4	6	4
			7	3
			8	2
			9	1
	2	2	2	4
			3	3, 5
			4	2, 2, 3, 4, 6
			5	1, 2, 2, 4, 4, 5
			6	0, 1, 1, 3, 3, 3, 4, 7
			7	0, 1, 1, 3, 3, 4
			8	0, 0, 1, 2, 4
			9	0, 2
			10	0
	2	3	4	6
			5	5, 5
			6	4, 4, 4, 7
			7	3, 3, 3, 5, 7
			8	2, 2, 3, 4, 6
			9	1, 2, 4, 5
			10	1, 3
			11	1
	2	4	7	7
			8	6
			9	5, 5
			10	4
			11	3
	3	3	6	8
			7	7, 7
			8	6, 6, 6
			9	5, 5, 5, 9
			10	4, 4, 7
			11	3, 5
			12	3
	3	4	9	9
			10	8
			11	7
			12	6
	4	4	12	10

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