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Intermediate- and extreme-sum processes

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Let $X_{1,n} \leq \cdots \leq X_{n,n}$ be the order statistics of n independent random variables with a common distribution function F and let k_n be positive numbers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. With suitable centering and norming, we investigate the weak convergence of the intermediate-sum process $\sum_{i=\lceil ak_n \rceil+1}^{\lceil tk_n \rceil} X_{n+1-i,n}$, $a \leq t \leq b$, where $0 < a < b < \infty$, and the weak convergence of the extreme-sum process $\sum_{i=1}^{\lceil tk_n \rceil} X_{n+1-i,n}$, $0 \leq t \leq b$. Convergence is with respect to the supremum norm and can take place along a subsequence of the positive integers $\{n\}$.

order statistics * intermediate-sum processes * extreme-sum processes * weak convergence * extreme-value domain of attraction

1. Introduction and statement of results

Let $X, X_1, X_2, \ldots,$ be a sequence of independent non-degenerate random variables with a common distribution function $F(x) = P\{X \leq x\}$, $x \in \mathbb{R}$, and for each integer $n \geq 1$ let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Let $\{k_n\}$ be a sequence of positive numbers such that

$$k_n \to \infty \text{ and } k_n/n \to 0 \quad \text{as } n \to \infty, \tag{1.1}$$

and consider the sum process

$$I_n(a, t) = I_n(a, t; k_n) = \sum_{i=\lceil ak_n \rceil+1}^{\lceil tk_n \rceil} X_{n+1-i,n}, \quad a \leq t \leq b, \tag{1.2}$$

of intermediate order statistics, where $0 < a < b$, and the sum process

$$E_n(t) = E_n(t; k_n) = \begin{cases} \sum_{i=1}^{\lceil tk_n \rceil} X_{n+1-i,n}, & 1/k_n \leq t \leq b, \\ 0, & 0 \leq t < 1/k_n, \end{cases} \tag{1.3}$$

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of extreme order statistics, where $\lceil x \rceil$ is the smallest integer not smaller than x and an empty sum is understood as zero.

The asymptotic distribution of the intermediate sum $I_n(a, b)$ for fixed $0<a<b$ has been thoroughly investigated in [3]. We found necessary and sufficient conditions for the existence of constants $A_n>0$ and $C_n \in \mathbb{R}$ such that $A_n^{-1}(I_n(a, b)-C_n)$ converges in distribution along subsequences of the positive integers $\{n\}$ to non-degenerate limits and completely described the possible subsequential limiting distributions. Exactly the same programme has been carried out previously in [2] for the extreme sums $E_n(1)$. The aim of the prsent paper is to investigate the weak convergence of the suitably centered and normalized processes $I_n(a, \cdot)$ and $E_n(\cdot)$ in the supremum norm on $[a, b]$ and $[0, b]$, respectively.

Consider the inverse or quantile function of F defined as

$$Q(s)=\inf\{x: F(x) \geqslant s\}, \quad 0<s \leqslant 1,$$

and introduce the associated left-continuous non-decreasing function

$$H(s)=-Q((1-s)-), \quad 0 \leqslant s<1. \tag{1.4}$$

Consider the centering functions

$$\mu_n(a, t)=\mu_n(a, t; k_n)=-n \int_{\lceil ak_n \rceil/n}^{\lceil tk_n \rceil/n} H(s)\, \mathrm{d}s, \quad 0 \leqslant a<t. \tag{1.5}$$

We say that a sequence $\{\xi_n(t): a \leqslant t \leqslant b\}_{n=1}^{\infty}$ of stochastic processes has a distributionally equivalent version $\{\eta_n(t): a \leqslant t \leqslant b\}_{n=1}^{\infty}$ if the distributional equality $\{\xi_n(t): a \leqslant t \leqslant b\} =_{\mathscr{D}} \{\eta_n(t): a \leqslant t \leqslant b\}$ holds for each $n \geqslant 1$, that is, all finite-dimensional distributions of $\xi_n(\cdot)$ and $\eta_n(\cdot)$ are the same on $[a, b]$ for each $n \geqslant 1$.

Theorem 1. *Let $\{k_n\}$ be a sequence as in (1.1) and fix $0<a_0<b_0<\infty$. Suppose that there exist a subsequence $\{n'\}$ of the positive integers and positive numbers $B_{n'}$ along it such that for a function φ continuous on $[a_0, b_0]$, necessarily non-negative, non-decreasing and satisfying $\varphi(a_0)=0$, we have*

$$\varphi_{n'}(a_0; x):=\int_{a_0}^{x} \mathrm{d}H\left(\frac{sk_{n'}}{n'}\right) \Big/ B_{n'} \to \varphi(x) \ \text{ at each } x \in [a_0, b_0] \quad \text{as } n' \to \infty. \tag{1.6}$$

Then on a suitable probability space, for any choice of $a_0<a<b<b_0$, there exist a sequence $\{\tilde{I}_n(a, t): a \leqslant t \leqslant b\}_{n=1}^{\infty}$ of distributionally equivalent versions of the sequence $\{I_n(a, t): a \leqslant t \leqslant b\}_{n=1}^{\infty}$ and a standard Wiener process $W(t)$, $t \geqslant 0$, such that

$$\sup_{a \leqslant t \leqslant b}\left|\frac{1}{\sqrt{k_{n'}}\, B_{n'}}\{\tilde{I}_{n'}(a, t)-\mu_{n'}(a, t)\}-\int_a^t W(s)\, \mathrm{d}\varphi(s)\right| \to 0 \quad a.s. \tag{1.7}$$

as $n' \to \infty$.

