JOURNAL OF ALGEBRA 145, 396–453 (1992)

The Projective Representations of the Hyperoctahedral Group

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Communicated by Gordon James

Received October 26, 1989

The hyperoctahedral group (the Weyl group of the root system B_n) has seven distinct nonsplit double covers (for $n \geqslant 4$), and hence, seven families of projective representations. We give constructions of all the irreducible representations of these seven double covers in terms of symmetric group representations, and determine the associated character tables. As a corollary, we also obtain the irreducible projective representations and characters of the Weyl group of the root system D_n.

INTRODUCTION

The hyperoctahedral group is the Weyl group of the root systems B_n and C_n, or equivalently, the symmetry group of an n-dimensional cube. For $n \geqslant 4$, this group (hereafter denoted W_n) has a Schur multiplier isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. This implies in particular that the projective representations of W_n (over the complex field) can be viewed as certain linear representations of seven distinct nonsplit double covers of W_n.

Given the close connections between the linear representations of W_n and the symmetric groups S_n, it is natural to expect that similar connections exist for projective representations. Indeed, the main result of this paper is the fact that the projective representations of W_n can be explicitly constructed from the linear and projective representations of symmetric groups. To be fair, we should modify this by adding two qualifications. First, one should add to this list the representation obtained by restricting the basic spin representation of O_n to W_n (embedded via the reflection representation). Second, to make the constructions completely explicit, one also needs to have at hand the irreducible decomposition of each

* Partially supported by an NSF Postdoctoral Research Fellowship.

0021-8693/92 $3.00

symmetric group module with respect to the alternating subgroup. In the projective case, this can be obtained as a corollary of Nazarov's work [N], but curiously, in the linear case, there does not appear to be any known solution in the vast literature on representations of the symmetric group.

There have been several papers written about projective representations of W_n or closely related groups such as the wreath products $\mathbf{Z}_l \wr S_n$ (of which W_n is the special case $l=2$), or more generally $G \wr S_n$ for arbitrary groups G. For example, see the papers by Read [R2], Morris [Mo2], and Hoffman and Humphreys [HH1–2]. These papers approach the subject from various other points of view. Furthermore, Read's paper is not as explicit, and the other papers not as complete in their treatment of W_n, as the approach we take here. There is little doubt that the methods of this paper could be applied to $\mathbf{Z}_l \wr S_n$, and to some extent $G \wr S_n$, but we have deliberately avoided this to preserve simplicity.

Another aspect of our approach that has not been addressed in previous papers is the description of the irreducible characters. In the course of constructing the irreducible representations of the various double covers of W_n we will also determine the corresponding characters. We should mention that in a recent paper [J2], Józefiak has also determined the characters of one of the double covers; namely, the one associated with the factor set labeled $[+1, +1, -1]$ (cf. Section 9).

The paper is organized as follows.

In Section 1 (a reformulation of results in [DM]), we construct the eight twisted group algebras corresponding to the eight factor sets (2-cocycles) of W_n; the fact that there are eight was first proved in a case-by-case analysis of finite reflection groups by Ihara and Yokonuma [IY]. A uniform approach for Coxeter groups has been given recently by Howlett [H].

In Section 2, we analyze the structure of the conjugacy classes of the seven nonsplit double covers of W_n (Theorem 2.1). Although one can, in principle, obtain the same information from an earlier paper by Read [R1], we have included proofs since we will also need more detailed information regarding the centralizers of individual elements (Lemma 2.2).

In Section 3 we present an elementary idea that leads to a tremendous simplification of the task of constructing the projective representations of W_n, or more generally, of any group G with a quotient isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. The fundamental idea is the observation that $\mathbf{Z}_2^2$ has a two-dimensional projective representation ρ (corresponding to the fact that there exist anticommuting involutions in GL_2), and so by extension, every covering group of $\mathbf{Z}_2^2$ has such a representation. It follows that the operation $V \mapsto \rho \otimes V$ allows one to pass between representations of W_n corresponding to two different factor sets, and the fact that ρ is two-dimensional prevents the creation of "too many" irreducible submodules in $\rho \otimes V$ when V is irreducible. We use a simple application of the methods of Clifford

Theory to classify the irreducible submodules of $\rho \otimes V$ (Theorem 3.2), and show that the irreducible characters of these submodules can be expressed in terms of the difference characters of W_n (or one of its double covers).

An added virtue of the approach we take in Section 3 is that it provides a simple explanation of the "twisted" outer tensor product that has figured prominently (in various guises) in many of the recent papers [HH1–2, Mo2, J1–2, St1–2] on projective representations of S_n and related groups. In our approach, one starts with irreducible modules V_1 and V_2 for two groups G_1 and G_2 with $\mathbf{Z}_2$-quotients (i.e., subgroups of index 2). Using the fact that $G_1 \times G_2$ has an obvious $\mathbf{Z}_2^2$-quotient, we define the twisted tensor product $V_1 \hat{\otimes} V_2$ to be one or more of the irreducible submodules of $\rho \otimes (V_1 \otimes V_2)$.

To complete the preliminary half of this paper, we summarize in Section 4 the basic techniques needed to use the algebra of symmetric functions to manipulate symmetric group characters (both linear and projective). More details can be found in [M1] (for the linear case) and [St1] (for the projective case). In Section 5 we recall the well-known construction of the irreducible linear representations of W_n, and show how the methods of Section 4 can be used to verify the validity of this construction. This technique will serve as the motivation behind our approach to one of the double covers of W_n in Section 7.

Finally, in Part II, we turn to the problem of constructing the modules and describing the characters for each of the seven families of projective W_n-representations. This task can be roughly divided into two phases. In the first phase, occupying Section 5, Section 7, and the first half of Section 9, we construct the modules and characters corresponding to four of the factor sets of W_n (counting the linear representations as one of the four cases). The only limiting factor in making our constructions explicit is the extent to which one wishes to make the structure of the modules and characters of the symmetric groups (linear and projective) explicit. In the second phase, occupying Section 6, Section 8, and the second half of Section 9, we use the methods of Section 3 to obtain the modules and characters for the remaining four factor sets. This task essentially amounts to constructing the "associators" (cf. (3.1)) and difference characters for the irreducible representations of the first phase. In this second phase, the modules are obtained as eigenspaces of certain involutions related to the associators.

We remark that W_n has three subgroups of index 2, along with a normal subgroup of index 4. For each of the factor sets of these groups that are restrictions of W_n-factor sets, one may obtain the corresponding irreducible representations and characters as immediate corollaries of the work in Part II. In the Appendix, we briefly consider the implications of this for one particular subgroup of index 2—the Weyl group of the root system D_n.

Notational Remarks

We will use P to denote the set of partitions; i.e., sequences of zero or more positive integers of the form $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_l)$. For the most part, we will use the same notation as [M1] for the various parameters and operations associated with partitions. In particular, we will use $|\lambda|$ to denote the sum of the parts, and $l(\lambda)$ to denote the number of parts. The modifiers E, O, and D will be used to indicate that the parts should be restricted to be even, odd, or distinct, respectively. For example, the notation DOP will thus refer to the set of partitions with distinct, odd parts. Also, we will use DP^+ (resp., DP^-) to refer to the partitions $\lambda \in DP$ for which $|\lambda| - l(\lambda)$ is even (resp., odd).

PART I: PRELIMINARIES

1. The Twisted Group Algebras of W_n

Let $s_1, ..., s_{n-1}$, t denote a set of Coxeter generators for W_n. We will assume that these generators are labeled so that for the reflection representation, s_i corresponds to interchanging coordinates i and $i+1$, and t corresponds to changing the sign of the first coordinate. Note that the Coxeter relations

$$s_i^2 = t^2 = (s_i s_{i+1})^3 = 1, \qquad (s_i s_j)^2 = 1 \ (|i-j| \geqslant 2),$$

$$(s_i t)^2 = 1 \ (i > 1), \qquad (s_1 t)^4 = 1 \tag{1.1}$$

constitute a presentation of W_n.

Let $L_n = \mathrm{Hom}(W_n, \mathbf{C}^*)$ denote the abelian group of linear characters of W_n. An easy application of the Coxeter relations shows that L_n is generated by the two characters ε and δ defined by

$$\varepsilon(s_i) = -1, \qquad \varepsilon(t) = +1, \qquad \delta(s_i) = +1, \qquad \delta(t) = -1.$$

This shows in particular that $L_n \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Let α be a factor set for W_n, and let $\mathbf{C}W_n^\alpha$ denote the corresponding complex twisted group algebra [CR, Sect. 8B]. To describe this algebra explicitly, let $\sigma_1, ..., \sigma_{n-1}, \tau$ denote generators for $\mathbf{C}W_n^\alpha$ corresponding to $s_1, ..., s_{n-1}, t$. Note that scalar multiplication of these generators has the same effect as replacing α with an equivalent factor set, so there will be no loss of generality in requiring that α be chosen so that $\sigma_i^2 = \tau^2 = 1$.

PROPOSITION 1.1. *The algebra $\mathbf{C}W_n^\alpha$ has a presentation of the form*

$$\begin{gathered}\sigma_i^2=\tau^2=(\sigma_i\sigma_{i+1})^3=1, \qquad (\sigma_i\sigma_j)^2=\varepsilon_1\ (|i-j|\geqslant 2),\\ (\sigma_i\tau)^2=\varepsilon_2\ (i>1), \qquad (\sigma_1\tau)^4=\varepsilon_3,\end{gathered} \tag{1.2}$$

for suitable scalars ε_1, ε_2, $\varepsilon_3=\pm 1$ (depending on α).

Proof. If $x, y\in \mathbf{C}W_n^\alpha$ are involutions, then so is $(xy)^k x$ for any integer k. If $(xy)^k$ is a scalar, it must therefore be ± 1. In particular, since (1.1) shows that each of $(\sigma_i\sigma_{i+1})^3$, $(\sigma_i\sigma_j)^2$ ($|i-j|\geqslant 2$), $(\sigma_i\tau)^2$ ($i>1$), and $(\sigma_1\tau)^4$ must be scalars, they must all be ± 1. In other words, α may be taken to be ± 1-valued.

We may now substitute $\sigma_i\to -\sigma_i$, if necessary, to ensure that $(\sigma_i\sigma_{i+1})^3=1$.

Let $W_n(\alpha)$ denote the double cover of W_n generated by $\sigma_1, ..., \sigma_{n-1}$, τ, and -1. If the images of $x, y\in W_n(\alpha)$ are conjugate in W_n, then there exists an element $z\in W_n(\alpha)$ such that $zxz^{-1}=\pm y$. In particular, this shows that x^2 and y^2 are $W_n(\alpha)$-conjugates. Therefore, since the involutions s_is_j ($|i-j|\geqslant 2$) are all W_n-conjugates, then the scalars $(\sigma_i\sigma_j)^2$ must all be $W_n(\alpha)$-conjugates. In other words, there exists a scalar $\varepsilon_1=\pm 1$ independent of i and j such that $(\sigma_i\sigma_j)^2=\varepsilon_1$. Similarly, the involutions s_it are conjugate for $i>1$, so there exists a scalar $\varepsilon_2=\pm 1$ so that $(\sigma_i\tau)^2=\varepsilon_2$ for all such i.

To complete the proof, note that since (1.1) defines a group of order $|W_n|$, then (1.2) defines an algebra of dimension at most $|W_n|$. Therefore, given that the relations (1.2) are satisfied by some algebra of dimension $|W_n|$ (namely, $\mathbf{C}W_n^\alpha$), then these relations must form a presentation of that algebra. ∎

An issue that this proposition does not address is whether there actually exist nonzero $\mathbf{C}$-algebras that satisfy the relations (1.2) for each of the eight possible choices for $\varepsilon_1, \varepsilon_2$, and ε_3. This can be settled by noting that in Part II, we will construct explicit representations for each of these choices. From this one may conclude that there do exist factor sets corresponding to each of the eight possibilities. We will use the expression $[\varepsilon_1, \varepsilon_2, \varepsilon_3]$ to denote the (equivalence class of the) W_n-factor set corresponding to ε_1, ε_2, ε_3.

Note that the relations $(\sigma_i\sigma_j)^2=\varepsilon_1$ exist only for $n\geqslant 4$, and the relations $(\sigma_i\tau)^2=\varepsilon_2$ exist only for $n\geqslant 3$. It follows that the factor sets $[\pm 1, \varepsilon_2, \varepsilon_3]$ are identical for $n=3$ and the factor sets $[\pm 1, \pm 1, \varepsilon_3]$ are identical for $n=2$. It is easy to check that these are the only isomorphisms between the factor sets, so the Schur multiplier of W_n is $\mathbf{Z}_2^3$ for $n\geqslant 4$, $\mathbf{Z}_2^2$ for $n=3$, and $\mathbf{Z}_2$ for $n=2$ (cf. [IY, H]).

It will be convenient to record here a few basic identities that will be needed later. To simplify the notation, let us define

$$\begin{aligned} t_i &= s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1} \in W_n \\ \tau_i &= \sigma_{i-1} \cdots \sigma_1 \tau \sigma_1 \cdots \sigma_{i-1} \in W_n(\alpha) \end{aligned} \tag{1.3}$$

for $1 \leqslant i \leqslant n$. Note that $t_1, ..., t_n$ are the reflections corresponding to the short roots of the root system B_n; we will refer to them as the "short reflections."

In the following, $[x, y] = xyx^{-1}y^{-1}$ denotes the group commutator in $W_n(\alpha)$.

PROPOSITION 1.2. *Assuming* $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3]$, *we have*

$$[\sigma_i, \sigma_j] = \varepsilon_1 \ (|i-j| \geqslant 2), \qquad [\sigma_i, \tau_j] = \varepsilon_2 \ (j - i \neq 0, 1),$$
$$[\tau_i, \tau_j] = \varepsilon_3 \ (i \neq j).$$

Proof. The first relation is just a restatement of (1.2). To prove the second relation, note that the involutions $s_i t_j$ $(j-i \neq 0, 1)$ are all W_n-conjugates, so by the same reasoning used in the proof of Proposition 1.1, the commutators $[\sigma_i, \tau_j]$ must be independent of i and j. Therefore, $[\sigma_i, \tau_j] = [\sigma_2, \tau] = \varepsilon_2$, by (1.2). To prove the third relation, note that the involutions $t_i t_j$ $(i \neq j)$ are W_n-conjugates, so $[\tau_i, \tau_j]$ is also independent of i and j. Therefore, $[\tau_i, \tau_j] = [\tau_2, \tau_1] = [\sigma_1 \tau \sigma_1, \tau] = (\sigma_1 \tau)^4 = \varepsilon_3$, again by (1.2). ∎

We remark that for factor sets α with $\varepsilon_1 = +1$, the subalgebra of $\mathbf{C}W_n^\alpha$ generated by $\sigma_1, ..., \sigma_{n-1}$ is the group algebra $\mathbf{C}S_n$; for $\varepsilon_1 = -1$, the corresponding subalgebra is a twisted group algebra for S_n. We will denote this latter algebra by $\mathbf{C}S'_n$. Also, we will use the notation $\tilde{S}_n$ for the corresponding double cover of S_n, i.e., the subgroup of $\mathbf{C}S'_n$ generated by $\sigma_1, ..., \sigma_{n-1}$ and -1.

2. CONJUGACY CLASSES

A projective representation of W_n can be regarded as a module for one of the twisted group algebras $\mathbf{C}W_n^\alpha$, or equivalently, as a linear representation of the corresponding double cover $W_n(\alpha)$ in which the central element $-1 \in W_n(\alpha)$ is represented faithfully. The latter will be referred to as *spin representations* of $W_n(\alpha)$. In this section, we will classify the conjugacy classes of each group $W_n(\alpha)$ in order to simplify the task of describing the irreducible characters. This has also been done more generally by Read [R1] for the wreath products $\mathbf{Z}_l \wr S_n$.

First, consider the conjugacy classes of W_n. These are indexed by ordered pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$. To describe the members of a given class, let us identify W_n with the group of $n \times n$ monomial matrices with entries chosen from $\mathbf{Z}_2$. References to the cycles of an element $w \in W_n$ will thus refer to the cycles of the underlying permutation matrix. We will say that a cycle of w is *positive* or *negative* according to whether the number of -1's in the matrix entries of the cycle is even or odd. In these terms, the class indexed by (λ, μ) consists of those $w \in W_n$ whose positive (resp., negative) cycles have lengths $\lambda_1, \lambda_2, \ldots$ (resp., $\mu_1, \mu_2, \ldots$). We remark that the values of the linear characters ε and δ on the (λ, μ)-class are $(-1)^{n-l(\lambda)-l(\mu)}$ and $(-1)^{l(\mu)}$, respectively. For further details, see [JK, Sect. 4.4] or [Z, Sect. 7].

Now, choose a particular W_n-factor set $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3]$, and let $C(\alpha)$ denote the inverse image in $W_n(\alpha)$ of some W_n-class C. If any $w \in C(\alpha)$ is conjugate to $-w$, then $C(\alpha)$ is a $W_n(\alpha)$-class; otherwise, $C(\alpha)$ splits into two such classes. Thus, the essential structure of the $W_n(\alpha)$-classes can be inferred from knowledge of which pairs (λ, μ) index W_n-classes that split in $W_n(\alpha)$.

THEOREM 2.1. *For each nontrivial factor set α, the pairs of partitions (λ, μ) that index split classes of $W_n(\alpha)$ can be found in Table I.*

The lists in Table I have been broken up into four columns according to the four possible values of ε and δ on a given W_n-class. For example, the entry in the fourth column for the factor set $[+1, -1, -1]$ is (DOP, DEP). This means that if (λ, μ) indexes a W_n-class with $\varepsilon = \delta = -1$, then the inverse image of this class splits in $W_n([+1, -1, -1])$ if and only if the parts of λ are distinct and odd and those of μ are distinct and even.

Proof. Let $w \mapsto \bar{w}$ denote the canonical epimorphism $W_n(\alpha) \to W_n$. The class of w will split if and only if the normalizer $N(\pm w)$ actually centralizes w. To decide when this occurs, it suffices to identify a set of generators for

TABLE I

α	$\varepsilon = +1, \delta = +1$	$\varepsilon = -1, \delta = +1$	$\varepsilon = +1, \delta = -1$	$\varepsilon = -1, \delta = -1$
$[+1, -1, +1]$	(P, P)	$(EP, \varnothing)$	(DOP, DOP)	$(\varnothing, EP)$
$[-1, +1, +1]$	(OP, OP)	(DP, DP)	(OP, OP)	(DP, DP)
$[-1, -1, +1]$	(OP, OP)	$(DEP, \varnothing)$	(DP, DP)	$(\varnothing, DEP)$
$[+1, +1, -1]$	$(OP, \varnothing)$	$\varnothing$	$(\varnothing, DP)$	$(\varnothing, DP)$
$[+1, -1, -1]$	$(OP, \varnothing)$	$(\varnothing, DP)$	$(\varnothing, OP)$	(DOP, DEP)
$[-1, +1, -1]$	(OP, EP)	$\varnothing$	$(\varnothing, DOP)$	$(\varnothing, P)$
$[-1, -1, -1]$	(OP, EP)	$(\varnothing, P)$	$(\varnothing, P)$	(OP, EP)

$N(\pm w)$ and determine if any of these generators fail to commute with w. To choose a canonical element $w = w_{\lambda\mu}$ so that $\bar{w}$ belongs to the class indexed by (λ, μ), let us define

$$w_{\lambda\mu} := w_1 w_2 \cdots w'_1 w'_2 \cdots, \tag{2.1}$$

where w_i and w'_i are

$$w_i = \sigma_{a_{i-1}+1} \cdots \sigma_{a_i-2} \sigma_{a_i-1} \qquad (a_i = \lambda_1 + \cdots + \lambda_i)$$

$$w'_i = \sigma_{b_{i-1}+1} \cdots \sigma_{b_i-2} \sigma_{b_i-1} \tau_{b_i} \qquad (b_i = |\lambda| + \mu_1 + \cdots + \mu_i).$$

Note that $\bar{w}_i$ (resp., $\bar{w}'_i$) is a positive λ_i-cycle (resp., negative μ_i-cycle).

The image of $N(\pm w_{\lambda\mu})$ in W_n is the centralizer of $\bar{w}_{\lambda\mu}$, which can be identified as a direct product of wreath products [JK, Chap. 4]. From this structural decomposition, it is not hard to deduce that a set of generators for this centralizer (as a group of monomial matrices) consists of: (1) the cycles of $\bar{w}_{\lambda\mu}$, (2) for each positive cycle of $\bar{w}_{\lambda\mu}$, the diagonal matrix with -1's in the positions indexed by the cycle, and (3) for each pair of positive (resp., negative) cycles of the same length, an involution that interchanges corresponding positions in the two cycles. Therefore, a set of generators for $N(\pm w_{\lambda\mu})$ consists of the central element -1 and

(1) The "cycles" w_i and w'_i.

(2) For each positive cycle w_i, the element $z_i = \tau_{a_{i-1}+1} \cdots \tau_{a_i-1} \tau_{a_i}$.

(3) For each adjacent pair of positive cycles of length k (so that $\lambda_i = \lambda_{i+1} = k$), the element $u_i = (\sigma_{a_{i-1}+1} \cdots \sigma_{a_{i+1}-2} \sigma_{a_{i+1}-1})^k$.

(3′) For each adjacent pair of negative cycles of length k (so that $\mu_i = \mu_{i+1} = k$), the element $u'_i = (\sigma_{b_{i-1}+1} \cdots \sigma_{b_{i+1}-2} \sigma_{b_{i+1}-1})^k$.

The following result determines which of these generators commute with $w_{\lambda\mu}$.

LEMMA 2.2. *Assume* $l = n - l(\lambda) - l(\mu)$.

(a) *If* $\lambda_i = k$, *then* $[w_{\lambda\mu}, w_i] = \varepsilon_1^{(k-1)(l-1)} \varepsilon_2^{(k-1)l(\mu)}$.

(b) *If* $\mu_i = k$, *then* $[w_{\lambda\mu}, w'_i] = \varepsilon_1^{(k-1)(l-1)} \varepsilon_2^{(k-1)l(\mu)+l} \varepsilon_3^{l(\mu)-1}$.