We note that by Theorem 1 in [3] for convergence in distribution of the process

$$Y_n(a, t):=\{I_n(a, t)-\mu_n(a, t)\}/\{\sqrt{k_n}\, B_n\}, \tag{1.8}$$

at a fixed point $a \leq t \leq b$ with $a_0 < a < b < b_0$ along a subsequence of $\{n\}$ we can always choose

$$B_n = \Delta_n(a_0, b_0) := \max(H(b_0 k_n/n) - H(a_0 k_n/n), 1) > 0.$$

Then, with this choice of B_n, for the non-decreasing, left-continuous functions $\varphi_n(a_0; x)$ we have $0 \leq \varphi_n(a_0; x) \leq 1$ on $[a_0, b_0]$. Hence by a Helly selection one can always find a subsequence $\{n'\} \subset \{n\}$ and a non-negative, non-decreasing, left-continuous function φ on (a_0, b_0) such that $\varphi_{n'}(a_0; x)$ converges to $\varphi(x)$ as $n' \to \infty$ at any continuity point x of φ. Theorem 1 in [3] shows that one can hope for weak convergence of $I_{n'}(\cdot)$ in the supremum norm on $[a, b]$ only in the case when φ is continuous on some interval containing $[a, b]$. This is the underlying reason for condition (1.6).

Now we trun to the weak-convergence problem of the extreme-sum process $E_n(t) = I_n(0, t), 0 \leq t \leq b$. Even though the problem is now the behavior in the vicinity of zero, we still need a reference point $a_0 > 0$ as in (1.6), which can in principle be chosen to be b_0.

Theorem 2. *Let $\{k_n\}$ be a sequence as in (1.1) and fix $0 < a_0 \leq b_0 < \infty$. Suppose there exist a subsequence $\{n'\}$ of the positive integers and positive numbers $B_{n'}$ such that for a function φ continuous on $(0, b_0]$, necessarily non-decreasing and satisfying $\varphi(a_0) = 0$, we have*

$$\varphi_{n'}(a_0; x) = \int_{a_0}^{x} \mathrm{d}H\left(\frac{sk_{n'}}{n'}\right) \Big/ B_{n'} \to \varphi(x) \quad \text{at each } x \in (0, b_0] \tag{1.9}$$

as $n' \to \infty$ and

$$\lim_{a \downarrow 0} \limsup_{n' \to \infty} \int_0^a \sqrt{x}\, \mathrm{d}\varphi_{n'}(a_0, x) = 0 \quad \text{and} \quad \lim_{a \downarrow 0} \int_0^a \sqrt{x}\, \mathrm{d}\varphi(x) = 0. \tag{1.10}$$

Then on a suitable probability space, for any choice of $0 < b < b_0$, there exist a sequence $\{\tilde{E}_n(t): 0 \leq t \leq b\}_{n=1}^{\infty}$ of distributionally equivalent versions of the sequence $\{E_n(t): 0 \leq t \leq b\}_{n=1}^{\infty}$ and a standard Wiener process $W(t)$, $t \geq 0$, such that

$$\sup_{0 \leq t \leq b} \left| \frac{1}{\sqrt{k_{n'}}\, B_{n'}} \{\tilde{E}_{n'}(t) - \mu_{n'}(0, t)\} - \int_0^t W(s)\, \mathrm{d}\varphi(s) \right| \to_{\mathrm{P}} 0$$

as $n' \to \infty$.

We note that it is easy to see using integration by parts that if (1.9) holds and there exists a constant $\beta > -\frac{1}{2}$ such that $|\varphi_{n'}(a_0, x)| < x^{\beta}$ for all n' large enough and all $x > 0$ small enough, then condition (1.10) is also satisfied.

Now we formulate a corollary to Theorems 1 and 2 under the classical condition of extreme value theory. We say that F is in the domain of attraction of an extreme value distribution if $(X_{n,n} - c_n)/a_n$ converges in distribution to a non-degenerate random variable Y, where $a_n > 0$ and $c_n \in \mathbb{R}$ are some constants. As pointed out in

[2], with earlier references, this happens if and only if there exists a constant $\gamma \in \mathbb{R}$ such that

$$\lim_{s \downarrow 0} \frac{H(sx)-H(sy)}{H(su)-H(sv)} = \begin{cases} (x^{-\gamma}-y^{-\gamma})/(u^{-\gamma}-v^{-\gamma}), & \text{if } \gamma \neq 0, \\ \log(x/y)/\log(u/v), & \text{if } \gamma = 0, \end{cases} \tag{1.11}$$

for all distinct $0<x, y, u, v<\infty$. In this case we write $F \in \Lambda_\gamma$, where, with suitable choices of a_n and c_n,

$$\Lambda_\gamma(y) = P\{Y \leq y\} = \begin{cases} \exp(-y^{1/\gamma}), \ y>0, & \text{if } \gamma>0, \\ \exp(-\exp(-y)), \ y \in \mathbb{R}, & \text{if } \gamma=0, \\ \exp(-(-y)^{1/\gamma}), \ y<0, & \text{if } \gamma<0. \end{cases}$$

For any $\gamma \in \mathbb{R}$, set

$$u_\gamma = \begin{cases} 1, & \text{if } \gamma>0, \\ \mathrm{e}, & \text{if } \gamma=0, \\ (1-\gamma)^{-1/\gamma}, & \text{if } \gamma<0, \end{cases} \quad \text{and} \quad v_\gamma = \begin{cases} (1+\gamma)^{-1/\gamma}, & \text{if } \gamma>0, \\ 1, & \text{if } \gamma=0, \\ 1, & \text{if } \gamma<0, \end{cases}$$

so that $u_\gamma^{-\gamma} - v_\gamma^{-\gamma} = -\gamma$ if $\gamma \neq 0$ and $\log(u_0/v_0)=1$. For a sequence $\{k_n\}$ satisfying (1.1) we define

$$\Delta_n(\gamma) = H(u_\gamma k_n/n) - H(v_\gamma k_n/n).$$

Choosing the reference point of Theorem 2 as $a_0=1$, introduce now

$$\varphi_n(x) = \varphi_n(1,x) = \int_1^x \mathrm{d}H\left(\frac{sk_n}{n}\right) \Big/ \Delta_n(\gamma)$$
$$= \left\{ H\left(\frac{xk_n}{n}\right) - H\left(\frac{k_n}{n}\right)\right\} \Big/ \Delta_n(\gamma), \tag{1.12}$$

which is well-defined for $0<x<n/k_n$. Then, if $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$, we obtain from (1.11) that

$$\varphi_n(x) \to \varphi_\gamma(x) := \begin{cases} (1-x^{-\gamma})/\gamma, & \text{if } \gamma \neq 0, \\ \log x, & \text{if } \gamma = 0, \end{cases} \quad \text{for any } x>0 \tag{1.13}$$

as $n \to \infty$, that is, we have (1.9) with the continuous function $\varphi = \varphi_\gamma$ along the whole $\{n\}$ and for any $b_0>0$, or, what is the same, (1.6) for any $0<a_0<b_0$. Hence the first statement of Corollary 1 below follows from Theorem 1 and the second statement will follow from Theorem 2 after proving (1.10) for the present φ_n and $\varphi = \varphi_\gamma$.

Corollary 1. *If $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$ and $\{k_n\}$ satisfies (1.1), then on a suitable probability space, for any choice of $b>0$, there exist a sequence $\{\tilde{E}_n(t)\colon 0 \leq t \leq b\}_{n=1}^{\infty}$ of distributionally equivalent versions of the sequence $\{E_n(t)\colon 0 \leq t \leq b\}_{n=1}^{\infty}$ and a standard Wiener process $W(t)$, $t \geq 0$, such that for the sequence $\{\tilde{I}_n(a,t) = \tilde{E}_n(t) -$*

$\tilde{E}_n(a)$: $a \leq t \leq b\}_{n=1}^{\infty}$, *being a distributionally equivalent version of the sequence* $\{I_n(a, t)$: $a \leq t \leq b\}_{n=1}^{\infty}$, *and for any* $0 < a < b$,

$$\sup_{a \leq t \leq b} \left| \frac{1}{\sqrt{k_n}\, \Delta_n(\gamma)} \left\{ \tilde{I}_n(a, t) - n \int_{[ak_n]/n}^{[tk_n]/n} Q(1-s)\, \mathrm{d}s \right\} - \int_a^t W(s) s^{-1-\gamma}\, \mathrm{d}s \right| \to 0$$

almost surely as $n \to \infty$. *Furthermore, if* $\gamma < \frac{1}{2}$, *then for any* $b > 0$,

$$\sup_{0 \leq t \leq b} \left| \frac{1}{\sqrt{k_n}\, \Delta_n(\gamma)} \left\{ \tilde{E}_n(t) - n \int_0^{[tk_n]/n} Q(1-s)\, \mathrm{d}s \right\} - \int_0^t W(s) s^{-1-\gamma}\, \mathrm{d}s \right| \to_{\mathrm{P}} 0$$

as $n \to \infty$.

Notice that if $F \in \Lambda_\gamma$ and $\gamma \geq \frac{1}{2}$, then we have (1.13) but condition (1.10) is not satisfied by the limiting function $\varphi = \varphi_\gamma$. In this case, according to Corollary 2 in [2], the appropriately centered extreme sums $E_n(t) - \mu_n(0+, t)$ require a norming sequence $A_n > 0$ to converge in distribution (denoted by $\to_{\mathscr{D}}$) to a non-degenerate random variable V that is heavier than the one needed by the centered intermediate sums. Namely, it follows from Corollary 2 in [2] tht if $F \in \Lambda_\gamma$ for some $\gamma \geq \frac{1}{2}$, then there is a sequence $A_n = A_n(k_n) > 0$, completely specified in [2], such that

$$\{I_n(a, b) - \mu_n(a, b)\}/A_n \to_{\mathrm{P}} 0 \quad \text{for all } 0 < a < b$$

and

$$\{E_n(t) - \mu_n(0+, t)\}/A_n \to_{\mathscr{D}} V \quad \text{for all } t > 0,$$

where if $\gamma = \frac{1}{2}$, then V is the *same* standard normal random variable for all $t > 0$, and if $\gamma > \frac{1}{2}$, then V is the *same* stable random variable with index $1/\gamma$ for all $t > 0$.