(c) *If* $\lambda_i = k$, *then* $[w_{\lambda\mu}, z_i] = \varepsilon_2^{kl} \varepsilon_3^{kl(\mu)+k-1}$.

(d) *If* $\lambda_i = \lambda_{i+1} = k$, *then* $[w_{\lambda\mu}, u_i] = \varepsilon_1^{kl+k-1} \varepsilon_2^{kl(\mu)}$.

(e) *If* $\mu_i = \mu_{i+1} = k$, *then* $[w_{\lambda\mu}, u'_i] = \varepsilon_1^{kl+k-1} \varepsilon_2^{kl(\mu)} \varepsilon_3$.

Given this lemma, one may easily verify each of the 28 entries in Table I. For example, consider the factor set $[-1, +1, -1]$, and suppose that $n - l(\lambda) - l(\mu)$ is even and $l(\mu)$ is odd (so that $\varepsilon(w_{\lambda\mu}) = +1$ and

$\delta(w_{\lambda\mu}) = -1$). In that case, the five commutators listed above simplify to $(-1)^{k-1}$, $(-1)^{k-1}$, -1, $(-1)^{k-1}$, and $(-1)^k$, respectively. This implies that one of the generators of $N(\pm w_{\lambda\mu})$ will fail to commute with $w_{\lambda\mu}$ if λ has any parts (by part (c)), if μ has any even parts (by part (b)), or if μ has any repeated odd parts (by part (e)). Conversely, if there are no such parts, it is easy to see that each of the five commutators is trivial. Thus, we conclude in this case that (λ, μ) indexes a split class if and only if $\lambda = \varnothing$ and $\mu \in DOP$, which agrees with the corresponding entry in Table I. The remaining cases follow by similar arguments; the tedious details are left to the reader. ∎

Proof of Lemma 2.2. If $x \in N(\pm w_{\lambda\mu})$ has the property that each of the commutators $[w_i, x]$ and $[w_i', x]$ are scalars ($i = 1, 2, \ldots$), then $[w_{\lambda\mu}, x]$ will factor as the product of these scalars. For example, by repeated application of Proposition 1.2, we have

$$\begin{aligned}
[w_j, w_i] &= \varepsilon_1^{(\lambda_j - 1)(\lambda_i - 1)} \qquad (i \neq j)\\
[w_j', w_i] &= \varepsilon^{(\mu_j - 1)(\lambda_i - 1)} \varepsilon_2^{(\lambda_i - 1)}\\
[w_j', w_i'] &= \varepsilon_1^{(\mu_j - 1)(\mu_i - 1)} \varepsilon_2^{(\mu_j - 1) + (\mu_i - 1)} \varepsilon_3 \qquad (i \neq j).
\end{aligned}$$

Therefore, since $[w_j, w_i]$ and $[w_j', w_i]$ are both scalars, it follows that

$$\begin{aligned}
[w_{\lambda\mu}, w_i] &= [w_1, w_i][w_2, w_i] \cdots [w_1', w_i][w_2', w_i] \cdots\\
&= \varepsilon_1^{(k-1)(l-k+1)} \varepsilon_2^{(k-1)l(\mu)},
\end{aligned}$$

where $k = \lambda_i$ and $l = n - l(\lambda) - l(\mu)$. This proves part (a) since $(k-1)(l-k+1) = (k-1)(l-1) \bmod 2$. One can prove part (b) similarly by noting that

$$\begin{aligned}
[w_{\lambda\mu}, w_i'] &= [w_1, w_i'][w_2, w_i'] \cdots [w_1', w_i'][w_2', w_i'] \cdots\\
&= \varepsilon_1^{(k-1)(l-k+1)} \varepsilon_2^{(l-k+1) + (k-1)(l(\mu) - 1)} \varepsilon_3^{l(\mu) - 1}\\
&= \varepsilon_1^{(k-1)(l-1)} \varepsilon_2^{(k-1)l(\mu) + l} \varepsilon_3^{l(\mu) - 1}.
\end{aligned}$$

For part (c), repeated application of Proposition 1.2 shows that $[w_j, z_i] = \varepsilon_2^{\lambda_i(\mu_j - 1)}$ for $i \neq j$, and $[w_j', z_i] = \varepsilon_2^{\lambda_i(\lambda_j - 1)} \varepsilon_3^{\lambda_i}$. Hence,

$$[w_{\lambda\mu}, z_i] = \varepsilon_2^{k(l-k+1)} \varepsilon_3^{kl(\mu)} [w_i, z_i] = \varepsilon_2^{kl} \varepsilon_3^{kl(\mu)} [w_i, z_i],$$

where $k = \lambda_i$. To evaluate $[w_i, z_i]$, let $z = \tau_1 \cdots \tau_k \in W_k(\alpha)$, and note that $[\sigma_1, z] = \varepsilon_2^{k-2} \varepsilon_3$, since $[\sigma_1, \tau_j] = \varepsilon_2$ for $j > 2$ and $[\sigma_1, \tau_1 \tau_2] = [\sigma_1, \tau\sigma_1\tau\sigma_1] = (\sigma_1 \tau)^4 = \varepsilon_3$. Furthermore, the W_k-images of $\sigma_j z$ $(1 \leqslant j < k)$ are all conjugate, so for the same reasons we used in the proof of

Proposition 1.1, the scalars $[\sigma_j, z]$ must be independent of j. Hence, $[w_i, z_i] = [\sigma_1 \cdots \sigma_{k-1}, z] = [\sigma_1, z]^{k-1} = \varepsilon_2^{(k-1)(k-2)} \varepsilon_3^{k-1} = \varepsilon_3^{k-1}$, and thus part (c) follows.

For the proof of (d), we claim that $[w_j, u_i] = \varepsilon_1^{k(\lambda_j - 1)}$ $(j - i \neq 0, 1)$, $[w_j', u_i] = \varepsilon_1^{k(\mu_j - 1)} \varepsilon_2^k$, and $[w_i w_{i+1}, u_i] = \varepsilon_1^{k-1}$, assuming $\lambda_i = \lambda_{i+1} = k$. The first two of these formulas are direct consequences of Proposition 1.2; the third is not quite as routine (see Lemma 2.5 of [St1] for a proof). We therefore have

$$\begin{aligned}[w_{\lambda\mu}, u_i] &= [w_i w_{i+1}, u_i] \prod_{j \neq i, i+1} [w_j, u_i] \prod_j [w_j', u_i] \\ &= \varepsilon_1^{k(l - 2k + 2) + k - 1} \varepsilon_2^{kl(\mu)} = \varepsilon_1^{kl + k - 1} \varepsilon_2^{kl(\mu)}.\end{aligned}$$

For the proof of (e), we claim that $[w_j, u_i'] = \varepsilon_1^{k(\lambda_j - 1)}$, $[w_j', u_i'] = \varepsilon_1^{k(\mu_j - 1)} \varepsilon_2^k$ $(j - i \neq 0, 1)$, and $[w_i' w_{i+1}', u_i'] = \varepsilon_1^{k-1} \varepsilon_3$, assuming $\mu_i = \mu_{i+1} = k$. Again, the first two of these formulas are easy consequences of Proposition 1.2; the third requires more explanation. If we define $w = \sigma_1 \cdots \sigma_{k-1} \sigma_{k+1} \cdots \sigma_{2k-1}$ and $u = (\sigma_1 \cdots \sigma_{2k-1})^k$, then $[w, u] = \varepsilon_1^{k-1}$ by (d). Furthermore, the W_n-images of the pairs $\{w_i' w_{i+1}', u_i'\}$ and $\{w \tau_k \tau_{2k}, u\}$ are simultaneously conjugate in W_n, so $[w_i' w_{i+1}', u_i'] = [w \tau_k \tau_{2k}, u]$. Since the W_n-image of u is a product of disjoint, positive 2-cycles, and $\tau_k \tau_{2k} \in N(\pm u)$, we have $[\tau_k \tau_{2k}, u] = \varepsilon_3$, by (c). Hence,

$$[w_i' w_{i+1}', u_i'] = [w \tau_k \tau_{2k}, u] = [w, u][\tau_k \tau_{2k}, u] = \varepsilon_1^{k-1} \varepsilon_3,$$

as claimed. The formula for $[w_{\lambda\mu}, u_i']$ may now be deduced by the same reasoning used in the proof of (d). ∎

We remark that a basis of the center of $\mathbf{C}W_n^\alpha$ can be generated by the $W_n(\alpha)$-conjugates of the $w_{\lambda\mu}$'s that split. This implies in particular that the number of irreducible spin representations of $W_n(\alpha)$ is the number of split pairs (λ, μ).

If χ is the character of any representation of $W_n(\alpha)$ (or in fact, *any* function defined on $W_n(\alpha)$), we will write $\chi(\lambda, \mu)$ as an abbreviation for $\chi(w_{\lambda\mu})$. Clearly, any such character is determined by its values on these elements. More particularly, the character of a spin representation is determined by its values on the split classes, since in that case, $\chi(-w) = -\chi(w) = 0$ whenever w and $-w$ are conjugate in $W_n(\alpha)$.

We will follow a similar convention for the characters of S_n and $\tilde{S}_n$. For each partition λ of n, define $w_\lambda = w_{\lambda, \varnothing}$, regarded as a member of S_n or $\tilde{S}_n$ according to whether $\varepsilon_1 = +1$ or $\varepsilon_1 = -1$. In these terms, given any character χ of S_n or $\tilde{S}_n$, we will write $\chi(\lambda)$ as an abbreviation for $\chi(w_\lambda)$.

3. Clifford Theory for $\mathbf{Z}_2^2$-Quotients

Let G be a finite group with a subgroup H of index 2, and let ε denote the "sign" character of the natural homomorphism $G \to G/H$. If V is an irreducible $\mathbf{C}G$-module, we will say that V is *self-associate* (with respect to ε) if $V \cong \varepsilon \otimes V$; otherwise, we will say that V and $\varepsilon \otimes V$ form an *associate pair* (with respect to ε).

If V is self-associate, there exists an endomorphism $S \in GL(V)$ such that

$$gSv = \varepsilon(g)\, Sgv \tag{3.1}$$

for all $v \in V$, $g \in G$. By Schur's Lemma, S^2 must be a scalar. In case $S^2 = 1$, we will refer to S as the *ε-associator* of V. This terminology is slightly imprecise, since it permits us to refer to both S and $-S$ as the associator. Assuming a particular choice for S, let $V = V^+ \oplus V^-$ denote the eigenspace decomposition of S on V, labeled so that V^+ corresponds to the eigenvalue $+1$. Note that (3.1) implies that V^+ and V^- are $\mathbf{C}H$-modules; in fact, they are irreducible and nonisomorphic as such modules. Otherwise, if V and $\varepsilon \otimes V$ are an associate pair, then both are irreducible (and isomorphic) as $\mathbf{C}H$-modules. (See Lemma 4.1 of [St1], for example.) Similar remarks also apply to modules for any twisted group algebra of G; one merely needs to replace G by a suitable central extension $\tilde{G}$, and H by its preimage in $\tilde{G}$.

Given a self-associate $\mathbf{C}G$-module V with associator S and character χ, the *difference character* $\Delta^\varepsilon \chi = \Delta\chi$ is the trace function

$$\Delta\chi(g) := \operatorname{tr}_V(Sg) = \operatorname{tr}_{V^+}(g) - \operatorname{tr}_{V^-}(g);$$

i.e., the difference between the characters of the $\mathbf{C}H$-submodules. Note that by substituting $-S$ for S, one obtains $-\Delta\chi$ as the difference character. We also remark that

$$\Delta\chi(xgx^{-1}) = \varepsilon(x)\, \Delta\chi(g) \tag{3.2}$$

is a simple consequence of (3.1).

We now consider a similar analysis for the case in which G has a normal subgroup H such that $G/H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. The principal examples for G we have in mind are W_n and its double covers $W_n(\alpha)$, but we also plan to apply this analysis to a double cover of $W_m \times W_n$, relative to the subgroup generated by the $\varepsilon = +1$ portions of W_m and W_n.

Let $L \cong \operatorname{Hom}(G/H, \mathbf{C}^*)$ denote the group of four linear characters of G that arise from the quotient of G by H, and let α be any factor set of G. Note that L acts on (isomorphism classes of) $\mathbf{C}G^\alpha$-modules via $V \mapsto \varepsilon \otimes V$ ($\varepsilon \in L$). We will use the notation $L_V = \{\varepsilon \in L\colon V \cong \varepsilon \otimes V\}$ for the stabilizer of V. The following result classifies the possible behaviors that can occur when a $\mathbf{C}G^\alpha$-module is restricted to $\mathbf{C}H^\alpha$.

THEOREM 3.1. *Let V be a finite dimensional, irreducible* $\mathbf{C}G^\alpha$-module.

(a) *If $L_V = \{1\}$, then V is an irreducible $\mathbf{C}H^\alpha$-module.*

(b) *If $L_V = \{1, \varepsilon\}$, then the eigenspaces of the ε-associator of V are irreducible, nonisomorphic $\mathbf{C}H^\alpha$-modules (and similarly for $L_V = \{1, \delta\}$ or $\{1, \varepsilon\delta\}$).*

(c) *If $L_V = L = \{1, \varepsilon, \delta, \varepsilon\delta\}$, and $S, T \in GL(V)$ are the associators of V for ε and δ, then $ST = \pm TS$. Moreover,*

(i) *If $ST = TS$, then V is the direct sum of 4 irreducible, nonisomorphic $\mathbf{C}H^\alpha$-modules; viz., the weight spaces (intersections of eigenspaces) of S and T on V.*

(ii) *If $ST = -TS$, then V is the direct sum of two copies of one irreducible $\mathbf{C}H^\alpha$-module; this module is isomorphic to all of the eigenspaces of S and T.*

Proof. Let $L = \{1, \varepsilon, \delta, \varepsilon\delta\}$. By passing to a suitable central extension of G, if necessary, we may assume that α is the trivial factor set. In these terms, V is an ordinary $\mathbf{C}G$-module, and we claim that

$$4\,\|\chi\|_H^2 = \|\chi(1+\varepsilon)(1+\delta)\|_G^2 = 4\,|L_V|, \tag{3.3}$$

where χ denotes the character of V. The first equality is a consequence of the fact that $(1+\varepsilon(g))(1+\delta(g)) = 4$ for $g \in H$ and is zero otherwise. The second equality follows from the fact that $\chi(1+\varepsilon)(1+\delta)$ is a sum of $|L_V|$ copies of $[L: L_V]$ distinct irreducible G-characters. We may thus conclude that $\|\chi\|_H^2 = |L_V|$.

Part (a) now follows immediately, since $|L_V| = 1$ implies $\|\chi\|_H^2 = 1$.

For (b), observe that if $L_V = \{1, \varepsilon\}$, then we have $\|\chi\|_H^2 = 2$, so V must be a direct sum of two irreducible, nonisomorphic $\mathbf{C}H$-modules. These submodules are clearly the eigenspaces of the ε-associator of V.

Now consider the case $L_V = L$, and let $S, T \in GL(V)$ be the associators for ε and δ. Since STS^{-1} also satisfies the definition for a δ-associator (i.e., (3.1)), Schur's Lemma implies $STS^{-1} = \pm T$. In case $ST = TS$, V must be the direct sum of the weight spaces $V^{\pm,\pm} = \{v \in V: Sv = \pm v, Tv = \pm v\}$; they are nonzero since G acts transitively on them. Therefore, since we already know that $\|\chi\|_H^2 = 4$ in this case, we are forced to conclude that the weight spaces are irreducible and nonisomorphic as $\mathbf{C}H$-modules. In case $ST = -TS$, let $V^\pm$ be the eigenspaces of S on V. Since $TST^{-1} = -S$, it follows that T permutes V^+ and V^-, and therefore, $V^+ \cong V^-$ as $\mathbf{C}H$-modules. Since we already know that $\|\chi\|_H^2 = 4$, we are forced to conclude that V^+ and V^- are irreducible. ∎

To construct a complete list of irreducible $\mathbf{C}H^\alpha$-modules, it suffices to choose one irreducible $\mathbf{C}G^\alpha$-module V from each L-orbit, and apply the above criteria. For a given associator S, the endomorphisms $1 \pm S$ are clearly eigenspace projections, so the extent to which this construction can be made explicit is governed only by the extent to which the action of S can be made explicit.

Similarly, the irreducible characters of H can be determined from those of G. For each irreducible character χ of G, let $L_\chi = \{\varepsilon \in L : \varepsilon\chi = \chi\}$ denote the χ-stabilizer. According to Theorem 3.1, there are essentially four possibilities for the irreducible constituents of $\chi \downarrow H$. If $L_\chi = \{1\}$ then χ itself is irreducible as an H-character. If $L_\chi = \{1, \varepsilon\}$ then there are two constituents; namely, $1/2(\chi \pm \Delta^\varepsilon \chi)$. If $L_\chi = L$ and the associators anticommute, then there is a single constituent, $\chi/2$; it occurs with multiplicity two. Finally, if $L_\chi = L = \{1, \varepsilon, \delta, \varepsilon\delta\}$ and the associators commute, there are four constituents. Assuming that the associators chosen for ε, δ, and $\varepsilon\delta$ are of the form S, T, and ST (rather than S, T, and $-ST$), we claim that the constituents are the four expressions of the form

$$\tfrac{1}{4}(\chi \pm \Delta^\varepsilon \chi \pm \Delta^\delta \chi \pm \Delta^{\varepsilon\delta} \chi) \tag{3.4}$$

in which an even number of "—" sign occur. To sketch the proof, note that S and T act as ± 1 on each constituent, so each of χ, $\Delta^\varepsilon\chi$, $\Delta^\delta\chi$, and $\Delta^{\varepsilon\delta}\chi$ can be written as linear combinations of the (unknown) constituents. We leave to the reader the easy task of inverting these linear combinations and thus verifying the above formula.

Any group G with a $\mathbf{Z}_2^2$-quotient has a two-dimensional projective representation arising from the fact that the dihedral group of order 8 doubly covers $\mathbf{Z}_2^2$. To be more precise, let $\rho : \mathbf{Z}_2^2 \to PGL_2$ denote the projective representation obtained from the reflection representation of the dihedral group modulo its center. An explicit realization of ρ may be obtained by assigning $\rho(-1, 1) = x$ and $\rho(1, -1) = y$, where x and y denote any pair of anticommuting involutions in GL_2, such as

$$x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.5}$$

We may thus obtain a projective G-representation, also to be denoted by ρ, via

$$G \to G/H \xrightarrow{\cong} \mathbf{Z}_2^2 \xrightarrow{\rho} PGL_2.$$

We remark that ρ is self-associate with respect to each of the linear characters; the associators are $\pm x$, $\pm y$, and $\pm ixy$. Note that the associators anticommute.

Let β denote the factor set of ρ, and observe that if V is any

CG^{α}-module, then $\rho \otimes V$ is a $CG^{\beta\alpha}$-module. The following result classifies the irreducible constituents of $\rho \otimes V$.

THEOREM 3.2. *Let V be a finite-dimensional, irreducible CG^{α}-module.*

(a) *If $L_V = \{1\}$, then $\rho \otimes V$ is an irreducible $CG^{\beta\alpha}$-*module.

(b) *If $L_V = \{1, \varepsilon\}$, and S_ρ and S are the ε-associators of ρ and V, then the eigenspaces of $S_\rho \otimes S$ on $\rho \otimes V$ are irreducible, nonisomorphic $CG^{\beta\alpha}$-modules.*

(c) *If $L_V = L = \{1, \varepsilon, \delta, \varepsilon\delta\}$, then let S_ρ and S denote the ε-associators, and let T_ρ and T denote the δ-associators for ρ and V.*

(i) *If $ST = TS$, then $\rho \otimes V$ is the direct sum of two copies of one irreducible $CG^{\beta\alpha}$-module; this module is isomorphic to all of the eigenspaces of $S_\rho \otimes S$ and $T_\rho \otimes T$.*

(ii) *If $ST = -TS$, then $\rho \otimes V$ is the direct sum of four irreducible, nonisomorphic $CG^{\beta\alpha}$-modules; namely, the weight spaces of $S_\rho \otimes S$ and $T_\rho \otimes T$.*

Proof. Let $\tilde{G}$ be a representation group of G [CR, Sect. 11E], and let $\tilde{H}$ be the preimage of H in $\tilde{G}$. We may regard V and ρ as $\tilde{G}$-representations. The central elements of $\tilde{G}$ will be represented by ρ as scalar roots of unity, so the $\tilde{G}$-character θ of ρ will be of absolute value 2 on $\tilde{H}$, but otherwise zero (cf. (3.5)). Hence, $|\theta(g)| = 1/2\, |(1+\varepsilon(g))(1+\delta(g))|$ for all $g \in \tilde{G}$, which implies $\|\theta\chi\|_{\tilde{G}}^2 = 1/4\, \|(1+\varepsilon)(1+\delta)\,\chi\|_{\tilde{G}}^2$, where χ denotes the $\tilde{G}$-character of V. We may therefore use (3.3) to conclude that $\|\theta\chi\|_{\tilde{G}}^2 = |L_\chi|$.

Parts (a) and (b) now follow by the same reasoning used in the proof of Theorem 3.1. For (b) in particular, note that $S_\rho \otimes S$ commutes with the $CG^{\beta\alpha}$-module structure of $\rho \otimes V$, so its eigenspaces are necessarily $CG^{\beta\alpha}$-modules.

For (c), observe that since $S_\rho T_\rho = -T_\rho S_\rho$, then $S_\rho \otimes S$ and $T_\rho \otimes T$ will commute if S and T anticommute, and conversely. The claimed conclusions now follow by the same reasoning used in the proof of Theorem 3.1(c), except for the following minor detail for (ii): One knows that the four weight spaces of $\rho \otimes V$ are nonzero from the fact that $1 \otimes S$ and $1 \otimes T$ permute them transitively. ∎

Every irreducible $CG^{\beta\alpha}$-module V is (isomorphic to) a submodule of $\rho \otimes V'$, for some CG^{α}-module V'. In fact, since ρ is self-dual, V is a summand of $\rho \otimes \rho \otimes V$. Therefore, a complete list of irreducible $CG^{\beta\alpha}$-modules can be constructed by choosing one irreducible CG^{α}-module V' from each L-orbit, and decomposing $\rho \otimes V'$ according to the above criteria. In Part II, we will apply this technique to four of the factor sets of W_n.