Our next corollary discloses a curious Darling-Erdős type behavior for the extreme-sum process. Whenever $F \in \Lambda_\gamma$ for some $\gamma < \frac{1}{2}$ and $\{k_n\}$ satisfies (1.1) write

$$e_n(t) := \sigma_\gamma t^{\gamma - 1/2} \frac{E_n(t) - n \int_0^{[tk_n]/n} Q(1-s)\, \mathrm{d}s}{\sqrt{k_n}\, \Delta_n(\gamma)},$$

where

$$\sigma_\gamma = (\tfrac{1}{2}(1-\gamma)(1-2\gamma))^{1/2},$$

and for $T > 0$ and $\gamma < \frac{1}{2}$ set

$$A(T) = (2 \log \max(T, \mathrm{e}))^{1/2} \tag{1.14}$$

and

$$B_\gamma(T) = A(T) + (\log(\sqrt{\lambda_\gamma}/2\pi))^{1/2}/A(T) \tag{1.15}$$

where

$$\lambda_\gamma = \tfrac{1}{4}(1-2\gamma). \tag{1.16}$$

This behavior will be a consequence of the weak convergence of a time-changed variant of $e_n(\cdot)$ to the stochastic process

$$V_\gamma(x) := \sigma_\gamma\, e^{(1/2-\gamma)x} \int_0^{e^{-x}} W(u) u^{-1-\gamma}\, du, \quad 0 \le x < \infty.$$

It is readily verified that $V_\gamma(\cdot)$ is a sample-continuous mean zero stationary Gaussian process with covariance function given, for $x \ge 0$ and $h \ge 0$, by

$$r_\gamma(h) = EV_\gamma(x+h)V_\gamma(x)$$
$$= \begin{cases} \dfrac{1-2\gamma}{\gamma}\left\{\dfrac{\exp((\gamma-\frac{1}{2})h)}{1-2\gamma} - \exp(-\tfrac{1}{2}h)\right\}, & \gamma < \tfrac{1}{2} \text{ and } \gamma \neq 0, \\ \tfrac{1}{2}(2+h)\exp(-\tfrac{1}{2}h), & \gamma = 0. \end{cases}$$

Corollary 2. *Assume that $F \in \Lambda_\gamma$ with $\gamma < \frac{1}{2}$ and let $\{k_n\}$ be a sequence satisfying* (1.1). *Then for any fixed $0 < c < 1$ the sequence $\{e_n(e^{-x}): 0 \le x \le \log(1/c)\}$ of processes converges weakly in the Skorohod space $D[0, \log(1/c)]$ to the process $\{V_\gamma(x): 0 \le x \le \log(1/c)\}$. Furthermore,*

$$\lim_{c \downarrow 0} \lim_{n\to\infty} P\left\{A(\log(1/c))\left\{\sup_{c \le t \le 1} e_n(t) - B_\gamma(\log(1/c))\right\} \le x\right\}$$
$$= \exp(-e^{-x})$$

for all $x \in \mathbb{R}$.

Finally we would like to connect Theorem 1 to the classical theory of domains of partial attraction for the whole sums $X_1 + \cdots + X_n$ and thereby show that Theorem 1 is not empty for any choice of $0 < a < b$ and non-negative, non-decreasing continuous function φ on $[a, b]$. In particular, we claim the following: Let $0 < a < b < \infty$ be arbitrary and let φ be any non-negative, non-decreasing, continuous function on $[a, b]$. Then there exist a distribution function F, a subsequence $\{n'\} = \{n_j'\}_{j=1}^\infty$, and a sequence $k_j' = k_{n_j'}$ satisfying $k_j' \to \infty$ and $k_j'/n_j' \to 0$ as $j \to \infty$ such that for the versions $\tilde{I}_{n'}$ of the intermediate sums $I_{n'}$ pertaining to F in Theorem 1 we have (1.7) as $n' \to \infty$. In fact, there is a universal F that does the job for all $0 < a < b$ and all functions φ on $[a, b]$ with the described properties.

Indeed, let $0 < a < b$ be arbitrary and φ be any continuous, non-negative, non-decreasing function on $[a, b]$. Choose $0 < a_0 < a < b < b_0 < \infty$ and extend the definition of φ so that the extended φ is continuous and non-decreasing on $[a_0, b_0]$ and $\varphi(a_0) = 0$. Now define

$$\psi(s) = \begin{cases} \varphi(a_0) - \varphi(b_0), & 0 < s \le a_0, \\ \varphi(s) - \varphi(b_0), & a_0 < s \le b_0, \\ 0, & s > b_0, \end{cases}$$

and introduce $R(x) = -\inf\{s > 0 : \psi(s) \geq -x\}$, $x > 0$. Consider the spectrally right-sided infinitely divisible distribution without a normal component, the right Lévy measure of which is $R(x)$, $x > 0$. Then by the classical theorem of Khinchin [4, p. 184] there is an F in the domain of partial attraction of this infinitely divisible law. Using Theorems 3 and 5 in [1], this means, in particular, that there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of the positive integers such that if Q denotes the quantile function belonging to F then we have

$$-Q((1 - s/n_j)-)/B_j' \to \psi(s), \quad s > 0, \tag{1.17}$$

as $j \to \infty$, where $B_j' > 0$ are some constants. Now define $n_j' = \lceil n_j^{3/2} \rceil$ and $k_j' = \lceil n_j^{3/2} \rceil / n_j$, $j = 1, 2, \ldots$. Then $k_j' \to \infty$ and $k_j'/n_j' \to 0$ as $j \to \infty$, and it follows from (1.17) that

$$\left\{ H\left(\frac{sk_j'}{n_j'}\right) - H\left(\frac{a_0 k_j'}{n_j'}\right) \right\} \Big/ B_j' \to \psi(s) - \psi(a_0) = \varphi(s)$$

for each $a_0 \leq s \leq b_0$. This means that condition (1.6) is satisfied and hence by Theorem 1 we obtain (1.7) along $\{n'\} = \{n_j'\}_{j=1}^{\infty}$, with $B_{n_j'}$ replaced by the present B_j'. A universal F is obtained by using the distribution function F of any of the universal laws of Doeblin [4, p. 189].