To describe the character analogue of Theorem 3.2, suppose that χ is an irreducible character of the representation group $\tilde{G}$. If $L_\chi = \{1\}$ then $\theta\chi$ is also irreducible. If $L_\chi = \{1, \varepsilon\}$ and V is a module for the character χ, then we have

$$\operatorname{tr}_{\rho \otimes V}((S_\rho \otimes S)\, g) = \Delta^\varepsilon \theta(g)\, \Delta^\varepsilon \chi(g),$$

where S_ρ and S denote associators for ρ and V. Since $S_\rho \otimes S$ is an involution that commutes with the action of $\tilde{G}$, it follows that $\Delta^\varepsilon\theta\Delta^\varepsilon\chi$ is the difference between two characters, so that $1/2(\theta\chi \pm \Delta^\varepsilon\theta\Delta^\varepsilon\chi)$ are the irreducible constituents of $\theta\chi$. If $L_\chi = L$ and the associators of χ commute, then $1/2\chi\theta$ is the only irreducible constituent of $\chi\theta$. Finally, if $L_\chi = L$ and the associators anticommute, then $\theta\chi$ has four constituents. Assuming that the difference characters are labeled so that $\Delta^{\varepsilon\delta}\theta(g)\, \Delta^{\varepsilon\delta}\chi(g)$ is the trace of $(S_\rho T_\rho \otimes ST)\, g$ (rather than its negative), these constituents are the four expressions of the form

$$\tfrac{1}{4}(\theta\chi \pm \Delta^\varepsilon\theta\Delta^\varepsilon\chi \pm \Delta^\delta\theta\Delta^\delta\chi \pm \Delta^{\varepsilon\delta}\theta\Delta^{\varepsilon\delta}\chi) \tag{3.6}$$

in which an even number of "—" signs appear (cf. (3.4)).

These remarks show that once the irreducible spin characters for one of the double covers $W_n(\alpha)$ have been determined, one only needs to construct the associated difference characters to obtain the spin characters for $W_n(\beta\alpha)$.

4. Symmetric Functions

Let $\Lambda = \bigoplus_n \Lambda^n$ denote the graded ring of symmetric functions in the variables $x_1, x_2, \ldots$, with coefficients in $\mathbf{Z}$, and let $\Lambda_{\mathbf{C}} = \mathbf{C} \otimes \Lambda$ denote the corresponding $\mathbf{C}$-algebra. For $f \in \Lambda$ we will sometimes write $f(x)$ for $f(x_1, x_2, \ldots)$, and similarly $f(y)$ for $f(y_1, y_2, \ldots)$. The two most important bases of $\Lambda_{\mathbf{C}}$ for our purposes are the power-sums p_λ and the Schur functions s_λ. For definitions and further details, see [M1].

There is a well-known inner product on $\Lambda_{\mathbf{C}}$; it is characterized by the fact that

$$\langle s_\lambda, s_\mu \rangle = z_\lambda^{-1} \langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu}, \tag{4.1}$$

where z_λ denotes the order of the centralizer of any permutation with cycle-type λ; i.e., $z_\lambda = m_1!\, 1^{m_1} \cdot m_2!\, 2^{m_2} \cdots$, where m_i denotes the multiplicity of i in λ. Closely related to the inner product $\langle \cdot, \cdot \rangle$ are the generating functions [M1, I.4]

$$\sum_\lambda s_\lambda(x)\, s_\lambda(y) = \sum_\lambda z_\lambda^{-1} p_\lambda(x)\, p_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}. \tag{4.2}$$

To explain the connection between these generating functions and the inner product, suppose that f_λ and g_λ are any pair of homogeneous bases of $\Lambda_{\mathbf{C}}$. In that case, the expression $\sum f_\lambda(x)\, g_\lambda(y)$ will depend only on the bilinear form B defined by $B(f_\lambda, g_\mu)=\delta_{\lambda\mu}$. We may thus refer to

$$F_B(x, y)=\sum_\lambda f_\lambda(x)\, g_\lambda(y)$$

as the generating function of B. If f'_λ and g'_λ are any other pair of families of homogeneous symmetric functions with $F_B(x, y)=\sum f'_\lambda(x)\, g'_\lambda(y)$, then they must form a pair of dual bases with respect to B [M1, I.4]. From this point of view, the information in (4.2) says that $\prod(1-x_i y_j)^{-1}$ is the generating function of $\langle\cdot,\cdot\rangle$ and it also implies the orthogonality relations in (4.1).

Let $\mathrm{cl}(S_n)$ denote the space of complex-valued class functions on S_n, and recall that $\chi(\alpha)$ is an abbreviation for $\chi(w_\alpha)$, according to the conventions established in Section 2. The inner product of any pair χ, $\varphi\in\mathrm{cl}(S_n)$ can therefore be expressed in the form

$$\langle\chi, \varphi\rangle=\sum_\alpha z_\alpha^{-1}\chi(\alpha)\,\bar{\varphi}(\alpha). \tag{4.3}$$

In addition to the internal ring structure of $\mathrm{cl}(S_n)$, there is also a natural graded ring structure for $\mathrm{cl}(S):=\bigoplus_n \mathrm{cl}(S_n)$. Given $\chi\in\mathrm{cl}(S_m)$ and $\varphi\in\mathrm{cl}(S_n)$, one defines their product by induction from $S_m\times S_n$ to S_{m+n} via

$$\chi\circ\varphi=(\chi\otimes\varphi)\uparrow S_{m+n}.$$

This product is well known to be commutative and associative.

Symmetric functions and S_n-characters are closely connected by means of the characteristic map of Frobenius. For $\chi\in\mathrm{cl}(S_n)$, the *characteristic* of χ is the symmetric function

$$\mathrm{ch}(\chi)=\sum_\alpha z_\alpha^{-1}\chi(\alpha)\, p_\alpha(x).$$

Clearly, $\mathrm{ch}:\mathrm{cl}(S_n)\to\Lambda^n_{\mathbf{C}}$ is an isomorphism of vector spaces, and a comparison of (4.1) and (4.3) shows that it is an isometry; i.e., $\langle\chi, \varphi\rangle=\langle\mathrm{ch}(\chi), \mathrm{ch}(\varphi)\rangle$ for χ, $\varphi\in\mathrm{cl}(S_n)$. In fact, the orthonormal basis of irreducible S_n-characters is mapped by ch to the orthonormal Schur function basis $\{s_\lambda:|\lambda|=n\}$. Furthermore, it can be shown that $\mathrm{ch}:\mathrm{cl}(S)\to\Lambda_{\mathbf{C}}$ is an algebra isomorphism; i.e.,

$$\mathrm{ch}(\chi\circ\varphi)=\mathrm{ch}(\chi)\cdot\mathrm{ch}(\varphi)$$

for $\chi\in\mathrm{cl}(S_m)$, $\varphi\in\mathrm{cl}(S_n)$ [M1, I.7]. We will exploit this equivalence between symmetric functions and characters in the next section.

There is also a close connection between symmetric functions and the spin characters of S_n. Let $\Omega_{\mathbf{C}} = \bigoplus_n \Omega_{\mathbf{C}}^n$ denote the graded subalgebra of $\Lambda_{\mathbf{C}}$ generated by 1 and the odd power-sums $p_1, p_3, p_5, \ldots$. Note that $\{p_\alpha : \alpha \in OP\}$ forms a basis for $\Omega_{\mathbf{C}}$. Another important basis for our purposes will be Schur's Q-functions, Q_λ, defined for $\lambda \in DP$. See [St1, Sect. 6] or [S] for their definition, or substitute $t = -1$ in the corresponding Hall–Littlewood function [M1].

There is a useful inner product $[\cdot, \cdot]$ on $\Omega_{\mathbf{C}}$; it can be defined by either

$$2^{-l(\lambda)}[Q_\lambda, Q_\mu] = \delta_{\lambda\mu} \qquad \text{or} \qquad 2^{l(\alpha)} z_\alpha^{-1}[p_\alpha, p_\beta] = \delta_{\alpha\beta} \tag{4.4}$$

for λ, $\mu \in DP$ and α, $\beta \in OP$ [St1]. Furthermore, in view of the identities [M1, III.4]

$$\sum_{\lambda \in DP} 2^{-l(\lambda)} Q_\lambda(x)\, Q_\lambda(y) = \sum_{\alpha \in OP} \frac{2^{l(\alpha)}}{z_\alpha} p_\alpha(x)\, p_\alpha(y) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j}, \tag{4.5}$$

we see that $\prod (1 + x_i y_j)(1 - x_i y_j)^{-1}$ is the generating function of $[\cdot, \cdot]$.

Let $\mathrm{cl}'(S_n)$ denote the complex vector space spanned by the spin characters of $\tilde{S}_n$, i.e., the space of $\mathbf{C}$-valued class functions χ on $\tilde{S}_n$ for which $\chi(-w) = -\chi(w)$. Since the split classes of $\tilde{S}_n$ are indexed by the partitions $\alpha \in OP \cup DP^-$ [St1, Sect. 2], it follows that the inner product of any pair χ, $\varphi \in \mathrm{cl}'(S_n)$ can be expressed in the form

$$\langle \chi, \varphi \rangle_{\tilde{S}_n} = \sum_{\alpha \in OP \cup DP^-} z_\alpha^{-1} \chi(\alpha)\, \bar{\varphi}(\alpha).$$

There is a "spin" characteristic $\mathrm{ch}' : \mathrm{cl}'(S_n) \to \Omega_{\mathbf{C}}^n$ that connects spin characters with symmetric functions [J1, St1]. Given $\chi \in \mathrm{cl}'(S_n)$, we define

$$\mathrm{ch}'(\chi) = \sum_{\alpha \in OP} (-1)^{(n - l(\alpha))/2} z_\alpha^{-1} 2^{l(\alpha)/2} \chi(\alpha)\, p_\alpha(x).$$

By a theorem of Schur, one knows that the irreducible spin characters of $\tilde{S}_n$ are mapped by ch′ to (scalar multiples of) the symmetric functions Q_λ [S] (see also [J1, St1]); we will have more to say about this in Section 7. Although ch′ is clearly surjective, it has a nonzero kernel containing those $\chi \in \mathrm{cl}'(S_n)$ whose restriction to $\tilde{A}_n$ (the preimage of the alternating group in $\tilde{S}_n$) is zero. In view of (4.4), we have

$$[\mathrm{ch}'(\chi), \mathrm{ch}'(\varphi)] = \langle \chi \downarrow \tilde{A}_n, \varphi \downarrow \tilde{A}_n \rangle_{\tilde{S}_n} = 1/2 \langle \chi, \varphi \rangle_{\tilde{A}_n},$$

so the spin characteristic is nearly isometric.

We remark that our definition of the spin characteristic includes a factor $(-1)^{(n - l(\alpha))/2}$ that is not present in the original definition in [St1]. It is needed to compensate for the fact that we are using characters of the double cover of S_n in which σ_j has order 2, whereas [St1] uses the double cover in which σ_j has order 4. To pass between representations of these two groups, one needs to multiply the representing matrices for $\sigma_1, ..., \sigma_{n-1}$ by the factor i. This multiplies the representing matrix for the canonical representative w_α by $i^{n-l(\alpha)}$.

Let $S_{(k,n-k)}$ denote the Young subgroup of S_n (isomorphic to $S_k \times S_{n-k}$) consisting of permutations that leave $\{1, ..., k\}$ and $\{k+1, ..., n\}$ invariant. We will use the notation $\tilde{S}_{(k,n-k)}$ for the inverse image of $S_{(k,n-k)}$ in $\tilde{S}_n$, and $\mathbf{C}S'_{(k,n-k)}$ for the corresponding subalgebra of $\mathbf{C}S'_n$. There is a natural way to define an operation $(V_1, V_2) \mapsto V_1 \hat{\otimes} V_2$ that creates irreducible modules for $\mathbf{C}S'_{(k,n-k)}$ from irreducible modules for $\mathbf{C}S'_k$ and $\mathbf{C}S'_{n-k}$. Once given, this will permit us to create a $\mathbf{C}S'_n$-module by induction from $\tilde{S}_{(k,n-k)}$ to $\tilde{S}_n$. We will use the notation

$$V_1 \hat{\circ} V_2 := (V_1 \hat{\otimes} V_2) \uparrow \tilde{S}_n$$

as an abbreviation for this operation, and we will also use $\hat{\circ}$ to denote the corresponding operation on characters.

To define the $\hat{\otimes}$-operation, let V_1 and V_2 be irreducible modules for $\mathbf{C}S'_k$ and $\mathbf{C}S'_{n-k}$, with $\mathbf{C}S'_{n-k}$ regarded as the subalgebra of $\mathbf{C}S'_n$ generated by $\sigma_{k+1}, ..., \sigma_{n-1}$. The usual tensor product $V_1 \otimes V_2$ is a module for $\mathbf{C}S'_k \otimes \mathbf{C}S'_{n-k}$; this will not be a module for $\mathbf{C}S'_{(k,n-k)}$ unless $k=1$ or $k=n-1$. The essential problem is that the generators of $\mathbf{C}S'_k$ and $\mathbf{C}S'_{n-k}$ will commute on $V_1 \otimes V_2$, whereas they must anticommute on $V_1 \hat{\otimes} V_2$. This problem can be fixed by recognizing that $S_k \times S_{n-k}$ has an obvious $\mathbf{Z}_2^2$-quotient, and thus, a two-dimensional projective representation ρ as in (3.5). It follows that we may impose a $\mathbf{C}S'_{(k,n-k)}$-module structure on $\rho \otimes (V_1 \otimes V_2)$ via

$$\sigma_j(v \otimes v_1 \otimes v_2) = \begin{cases} xv \otimes \sigma_j v_1 \otimes v_2 & \text{if } j<k \\ yv \otimes v_1 \otimes \sigma_j v_2 & \text{if } j>k, \end{cases}$$

with x and y as in (3.5). We now define $V_1 \hat{\otimes} V_2$ to be any of the irreducible submodules of $\rho \otimes (V_1 \otimes V_2)$. By Theorem 3.2, there are essentially three possible types of structure:

Case 1. If neither V_1 nor V_2 is self-associate with respect to the sign character ε, then $\rho \otimes (V_1 \otimes V_2)$ is irreducible, so in this case, we have $V_1 \hat{\otimes} V_2 = \rho \otimes (V_1 \otimes V_2)$.

Case 2. If only one of V_1 or V_2 is self-associate, then there are two distinct irreducible submodules of $\rho \otimes (V_1 \otimes V_2)$ (cf. Theorem 3.2(b)). The $\hat{\otimes}$-notation is somewhat sloppy in this case as there are two choices for the product. In situations where we need to emphasize the existence of these two choices, we will write $(V_1 \hat{\otimes} V_2)_\pm$.

Case 3. If both V_1 and V_2 are self-associate, with associators S_1 and S_2, then $S_1 \otimes 1$ and $1 \otimes S_2$ are a pair of commuting associators for $V_1 \otimes V_2$. Consequently, Theorem 3.2(c) implies that $\rho \otimes (V_1 \otimes V_2)$ is a direct sum of two copies of one irreducible module, so in this case there is only one choice for $V_1 \hat{\otimes} V_2$ (up to isomorphism).

The same construction of the $\hat{\otimes}$-product appears in [St1–2], and a similar construction has been given by Hoffman and Humphreys [HH1]. If we ignore the fact that $\hat{\otimes}$ is sometimes multi-valued (cf. Case 2), the $\hat{\circ}$-operation would give $\mathrm{cl}'(S) = \bigoplus_n \mathrm{cl}'(S_n)$ a graded algebra structure. Furthermore, $\mathrm{ch}' : \mathrm{cl}'(S) \to \Omega_{\mathbf{C}}$ is nearly an isomorphism of this pseudo-algebra in the following sense:

$$\mathrm{ch}'(\chi \mathbin{\hat{\circ}} \varphi) = \begin{cases} 2\mathrm{ch}'(\chi)\,\mathrm{ch}'(\varphi) & \text{if } \chi \text{ and } \varphi \text{ are not self-associate} \\ \mathrm{ch}'(\chi)\,\mathrm{ch}'(\varphi) & \text{otherwise,} \end{cases} \tag{4.6}$$

for irreducible spin characters χ and φ [St1, Sect. 5].

5. The Linear Representations of W_n

For each partition λ of n, let X^λ denote the irreducible representation of S_n indexed by λ, and let χ^λ denote the corresponding character. The irreducible linear representations of W_n are indexed by ordered pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$ (e.g., see [**Z**]). We will write $X^{\lambda,\mu}$ for the representation and $\chi^{\lambda,\mu}$ for the character indexed by the pair (λ, μ). In the usual parameterization, $X^{\lambda,\varnothing}$ is the extension of X^λ from S_n to W_n in which the short reflections t_i act trivially, and $X^{\varnothing,\lambda}$ is $\delta \otimes X^{\lambda,\varnothing}$. (Recall the definition of δ in Section 1.) In the general case, assuming $|\lambda| = k$ and $|\mu| = n-k$, one defines

$$X^{\lambda,\mu} = X^{\lambda,\varnothing} \circ X^{\varnothing,\mu},$$

where $\circ$ denotes induction from $W_k \times W_{n-k}$ to W_n; i.e.,

$$V_1 \circ V_2 = (V_1 \otimes V_2) \uparrow W_n,$$

for any $\mathbf{C}W_k$-module V_1 and $\mathbf{C}W_{n-k}$-module V_2. We will also use $\circ$ to denote the corresponding operation on characters, so that $\chi^{\lambda,\mu} = \chi^{\lambda,\varnothing} \circ \chi^{\varnothing,\mu}$.

Let $\mathrm{cl}(W_n)$ denote the space of W_n-class functions, and let $\mathrm{cl}(W) := \bigoplus_n \mathrm{cl}(W_n)$ denote the graded $\mathbf{C}$-algebra structure associated with the $\circ$-product. The inner product of any pair $\chi, \varphi \in \mathrm{cl}(W_n)$ can be expressed in the form

$$\langle \chi, \varphi \rangle_{W_n} = \sum_{\alpha,\beta} z_{\alpha,\beta}^{-1} \chi(\alpha, \beta)\, \bar{\varphi}(\alpha, \beta), \tag{5.1}$$

where $z_{\alpha,\beta}=2^{l(\alpha)+l(\beta)}z_\alpha z_\beta$ denotes the order of the centralizer common to the class indexed by (α, β).

There is an analogue of the characteristic map for W_n that maps a given class function $\chi \in \mathrm{cl}(W_n)$ to $\mathrm{ch}(\chi) \in \Lambda \otimes \Lambda$ via

$$\mathrm{ch}(\chi)=\sum_{\alpha,\beta} \frac{1}{z_\alpha z_\beta} \chi(\alpha, \beta)\, p_\alpha(x)\, p_\beta(x'),$$

using the obvious embedding of $\Lambda \otimes \Lambda$ in $\mathbf{C}[[x, x']]$ in which the two copies of Λ are identified with the symmetric functions of $x_1, x_2, \ldots$ and $x'_1, x'_2, \ldots$, respectively. A similar map has been defined by Macdonald for all wreath products of the form $G \wr S_n$ [M2].

It is easy to see that $\mathrm{ch}: \mathrm{cl}(W) \to \Lambda \otimes \Lambda$ is a vector space isomorphism. We may therefore define an inner product $\langle \cdot, \cdot \rangle$ on $\Lambda \otimes \Lambda$ by transferring the corresponding structure from $\mathrm{cl}(W_n)$. The power sums $p_\alpha(x)\, p_\beta(x')$ are orthogonal with respect to this inner product, and furthermore (cf. (5.1))

$$\langle p_\alpha(x)\, p_\beta(x'), p_\alpha(x)\, p_\beta(x') \rangle = z_\alpha z_\beta / 2^{l(\alpha)+l(\beta)}.$$

We remark that by (4.2), one may deduce that

$$\sum_{\alpha,\beta} 2^{l(\alpha)+l(\beta)} z_\alpha^{-1} z_\beta^{-1} p_\alpha(x)\, p_\alpha(y)\, p_\beta(x')\, p_\beta(y')$$

$$= \prod_{i,j} \frac{1}{(1-x_i y_j)^2} \frac{1}{(1-x'_i y'_j)^2} \tag{5.2}$$

is the generating function of this inner product.

We also claim that ch is an algebra isomorphism between $\mathrm{cl}(W)$ and $\Lambda \otimes \Lambda$; i.e.,

$$\mathrm{ch}(\chi \circ \varphi) = \mathrm{ch}(\chi)\, \mathrm{ch}(\varphi) \tag{5.3}$$

for $\chi \in \mathrm{cl}(W_k)$, $\varphi \in \mathrm{cl}(W_{n-k})$. The proof of this is similar to the argument in [M1], so we merely provide a sketch. One defines a $\Lambda \otimes \Lambda$-valued class function π_n on W_n by setting

$$\pi_n(w) = 2^{l(\alpha)+l(\beta)} p_\alpha(x)\, p_\beta(x')$$

for w belonging to the class indexed by (α, β). Since the restriction of π_n to $W_k \times W_{n-k}$ is $\pi_k \otimes \pi_{n-k}$, it follows that

$$\begin{aligned} \mathrm{ch}(\chi \circ \varphi) &= \langle \chi \circ \varphi, \pi_n \rangle_{W_n} = \langle \chi \otimes \varphi, \pi_n \rangle_{W_k \times W_{n-k}} \\ &= \langle \chi, \pi_k \rangle \langle \varphi, \pi_{n-k} \rangle = \mathrm{ch}(\chi)\, \mathrm{ch}(\varphi), \end{aligned}$$

by Frobenius reciprocity.

It is well known that the Schur function s_λ is the image of χ^λ under the characteristic map [M1]. To describe the analogue for W_n-characters, let us define an algebra automorphism $f(x, x') \mapsto f^*(x, x')$ of $\Lambda \otimes \Lambda$ by setting $p_r^*(x) = p_r(x)$ and $p_r^*(x') = -p_r(x')$. (In the notation of [M1], one has $f^*(x') = \omega f(-x')$.) Note that since $(p_\alpha(x)\, p_\beta(x'))^* = (-1)^{l(\beta)} p_\alpha(x)\, p_\beta(x') = \delta(\alpha, \beta)\, p_\alpha(x)\, p_\beta(x')$, it follows that this involution is isomorphic to the action of δ on $\mathrm{cl}(W_n)$; i.e.,

$$\mathrm{ch}(\delta\chi)(x, x') = \mathrm{ch}(\chi)^*(x, x'). \tag{5.4}$$

PROPOSITION 5.1. *We have* $\mathrm{ch}(\chi^{\lambda,\mu}) = s_\lambda(x, x')\, s_\mu^*(x, x')$.

Proof. In view of (5.3), (5.4), and the fact that $\chi^{\lambda,\mu} = \chi^{\lambda,\varnothing} \circ \chi^{\varnothing,\mu}$, it suffices to prove $\mathrm{ch}(\chi^{\lambda,\varnothing}) = s_\lambda(x, x')$. For this, note that since $\chi^{\lambda,\varnothing}(\alpha, \beta) = \chi^\lambda(\alpha \cup \beta)$, we have

$$\begin{aligned} \mathrm{ch}(\chi^{\lambda,\varnothing}) &= \sum_{\alpha,\beta} \frac{1}{z_\alpha z_\beta} \chi^\lambda(\alpha \cup \beta)\, p_\alpha(x)\, p_\beta(x') \\ &= \sum_\gamma \frac{1}{z_\gamma} \chi^\lambda(\gamma) \sum_{\gamma = \alpha \cup \beta} \frac{z_\gamma}{z_\alpha z_\beta} p_\alpha(x)\, p_\beta(x'). \end{aligned}$$

If m_i (resp., n_i) denotes the multiplicity of i in α (resp., β), then we have $z_{\alpha\cup\beta}/z_\alpha z_\beta = \prod \binom{m_i+n_i}{m_i}$. It follows that the inner sum in the above expression simplifies to $p_\gamma(x, x')$, and so we have

$$\mathrm{ch}(\chi^{\lambda,\varnothing}) = \sum_\gamma z_\gamma^{-1} \chi^\lambda(\gamma)\, p_\gamma(x, x') = \mathrm{ch}(\chi^\lambda)(x, x').$$

The claimed formula is now a consequence of the fact that $\mathrm{ch}(\chi^\lambda) = s_\lambda$. ∎

We remark that a corollary of this result is the fact that the class functions $\chi^{\lambda,\mu}$ are indeed the irreducible characters of W_n. Certainly they are characters, so it suffices merely to prove that they are an orthonormal basis of $\mathrm{cl}(W_n)$. For this, observe that (4.2) implies

$$\sum_\lambda s_\lambda(x, x')\, s_\lambda(y, y') = \Pi(x; y)\, \Pi(x'; y)\, \Pi(x; y')\, \Pi(x'; y'),$$

where $\Pi(x; y) = \prod(1 - x_i y_j)^{-1}$, and similarly, (3.8) and (4.3′) of [M1] imply

$$\sum_\mu s_\mu^*(x, x')\, s_\mu^*(y, y') = \Pi(x; y)\, \Pi(x'; y')/\Pi(x; y')\, \Pi(x'; y).$$

By Proposition 5.1, we know that the generating function

$$\sum_{\lambda,\mu} \mathrm{ch}\chi^{\lambda,\mu}(x, x')\, \mathrm{ch}\chi^{\lambda,\mu}(y, y')$$

is the product of the two previous expressions; i.e., $\Pi(x; y)^2\, \Pi(x', y')^2$. Since this is the generating function of the inner product $\langle \cdot, \cdot \rangle$ (cf. (5.2)), it follows that the symmetric functions $\mathrm{ch}(\chi^{\lambda,\mu})$ are an orthonormal basis of $\Lambda \otimes \Lambda$, and hence the characters $\chi^{\lambda,\mu}$ must form an orthonormal basis of $\mathrm{cl}(W)$.

As a second remark, we mention that there is a simple way to evaluate the characters $\chi^{\lambda,\mu}$. Given any partition α and any subset $I \subset \{1, ..., l(\alpha)\}$, we will use the notation α_I for the subpartition of α indexed by I, i.e., the partition obtained by selecting the ith part of α for each $i \in I$. Also, we will write α_I^c for the complementary subpartition, so that $\alpha = \alpha_I \cup \alpha_I^c$. Using standard techniques for evaluating induced characters (e.g., [CR, Sect. 10]), one may show that

$$\chi^{\lambda,\mu}(\alpha, \beta) = \sum_{I,J} (-1)^{|J^c|} \chi^\lambda(\alpha_I \cup \beta_J)\, \chi^\mu(\alpha_I^c \cup \beta_J^c), \tag{5.5}$$

where I and J are restricted to those subsets for which $|\alpha_I \cup \beta_J| = |\lambda|$. By combining this with the Murnaghan–Nakayama rule for symmetric group characters [JK], it is possible to give a combinatorial rule for evaluating $\chi^{\lambda,\mu}$. (See for example [St3, Sect. 7], where a more general rule is given for characters of the groups $G \wr S_n$.)

PART II: REPRESENTATIONS

6. The Factor Set $[+1, -1, +1]$

Let $\Theta: W_n \to PGL_2$ denote the projective representation obtained by composing the natural homomorphism $W_n \to \mathbf{Z}_2^2$ with the representation $\rho: \mathbf{Z}_2^2 \to PGL_2$ of Section 3. One may define Θ more explicitly by setting $\Theta(\sigma_j) = x$ and $\Theta(\tau) = y$, where x and y are any pair of anticommuting involutions, such as (3.5). By Proposition 1.2 we see that $[+1, -1, +1]$ is the factor set of Θ. In this section, we consider the irreducible decompositions of the representations $\Theta \otimes X^{\lambda,\mu}$. According to the program laid out in Section 3, the constituents of these representations (if not already irreducible) can be constructed from the associators of Θ and $X^{\lambda,\mu}$ with respect to each of the linear characters of W_n.

First consider Θ and its character θ. The associators of Θ with respect to ε, δ, and $\varepsilon\delta$ are $\pm y$, $\pm x$, and $\pm ixy$. In the following description of the difference characters, we have specifically chosen $(-1)^{n-1} y$, x, and $(-1)^n ixy$ as the associators.

PROPOSITION 6.1. *The only nonzero values of θ and its difference characters are*

$$\begin{aligned}
\theta(\alpha, \beta) &= 2(-1)^{l(\beta)/2} && \text{if } \varepsilon(\alpha, \beta) = +1 \text{ and } \delta(\alpha, \beta) = +1 \\
\Delta^{\varepsilon}\theta(\alpha, \beta) &= 2(-1)^{(l(\beta)-1)/2} && \text{if } \varepsilon(\alpha, \beta) = +1 \text{ and } \delta(\alpha, \beta) = -1 \\
\Delta^{\delta}\theta(\alpha, \beta) &= 2(-1)^{l(\beta)/2} && \text{if } \varepsilon(\alpha, \beta) = -1 \text{ and } \delta(\alpha, \beta) = +1 \\
\Delta^{\varepsilon\delta}\theta(\alpha, \beta) &= 2i(-1)^{(l(\beta)-1)/2} && \text{if } \varepsilon(\alpha, \beta) = -1 \text{ and } \delta(\alpha, \beta) = -1.
\end{aligned}$$

Proof. From the definition of $w_{\alpha\beta}$ (2.1) and the fact that $\Theta(\tau_j) = (-1)^{j-1}\, y$ (cf. (1.3)), it is easy to show that

$$\Theta(w_{\alpha\beta}) = (-1)^{l_2(2n-l_2-1)/2} x^{l_1} y^{l_2},$$

where $l_1 = n - l(\alpha) - l(\beta)$ and $l_2 = l(\beta)$. Therefore, $\theta(w_{\alpha\beta}) = 0$ unless l_1 and l_2 are both even. In that case, we have $l_2(2n - l_2 - 1)/2 = l(\beta)/2 \bmod 2$, which yields the claimed formula for θ. The difference characters can be treated similarly. For example, since the $\varepsilon\delta$-associator S of Θ is $(-1)^n\, ixy$, we have

$$S\Theta(w_{\alpha\beta}) = (-1)^{l_2(2n-l_2-1)/2 + l_1 + n} i x^{l_1 - 1} y^{l_2 - 1},$$

which has nonzero trace only if l_1 and l_2 are both odd. Under these circumstances, we have $l_2(2n - l_2 - 1)/2 + l_1 + n = (l(\beta) - 1)/2 \bmod 2$, and so the formula for $\Delta^{\varepsilon\delta}\theta$ follows. ▌

Now consider the problem of constructing the associators of the linear representations $X^{\lambda,\mu}$ of W_n. It will be convenient in what follows to have an explicit description of a module for $X^{\lambda,\mu}$ in terms of modules for X^{λ} and X^{μ}. For this we need to specify a particular embedding of $W_k \times W_{n-k}$ in W_n; the most obvious choice is the inverse image $W_{(k,n-k)}$ of the Young subgroup $S_{(k,n-k)}$ (cf. Section 4) in W_n. Given $w_1 \in W_k$ and $w_2 \in W_{n-k}$, we will identify (w_1, w_2) with the corresponding element of $W_{(k,n-k)}$. As a collection of (left) coset representatives for $W_{(k,n-k)}$ in W_n, we will use the set $W^k \subset S_n$ consisting of all permutations w of $1, \ldots, n$ such that $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$. Now, given modules V_1 and V_2 for $\mathbf{C}W_k$ and $\mathbf{C}W_{n-k}$ with characters $\chi^{\lambda,\varnothing}$ and $\chi^{\mu,\varnothing}$, we may impose the module structure of $X^{\lambda,\mu}$ on the vector space $\mathbf{C}W^k \otimes V_1 \otimes V_2$ by defining

$$w(w_0 \otimes v_1 \otimes v_2) = \delta(w_2)\, w_0' \otimes w_1 v_1 \otimes w_2 v_2, \tag{6.1}$$

for all $v_i \in V_i$, where $w_0, w_0' \in W^k$, $w w_0 = w_0'(w_1, w_2)$, and $(w_1, w_2) \in W_{(k,n-k)}$.

The ε-Associators

It is well known that $\varepsilon\chi^{\nu} = \chi^{\nu'}$, where ν' denotes the partition conjugate to ν [M1]. Hence, X^{ν} is self-associate with respect to ε if and only if $\nu \in SC$, where SC denotes the set of self-conjugate partitions (i.e., partitions ν such that $\nu = \nu'$). Similarly, $X^{\lambda,\mu}$ is self-associate with respect to ε if and only if λ and μ are both in SC.

Although there are known formulas for the difference characters $\Delta\chi^{\nu}$ (for $\nu \in SC$), there does not seem to be any explicit construction known for the corresponding associators. This would be equivalent to an explicit decomposition of $X^{\nu} \downarrow A_n$ into its irreducible constituents. Nevertheless, we will show that the ε-associator of $X^{\lambda,\mu}$ can be constructed in terms of the associators of X^{λ} and X^{μ}, and from this we will obtain formulas for the difference characters $\Delta^{\varepsilon}\chi^{\lambda,\mu}$.

To describe $\Delta\chi^{\nu}$, recall that there is a well-known bijection $SC \to DOP$ in which the self-conjugate partition ν is paired with the partition ν^* whose jth part is the (evidently odd) hooklength of the jth node on the main diagonal of the diagram of ν. In these terms, we have $\Delta\chi^{\nu}(w) = 0$ unless w is of cycle-type ν^*, and in that case,

$$\Delta\chi^{\nu}(\nu^*) = i^{(n - d(\nu))/2} \sqrt{z_{\nu^*}}, \tag{6.2}$$

where $d(\nu) := l(\nu^*)$ denotes the number of nodes on the main diagonal of ν [JK]. Note that this formula presupposes a particular choice for the associator of X^{ν}.

To describe the ε-difference characters of W_n, we will need a parameter $\varepsilon^{\lambda,\mu}_{\alpha,\beta}$ indexed by four partitions λ, μ, α, $\beta \in DP$ with $\lambda \cup \mu = \alpha \cup \beta$. To define this parameter, let $(a_1, ..., a_l)$ (resp., $(b_1, ..., b_l)$) denote the parts of (λ, μ) (resp., (α, β)), labeled so that

$$\begin{aligned}(a_1, ..., a_l) &= (\lambda_1, \lambda_2, ..., \mu_1, \mu_2, ...)\\ (b_1, ..., b_l) &= (\alpha_1, \alpha_2, ..., \beta_1, \beta_2, ...),\end{aligned} \tag{6.3}$$

where $l = l(\lambda) + l(\mu) = l(\alpha) + l(\beta)$. By assumption, there exists a permutation π of $1, ..., l$ such that $b_{\pi(i)} = a_i$; this provides a matching between the parts of (λ, μ) and (α, β). Let $n_1(\pi)$ denote the number of inversions in π involving parts of the same parity, i.e.,

$$n_1(\pi) = |\{(i, j): i < j, \pi(i) > \pi(j) \text{ and } a_i = a_j \bmod 2\}|,$$

and let $n_2(\pi)$ denote the number of parts of μ assigned to β by π. We define

$$\varepsilon^{\lambda,\mu}_{\alpha,\beta} = (-1)^{n_1(\pi) + n_2(\pi)}.$$

Since the partitions involved have no repeated parts, a given integer r can appear at most twice among the a_i's and twice among the b_j's. For each such integer r this allows the possibility of choosing π so that the r's are matched by π in an inverted or noninverted way. Since these possibilities change the parity of both $n_1(\pi)$ and $n_2(\pi)$, it follows that $\varepsilon^{\lambda,\mu}_{\alpha,\beta}$ is well-defined. For example, if $(\lambda, \mu) = (973, 741)$ and $(\alpha, \beta) = (743, 971)$, then we may choose $\pi = (1452)(3)(6)$, so that $n_1(\pi) = n_2(\pi) = 2$ and $\varepsilon^{\lambda,\mu}_{\alpha,\beta} = +1$.

Let us assume henceforth that λ, $\mu \in SC$, and let S_1 and S_2 denote ε-associators for V_1 and V_2 as symmetric group representations. We claim that the ε-associator $S^{\lambda,\mu}$ of $X^{\lambda,\mu}$ can be defined on $\mathbf{C}W^k \otimes V_1 \otimes V_2$ via

$$S^{\lambda,\mu}(w_0 \otimes v_1 \otimes v_2) = \varepsilon(w_0)\, w_0 \otimes S_1 v_1 \otimes S_2 v_2 \tag{6.4}$$

for $w_0 \in W^k$ and $v_i \in V_i$.

THEOREM 6.2. (a) *$S^{\lambda,\mu}$ is the ε-associator of $X^{\lambda,\mu}$.*

(b) *The difference character $\Delta^\varepsilon \chi^{\lambda,\mu}(w)$ vanishes unless w belongs to a W_n-class of the form (α, β) with α, $\beta \in DOP$ and $\lambda^* \cup \mu^* = \alpha \cup \beta$. In that case,*

$$\Delta^\varepsilon \chi^{\lambda,\mu}(\alpha, \beta) = 2^{l(\lambda^* \cap \mu^*)} \varepsilon^{\lambda^*,\mu^*}_{\alpha,\beta}\, i^{(n - d(\lambda) - d(\mu))/2} \sqrt{z_{\lambda^*} z_{\mu^*}}.$$

We remark that a simpler definition of $\varepsilon^{\lambda^*,\mu^*}_{\alpha,\beta}$ could have been given that takes advantage of that fact that all relevant cycle lengths are odd. However, this parameter will also be needed later in situations with both odd and even cycle lengths.

Proof. For any $w \in W_n$ and any coset representative $w_0 \in W^k$, (6.1) and (6.4) imply

$$S^{\lambda,\mu} w(w_0 \otimes v_1 \otimes v_2) = \varepsilon(w_0')\, \delta(w_2)\, w_0' \otimes S_1 w_1 v_1 \otimes S_2 w_2 v_2$$
$$w S^{\lambda,\mu}(w_0 \otimes v_1 \otimes v_2) = \varepsilon(w_0)\, \delta(w_2)\, w_0' \otimes w_1 S_1 v_1 \otimes w_2 S_2 v_2,$$

where $w w_0 = w_0'(w_1, w_2)$. Since $w_1 S_1 v_1 = \varepsilon(w_1)\, S_1 w_1 v_1$ (and similarly for w_2), it follows that the actions of $S^{\lambda,\mu} w$ and $w S^{\lambda,\mu}$ differ by a factor of $\varepsilon(w_0 w_0'(w_1, w_2)) = \varepsilon(w)$. Since $S^{\lambda,\mu}$ is clearly an involution, we may conclude that it is indeed the ε-associator.

The only subspaces $\mathbf{C}w_0 \otimes V_1 \otimes V_2$ that contribute to the trace of $S^{\lambda,\mu} w$ are those for which $w_0 = w_0'$; i.e., those for which $w_0^{-1} w w_0 \in W_{(k, n-k)}$. By adding the contributions of these subspaces, we obtain

$$\begin{aligned}\Delta^\varepsilon \chi^{\lambda,\mu}(w) &= \sum_{w_0 \in W^k} \varepsilon(w_0)\, \delta(w_2)\, \mathrm{tr}_{V_1}(S_1 w_1)\, \mathrm{tr}_{V_2}(S_2 w_2) \\ &= \sum_{w_0 \in W^k} \varepsilon(w_0)\, \delta(w_2)\, \Delta^\varepsilon \chi^{\lambda,\varnothing}(w_1)\, \Delta^\varepsilon \chi^{\mu,\varnothing}(w_2),\end{aligned} \tag{6.5}$$

where the sums are restricted to those w_0 for which $w_0^{-1}ww_0 = (w_1, w_2) \in W_{(k,n-k)}$.

For the remainder of the proof, we will assume that $w = w_{\alpha\beta}$ is the canonical representative of some W_n-class (α, β). Since $\Delta\chi^{\nu}$ is nonzero only for cycle-type ν^*, (6.5) implies that $\Delta^{\varepsilon}\chi^{\lambda,\mu}(w) = 0$ unless $\lambda^* \cup \mu^* = \alpha \cup \beta$ (and thus, α, $\beta \in OP$). If there are any repeated parts in α or β, then there is an involution u that centralizes w by interchanging positions in the corresponding cycles. Since $\varepsilon(u) = -1$ (the cycles necessarily have odd length), (3.2) would imply $\Delta^{\varepsilon}\chi^{\lambda,\mu}(w) = 0$. Therefore, we may further assume that α, $\beta \in DOP$ and $\lambda^* \cup \mu^* = \alpha \cup \beta$.

Let $l = \ell(\alpha) + \ell(\beta)$. For each permutation π of the parts that sends the l-tuple (α, β) to (λ^*, μ^*), there is a corresponding coset representative $w_0 = w_0(\pi)$ with $w_0^{-1}ww_0 = (w_1, w_2)$ in which the S_n-image of (w_1, w_2) is $(w_{\lambda^*}, w_{\mu^*})$, and conversely. Furthermore, each inversion in π involving two parts l_1 and l_2 will add 1 to $n_1(\pi)$ and add an odd number of inversions (namely, $l_1 l_2$) to $w_0(\pi)$; i.e., $\varepsilon(w_0(\pi)) = (-1)^{n_1(\pi)}$. Since $\delta(w_2(\pi)) = (-1)^{n_2(\pi)}$, we may conclude that $\varepsilon_{\alpha,\beta}^{\lambda^*,\mu^*} = \varepsilon(w_0(\pi))\,\delta(w_2(\pi))$, so (6.2) and (6.5) imply

$$\Delta^{\varepsilon}\chi^{\lambda,\mu}(\alpha, \beta) = \sum_{\pi} i^{(n - d(\lambda) - d(\mu))/2} \varepsilon_{\alpha,\beta}^{\lambda^*,\mu^*} \sqrt{z_{\lambda^*} z_{\mu^*}}.$$

Since each summand is independent of π and there are $2^{l(\lambda^* \cap \mu^*)} = 2^{l(\alpha \cap \beta)}$ such summands, the result follows. ∎

The δ-Associators

Since $\delta\chi^{\lambda,\varnothing} = \chi^{\varnothing,\lambda}$ and $\chi^{\lambda,\mu} = \chi^{\lambda,\varnothing} \circ \chi^{\varnothing,\mu}$ (by definition), it follows in general that $\delta\chi^{\lambda,\mu} = \chi^{\mu,\lambda}$. Hence, $X^{\lambda,\mu}$ is self-associate with respect to δ if and only if $\lambda = \mu$. An explicit construction of the δ-associators and difference characters for $X^{\lambda,\lambda}$ can be found in [St3, Sect. 7]; we will restate the results here for the sake of completeness. Note that this amounts to a construction of the representations and characters of the Weyl group of the root system D_n (i.e., the kernel of δ).

In this case, we may assume that λ is a fixed partition of $k = n/2$, and $V_1 = V_2 = V$. Let $u \in S_n$ denote the involution $(1, k+1)(2, k+2) \cdots (k, 2k)$, and define an endomorphism T^{λ} on $\mathbf{C}W^{n/2} \otimes V \otimes V$ by setting

$$T^{\lambda}(w_0 \otimes v_1 \otimes v_2) = w_0 u \otimes v_2 \otimes v_1,$$

for $w_0 \in W^{n/2}$ and $v_j \in V$.