In order to give a flavor of the content of Theorems 1 and 2, we close this section by an illustrative example. Set

$$Q(1 - s) = \{\beta + \sin(\log s)\} s^{-\gamma}, \quad 0 < s \leq 1,$$

where $\gamma > 0$ and $\beta > (1 + \gamma)/\gamma$. Differentiation shows that Q is an actual quantile function. For $j = 1, 2, \ldots$, set

$$n_j' = \lceil \exp(4\pi j) \rceil \quad \text{and} \quad k_j' = k_{n_j'} = n_j' \exp(-2\pi j),$$

so that $k_j'/n_j' = \exp(-2\pi j)$, $j = 1, 2, \ldots$. Also let

$$B_j' = B_{n_j'} = \exp(2\pi\gamma j), \quad j = 1, 2, \ldots,$$

and choose $a_0 > 0$ arbitrarily. Then for any $a_0 \leq x < \infty$ and all j large enough,

$$\begin{aligned} \varphi_{n_j'}(a_0; x) &= \left\{ Q\left(1 - x\frac{k_j'}{n_j'}\right) - Q\left(1 - a_0\frac{k_j'}{n_j'}\right) \right\} \Big/ B_j' \\ &= \{\beta + \sin(\log s)\} x^{-\gamma} - \{\beta + \sin(\log a_0)\} a_0^{-\gamma} \\ &=: \varphi(x). \end{aligned}$$

We see that Theorem 1 applies along $\{n'\} = \{n_j'\}_{j=1}^{\infty}$ for all $\gamma > 0$ and all $a_0 < a < b < \infty$ and, moreover, it is easily checked that Theorem 2 is also applicable when $0 < \gamma < \frac{1}{2}$. Notice that by (1.11) the distribution function corresponding to such a Q is not in the domain of attraction of Λ_γ for any γ.

2. Proofs

Let $U_1, U_2, \ldots$, be independent random variables uniformly distributed on (0, 1) with corresponding order statistics $U_{1,n} \leq \cdots \leq U_{n,n}$. Consider the uniform empirical and quantile processes $\alpha_n(t)=\sqrt{n}(G_n(t)-t)$ and $\beta_n(t)=\sqrt{n}(t-U_n(t))$, $0\leq t\leq 1$, where $G_n(t)=n^{-1}\#\{1\leq k\leq n\colon U_k\leq t\}$, $0\leq t\leq 1$, and $U_n(t)=\inf\{0\leq s\leq 1\colon G_n(s)\geq t\}$, $0<t\leq 1$, $U_n(0)=U_{1,n}$, so that $U_n(t)=U_{k,n}$ if $(k-1)/n<t\leq k/n$, $k=1,\ldots,n$. The tail empirical and quantile processes pertaining to the given sequence $\{k_n\}$ satisfying (1.1) are defined as

$$w_n(s)=(n/k_n)^{1/2}\alpha_n(sk_n/n)$$

and

$$v_n(s)=(n/k_n)^{1/2}\beta_n(sk_n/n),\quad 0\leq s\leq n/k_n.$$

As pointed out in [6], $w_n(\cdot)$ converges weakly in the Skorohod space $D[0,T]$, for any $T>0$, to a standard Wiener process. Then by a Skorohod construction and the left-continuous version of Lemma 1 of Vervaat [8] (both also in [7]) we see that the sequences $\{w_n(\cdot)\}_{n=1}^{\infty}$ and $\{v_n(\cdot)\}_{n=1}^{\infty}$ have distributionally equivalent versions $\{\tilde{w}_n(\cdot)\}_{n=1}^{\infty}$ and $\{\tilde{v}_n(\cdot)\}_{n=1}^{\infty}$ on a rich enough probability space that carries a standard Wiener process W such that

$$\sup_{0\leq s\leq T}|\tilde{w}_n(s)-W(s)|\to 0 \text{ and } \sup_{0\leq s\leq T}|\tilde{v}_n(s)-W(s)|\to 0 \quad \text{a.s.} \tag{2.1}$$

as $n\to\infty$.

In order to obtain the distributionally equivalent copies $\tilde{I}_n$ of Theorem 1, we first note that from (1.4),

$$(X_{1,n},\ldots,X_{n,n})=_{\mathscr{D}}(-H(U_{n,n}),\ldots,-H(U_{1,n}))\quad \text{for each } n\geq 1.$$