THEOREM 6.3 [St3]. (a) *T^{λ} is the δ-associator of $X^{\lambda,\lambda}$.*

(b) *The difference character* $\Delta^{\delta}\chi^{\lambda,\lambda}(w)$ *vanishes unless* w *belongs to a* W_n*-class of the form* $(2\alpha, \varnothing)$ *for some partition* α *of* $n/2$. *In that case, we have*

$$\Delta^{\delta}\chi^{\lambda,\lambda}(2\alpha, \varnothing) = 2^{\ell(\alpha)}\chi^{\lambda}(\alpha).$$

The $\varepsilon\delta$*-Associators*

We have $\varepsilon\delta\chi^{\lambda,\mu} = \chi^{\mu',\lambda'}$, so $X^{\lambda,\mu}$ is self-associate with respect to $\varepsilon\delta$ if and only if $\lambda = \mu'$. As in the previous case, we assume that λ is a fixed partition of $k = n/2$, and let $V = V_1$ be a module with character $\chi^{\lambda,\varnothing}$. To impose the module structure of $X^{\lambda,\lambda'}$ on $\mathbf{C}W^{n/2} \otimes V \otimes V$, we need to modify (6.1) to take into account the fact that $V_2 = \varepsilon \otimes V_1$. In these terms, the action of $w \in W_n$ can be realized via

$$w(w_0 \otimes v_1 \otimes v_2) = \varepsilon\delta(w_2)\, w_0' \otimes w_1 v_1 \otimes w_2 v_2,$$

where $ww_0 = w_0'(w_1, w_2)$, as usual. We claim that the $\varepsilon\delta$-associator U^{λ} of $X^{\lambda,\lambda'}$ can be defined with respect to this basis via

$$U^{\lambda}(w_0 \otimes v_1 \otimes v_2) = i^{n/2}\varepsilon\delta(w_0)\, w_0 u \otimes v_2 \otimes v_1,$$

where u is the involution defined above.

THEOREM 6.4. (a) *U^{λ} is the $\varepsilon\delta$-associator of $X^{\lambda,\lambda'}$.*

(b) *The difference character* $\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda'}(w)$ *vanishes unless* w *belongs to a* W_n*-class of the form* $(\varnothing, 2\beta)$ *for some partition* β *of* $n/2$. *In that case, we have*

$$\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda'}(\varnothing, 2\beta) = i^{n^2/4}2^{\ell(\beta)}\chi^{\lambda}(\beta).$$

Proof. Continuing the notation defined above, we have

$$U^{\lambda}w(w_0 \otimes v_1 \otimes v_2) = i^{n/2}\varepsilon\delta(w_0' w_2)\, w_0' u \otimes w_2 v_2 \otimes w_1 v_1, \tag{6.6}$$

where $ww_0 = w_0'(w_1, w_2)$. Conversely, to compare this with the action of wU^{λ}, note that $u(w_1, w_2)\, u = (w_2, w_1)$, so $ww_0 u = (w_0' u)(w_2, w_1)$. This shows that $ww_0 u$ belongs to the coset of $w_0' u$, and hence

$$wU^{\lambda}(w_0 \otimes v_1 \otimes v_2) = i^{n/2}\varepsilon\delta(w_0 w_1)\, w_0' u \otimes w_2 v_2 \otimes w_1 v_1,$$

so $wU^{\lambda} = \varepsilon\delta(w)\, U^{\lambda} w$ on $\mathbf{C}W^{n/2} \otimes V \otimes V$. Since $\varepsilon\delta(u) = \varepsilon(u) = (-1)^{n/2}$, it follows that U^{λ} is an involution, so it must be the $\varepsilon\delta$-associator.

Observe that (6.6) implies that the only subspaces $\mathbf{C}w_0 \otimes V \otimes V$ that contribute to the trace of $U^{\lambda}w$ are those for which $w_0 = w_0' u$. Since the trace of any endomorphism of the form $v_1 \otimes v_2 \mapsto Bv_2 \otimes Av_1$ is $\operatorname{tr}(AB)$, it follows that

$$\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda'}(w) = i^{n/2} \sum_{w_0 \in W^{n/2}} \varepsilon\delta(w_0 w_1)\, \chi^{\lambda,\varnothing}(w_1 w_2), \tag{6.7}$$

where w_0 is restricted to those cases for which $w_0 = w_0' u$, i.e., to those cases for which $uw_0^{-1} w w_0 = (w_1, w_2) \in W_{(n/2, n/2)}$.

If w is not a zero of $\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda'}$, (3.2) would imply that there cannot exist any $z \in W_n$ with $\varepsilon\delta(z) = -1$ that centralize w. Since the cycles of w centralize w, this eliminates the possibility of any positive cycles of even length or negative cycles of odd length. One may also eliminate positive cycles of odd length, by using the centralizing elements that appear in Lemma 2.2(c). Thus, we may assume for the remainder of the proof that w is the canonical representative of some class $(\varnothing, 2\beta)$ where β is a partition of $k = n/2$.

To determine the representatives for which $w_0 = w_0' u$, first consider the case in which $\ell(\beta) = 1$, so that w is a negative $2k$-cycle whose S_n-image is $(1, 2, ..., 2k)$. One finds that there are two choices for w_0; namely, the permutations whose one-line notations are $2, 4, ..., 2k, 1, 3, ..., 2k-1$ and $1, 3, ..., 2k-1, 2, 4, ..., 2k$. In the first case, we have $uw_0^{-1} w w_0 = (x, 1)$, where x is a negative k-cycle whose S_n-image is $(1, 2, ..., k)$. Since w_0 has $\binom{k+1}{2}$ inversions and $\varepsilon\delta(x) = (-1)^k$, it follows that $\varepsilon\delta(w_0 w_1) = (-1)^{\binom{k}{2}}$ in this case. In the second case, w_0 has $\binom{k}{2}$ inversions and we have $uw_0^{-1} w w_0 = (1, x)$, so $\varepsilon\delta(w_0 w_1)$ is again $(-1)^{\binom{k}{2}}$. It follows that both choices contribute $i^k (-1)^{\binom{k}{2}} \chi^\lambda(x) = i^{k^2} \chi^\lambda(x)$ to (6.7).

In the general case, each cycle of w has an S_n-image of the form $(2s+1, 2s+2, ..., 2s+2r)$ for suitable r and s. The above analysis shows that if $uw_0^{-1} w w_0 \in W_{(n/2, n/2)}$, then there are two choices for arranging the letters $2s+1, ..., 2s+2r$ in the one-line form of w_0; either the even letters $2s+2, 2s+4, ...$ must appear in the left half and the odd letters $2s+1, 2s+3, ...$ in the right half, or vice versa. The first case will contribute a negative r-cycle to w_1; the second will contribute a negative r-cycle to w_2. It follows that the choices for w_0 may be indexed by subsets I of $\{1, ..., \ell(\beta)\}$, so that $i \in I$ iff the first alternative is used for the ith cycle of w. In these terms, we have $\varepsilon\delta(w_1) = (-1)^{|\beta_I|}$, and it is not hard to show that the number of inversions in w_0 is $\binom{k}{2} + |\beta_I|$. Since the $S_{n/2}$-image of $w_1 w_2$ will be of cycle-type β in all cases, it follows that each of the $2^{\ell(\beta)}$ choices for w_0 contributes $i^k (-1)^{\binom{k}{2}} \chi^\lambda(\beta) = i^{k^2} \chi^\lambda(\beta)$ to (6.7). ∎

Observe that $X^{\lambda,\mu}$ is self-associate with respect to both ε and δ if and only if $\lambda = \mu \in SC$. In that case, we have

$$S^{\lambda,\lambda} T^\lambda (w_0 \otimes v_1 \otimes v_2) = \varepsilon(w_0 u)\, w_0 u \otimes S v_2 \otimes S v_1,$$

where $S = S_1 = S_2$ denotes the ε-associator of $V = V_1 = V_2$. It follows that $(S^{\lambda,\lambda} T^\lambda)^2 = \varepsilon(u) = (-1)^{n/2}$; i.e., $S^{\lambda,\lambda}$ and T^λ commute when $n/2$ is even and anticommute when $n/2$ is odd. Therefore, Theorem 3.2(c) implies that $\Theta \otimes X^{\lambda,\lambda}$ is the direct sum of two copies of one irreducible representation when $n = 0 \bmod 4$, and four distinct irreducible representations when $n = 2 \bmod 4$.

We are now in a position to classify the irreducible representations of W_n for the factor set $[+1, -1, +1]$. First, observe that the action of L_n on the representations $X^{\lambda,\mu}$ induces an action of $\mathbf{Z}_2^2$ on the labels (λ, μ). It will be convenient to use the orbit of (λ, μ) to label each of the (isomorphism classes of) irreducible submodules in $\Theta \otimes X^{\lambda,\mu}$. A given irreducible module for the corresponding twisted group algebra will be labeled by only one orbit, and two modules will receive the same label if and only if they are both summands of one of the representations $\Theta \otimes X^{\lambda,\mu}$.

The following result is a corollary of Theorems 3.2 and Theorems 6.2–6.4; it summarizes the overall structure of the irreducible representations for the factor set $[+1, -1, +1]$.

COROLLARY 6.5. *The irreducible spin representations of $W_n([+1, -1, +1])$ can be labelled by L_n-orbits of pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$, so that the orbit of (λ, μ) labels submodules of $\Theta \otimes X^{\lambda,\mu}$. In the following table, $n_{\lambda,\mu}$ denotes the number of modules indexed by (λ, μ), $m_{\lambda,\mu}$ denotes their multiplicity in $\Theta \otimes X^{\lambda,\mu}$, and $o_{\lambda,\mu}$ denotes the size of the orbit of (λ, μ).*

$n_{\lambda,\mu}$	$o_{\lambda,\mu}$	$m_{\lambda,\mu}$	
1	1	2	*if* $\lambda = \mu \in SC$, $n = 0 \bmod 4$
4	1	1	*if* $\lambda = \mu \in SC$, $n = 2 \bmod 4$
2	2	1	*if* $\lambda, \mu \in SC$, *or* $\lambda = \mu$, *or* $\lambda = \mu'$, *but not* $\lambda = \mu \in SC$
1	4	1	*otherwise.*

The irreducible spin characters for the factor set $[+1, -1, +1]$ can be easily expressed in terms of the characters $\chi^{\lambda,\mu}$ and θ, together with the formulas for the difference characters in Proposition 6.1 and Theorems 6.2–6.4 (cf. the discussion following Theorem 3.2). However, we need to modify (3.6), since it was derived under the assumption that $\Delta^{\varepsilon\delta}\theta\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda}$ represents the trace of $(S_\theta S^{\lambda,\lambda} \otimes T_\theta T^\lambda)\, w$, rather than its negative.

To determine whether this is true for the particular choices we made, note that

$$S^{\lambda,\lambda} T^\lambda w(w_0 \otimes v_1 \otimes v_2) = \varepsilon(w_0' u)\, \delta(w_2)\, w_0' u \otimes S w_2 v_2 \otimes S w_1 v_1,$$

assuming $\lambda = \mu \in SC$. By reasoning similar to the proof of Theorem 6.4, it follows that

$$\operatorname{tr}(S^{\lambda,\lambda} T^\lambda w) = \sum \varepsilon(w_0)\, \varepsilon\delta(w_2)\, \chi^{\lambda,\varnothing}(w_1 w_2),$$

summed over all $w_0 \in W^{n/2}$ such that $uw_0^{-1}ww_0 = (w_1, w_2) \in W_{(n/2,n/2)}$. Assuming $\varepsilon\delta(w) = 1$, we have $\varepsilon\delta(w_1 w_2) = \varepsilon\delta(u) = (-1)^{n/2}$, and so by comparison with (6.7),

$$\operatorname{tr}(S^{\lambda,\lambda}T^{\lambda}w) = i^{n/2}\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda}(w). \tag{6.8}$$

Next, recall that the ε, δ, and $\varepsilon\delta$-associators used for Θ in Proposition 6.1 were of the form S_θ, T_θ, and $iS_\theta T_\theta$. Therefore $\operatorname{tr}(S_\theta T_\theta w) = -i\Delta^{\varepsilon\delta}\theta(w)$. Together with (6.8), this implies

$$\operatorname{tr}((S_\theta S^{\lambda,\lambda} \otimes T_\theta T^{\lambda})\, w) = (-1)^{(n-2)/4}\, \Delta^{\varepsilon\delta}\theta(w)\, \Delta^{\varepsilon\delta}\chi^{\lambda,\lambda}(w),$$

so the characters of the four constituents of $\Theta \otimes X^{\lambda,\lambda}$ that occur in the case $n = 2 \bmod 4$ are the four expressions

$$\tfrac{1}{4}(\theta\chi^{\lambda,\lambda} \pm \Delta^{\varepsilon}\theta\Delta^{\varepsilon}\chi^{\lambda,\lambda} \pm \Delta^{\delta}\theta\Delta^{\delta}\chi^{\lambda,\lambda} \pm (-1)^{(n-2)/4}\Delta^{\varepsilon\delta}\theta\Delta^{\varepsilon\delta}\chi^{\lambda,\lambda})$$

in which an even number of the "$\pm$"-choices are "$-$" (cf. (3.6)).

When $\lambda \in SC$ and $n = 0 \bmod 4$, a similar modification of (3.4) is needed to describe the characters of the four irreducible constituents that occur when $X^{\lambda,\lambda}$ is restricted to the index-four subgroup of W_n. According to the information in (6.8), these characters are the four expressions of the form

$$\tfrac{1}{4}(\chi^{\lambda,\lambda} \pm \Delta^{\varepsilon}\chi^{\lambda,\lambda} \pm \Delta^{\delta}\chi^{\lambda,\lambda} \pm (-1)^{n/4}\, \Delta^{\varepsilon\delta}\chi^{\lambda,\lambda})$$

in which an even number of the "$\pm$"-choices are "$-$."

7. The Factor Set $[-1, +1, +1]$

The spin representations of $\tilde{S}_n$ are labeled by the partitions $\lambda \in DP$ of size n. We will use the notation Φ^λ for the representation indexed by λ and φ^λ for the character. An explicit construction of the representations can be found in [N] and a recurrence for the characters can be found in [Mo1, MY, St2]. In case $\lambda \in DP^-$ (i.e., $n - \ell(\lambda)$ is odd), there is actually a pair of ε-associate representations indexed by λ; we will write $\Phi^\lambda_\pm$ in situations where we need to emphasize that there are two associates. For $\lambda \in DP^+$, there is only one representation Φ^λ; it is self-associate with respect to ε. The associators of these representations have been constructed by Nazarov [N, Sect. 5].

Let us agree to use $\tilde{W}_n$ as an abbreviation for the double cover $W_n([-1, +1, +1])$ and $\mathbf{C}W'_n$ for the corresponding twisted group algebra. This is to remind us of the fact (first noted in Section 1) that $\tilde{S}_n$ is a subgroup of $\tilde{W}_n$ and $\mathbf{C}S'_n$ is a subalgebra of $\mathbf{C}W'_n$. We will also use $\tilde{W}_{(k,n-k)}$ to denote the $\tilde{W}_n$-preimage of $W_{(k,n-k)}$, and $\mathbf{C}W'_{(k,n-k)}$ for the corresponding subalgebra of $\mathbf{C}W'_n$. Given $w_1 \in \tilde{W}_k$ and $w_2 \in \tilde{W}_{n-k}$, we will use (w_1, w_2) to denote the element $w_1 \hat{w}_2 \in \tilde{W}_{(k,n-k)}$ obtained by substituting σ_{j+k} for σ_j and τ_{j+k} for τ_j $(j = 1, 2, ...)$ in any expression for w_2 in terms of the generators σ_j and τ_j of $\tilde{W}_{n-k}$.

Extend the projective representation ρ of Section 4 from $S_{(k,n-k)}$ to $W_{(k,n-k)}$ by insisting that the short reflections t_i act trivially. In these terms, $\rho \otimes (V_1 \otimes V_2)$ is a module for $\mathbf{C}W'_{(k,n-k)}$ whenever V_1 and V_2 are modules for $\mathbf{C}W'_k$ and $\mathbf{C}W'_{n-k}$. We may thus extend the $\hat{\otimes}$-product of Section 4 by defining $V_1 \hat{\otimes} V_2$ to be any of the (isomorphism classes of) irreducible constituents of $\rho \otimes (V_1 \otimes V_2)$. Note in particular that there is actually a pair of ε-associate modules $(V_1 \hat{\otimes} V_2)_\pm$ if and only if one (but not both) of V_1 or V_2 is self-associate with respect to ε.

We claim that the irreducible spin representations of $\tilde{W}_n$ can be constructed in terms of this product. Given $\lambda \in DP$, we define $\Phi^{\lambda,\varnothing}$ to be the representation obtained by extending Φ^λ from $\tilde{S}_n$ to $\tilde{W}_n$ by having the τ_i's act trivially, and we define $\Phi^{\varnothing,\lambda}$ to be $\delta \otimes \Phi^{\lambda,\varnothing}$. In the general case, given $\lambda, \mu \in DP$ with $|\lambda| = k$ and $|\mu| = n-k$, we define

$$\Phi^{\lambda,\mu} = \Phi^{\lambda,\varnothing} \hat{\circ} \Phi^{\varnothing,\mu},$$

where $V_1 \hat{\circ} V_2$ denotes the induction of $V_1 \hat{\otimes} V_2$ from $\tilde{W}_{(k,n-k)}$ to $\tilde{W}_n$. If $\varepsilon(\lambda, \mu) = -1$ then either $\lambda \in DP^-$ or $\mu \in DP^-$, so there is actually a pair of ε-associates $\Phi^{\lambda,\mu}_\pm$ for this case. Otherwise, if $\varepsilon(\lambda, \mu) = +1$ then $\Phi^{\lambda,\mu}$ is self-associate with respect to ε.

THEOREM 7.1. *The irreducible spin representations of* $\tilde{W}_n$ *are* $\Phi^{\lambda,\mu}$ *(for* $\varepsilon(\lambda, \mu) = +1$*) and* $\Phi^{\lambda,\mu}_\pm$ *(for* $\varepsilon(\lambda, \mu) = -1$*), where* $\lambda, \mu \in DP$ *and* $|\lambda| + |\mu| = n$.

A more general version of this result for $G \wr S_n$ is due to Hoffman and Humphreys [HH1]. Our proof will be structured along the lines of Section 5.

According to Theorem 2.1, the split classes of $\tilde{W}_n$ are indexed by the pairs (α, β) with $\alpha, \beta \in OP$ along with those $\alpha, \beta \in DP$ with $\varepsilon(\alpha, \beta) = -1$. Although the character of any spin representation of $\tilde{W}_n$ is completely determined by its values on (the canonical representatives of) these classes, it will be convenient in what follows to know the values of characters at certain non-canonical elements $w \in \tilde{W}_n$; this amounts to determining whether w is conjugate to a canonical element $w_{\alpha\beta}$ or its negative.

To resolve questions like these, we will use a Clifford algebra representation of $\mathbf{C}W'_n$. To describe this representation, let $\xi_1, ..., \xi_n$ denote a set of anticommuting involutions that generate the Clifford algebra $\mathscr{C}_n$ (cf. [St1, Sect. 3]), and define

$$[\sigma_j] = \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}), \qquad [\tau_j] = 1.$$

By Proposition 1.1, this extends to a unique algebra homomorphism $[\cdot]$: $\mathbf{C}W_n' \to \mathscr{C}_n$. Although this representation ignores the short reflections, it contains sufficient information for our purposes.

LEMMA 7.2. *Let $w, u \in \tilde{W}_n$, and suppose that $[w] = p(\xi_1, ..., \xi_n)$ is some polynomial expression for $[w]$. If π is the S_n-image of u, then we have*

$$[uwu^{-1}] = \varepsilon(w)^l \, p(\xi_{\pi(1)}, ..., \xi_{\pi(n)}),$$

where l denotes the number of inversions in π.

Proof. It suffices to prove this when u is one of the generators of $\tilde{W}_n$. In case $u = \tau_j$, the result is obvious. For the case $u = \sigma_i$, a simple calculation will show that

$$\tfrac{1}{2}(\xi_i - \xi_{i+1})\, \xi_j (\xi_i - \xi_{i+1}) = \begin{cases} -\xi_{i+1} & \text{if } j = i \\ -\xi_i & \text{if } j = i+1 \\ -\xi_j & \text{otherwise.} \end{cases}$$

Therefore, the effect of conjugation by σ_i on $\mathscr{C}_n$ is to change the signs of the generators and permute ξ_i and ξ_{i+1}. Since p must be homogeneous of degree $\varepsilon(w)$ (with respect to the usual $\mathbf{Z}_2$-grading of $\mathscr{C}_n$), it follows that the sign change introduces a factor of $\varepsilon(w)$, as in the claimed formula. ∎

Observe that the W_n-image of the expression $\sigma_{j+1}\sigma_{j+2} \cdots \sigma_{j+l-1} \in \tilde{W}_n$ is a positive l-cycle that permutes an interval of consecutive coordinates. Similarly, the W_n-image of the expression $\sigma_{j+1}\sigma_{j+2} \cdots \sigma_{j+l-1}\tau_{j+l}$ is a negative l-cycle that permutes the same interval. We will refer to these particular expressions as *canonical cycles.* Note that the canonical elements $w_{\alpha\beta}$ were defined in Section 2 as products of disjoint canonical cycles.

LEMMA 7.3. *Any product of disjoint, odd-length canonical cycles is $\tilde{W}_n$-conjugate to a canonical representative.*

Proof. Let w be a canonical cycle of odd length. Since $\varepsilon(w) = 1$, Lemma 7.2 implies that the $\tilde{W}_n$-conjugates of $[w]$ may be determined by letting S_n act on the subscripts of $\xi_1, ..., \xi_n$. From this it follows that w is conjugate to any other canonical cycle of the same length and parity.