Define $B_n>0$ arbitrarily for an n which is not a member of $\{n'\}$ in (1.6). Then, using the notations in (1.2) and (1.5), starting out from the equality

$$\{I_n(a,t)-\mu_n(a,t)\colon a\leq t\leq b\}$$
$$=_{\mathscr{D}}\left\{-\int_{U_n(ak_n/n)}^{U_n(tk_n/n)} nH(u)\,\mathrm{d}G_n(u)+\int_{\lceil ak_n\rceil/n}^{\lceil tk_n\rceil/n} H(u)\,\mathrm{d}u\colon a\leq t\leq b\right\},\quad n\geq 1,$$

and then integrating by parts, for the processes $Y_n(a,t)$ in (1.8) and for

$$Y_n^*(a,t):=M_n^*(a,t)-R_n^*(a)+R_n^*(t),$$

where

$$M_n^*(a,t)=\frac{n}{\sqrt{k_n}\,B_n}\int_{\lceil ak_n\rceil/n}^{\lceil tk_n\rceil/n}(G_n(u)-u)\,\mathrm{d}H(u)$$

and

$$R_n^*(t)=\frac{n}{\sqrt{k_n}\,B_n}\int_{\lceil tk_n\rceil/n}^{U_n(tk_n/n)}(G_n(u)-u)\,\mathrm{d}H(u)$$

we obtain

$$\{Y_n(a,t)\colon a\leq t\leq b\}$$
$$=_{\mathscr{D}}\{Y_n^*(a,t)=M_n^*(a,t)-R_n^*(a)+R_n^*(t)\colon a\leq t\leq b\},\quad n\geq 1. \tag{2.2}$$

Substituting now $u = sk_n/n$ and transferring to the probability space of the versions $\tilde{w}_n$ and $\tilde{v}_n$ in (2.1), we get

$$\{Y_n(a,t): a \leq t \leq b\}$$
$$=_{\mathscr{D}} \{\tilde{Y}_n(a,t) := \tilde{M}_n(a,t) - \tilde{R}_n(a) + \tilde{R}_n(t): a \leq t \leq b\}, \quad n \geq 1. \qquad (2.3)$$

where, with the obvious extension of the definition of $\varphi_{n'}$ for an arbitrary n,

$$\tilde{M}_n(a,t) = \int_{\lceil ak_n \rceil/k_n}^{\lceil tk_n \rceil/k_n} \tilde{w}_n(s)\,\mathrm{d}\varphi_n(a_0; s)$$

and

$$\tilde{R}_n(t) = \int_{\lceil tk_n \rceil/n}^{-(1/\sqrt{k_n})\tilde{v}_n(\lceil tk_n \rceil/k_n) + \lceil tk_n \rceil/k_n} \left\{\tilde{w}_n(s) + s\sqrt{k_n} - \frac{\lceil tk_n \rceil}{\sqrt{k_n}}\right\} \mathrm{d}\,\frac{H(sk_n/n)}{B_n}$$
$$= \int_0^{-\tilde{v}_n(\lceil tk_n \rceil/k_n)} \left\{\tilde{w}_n\left(\frac{\lceil tk_n \rceil}{k_n} + \frac{x}{\sqrt{k_n}}\right) + x\right\} \mathrm{d}\,\frac{H(\lceil tk_n \rceil/n + x\sqrt{k_n}/n)}{B_n}.$$

Proof of Theorem 1. Relations (1.8) and (2.3) show the existence of versions $\{\tilde{I}_n(a,t): a \leq t \leq b\}$ of $\{I_n(a,t): a \leq t \leq b\}$ as claimed in the statement in the theorem once we prove that

$$\sup_{a \leq t \leq b} |\tilde{R}_{n'}(t)| \to 0 \quad \text{a.s.} \quad \text{as } n' \to \infty \qquad (2.4)$$

and

$$\sup_{a \leq t \leq b} \left|\tilde{M}_{n'}(a,t) - \int_a^t W(s)\,\mathrm{d}\varphi(s)\right| \to 0 \quad \text{a.s.} \quad \text{as } n' \to \infty. \qquad (2.5)$$

By (2.1) and the fact that $|W(\cdot)|$ is bounded on any finite interval with probability 1, there exists an almost surely finite random variable $K > 0$ such that for all n' large enough,

$$\sup_{a \leq t \leq b} |\tilde{R}_{n'}(t)|$$
$$\leq K \sup_{a \leq t \leq b} \int_{-K}^{K} \mathrm{d}\,\frac{H(\lceil tk_{n'} \rceil/n' + x\sqrt{k_{n'}}/n')}{B_{n'}}$$
$$= K \sup_{a \leq t \leq b} \frac{H(\lceil tk_{n'} \rceil/n' + K\sqrt{k_{n'}}/n') - H(\lceil tk_{n'} \rceil/n' - K\sqrt{k_{n'}}/n')}{B_{n'}}$$
$$= K \sup_{a \leq t \leq b} \left\{\varphi_{n'}\left(a_0, \frac{\lceil tk_{n'} \rceil}{k_{n'}} + \frac{K}{\sqrt{k_{n'}}}\right) - \varphi_{n'}\left(a_0, \frac{\lceil tk_{n'} \rceil}{k_{n'}} - \frac{K}{\sqrt{k_{n'}}}\right)\right\}$$

almost surely. Since $\varphi_{n'}(a_0, \cdot)$ is non-decreasing, by condition (1.6) and the continuity of φ this bound goes to zero as $n' \to \infty$ and hence we have (2.4).