Now consider any product $w = w_1 \cdots w_l$ of disjoint, odd-length canonical cycles. Certainly there exists an element $u \in \tilde{W}_n$ such that $uwu^{-1} = \pm w_{\alpha\beta}$ for some $\alpha, \beta \in OP$. For such u, the above reasoning shows that the expressions uw_ju^{-1} must be the canonical cycles that appear in the defining factorization of $w_{\alpha\beta}$, except possibly for the fact that they might appear in a permuted order. However, since the w_j's are odd-length cycles, they must commute, so the order of the factors is immaterial. ∎

Let $\varphi^{\lambda,\mu}$ (resp., $\varphi^{\lambda,\mu}_{\pm}$) denote the character of $\Phi^{\lambda,\mu}$ (resp., $\Phi^{\lambda,\mu}_{\pm}$). Note that a corollary of Lemma 7.3 is the fact that the $\tilde{S}_n$-image of $w_{\alpha\beta}$ is conjugate to $w_{\alpha\cup\beta}$ for $\alpha, \beta \in OP$. In particular, it follows that for $\alpha, \beta \in OP$, we have

$$\varphi^{\lambda,\varnothing}(\alpha, \beta) = \varphi^{\lambda}(\alpha \cup \beta). \tag{7.1}$$

We now define an analogue of the spin characteristic for $\tilde{W}_n$ that maps a given spin character χ to $\mathrm{ch}'(\chi) \in \Omega_{\mathbf{C}} \otimes \Omega_{\mathbf{C}}$ via

$$\mathrm{ch}'(\chi) = \sum_{\alpha,\beta \in OP} (-1)^{(n-\ell(\alpha)-\ell(\beta))/2} z_\alpha^{-1} z_\beta^{-1} 2^{(\ell(\alpha)+\ell(\beta))/2} \chi(\alpha, \beta)\, p_\alpha(x)\, p_\beta(x').$$

This map fails to be injective, since it ignores the behavior of χ on the classes with $\varepsilon = -1$. In order to create an inner product $[\cdot, \cdot]$ on $\Omega_{\mathbf{C}} \otimes \Omega_{\mathbf{C}}$ that makes ch′ nearly isometric, we insist that the power sums $p_\alpha(x)\, p_\beta(x')$ $(\alpha, \beta \in OP)$ be orthogonal, and we define

$$[p_\alpha(x)\, p_\beta(x'), p_\alpha(x)\, p_\beta(x')] = z_\alpha z_\beta / 2^{2\ell(\alpha)+2\ell(\beta)}.$$

The inner product of two spin characters for $\tilde{W}_n$ can be expressed in the form

$$\langle \chi, \varphi \rangle_{\tilde{W}_n} = \sum_{\alpha,\beta} z_{\alpha,\beta}^{-1} \chi(\alpha, \beta)\, \bar{\varphi}(\alpha, \beta),$$

with the sum is restricted to (OP, OP) and the $\varepsilon = -1$ portion of (DP, DP) (cf. (5.1)). It follows that if χ or φ vanishes on the $\varepsilon = -1$ portion of $\tilde{W}_n$, then

$$\langle \chi, \varphi \rangle_{\tilde{W}_n} = [\mathrm{ch}'(\chi), \mathrm{ch}'(\varphi)]. \tag{7.2}$$

We also remark that a consequence of (4.5) is the identity

$$\prod_{i,j} \left(\frac{1 + x_i y_j}{1 - x_i y_j} \cdot \frac{1 + x'_i y'_j}{1 - x'_i y'_j} \right)^2 = \sum_{\alpha,\beta \in OP} \frac{2^{2\ell(\alpha)+2\ell(\beta)}}{z_\alpha z_\beta} p_\alpha(x)\, p_\beta(x')\, p_\alpha(y)\, p_\beta(y'), \tag{7.3}$$

which may be identified as the generating function of the inner product $[\cdot, \cdot]$.

The following is an analogue of (4.6) for $\tilde{W}_n$.

LEMMA 7.4. *If χ and φ are irreducible spin characters of $\tilde{W}_n$, then*

$$\mathrm{ch}'(\chi \,\hat{\circ}\, \varphi) = \begin{cases} 2\mathrm{ch}'(\chi)\, \mathrm{ch}'(\varphi) & \textit{if } \chi \textit{ and } \varphi \textit{ are not } \varepsilon \textit{ self-associate} \\ \mathrm{ch}'(\chi)\, \mathrm{ch}'(\varphi) & \textit{otherwise.} \end{cases}$$

Proof. Let V_1 and V_2 be modules for $\mathbf{C}W'_k$ and $\mathbf{C}W'_{n-k}$ with characters χ and φ, and let ω denote the $\tilde{W}_{(k,n-k)}$-character of $\rho\otimes(V_1\otimes V_2)$. Since the ε-associates of a given spin character have the same image under ch′, it suffices to prove that $\mathrm{ch}'(\omega\uparrow\tilde{W}_n)=2\mathrm{ch}'(\chi)\,\mathrm{ch}'(\varphi)$. For this we define an $\Omega_{\mathbf{C}}\otimes\Omega_{\mathbf{C}}$-valued class function π_n on $\tilde{W}_n$ by setting $\pi_n(w)=0$ for $\varepsilon(w)=-1$, $\pi_n(-w)=-\pi_n(w)$, and

$$\pi_n(\alpha,\beta)=(-1)^{(n-\ell(\alpha)-\ell(\beta))/2}\,2^{(\ell(\alpha)+\ell(\beta))/2}p_\alpha(x)\,p_\beta(x')$$

for $\alpha,\beta\in OP$. In these terms, we have

$$\mathrm{ch}'(\omega\uparrow\tilde{W}_n)=\langle\omega\uparrow\tilde{W}_n,\pi_n\rangle_{\tilde{W}_n}=\langle\omega,\pi_n\rangle_{\tilde{W}_{(k,n-k)}},$$

by Frobenius reciprocity. Also, by Lemma 7.3 it follows that $\pi_n(w_1,w_2)=\pi_k(w_1)\,\pi_{n-k}(w_2)$ for $w_1\in\tilde{W}_k$ and $w_2\in\tilde{W}_{n-k}$. Similarly, we have $\omega(w_1,w_2)=2\chi(w_1)\,\varphi(w_2)$ provided that $\varepsilon(w_1)=\varepsilon(w_2)=+1$. Therefore,

$$\langle\omega,\pi_n\rangle_{\tilde{W}_{(k,n-k)}}=2\langle\chi,\pi_k\rangle_{\tilde{W}_k}\langle\varphi,\pi_{n-k}\rangle_{\tilde{W}_{n-k}}=2\mathrm{ch}'(\chi)\,\mathrm{ch}'(\varphi).\quad\blacksquare$$

We remarked in Section 4 that the spin characteristic of φ^λ is a scalar multiple of the corresponding Schur Q-function. The precise relationship is

$$\mathrm{ch}'(\varphi^\lambda)=\eta_\lambda^{-1}2^{-\ell(\lambda)/2}Q_\lambda(x),\tag{7.4}$$

where $\eta_\lambda=\sqrt{2}$ for $\lambda\in DP^-$ and $\eta_\lambda=1$ for $\lambda\in DP^+$ [St1, Sect. 7].

LEMMA 7.5. *We have*

$$\mathrm{ch}'(\varphi^{\lambda,\mu})=\eta_{\lambda,\mu}^{-1}2^{-(\ell(\lambda)+\ell(\mu))/2}Q_\lambda(x,x')\,Q_\mu(x,-x'),$$

where $\eta_{\lambda,\mu}=\sqrt{2}$ *for* $\varepsilon(\lambda,\mu)=-1$ *and* $\eta_{\lambda,\mu}=1$ *for* $\varepsilon(\lambda,\mu)=+1$.

Proof. Recall that in Section 5, we defined the algebra automorphism $*$ of $\Lambda\otimes\Lambda$ by setting $p_r^*(x)=p_r(x)$ and $p_r^*(x')=-p_r(x')$. Since $\Omega_{\mathbf{C}}$ is generated by the odd power sums, it follows that for $f\in\Omega_{\mathbf{C}}\otimes\Omega_{\mathbf{C}}$, we have $f^*(x,x')=f(x,-x')$. In particular, it follows that $\mathrm{ch}'(\delta\chi)(x,x')=\mathrm{ch}'(\chi)(x,-x')$ for any spin character χ of $\tilde{W}_n$ (cf. (5.4)). In view of (7.4) and Lemma 7.4, it therefore suffices to prove that $\mathrm{ch}'(\varphi^{\lambda,\varnothing})=\mathrm{ch}'(\varphi^\lambda)(x,x')$. For this, note that by (7.1), we have

$$\begin{aligned}\mathrm{ch}'(\varphi^{\lambda,\varnothing})&=\sum_{\alpha,\beta\in OP}(-1)^{(n-\ell(\alpha)-\ell(\beta))/2}z_\alpha^{-1}z_\beta^{-1}2^{(\ell(\alpha)+\ell(\beta))/2}\varphi^\lambda(\alpha\cup\beta)\\&\quad\times p_\alpha(x)\,p_\beta(x')\\&=\sum_{\gamma\in OP}(-1)^{(n-\ell(\gamma))/2}z_\gamma^{-1}2^{\ell(\gamma)/2}\varphi^\lambda(\gamma)\sum_{\gamma=\alpha\cup\beta}\frac{z_\gamma}{z_\alpha z_\beta}p_\alpha(x)\,p_\beta(x').\end{aligned}$$

As in the proof of Proposition 5.1, the inner sum simplifies to $p_\gamma(x, x')$, and so we have

$$\mathrm{ch}'(\varphi^{\lambda,\varnothing}) = \sum_{\gamma \in OP} (-1)^{(n-\ell(\gamma))/2} z_\gamma^{-1} 2^{\ell(\gamma)/2} \varphi^\lambda(\gamma)\, p_\gamma(x, x') = \mathrm{ch}'(\varphi^\lambda)(x, x'). \quad \blacksquare$$

Proof of Theorem 7.1. Recall that we have an ε-associate pair of representations indexed by (λ, μ) in case $\varepsilon(\lambda, \mu) = -1$. In that case, since $\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu}$ vanishes on the $\varepsilon = -1$ portion of $\tilde{W}_n$, (7.2) implies that $\Phi_+^{\lambda,\mu}$ and $\Phi_-^{\lambda,\mu}$ are irreducible if and only if the norm of $\mathrm{ch}'(\varphi^{\lambda,\mu})$ is $1/\sqrt{2}$. Hence, to prove that the representations $\Phi^{\lambda,\mu}$ are irreducible and form a complete list thereof, it suffices to show that the characteristics $\mathrm{ch}'(\varphi^{\lambda,\mu})$ are of length $1/\eta_{\lambda,\mu}$ and form an orthogonal basis of $\Omega_{\mathbf{C}} \otimes \Omega_{\mathbf{C}}$. In other words, we want to show that

$$\sum_{\lambda,\mu \in DP} \eta_{\lambda,\mu}^2 \mathrm{ch}'(\varphi^{\lambda,\mu})(x, x') \cdot \mathrm{ch}'(\varphi^{\lambda,\mu})(y, y') \tag{7.5}$$

agrees with the generating function (7.3) of the inner product $[\cdot, \cdot]$.

To prove this, note that by (4.5), we have

$$\sum_{\lambda \in DP} 2^{-\ell(\lambda)} Q_\lambda(x, x')\, Q_\lambda(y, y') = \Pi(x; y)\, \Pi(x'; y)\, \Pi(x; y')\, \Pi(x'; y'),$$

where $\Pi(x; y) = \prod_{i,j} (1 + x_i y_j)(1 - x_i y_j)^{-1}$, and similarly

$$\sum_{\mu \in DP} 2^{-\ell(\mu)} Q_\mu(x, -x')\, Q_\mu(y, -y') = \Pi(x; y)\, \Pi(x'; y')/\Pi(x'; y)\, \Pi(x; y').$$

The product of the two previous expressions is (7.3), the generating function of the inner product. Using Lemma 7.5, we see that this agrees with (7.5), so the proof is complete. $\blacksquare$

The following result provides a description of the irreducible characters. Note that part (a) is an analogue of (5.5). We remark that it is possible to deduce from this a combinatorial rule for evaluating $\varphi^{\lambda,\mu}$, although we will not pursue the details here.

THEOREM 7.6. *Define $c_{\lambda,\mu} = 2$ for λ, $\mu \in DP^-$, and $c_{\lambda,\mu} = 1$ otherwise.*

(a) *If α, $\beta \in OP$, then*

$$\varphi^{\lambda,\mu}(\alpha, \beta) = c_{\lambda,\mu} \sum_{I,J} (-1)^{|J^c|}\, \varphi^\lambda(\alpha_I \cup \beta_J)\, \varphi^\mu(\alpha_I^c \cup \beta_J^c),$$

summed over subsets I and J such that $|\alpha_I \cup \beta_J| = |\lambda|$.

(b) *If* $\alpha, \beta \in DP$ *and* $\varepsilon(\alpha, \beta) = -1$, *then* $\varphi_{\pm}^{\lambda,\mu}(\alpha, \beta) = 0$ *unless* $\lambda \cup \mu = \alpha \cup \beta$. *In that case, we have*

$$\varphi_{\pm}^{\lambda,\mu}(\alpha, \beta) = \pm 2^{\ell(\lambda \cap \mu)} \varepsilon_{\alpha,\beta}^{\lambda,\mu} i^{(n - \ell(\lambda) - \ell(\mu) - 1)/2} \sqrt{\tfrac{1}{2} z_\lambda z_\mu},$$

where $\varepsilon_{\alpha,\beta}^{\lambda,\mu}$ *denotes the parameter defined in Section* 6.

Proof. Let V_1 and V_2 be modules for $\mathbf{C}W'_k$ and $\mathbf{C}W'_{n-k}$ with characters $\varphi^{\lambda,\varnothing}$ and $\varphi^{\varnothing,\mu}$. Let ω denote the $\tilde{W}_{(k,n-k)}$-character of $\rho \otimes (V_1 \otimes V_2)$. Note that $\omega \uparrow \tilde{W}_n$ is either $\varphi^{\lambda,\mu}$, $\varphi_+^{\lambda,\mu} + \varphi_-^{\lambda,\mu}$, or $2\varphi^{\lambda,\mu}$, according to whether neither, one, or both of λ and μ belong to DP^+. This implies that $c_{\lambda,\mu}(\omega \uparrow \tilde{W}_n) = 2\varphi^{\lambda,\mu}$ on the $\varepsilon = +1$ portion of $\tilde{W}_n$.

Let w be the canonical representative of some class (α, β).

Recall that in Section 6, we defined a collection W^k of coset representatives for $W_{(k,n-k)}$ in W_n. By choosing one of the two $\tilde{W}_n$-preimages for each $w_0 \in W^k$, we may thus form a collection $\tilde{W}^k$ of coset representatives for $\tilde{W}_{(k,n-k)}$. Now consider the possible choices for $w_0 \in \tilde{W}^k$ with $w_0^{-1} w w_0 = (w_1, w_2) \in \tilde{W}_{(k,n-k)}$. These choices correspond to each of the possible ways to assign a subset of the cycles of w to w_1, with the remaining cycles being assigned to w_2. (Of course, we must also insist that k be the sum of the lengths of the cycles assigned to w_1.) The class of w_1 will thus be of the form (α_I, β_J) for some suitable subpartitions of α and β; similarly, the class of w_2 will be of the form (α_I^c, β_J^c). Note that by our choice of coset representatives, both w_1 and w_2 must be products of disjoint canonical cycles, modulo ± 1.

In case $\alpha, \beta \in OP$, Lemma 7.3 implies that w_1 and w_2 must be conjugate to the canonical representatives of (α_I, β_J) and (α_I^c, β_J^c). In particular, it follows that

$$\omega(w_1, w_2) = 2\delta(w_2)\, \varphi^\lambda(\alpha_I \cup \beta_J)\, \varphi^\mu(\alpha_I^c \cup \beta_J^c).$$

By the usual formulas for induced characters, we therefore have

$$2\varphi^{\lambda,\mu}(\alpha, \beta) = c_{\lambda,\mu}(\omega \uparrow \tilde{W}_n)(\alpha, \beta) = 2c_{\lambda,\mu} \sum_{I,J} (-1)^{|J^c|} \varphi^\lambda(\alpha_I \cup \beta_J)\, \varphi^\mu(\alpha_I^c \cup \beta_J^c),$$

and thus (a) follows.

Now consider the case in which $\varepsilon(w) = -1$. For this, we may assume $\alpha, \beta \in DP$, since this is necessary for the class of (α, β) to be split. We may further assume that $\varphi^{\lambda,\mu}$ indexes a pair of ε-associates (i.e., $\varepsilon(\lambda, \mu) = -1$), since $\varphi^{\lambda,\mu}(w)$ would otherwise be zero. In that case, either V_1 or V_2 must be self-associate with respect to ε (not both), and ω has two irreducible constituents, say ω_+ and ω_-. For simplicity, we will assume that V_1 is self-associate (and hence, $\lambda \in DP^+$, $\mu \in DP^-$) and let the reader supply the details when V_2 is self-associate.

The action of $(w_1, w_2) \in \tilde{W}_{(k,n-k)}$ on $\rho \otimes (V_1 \otimes V_2)$ can be represented via

$$(w_1, w_2)(v_0 \otimes v_1 \otimes v_2) = x^{l_1} y^{l_2} v_0 \otimes w_1 v_1 \otimes w_2 v_2, \tag{7.6}$$

where $\varepsilon(w_1) = (-1)^{l_1}$, $\varepsilon(w_2) = (-1)^{l_2}$, and x and y denote the anticommuting involutions on $\mathbf{C}^2$ used to represent ρ, as in Section 3.

Let S denote the ε-associator of V_1. Since y is the ε-associator of ρ on $\tilde{W}_k$, it follows that $y \otimes S \otimes 1$ is a nontrivial involution that commutes with $\tilde{W}_{(k,n-k)}$, and hence,

$$(\omega_+ - \omega_-)(w_1, w_2) = \operatorname{tr}((y \otimes S \otimes 1)\, w_1 w_2).$$

If we compose the action of $y \otimes S \otimes 1$ with (7.6), it is easy to see that there will be a nonzero trace only if $\varepsilon(w_1) = +1$ and $\varepsilon(w_2) = -1$; in that case, we get

$$2\omega_+(w_1, w_2) = (\omega_+ - \omega_-)(w_1, w_2) = 2\Delta^\varepsilon \varphi^{\lambda, \varnothing}(w_1)\, \varphi^{\varnothing, \mu}(w_2).$$

Therefore, since $\omega_\pm \uparrow \tilde{W}_n = \varphi_\pm^{\lambda,\mu}$, we may conclude that for $\varepsilon(w) = -1$ we have

$$\varphi_\pm^{\lambda,\mu}(w) = \pm \sum_{w_0 \in \tilde{W}^k} \delta(w_2)\, \Delta^\varepsilon \varphi^{\lambda,\varnothing}(w_1)\, \varphi^{\mu,\varnothing}(w_2), \tag{7.7}$$

where $w_0^{-1} w w_0 = (w_1, w_2) \in \tilde{W}_{(k,n-k)}$ as usual. It should be emphasized that the sign depends only on the choice of associate, not on w.

To evaluate the terms in this sum, we will need to know the $\varepsilon = -1$ portion of the character table of $\tilde{S}_n$, as well as the difference characters $\Delta\varphi^\lambda$. If $\nu \in DP^-$ is a partition of n, then we have $\varphi_\pm^\nu(w) = 0$ for all $w \in \tilde{S}_n$ with $\varepsilon(w) = -1$, except for those w of cycle-type ν. In that case, we have

$$\varphi_+^\nu(\nu) = i^{(n - \ell(\nu) - 1)/2} \sqrt{\tfrac{1}{2} z_\nu}, \tag{7.8}$$

and this defines a particular labeling of the two associates Φ_+^ν and Φ_-^ν. (For a proof, see, for example, [St1, Sect. 7], but remember to adjust for the fact that a different covering group of S_n is being used.) In case $\nu \in DP^+$, so that Φ^ν is self-associate, we have $\Delta\varphi^\nu(w) = 0$ for all $w \in \tilde{S}_n$, unless w is of cycle-type ν. In that case, we have

$$\Delta\varphi^\nu(\nu) = i^{(n - \ell(\nu))/2} \sqrt{z_\nu}, \tag{7.9}$$

and this defines a particular choice of associator for Φ^ν [St1, Sect. 7].

Note that (7.8) and (7.9) imply that a given summand of (7.7) will vanish unless w_1 has cycle-type λ and w_2 has cycle-type μ. Hence, for the remainder of the proof we may assume $\lambda \cup \mu = \alpha \cup \beta$, since $\varphi_\pm^{\lambda,\mu}(\alpha, \beta)$ would otherwise be zero.

Let $(a_1, ..., a_l)$ and $(b_1, ..., b_l)$ denote the sequence of parts in (λ, μ) and (α, β), as in (6.3). Following the proof of Theorem 6.2, we know that for each permutation π such that $b_{\pi(i)} = a_i$, there is a corresponding coset representative $w_0(\pi)$ such that the elements $w_1(\pi)$ and $w_2(\pi)$ appearing in (7.7) have $\tilde{S}_k$-image $\pm w_\lambda$ and $\tilde{S}_{n-k}$-image $\pm w_\mu$, respectively. Each pair of odd-length cycles in w that are inverted by π will contribute an odd number of inversions to $w_0(\pi)$; the odd-even and even-even pairs contribute an even number of inversions. Also, in order to sort the cycles of $w_0(\pi)^{-1} w w_0(\pi)$ so that they appear in the same order as they do in the definitions of w_λ and w_μ, it is necessary to introduce a factor of -1 for each pair of even-length cycles that are inverted by π, since these cycles anticommute. An application of Lemma 7.2 therefore implies that the $\tilde{S}_n$-image of $w_0(\pi)^{-1} w w_0(\pi)$ agrees with the $\tilde{S}_n$-image of $(-1)^{n_1(\pi)} (w_\lambda, w_\mu)$, where $n_1(\pi)$, as in Section 6, denotes the number of inversions in π involving parts of the same parity. Hence, by (7.8), (7.9), and the fact that $\varepsilon^{\lambda,\mu}_{\alpha,\beta} = (-1)^{n_1(\pi)} \delta(w_2)$, (7.7) can be rewritten in the form

$$\varphi^{\lambda,\mu}_{\pm}(w) = \pm \sum_{\pi} \varepsilon^{\lambda,\mu}_{\alpha,\beta} i^{(n-\ell(\lambda)-\ell(\mu)-1)/2} \sqrt{\tfrac{1}{2} z_\lambda z_\mu}.$$

Since each summand is independent of π, and there are $2^{\ell(\alpha \cap \beta)} = 2^{\ell(\lambda \cap \mu)}$ such summands, the result claimed in part (b) follows. ∎

8. The Factor Set $[-1, -1, +1]$

Following Section 3, we know that the irreducible projective representations of W_n with factor set $[-1, -1, +1]$ can be obtained as the irreducible constituents of $\Theta \otimes \Phi^{\lambda,\mu}$ for $\lambda, \mu \in DP$. By Theorem 3.2, these constituents and their characters can be constructed from the associators and difference characters of the representations $\Phi^{\lambda,\mu}$ that are self-associate. Note that for the character ε, the self-associate cases occur for $\lambda, \mu \in DP^+$ and $\lambda, \mu \in DP^-$. For the character δ, we have $\delta \otimes \Phi^{\lambda,\mu} \cong \Phi^{\mu,\lambda}$, so $\Phi^{\lambda,\mu}$ will be self-associate with respect to δ if and only if $\lambda = \mu$. In that case $\Phi^{\lambda,\mu}$ is also self-associate with respect to ε and $\varepsilon\delta$. Similarly $\Phi^{\lambda,\mu}$ will be self-associate with respect to $\varepsilon\delta$ only if it is also self-associate with respect to ε and δ, i.e., only if $\lambda = \mu$.