To prove (2.5), first notice that (2.1) and (1.6) easily imply that

$$\sup_{a\le t\le b}\left|\tilde{M}_{n'}(a,t)-\int_{[ak_{n'}]/k_{n'}}^{[tk_{n'}]/k_{n'}} W(s)\,\mathrm{d}\varphi_{n'}(a_0,s)\right|\to 0 \quad \text{a.s.} \quad \text{as } n'\to\infty.$$

Next, for all n' large enough,

$$\sup_{a\le t\le b}\left|\int_{[ak_{n'}]/k_{n'}}^{[tk_{n'}]/k_{n'}} W(s)\,\mathrm{d}\varphi_{n'}(a_0,s)-\int_a^t W(s)\,\mathrm{d}\varphi_{n'}(a_0,s)\right|$$

$$\le 2\sup_{a_0\le s\le b_0}|W(s)|\sup_{a\le t\le b}\left|\varphi_{n'}\left(a_0,\frac{[tk_{n'}]}{k_{n'}}\right)-\varphi_{n'}(a_0,t)\right|,$$

and this bound goes to zero again by (1.6) and the continuity of φ as $n'\to\infty$. Finally, (2.5) and hence the theorem will follow from these relations if we show that

$$\sup_{a\le t\le b}\left|\int_a^t W(s)\,\mathrm{d}\varphi_{n'}(a_0,s)-\int_a^t W(s)\,\mathrm{d}\varphi(s)\right|\to 0 \quad \text{a.s.} \tag{2.6}$$

as $n'\to\infty$.

To verify this, notice that with probability 1 for any given $\varepsilon>0$ there exists a (random) partition $a=t_0<t_1<\cdots<t_m<t_{m+1}=b$ such that

$$\left|\int_a^{t_{i+1}} W(s)\,\mathrm{d}\varphi_{n'}(a_0,s)-\int_a^{t_{i+1}} W(s)\,\mathrm{d}\varphi(s)\right|<\tfrac{1}{2}\varepsilon$$

and

$$\int_{t_i}^{t_{i+1}} |W(s)|\,\mathrm{d}\varphi_{n'}(a_0,s)+\int_{t_i}^{t_{i+1}} |W(s)|\,\mathrm{d}\varphi(s)<\tfrac{1}{2}\varepsilon$$

for all $i=0,\ldots,m$ and n' large enough, where m does not depend on n'. Thus for any $t_i\le t\le t_{i+1}$ and $i=0,\ldots,m$,

$$\left|\int_a^t W(s)\,\mathrm{d}\varphi_{n'}(a_0,s)-\int_a^t W(s)\,\mathrm{d}\varphi(s)\right|<\varepsilon$$

almost surely for all n' large enough, proving (2.6). □

Proof of Theorem 2. First we note that it is easily checked that (1.10) implies the finiteness of $\mu_n(0,b)$ for all n large enough, so that the representation (2.2) holds true for $a=0$. Hence, again defining $B_n>0$ arbitrarily for an $n\notin\{n'\}$,

$$\left\{\frac{E_n(t)-\mu_n(0,t)}{\sqrt{k_n}\,B_n}:\ 0\le t\le b\right\}$$

$$=_{\mathscr{D}}\{M_n^*(0,a)+M_n^*(a,t)-R_n^*(0)+R_n^*(t):\ 0\le a\le t\le b\}$$

for each $n\ge 1$. Furthermore, it follows from the derivation of (2.2) and (2.3) that for each $n\ge 1$,

$$\{(M_n^*(0,a),\ M_n^*(a,t),\ R_n^*(0),\ R_n^*(t)):\ 0\le a\le t\le b\}$$

$$=_{\mathscr{D}}\{(\tilde{M}_n(0,a),\ \tilde{M}_n(a,t),\ \tilde{R}_n(0),\ \tilde{R}_n(t)):\ 0\le a\le t\le b\}.$$

Hence, in view of the fact that now we have (2.4) and (2.5) for any fixed $0<a<b<b_0$, it suffices to prove

$$\lim_{a\downarrow 0}\limsup_{n'\to\infty} P\left\{\sup_{0\le t\le a}|M^*_{n'}(0,t)|>\varepsilon\right\}=0, \tag{2.7}$$

$$\lim_{a\downarrow 0} P\left\{\sup_{0\le t\le a}\left|\int_0^t W(s)\,\mathrm{d}\varphi(s)\right|>\varepsilon\right\}=0, \tag{2.8}$$

and

$$\lim_{a\downarrow 0}\limsup_{n'\to\infty} P\left\{\sup_{0\le t\le a}|R^*_{n'}(t)|>\varepsilon\right\}=0, \tag{2.9}$$

where $\varepsilon>0$ is arbitrary.

We have

$$\begin{aligned} E\left(\sup_{0\le t\le a}|M^*_{n'}(0,t)|\right) &\le \frac{1}{\sqrt{k_{n'}}\,B_{n'}} E\int_0^{[ak_{n'}]/n'} n'|G_{n'}(u)-u|\,\mathrm{d}H(u) \\ &\le \frac{\sqrt{n'}}{\sqrt{k_{n'}}\,B_{n'}}\int_0^{[ak_{n'}]/k_{n'}} \sqrt{u}\,\mathrm{d}H(u) \\ &= \int_0^{[ak_{n'}]/k_{n'}} \sqrt{s}\,\mathrm{d}\varphi_{n'}(a_0,s) \\ &\le \int_0^{2a} \sqrt{s}\,\mathrm{d}\varphi_{n'}(a_0,s), \end{aligned}$$

where the last inequality holds for all n' large enough. Hence, using condition (1.10), by the Markov inequality we obtain (2.7).

Also,

$$E\left(\sup_{0\le t\le a}\left|\int_0^t W(s)\,\mathrm{d}\varphi(s)\right|\right)\le E\int_0^a |W(s)|\,\mathrm{d}\varphi(s)\le \int_0^a \sqrt{s}\,\mathrm{d}\varphi(s),$$

and hence condition (1.10) and the Markov inequality again imply (2.8).