It will be convenient to explicitly describe the module structure of $\Phi^{\lambda,\mu}$ in terms of the module structures of Φ^λ and Φ^μ. For this, assume $|\lambda| = k$, $|\mu| = n-k$, and let V_1 and V_2 denote modules for $\mathbf{C}W'_k$ and $\mathbf{C}W'_{n-k}$ with characters $\varphi^{\lambda,\varnothing}$ and $\varphi^{\mu,\varnothing}$. (In case $\lambda, \mu \in DP^-$, assume that V_1 and V_2 have characters $\varphi^{\lambda,\varnothing}_+$ and $\varphi^{\mu,\varnothing}_+$ in the labeling imposed by (7.8).) Recall from Section 7 that we defined a set $\tilde{W}^k$ of coset representatives for

$\tilde{W}_{(k,n-k)}$ in $\tilde{W}_n$. We may thus impose a $\mathbf{C}W'_n$-module structure on $\mathbf{C}\tilde{W}^k \otimes \mathbf{C}^2 \otimes V_1 \otimes V_2$ by defining

$$w(w_0 \otimes v_0 \otimes v_1 \otimes v_2) = \delta(w_2)\, w'_0 \otimes x^{l_1} y^{l_2} v_0 \otimes w_1 v_1 \otimes w_2 v_2, \tag{8.1}$$

for all $w \in \tilde{W}_n$, where $w_0, w'_0 \in \tilde{W}^k$, $ww_0 = w'_0(w_1, w_2)$, $\varepsilon(w_1) = (-1)^{l_1}$, $\varepsilon(w_2) = (-1)^{l_2}$, and x and y denote the anticommuting involutions in GL_2 used to represent ρ, as in Section 7. In case $\lambda, \mu \in DP^-$, this defines the module structure of $\Phi^{\lambda,\mu}$; if $\lambda, \mu \in DP^+$, then this is isomorphic to the sum of two copies of $\Phi^{\lambda,\mu}$.

In case $\lambda, \mu \in DP^+$, then both V_1 and V_2 are self-associate with respect to ε; we will denote their ε-associators by S_1 and S_2. For this case, we will also need to define a pair of involutions $E_\pm^{\lambda,\mu}$ on $\mathbf{C}\tilde{W}^k \otimes \mathbf{C}^2 \otimes V_1 \otimes V_2$ via

$$E_\pm^{\lambda,\mu} = \frac{1}{\sqrt{2}}(1 \otimes x \otimes 1 \otimes S_2 \pm 1 \otimes y \otimes S_1 \otimes 1).$$

It is easy to see that both involutions commute with the action of $\tilde{W}_n$. In fact, the algebra of endomorphisms that commute with $\tilde{W}_n$ is generated by $1 \otimes y \otimes S_1 \otimes 1$ and $1 \otimes x \otimes 1 \otimes S_2$. It follows in particular that the eigenspaces of $E_+^{\lambda,\mu}$ and $E_-^{\lambda,\mu}$ all carry the module structure of $\Phi^{\lambda,\mu}$ for the case $\lambda, \mu \in DP^+$.

The ε-Associators

Define an involution $S^{\lambda,\mu}$ on $\mathbf{C}\tilde{W}^k \otimes \mathbf{C}^2 \otimes V_1 \otimes V_2$ via

$$S^{\lambda,\mu}(w_0 \otimes v_0 \otimes v_1 \otimes v_2) = \begin{cases} i\varepsilon(w_0)\, w_0 \otimes yxv_0 \otimes v_1 \otimes v_2 & \text{if } \lambda, \mu \in DP^- \\ \varepsilon(w_0)\, w_0 \otimes v_0 \otimes S_1 v_1 \otimes S_2 v_2 & \text{if } \lambda, \mu \in DP^+. \end{cases}$$

THEOREM 8.1. (a) *For $\lambda, \mu \in DP^-$, the ε-associator of $\Phi^{\lambda,\mu}$ is $S^{\lambda,\mu}$. For $\lambda, \mu \in DP^+$, the ε-associator is the restriction of $S^{\lambda,\mu}$ to either of the eigenspaces of $E_+^{\lambda,\mu}$ (or $E_-^{\lambda,\mu}$).*

(b) *The difference character $\Delta^\varepsilon \varphi^{\lambda,\mu}(w)$ vanishes unless w belongs to a class (α, β) such that $\alpha, \beta \in DP$ and $\lambda \cup \mu = \alpha \cup \beta$. In that case, we have*

$$\Delta^\varepsilon \varphi^{\lambda,\mu}(\alpha, \beta) = 2^{\ell(\lambda \cap \mu)} \varepsilon_{\alpha,\beta}^{\lambda,\mu}\, i^{(n - \ell(\lambda) - \ell(\mu))/2} \sqrt{z_\lambda z_\mu}.$$

Proof. First consider the case $\lambda, \mu \in DP^-$. Using the notation of (8.1), we have

$$S^{\lambda,\mu} w(w_0 \otimes v_0 \otimes v_1 \otimes v_2)$$
$$= i\varepsilon(w'_0)\, \delta(w_2)\, w'_0 \otimes (yx)\, x^{l_1} y^{l_2} v_0 \otimes w_1 v_1 \otimes w_2 v_2$$
$$wS^{\lambda,\mu}(w_0 \otimes v_0 \otimes v_1 \otimes v_2)$$
$$= i\varepsilon(w_0)\, \delta(w_2)\, w'_0 \otimes x^{l_1} y^{l_2} (yx)\, v_0 \otimes w_1 v_1 \otimes w_2 v_2.$$

Since the (group) commutator of yx and $x^{l_1}y^{l_2}$ is $(-1)^{l_1+l_2}=\varepsilon(w_1w_2)$, it follows that the actions of $S^{\lambda,\mu}w$ and $\varepsilon(w)\,wS^{\lambda,\mu}$ agree, so $S^{\lambda,\mu}$ is indeed the associator.

For part (b), note that the only subspaces $\mathbf{C}w_0\otimes\mathbf{C}^2\otimes V_1\otimes V_2$ that contribute to the trace of $S^{\lambda,\mu}w$ are those for which $w_0=w_0'$; i.e., $w_0^{-1}ww_0=(w_1,w_2)\in\tilde{W}_{(k,n-k)}$. As a further condition for nonzero trace, we must have $\varepsilon(w_1)=\varepsilon(w_2)=-1$, so that $l_1=l_2=1 \bmod 2$ and $(yx)\,x^{l_1}y^{l_2}$ has trace 2. Under these conditions, we obtain

$$\Delta^{\varepsilon}\varphi^{\lambda,\mu}(w)=2i\sum_{w_0\in\tilde{W}^k}\varepsilon(w_0)\,\delta(w_2)\,\varphi_+^{\lambda,\varnothing}(w_1)\,\varphi_+^{\mu,\varnothing}(w_2),\tag{8.2}$$

with the sum restricted to those representatives w_0 for which $w_0^{-1}ww_0\in\tilde{W}_{(k,n-k)}$ and $\varepsilon(w_1)=\varepsilon(w_2)=-1$.

By (7.8), we know that the terms of (8.2) will vanish unless w_1 and w_2 have cycle types λ and μ. Therefore, assuming that w belongs to the W_n-class (α,β), we will have $\Delta^{\varepsilon}\varphi^{\lambda,\mu}(w)=0$ unless $\lambda\cup\mu=\alpha\cup\beta$. In case α or β has a pair of repeated parts of some length r, then parts (d) and (e) of Lemma 2.2 show that there will exist an element $u\in\tilde{W}_n$ with $\varepsilon(u)=(-1)^r$ such that $uwu^{-1}=(-1)^{r-1}w$. In that case, (3.2) would imply $\Delta^{\varepsilon}\varphi^{\lambda,\mu}(w)=0$. Thus, we may assume for the remainder of the proof that w is the canonical representative of some class (α,β) such that $\lambda\cup\mu=\alpha\cup\beta$ and $\alpha,\beta\in DP$.

As in the proof of Theorem 7.6(b), we know that for each permutation π that maps the sequence (α,β) to (λ,μ), there is a corresponding coset of $\tilde{W}_{(k,n-k)}$ for which the elements w_1 and w_2 have cycle types λ and μ. In this case, each pair of odd-length cycles in w that are inverted by π will contribute an odd number of inversions to $w_0=w_0(\pi)$, and each pair of inverted, even-length cycles will also introduce a factor of -1, since these cycles anticommute. For each choice of π we therefore have

$$\begin{aligned}\varepsilon(w_0)\,\delta(w_2)\,\varphi_+^{\lambda,\varnothing}(w_1)\,\varphi_+^{\mu,\varnothing}(w_2)&=\varepsilon_{\alpha,\beta}^{\lambda,\mu}\varphi_+^{\lambda}(\lambda)\,\varphi_+^{\mu}(\mu)\\&=\frac{1}{2i}\varepsilon_{\alpha,\beta}^{\lambda,\mu}i^{(n-\ell(\lambda)-\ell(\mu))/2}\sqrt{z_\lambda z_\mu},\end{aligned}$$

the latter equality being a consequence of (7.8). Since these terms are independent of π, and there are $2^{\ell(\lambda\cap\mu)}=2^{\ell(\alpha\cap\beta)}$ such terms, we thus obtain the claimed formula.

Now consider the case $\lambda,\mu\in DP^+$. Using the notation of (8.1), we have

$$S^{\lambda,\mu}w(w_0\otimes v_0\otimes v_1\otimes v_2)=\varepsilon(w_0')\,\delta(w_2)\,w_0'\otimes x^{l_1}y^{l_2}v_0\otimes S_1w_1v_1\otimes S_2w_2v_2$$

$$wS^{\lambda,\mu}(w_0\otimes v_0\otimes v_1\otimes v_2)=\varepsilon(w_0)\,\delta(w_2)\,w_0'\otimes x^{l_1}y^{l_2}v_0\otimes w_1S_1v_1\otimes w_2S_2v_2.$$

Since $ww_0 = w_0'(w_1, w_2)$ and $w_j S_j = \varepsilon(w_j) S_j w_j$ on V_j, it follows that the actions of $S^{\lambda,\mu}w$ and $\varepsilon(w) wS^{\lambda,\mu}$ agree. Since $S^{\lambda,\mu}$ clearly commutes with $E_{\pm}^{\lambda,\mu}$, it follows that $S^{\lambda,\mu}$ also acts as the associator of $\Phi^{\lambda,\mu}$ on each eigenspace of $E_{\pm}^{\lambda,\mu}$.

The trace of $S^{\lambda,\mu}w$ on $\mathbf{C}\tilde{W}^k \otimes \mathbf{C}^2 \otimes V_1 \otimes V_2$ will either be $2\Delta^\varepsilon \varphi^{\lambda,\mu}(w)$ or else identically zero, depending on whether $S^{\lambda,\mu}$ acts as the same associator on both eigenspaces or as a pair of opposite sign. As we shall see below, the trace of $S^{\lambda,\mu}w$ is not identically zero, so it must be that the former alternative occurs.

To compute this trace, we may restrict our attention to those coset representatives w_0 for which $w_0^{-1}ww_0 = (w_1, w_2) \in \tilde{W}_{(k,n-k)}$, as usual. From the above expression for $S^{\lambda,\mu}w$, we see that as a further condition for nonzero trace, we must have $\varepsilon(w_1) = \varepsilon(w_2) = +1$ (so that $l_1 = l_2 = 0 \bmod 2$). We therefore have

$$\Delta^\varepsilon \varphi^{\lambda,\mu}(w) = \sum_{w_0} \varepsilon(w_0)\, \delta(w_2)\, \Delta^\varepsilon \varphi^{\lambda,\varnothing}(w_1)\, \Delta^\varepsilon \varphi^{\mu,\varnothing}(w_2),$$

with the usual restrictions on w_0. Note that we need not explicitly require $\varepsilon(w_1) = \varepsilon(w_2) = 1$, since the difference characters of φ^λ and φ^μ vanish when $\varepsilon = -1$. The remainder of the proof is now the same as the previous case, except that (7.9) should now be used in place of (7.8). ∎

The remaining associators and difference characters arise only in the case $\lambda = \mu$. Thus, we will assume that $\lambda \in DP$ is a fixed partition of $k = n/2$, and that $V = V_1 = V_2$ carries the $\mathbf{C}W'_{n/2}$-module with character $\varphi^{\lambda,\varnothing}$ (or $\varphi_+^{\lambda,\varnothing}$, if $\lambda \in DP^-$). In the case $\lambda \in DP^+$, we will also write $S = S_1 = S_2$ for the associator of V.

To define the associators in these cases, we will need to make use of an element $u \in \tilde{S}_n$ whose S_n-image is the involution $(1, k+1)(2, k+2) \cdots (k, 2k)$ of Section 6. Note that u is conjugate to either $\sigma_1 \sigma_3 \cdots \sigma_{2k-1}$ or its negative, so we have

$$u^2 = (\sigma_1 \sigma_3 \cdots \sigma_{2k-1})^2 = (-1)^{\binom{k}{2}} = (-1)^{n(n-2)/8}.$$

Of course, there are two possibilities for u, so to make our formulas for the difference characters precise, we need to specify a particular choice for u. For this, note that if $\sigma \in \tilde{S}_n$ is one of the two preimages of the transposition (i, j), then we have $[\sigma] = \pm(\xi_i - \xi_j)/\sqrt{2}$, where $[\cdot]\colon \mathbf{C}W'_n \to \mathscr{C}_n$ denotes the Clifford algebra representation of Section 7. In these terms, we may define u by insisting that

$$[u] = 2^{-k/2}(\xi_1 - \xi_{k+1})(\xi_2 - \xi_{k+2}) \cdots (\xi_k - \xi_{2k}).$$

To evaluate the difference characters at a given element w, we will need to analyze the coset representatives $w_0 \in \tilde{W}^{n/2}$ such that $uw_0^{-1}ww_0 \in$

$\tilde{W}_{(n/2,n/2)}$. These representatives can be obtained as the $\tilde{W}_n$-preimages of the corresponding coset representatives that were determined during the proof of Theorem 6.4(b). It follows that there are no such representatives unless the cycles of w are all of even length; in that case, the representatives may be indexed by subsets I of $\{1, ..., l\}$, where l denotes the number of cycles of w.

To describe the coset representative w_0 indexed by I more precisely, let $w \in \tilde{W}_n$ be the canonical representative of the class $(2\gamma, \varnothing)$ for some partition γ of $k = n/2$. The ith cycle of w has an S_n-image of the form $(2s+1, ..., 2s+2r)$ for suitable r and s (namely, $r = \gamma_i$ and $s = \gamma_1 + \cdots + \gamma_{i-1}$). The membership of i in I (or lack thereof) forces the S_n-image of w_0 to assign the elements $s+j$ and $k+s+j$ $(1 \leqslant j \leqslant r)$ to one of two possible permutations of $2s+1, ..., 2s+2r$, as described in the following table:

	$s+1$	$s+2 \cdots s+r$	$k+s+1$	$k+s+2 \cdots k+s+r$	
$i \in I$	$2s+2$	$2s+4 \cdots 2s+2r$	$2s+1$	$2s+3$	$\cdots 2s+2r-1$
$i \notin I$	$2s+1$	$2s+3 \cdots 2s+2r-1$	$2s+2$	$2s+4$	$\cdots 2s+2r.$

These constraints completely determine the W_n-image of $w_0 = w_0(I)$; we may thus use either of the $\tilde{W}_n$-preimages since the expression $uw_0^{-1}ww_0$ is independent of this choice.

LEMMA 8.2. *Let w be the canonical representative of some class $(2\gamma, \varnothing)$, and define $n(\gamma) = \sum (i-1)\gamma_i$. If $w_0 = w_0(I)$, then $uw_0^{-1}ww_0 = (w_1, w_2)$, where*

$$w_1 w_2 = \begin{cases} (-1)^{n(\gamma)+|\gamma_I|} w_\gamma & \textit{if} \quad \varepsilon(w) = +1 \\ (-1)^{n(n-2)/8+n(\gamma)} w_\gamma & \textit{if} \quad \varepsilon(w) = -1. \end{cases}$$

In either case, w_1 (resp., w_2) has cycle-type γ_I (resp., γ_I^c), and $\varepsilon(w_0) = (-1)^{n(n-2)/8+|\gamma_I|}$.

Proof. The number of inversions in the permutation $2, 4, ..., 2r, 1, 3, ..., 2r-1$ is $\binom{r}{2}+r$; in the permutation $1, 3, ..., 2r-1, 2, 4, ..., 2r$ there are $\binom{r}{2}$ inversions. It follows that the total number of inversions in w_0 is

$$\sum_{i<j} \gamma_i\gamma_j + \sum_i \binom{\gamma_i}{2} + \sum_{i \in I} \gamma_i = \binom{k}{2} + |\gamma_I|,$$

which agrees with the result claimed for $\varepsilon(w_0)$.

Now let $w = x_1 \cdots x_l$ and $w_\gamma = y_1 \cdots y_l \in \tilde{S}_{n/2}$ denote the defining factorizations of the canonical representatives for $(2\gamma, \varnothing)$ and $(\gamma, \varnothing)$ as products of cycles. Assuming that the S_n-image of x_i is the $2r$-cycle $(2s+1, ..., 2s+2r)$, we have

$$[x_i] = 2^{-(2r-1)/2}(\xi_{2s+1} - \xi_{2s+2}) \cdots (\xi_{2s+2r-1} - \xi_{2s+2r}).$$

The involution $(s+1, k+s+1) \cdots (s+r, k+s+r) \in S_n$ has two $\tilde{S}_n$-preimages; we define u_i to be the particular preimage for which

$$[u_i] = 2^{-r/2}(\xi_{s+1} - \xi_{k+s+1}) \cdots (\xi_{s+r} - \xi_{k+s+r}), \tag{8.3}$$

so that $u = u_i \cdots u_l$. Under these circumstances, we claim that

$$\varepsilon(w_0)\, u_i w_0^{-1} x_i w_0 = \begin{cases} (-1)^r (y_i, 1) & \text{if} \quad i \in I \\ (1, y_i) & \text{if} \quad i \notin I. \end{cases} \tag{8.4}$$

To prove this, let us first suppose that $i \in I$. By Lemma 7.2, we have

$$[w_0^{-1} x_i w_0] = \varepsilon(w_0)\, 2^{-(2r-1)/2}\, (\xi_{k+s+1} - \xi_{s+1}) \times \prod_{j=2}^{r} (\xi_{s+j-1} - \xi_{k+s+j})(\xi_{k+s+j} - \xi_{s+j}).$$

After a rearrangement of factors, we obtain

$$[u_i w_0^{-1} x_i w_0] = -\varepsilon(w_0)\, 2^{-(3r-1)/2}\, (\xi_{k+s+1} - \xi_{s+1})^2 \times \prod_{j=2}^{r} (\xi_{k+s+j} - \xi_{s+j})(\xi_{s+j-1} - \xi_{k+s+j})(\xi_{k+s+j} - \xi_{s+j}).$$

The presence of the extra "$-$" sign is accounted for by the fact that we have reversed the internal order of each of the terms in (8.3) (thus introducing a factor of $(-1)^r$), and every term in (8.3) except the first was moved past an odd number of the terms in $[w_0^{-1} x_i w_0]$ to reach its position in the above expression (thus introducing a factor of $(-1)^{r-1}$). Since the jth term simplifies to $-2(\xi_{s+j-1} - \xi_{s+j})$, we therefore have

$$\varepsilon(w_0)[u_i w_0^{-1} x_i w_0] = (-1)^r\, 2^{-(r-1)/2}\, (\xi_{s+1} - \xi_{s+2}) \cdots (\xi_{s+r-1} - \xi_{s+r}),$$

which may be identified as the $\mathscr{C}_n$-image of $(-1)^r (y_i, 1)$. The proof for the case $i \notin I$ is similar; we omit the details.