Finally, using the fact that for any $a>0$,

$$\frac{n}{k_n}U_n\left(\frac{ak_n}{n}\right)\to_{\mathrm{P}} a \quad \text{as } n\to\infty,$$

which follows for example from (2.1), we obtain that for each $a>0$,

$$\begin{aligned} \sup_{0\le t\le a}|R^*_{n'}(t)| &\le \frac{1}{\sqrt{k_{n'}}\,B_{n'}}\int_0^{U_{n'}(ak_{n'}/n')} n'|G_{n'}(u)-u|\,\mathrm{d}H(u) \\ &= \frac{1}{\sqrt{k_{n'}}}\int_0^{n'U_{n'}(ak_{n'}/n')/k_{n'}} n'\left|G_{n'}\left(\frac{sk_{n'}}{n'}\right)-\frac{sk_{n'}}{n'}\right|\,\mathrm{d}\varphi_{n'}(a_0,s) \\ &\le \frac{1}{\sqrt{k_{n'}}}\int_0^{2a} n'\left|G_{n'}\left(\frac{sk_{n'}}{n'}\right)-\frac{sk_{n'}}{n'}\right|\,\mathrm{d}\varphi_{n'}(a_0,s)+o_{\mathrm{P}}(1) \end{aligned}$$

as $n' \to \infty$. But the expectation of the first term here is not greater than

$$\int_0^{2a} \sqrt{s}\, \mathrm{d}\varphi_{n'}(a_0, s),$$

and hence we obtain (2.9) as above. □

Proof of Corollary 1. Since

$$\int_0^a \sqrt{s}\, \mathrm{d}\varphi_\gamma(s) = \int_0^a s^{-\gamma-1/2}\, \mathrm{d}s \to 0 \quad \text{as } a\downarrow 0$$

whenever $\gamma < \frac{1}{2}$, we only have to prove that

$$\lim_{a\downarrow 0} \limsup_{n\to\infty} \int_0^a \sqrt{s}\, \mathrm{d}\varphi_n(s) = 0 \quad \text{if } \gamma < \tfrac{1}{2}, \tag{2.10}$$

where φ_n is given in (1.12).

Fix $0 < a < 1$. Then we have

$$\int_0^a \sqrt{s}\, \mathrm{d}\varphi_n(s) = \sum_{i=0}^{\infty} \int_{a/2^{i+1}}^{a/2^i} \sqrt{s}\, \mathrm{d}\varphi_n(s) \le \sqrt{a} \sum_{i=0}^{\infty} r_i(n, a),$$

where, using (1.11),

$$\begin{aligned} r_i(n, a) &= 2^{-i/2}\{\varphi_n(a/2^i) - \varphi_n(a/2^{i+1})\} \\ &= 2^{-i/2} \frac{H(2^{-(i+1)}ak_n/n) - H(2^{-i}ak_n/n)}{H(u_\gamma k_n/n) - H(v_\gamma k_n/n)} \\ &= 2^{-i/2} \frac{H(2^{-1}ak_n/n) - H(ak_n/n)}{H(u_\gamma k_n/n) - H(v_\gamma k_n/n)} \\ &\quad \times \prod_{m=1}^{i} \frac{H(2^{-(m+1)}ak_n/n) - H(2^{-m}ak_n/n)}{H(2^{-m}ak_n/n) - H(2^{-(m-1)}ak_n/n)} \\ &\le 2^{-i/2}\left\{a^{-\gamma} \frac{|(\frac{1}{2})^{-\gamma} - 1|}{|\gamma|} + a^{1/2}\right\}\{2^\gamma + a\}^i \\ &= \left\{a^{-\gamma} \frac{|2^\gamma - 1|}{|\gamma|} + a^{1/2}\right\}\{2^{\gamma-1/2} + a2^{-1/2}\}^i \end{aligned}$$

for all n large enough, where we use the convention that $|((\frac{1}{2})^{-\gamma} - 1)/\gamma| = \log 2$ if $\gamma = 0$. Hence for all n large enough and all $a > 0$ small enough,

$$\int_0^a \sqrt{s}\, \mathrm{d}\varphi_n(s) \le \left(a + a^{1/2-\gamma} \frac{|2^\gamma - 1|}{|\gamma|}\right)(1 - 2^{\gamma-1/2} - a2^{-1/2})^{-1}.$$

Since this bound goes to zero as $a\downarrow 0$, (2.10) follows. □

Proof of Corollary 2. The first statement follows directly from Corollary 1.

Calculation shows that for the covariance function $r_\gamma(\cdot)$ of the process $V_\gamma(\cdot)$ we have

$$r_\gamma(h) = 1 - \tfrac{1}{2}\lambda_\gamma h^2 + o(h^2) \quad \text{as } h \to \infty,$$

where λ_γ is as in (1.16). Applying now Theorem 8.2.6 in Leadbetter, Lindgren and Rootzén [5], we get that for all $x \in \mathbb{R}$,

$$\lim_{T\to\infty} P\left\{A(T)\left\{\sup_{0\le x\le T} V_\gamma(x) - B_\gamma(T)\right\} \le x\right\} = \exp(-e^{-x}),$$

where $A(T)$ and $B_\gamma(T)$ are given in (1.15) and (1.16). This and the first statement now easily imply the second statement after a time change. □

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