Since the commutator of u_i and $w_0^{-1} x_j w_0$ is $\varepsilon(u_i) = (-1)^{y_i}$ for $i \neq j$, it follows that

$$u w_0^{-1} w w_0 = (-1)^{n(y)} \prod_{i=1}^{l} u_i w_0^{-1} x_i w_0.$$

By repeated application of (8.4), we thus obtain $uw_0^{-1}ww_0=(w_1, w_2)$, where w_1 has cycle-type γ_I, w_2 has cycle-type γ_I^c, and $w_1w_2=\varepsilon(w_0)^l(-1)^{n(\gamma)+|\gamma_I|}w_\gamma$. The claimed results now follow from the formula for $\varepsilon(w_0)$ and the fact that $\varepsilon(w)=(-1)^l$. ∎

The δ-Associators

Define a pair of endomorphisms $T_\pm^\lambda$ on $\mathbf{C}\tilde{W}^{n/2}\otimes\mathbf{C}^2V\otimes V$ via

$$T_\pm^\lambda(w_0\otimes v_0\otimes v_1\otimes v_2)=\frac{1}{\sqrt{2}}\,w_0u\otimes(x\pm y)\,v_0\otimes v_2\otimes v_1.$$

We claim that if $\lambda\in DP^-$, then either T_+^λ or T_-^λ is a δ-associator (modulo scalar factors), according to the parity of $n/2$. If $\lambda\in DP^+$, then the restriction of either T_+^λ or T_-^λ to the respective eigenspaces of either $E_+^{\lambda,\mu}$ or $E_-^{\lambda,\mu}$ will be a δ-associator. In this latter case, it will develop that the restrictions yield two associators of opposite sign, so it will be more natural to treat $E_\pm^{\lambda,\mu}T_\pm^\lambda$ as the associator, since it will act as the same endomorphism on both eigenspaces. To account for these various possibilities, we therefore define

$$T^\lambda=\begin{cases} i^{n/4}T_+^\lambda & \text{if}\quad n=0 \bmod 4 \text{ and } \lambda\in DP^- \\ i^{n/4}E_+^{\lambda,\lambda}T_+^\lambda & \text{if}\quad n=0 \bmod 4 \text{ and } \lambda\in DP^+ \\ i^{(n-2)/4}T_-^\lambda & \text{if}\quad n=2 \bmod 4 \text{ and } \lambda\in DP^- \\ i^{(n-2)/4}E_-^{\lambda,\lambda}T_-^\lambda & \text{if}\quad n=2 \bmod 4 \text{ and } \lambda\in DP^+ \end{cases}$$

THEOREM 8.3. (a) *If $\lambda\in DP^-$, then T^λ is the δ-associator of $\Phi^{\lambda,\lambda}$; if $\lambda\in DP^+$, then the δ-associator of $\Phi^{\lambda,\lambda}$ is the restriction of T^λ to the eigenspaces of $E_+^{\lambda,\lambda}$ (for $n=0 \bmod 4$) or $E_-^{\lambda,\lambda}$ (for $n=2 \bmod 4$).*

(b) *The only nonzero values of the difference characters are*

$$\Delta^\delta\varphi^{\lambda,\lambda}(2\lambda,\varnothing)=(-i)^{(\ell(\lambda)+1)/2}(-1)^{n(\lambda)}2^{\ell(\lambda)}\sqrt{z_\lambda}$$

if $n=0 \bmod 4$ *and* $\lambda\in DP^-$

$$\Delta^\delta\varphi^{\lambda,\lambda}(\varnothing,2\lambda)=i^{(n-\ell(\lambda))/2}(-1)^{n(\lambda)}2^{\ell(\lambda)}\sqrt{z_\lambda}$$

if $n=0 \bmod 4$ *and* $\lambda\in DP^+$

$$\Delta^\delta\varphi^{\lambda,\lambda}(\varnothing,2\lambda)=i^{(n-\ell(\lambda)+2)/2}(-1)^{n(\lambda)}\,2^{\ell(\lambda)}\sqrt{z_\lambda}$$

if $n=2 \bmod 4$ *and* $\lambda\in DP^-$

$$\Delta^\delta\varphi^{\lambda,\lambda}(2\lambda,\varnothing)=(-i)^{(\ell(\lambda)-1)/2}(-1)^{n(\lambda)}2^{\ell(\lambda)}\sqrt{z_\lambda}$$

if $n=2 \bmod 4$ *and* $\lambda\in DP^+$.

Proof. Since $(T^\lambda_\pm)^2 = u^2 = (-1)^{n(n-2)/8}$, it is easy to see that $i^{n/4} T^\lambda_\pm$ is an involution for $n = 0 \bmod 4$, and that $i^{(n-2)/4} T^\lambda_\pm$ is an involution for $n = 2 \bmod 4$.

For $w \in \tilde{W}_n$, we have

$$\begin{aligned} T^\lambda_\pm w(w_0 \otimes v_0 \otimes v_1 \otimes v_2) \\ = \frac{1}{\sqrt{2}} \delta(w_2)\, w_0' u \otimes (x \pm y)\, x^{l_1} y^{l_2} v_0 \otimes w_2 v_2 \otimes w_1 v_1, \end{aligned} \tag{8.5}$$

using the notation of (8.1). By Lemma 7.2, we know that $u^{-1}(\sigma_j, 1)\, u = \varepsilon(u)(1, \sigma_j)$ and $u^{-1}(\tau_j, 1)\, u = (1, \tau_j)$, so for $w_1, w_2 \in \tilde{W}_{n/2}$, we have

$$u^{-1}(w_1, w_2)\, u = \varepsilon(u)^{l_1 + l_2}\, (1, w_1)(w_2, 1).$$

Since $w w_0 u = w_0' u \cdot u^{-1}(w_1, w_2)\, u$, it follows that

$$\begin{aligned} w T^\lambda_\pm (w_0 \otimes v_0 \otimes v_1 \otimes v_2) \\ = \frac{1}{\sqrt{2}} \delta(w_1)\, \varepsilon(u)^{l_1 + l_2}\, w_0' u \otimes y^{l_1} x^{l_2} (x \pm y)\, v_0 \otimes w_2 v_2 \otimes w_1 v_1. \end{aligned}$$

Observe that $\delta(w) = \delta(w_1 w_2)$ and $\varepsilon(u) = (-1)^{n/2}$. Since $y^{l_1} x^{l_2}(x + y) = (x + y)\, x^{l_1} y^{l_2}$, a comparison of $T^\lambda_\pm w$ and $w T^\lambda_\pm$ shows that $w T^\lambda_+ = \delta(w)\, T^\lambda_+ w$ in case $n = 0 \bmod 4$. Similarly, when $n = 2 \bmod 4$, we have $\varepsilon(u) = -1$ and $y^{l_1} x^{l_2}(x - y) = (-1)^{l_1 + l_2}\, (x - y)\, x^{l_1} y^{l_2}$, so $w T^\lambda_- = \delta(w)\, T^\lambda_- w$. It follows that T^λ is indeed the associator for $\lambda \in DP^-$. In the case $\lambda \in DP^+$, one also needs to verify that $E^{\lambda,\lambda}_+$ commutes with T^λ_+, and that $E^{\lambda,\lambda}_-$ commutes with T^λ_-; we leave this easy exercise to the reader. Once verified, it follows that T^λ acts as the associator on the eigenspaces of $E^{\lambda,\lambda}_+$ (for $n = 0 \bmod 4$) or $E^{\lambda,\lambda}_-$ (for $n = 2 \bmod 4$).

To evaluate the difference characters, first consider the case in which $n = 0 \bmod 4$ and $\lambda \in DP^-$. Note that $(x + y)\, x^{l_1} y^{l_2}$ has nonzero trace (equal to 2) only if $\varepsilon(w_1 w_2) = -1$ (i.e., $l_1 + l_2 = 1 \bmod 2$). Since the trace of $v_1 \otimes v_2 \mapsto B v_2 \otimes A v_1$ is $\operatorname{tr}(AB)$, (8.5) implies

$$\Delta^\delta \varphi^{\lambda,\lambda}(w) = i^{n/4} \sqrt{2} \sum_{w_0 \in \tilde{W}^{n/2}} \delta(w_2)\, \varphi^{\lambda,\varnothing}_+(w_1 w_2), \tag{8.6}$$

with the sum restricted so that $u w_0^{-1} w w_0 = (w_1, w_2) \in \tilde{W}_{n/2,n/2}$ and $\varepsilon(w_1 w_2) = -1$. Since $\varepsilon(u) = 1$, we may ignore the restriction on $\varepsilon(w_1 w_2)$ by insisting that $\varepsilon(w) = -1$.

Given that $\varepsilon(w) = -1$, Lemma 2.2(b) implies that if w has any negative cycles, then there will exist an element z with $\delta(z) = -1$ that centralizes w. Such z will also exist if w has any positive cycles of odd length, by

Lemma 2.2(c). Therefore, we may assume that w is the canonical representative of some class $(2\gamma, \varnothing)$, since (3.2) would otherwise imply $\Delta^\delta\varphi^{\lambda,\lambda}(w)=0$.

According to Lemma 8.2, the choices for w_0 are indexed by $I\subset\{1, ..., \ell(\gamma)\}$, and for each such choice we have

$$\varphi_+^{\lambda,\varnothing}(w_1 w_2) = (-1)^{n/4+n(\gamma)}\varphi_+^\lambda(\gamma),$$

using the fact that $n(n-2)/8 = n/4 \bmod 2$ when $n/2$ is even. However, by (7.8), we know $\varphi_+^\lambda(\gamma)=0$ unless $\lambda=\gamma$. Under these circumstances, (8.6) becomes

$$\Delta^\delta\varphi^{\lambda,\lambda}(2\lambda, \varnothing) = \sum_I (-i)^{(\ell(\lambda)+1)/2}(-1)^{n(\lambda)}\sqrt{z_\lambda}.$$

Since each summand is independent of I and there are $2^{\ell(\lambda)}$ such summands, the claimed formula follows.

Now consider the case in which $n=2 \bmod 4$ and $\lambda\in DP^-$. Note that $(x-y)\,x$ has trace 2, and $(x-y)\,y$ has trace -2; otherwise, $(x-y)\,x^{l_1}y^{l_2}$ has trace zero. The analogue of (8.6) can thus be written in the form

$$\Delta^\delta\varphi^{\lambda,\lambda}(w) = i^{(n-2)/4}\sqrt{2}\sum_{w_0\in\tilde W^{n/2}} \varepsilon\delta(w_2)\,\varphi_+^{\lambda,\varnothing}(w_1 w_2), \tag{8.7}$$

with the usual restrictions on w_0, w_1, and w_2; i.e., $uw_0^{-1}ww_0 = (w_1, w_2)\in \tilde W_{(n/2,n/2)}$ and $\varepsilon(w_1w_2) = -1$. Since $\varepsilon(u) = -1$ in this case, we may ignore the restriction on $\varepsilon(w_1w_2)$ by insisting that $\varepsilon(w) = +1$.

Given that $\varepsilon(w) = +1$, Lemma 2.2(b) implies that if w has any negative cycles of odd length, then there will exist an element z with $\delta(z) = -1$ that centralizes w. Similarly, by use of Lemmas 2.2(a) and 2.2(c), one can show that if w has any positive cycles, there will exist either a similar z or else a z' with $\delta(z') = +1$ that anticommutes with w. In any of these cases, (3.2) would imply $\Delta^\delta\varphi^{\lambda,\lambda}(w)=0$. We may therefore assume that w is the canonical representative of some class $(\varnothing, 2\gamma)$.

As in the previous case, the choices for w_0 are indexed by subsets I of $\{1, ..., \ell(\gamma)\}$. Since $w = w_{\varnothing,2\gamma}$ is no longer in $\tilde S_n$, we need to modify the conclusion of Lemma 8.2 slightly. Assuming $w_0 = w_0(I)$, we still have $uw_0^{-1}ww_0 = (w_1, w_2)$ where the $\tilde S_{n/2}$-image of w_1w_2 is $(-1)^{n(\gamma)+|\gamma_I|}w_\gamma$, but w_1 and w_2 are now products of *negative* cycles. In particular, the $W_{n/2}$-class of w_2 is $(\varnothing, \gamma_I^c)$, and so we have

$$\varepsilon\delta(w_2)\,\varphi_+^{\lambda,\varnothing}(w_1w_2) = (-1)^{n(\gamma)+|\gamma_I|+|\gamma_I^c|}\,\varphi_+^{\lambda,\varnothing}(w_1w_2) = -(-1)^{n(\gamma)}\varphi_+^\lambda(\gamma).$$

Again, (7.8) implies that $\Delta^\delta\varphi^{\lambda,\lambda}(w)=0$ unless $\lambda=\gamma$; in that case, (8.7) becomes

$$\Delta^\delta\varphi^{\lambda,\lambda}(\varnothing, 2\lambda) = \sum_I i^{(n-\ell(\lambda)+2)/2}(-1)^{n(\lambda)}\sqrt{z_\lambda}.$$

Since each summand is independent of I and there are $2^{\ell(\lambda)}$ such summands, the claimed formula follows.

Now consider the cases with $\lambda \in DP^+$. Although (8.6) and (8.7) still represent the traces of $T^\lambda_+ w$ and $T^\lambda_- w$, these expressions are identically zero in this case, since $\varphi^\lambda = 0$ on the $\varepsilon = -1$ portion of $\tilde{S}_{n/2}$. Hence, the two associators obtained by restricting $T^\lambda_\pm$ to the eigenspaces of $E^{\lambda,\lambda}_\pm$ are of opposite sign. Since we defined T^λ as (a scalar multiple of) $E^{\lambda,\lambda}_\pm T^\lambda_\pm$, we thus obtain $2\Delta^\delta\varphi^{\lambda,\lambda}(w)$ as the trace of $T^\lambda w$ on $\mathbf{C}\tilde{W}^{n/2} \otimes \mathbf{C}^2 \otimes V \otimes V$.

To determine the trace of $T^\lambda w$ under these circumstances, note that

$$\begin{aligned} &E^{\lambda,\lambda}_\pm T^\lambda_\pm w(w_0 \otimes v_0 \otimes v_1 \otimes v_2) \\ &\quad = \tfrac{1}{2}\delta(w_2)\, w_0' u \otimes [x(x \pm y)\, x^{l_1} y^{l_2} v_0 \otimes w_2 v_2 \otimes S w_1 v_1 \\ &\qquad + y(x \pm y)\, x^{l_1} y^{l_2} v_0 \otimes S w_2 v_2 \otimes w_1 v_1], \end{aligned}$$

using "+" throughout for $n = 0 \mod 4$ and "−" throughout for $n = 2 \mod 4$. From this it is clear that we must have $\varepsilon(w_1 w_2) = +1$ (so that $l_1 = l_2 \mod 2$) to obtain a nonzero trace. This forces $\varepsilon(w) = +1$ for $n = 0 \mod 4$ and $\varepsilon(w) = -1$ for $n = 2 \mod 4$. By the same reasoning used in the previous cases, we may therefore assume that w is the canonical representative of some class $(\varnothing, 2\gamma)$ or $(2\gamma, \varnothing)$, according to whether $n = 0 \mod 4$ or $n = 2 \mod 4$. We leave the reader to evaluate the traces of $x(x \pm y)\, x^{l_1} y^{l_2}$ and $y(x \pm y)\, x^{l_1} y^{l_2}$, and thereby obtain

$$\Delta^\delta \varphi^{\lambda,\lambda}(w) = \begin{cases} \sum\limits_{w_0} i^{n/4} \varepsilon\delta(w_2)\, \Delta^\varepsilon \varphi^{\lambda,\varnothing}(w_1 w_2) & \text{if } n = 0 \mod 4 \\ \sum\limits_{w_0} i^{(n-2)/4} \delta(w_2)\, \Delta^\varepsilon \varphi^{\lambda,\varnothing}(w_1 w_2) & \text{if } n = 2 \mod 4, \end{cases}$$

with the usual restrictions on w_0. Here one also needs to make use of the fact that the trace of $v_1 \mapsto w_2 v_2 \otimes S w_1 v_1$ is $\Delta^\varepsilon \varphi^{\lambda,\varnothing}(w_1 w_2)$, and the trace of $v_1 \otimes v_2 \mapsto S w_2 v_2 \otimes w_1 v_1$ is $\varepsilon(w_2)\, \Delta^\varepsilon \varphi^{\lambda,\varnothing}(w_1 w_2)$.

The remainder of the proof is similar to the previous cases. Use (7.9) to deduce that $\Delta^\delta \varphi^{\lambda,\lambda}(w)$ will vanish unless $\lambda = \gamma$, and then apply Lemma 8.2 to evaluate each of the terms appearing in the above sums. ∎

The $\varepsilon\delta$-Associators

Of course, the $\varepsilon\delta$-associator U^λ of $\Phi^{\lambda,\lambda}$ can be obtained (modulo scalar multiples) by composing $S^{\lambda,\lambda}$ and T^λ. From their definitions, it is easy to see that the commutator of $S^{\lambda,\lambda}$ and T^λ is $-\varepsilon(u)$ for $\lambda \in DP^-$, and $\varepsilon(u)$ for $\lambda \in DP^+$. Hence, $S^{\lambda,\lambda}$ and T^λ will anticommute when $n = 2 \mod 4$ and $\lambda \in DP^+$, or when $n = 0 \mod 4$ and $\lambda \in DP^-$. By Theorem 3.2, it follows that $\Theta \otimes \Phi^{\lambda,\lambda}$ will split into four irreducible constituents precisely in these latter two cases; i.e., when $\ell(\lambda)$ is odd.

In case $S^{\lambda,\lambda}$ and T^{λ} commute (i.e., $\ell(\lambda)$ is even), our choices for U^{λ} are $\pm S^{\lambda,\lambda}T^{\lambda}$. Hence, to maintain the validity of (3.4), we will define U^{λ} to be $S^{\lambda,\lambda}T^{\lambda}$. In case $S^{\lambda,\lambda}$ and T^{λ} anticommute, our choices for U^{λ} are $\pm iS^{\lambda,\lambda}T^{\lambda}$. Since the difference characters of Θ described in Proposition 6.1 were obtained by choosing associators for ε, δ, and $\varepsilon\delta$ of the form S_θ, T_θ, and $iS_\theta T_\theta$, it follows that we may preserve the validity of (3.6) by choosing U^{λ} to be $-iS^{\lambda,\lambda}T^{\lambda}$. In summary, we define the $\varepsilon\delta$-associators via

$$U^{\lambda}=\begin{cases} i^{(n-4)/4}S^{\lambda,\lambda}T^{\lambda}_{+} & \text{if } \ n=0 \bmod 4 \text{ and } \lambda\in DP^{-}\\ i^{n/4}E^{\lambda,\lambda}_{+}S^{\lambda,\lambda}T^{\lambda}_{+} & \text{if } \ n=0 \bmod 4 \text{ and } \lambda\in DP^{+}\\ i^{(n-2)/4}S^{\lambda,\lambda}T^{\lambda}_{-} & \text{if } \ n=2 \bmod 4 \text{ and } \lambda\in DP^{-}\\ i^{(n-6)/4}E^{\lambda,\lambda}_{-}S^{\lambda,\lambda}T^{\lambda}_{-} & \text{if } \ n=2 \bmod 4 \text{ and } \lambda\in DP^{+}. \end{cases}$$

THEOREM 8.4. *The only nonzero values of $\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}$ are*

$$\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(\varnothing,2\lambda)=i^{(n-\ell(\lambda)+3)/2}(-1)^{n(\lambda)}2^{\ell(\lambda)}\sqrt{z_\lambda}$$
$$\textit{if} \quad n=0 \bmod 4 \textit{ and } \lambda\in DP^{-}$$
$$\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(2\lambda,\varnothing)=(-i)^{\ell(\lambda)/2}(-1)^{n(\lambda)}\,2^{\ell(\lambda)}\sqrt{z_\lambda}$$
$$\textit{if} \quad n=0 \bmod 4 \textit{ and } \lambda\in DP^{+}$$
$$\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(2\lambda,\varnothing)=(-i)^{(\ell(\lambda)-2)/2}\,(-1)^{n(\lambda)}\,2^{\ell(\lambda)}\sqrt{z_\lambda}$$
$$\textit{if} \quad n=2 \bmod 4 \textit{ and } \lambda\in DP^{-}$$
$$\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(\varnothing,2\lambda)=i^{(n-\ell(\lambda)+1)/2}(-1)^{n(\lambda)}\,2^{\ell(\lambda)}\sqrt{z_\lambda}$$
$$\textit{if} \quad n=2 \bmod 4 \textit{ and } \lambda\in DP^{+}.$$

Proof. First consider the case $\lambda\in DP^{-}$. Using the notation of (8.1), we have

$$S^{\lambda,\lambda}T^{\lambda}_{\pm}\,w(w_0\otimes v_0\otimes v_1\otimes v_2)$$
$$=\frac{i}{\sqrt{2}}\delta(w_2)\,\varepsilon(w_0'u)\,w_0'u\otimes yx(x\pm y)\,x^{l_1}y^{l_2}v_0\otimes w_2v_2\otimes w_1v_1.$$

We note that the respective traces of $yx(x+y)x$, $yx(x+y)y$, $yx(x-y)x$, and $yx(x-y)y$ are -2, 2, 2, and 2; in all other cases, $yx(x+y)x^{l_1}y^{l_2}$ has trace zero. It follows that

$$\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w)=\begin{cases}\sqrt{2}\sum_{w_0} i^{n/4}\varepsilon(w_0w_1)\,\delta(w_2)\,\varphi_{+}^{\lambda,\varnothing}(w_1w_2) & \text{if } \ n=0 \bmod 4\\ \sqrt{2}\sum_{w_0} i^{(n+2)/4}\varepsilon(w_0)\,\delta(w_2)\,\varphi_{+}^{\lambda,\varnothing}(w_1w_2) & \text{if } \ n=2 \bmod 4,\end{cases} \tag{8.8}$$

with the usual restrictions on w_0, w_1, and w_2; i.e., $uw_0^{-1}ww_0=(w_1, w_2)$ and $\varepsilon(w_1 w_2)=-1$. We may drop the latter condition by insisting that $\varepsilon(w)=-1$ for $n=0 \bmod 4$ and $\varepsilon(w)=+1$ for $n=2 \bmod 4$. In case $\varepsilon(w)=-1$, one may apply Lemma 2.2 to show that if w has any positive cycles, or any odd-length negative cycles, then there will exist elements z with $\varepsilon\delta(z)=-1$ that centralize w, and hence force $\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w)$ to vanish. One can similarly use Lemma 2.2 to show that if $\varepsilon(w)=+1$, then $\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w)$ will vanish unless all cycles of w are positive and of even length. We may therefore assume that w is the canonical representative of either $(\varnothing, 2\gamma)$ or $(2\gamma, \varnothing)$, according to whether $n=0 \bmod 4$ or $n=2 \bmod 4$.

As in the proof of Theorem 8.3, the only coset representatives that contribute to the trace of $U^\lambda w$ are those of the form $w_0(I)$. For these cosets, Lemma 8.2 implies

$$\begin{aligned}
&\varepsilon(w_0 w_1)\,\delta(w_2)\,\varphi_+^{\lambda,\varnothing}(w_1 w_2)\\
&\quad=(-1)^{n(n-2)/8+|\gamma/|}\cdot(-1)^{|\gamma/|+1}\cdot(-1)^{n(n-2)/8+n(\gamma)}\varphi_+^\lambda(\gamma)\\
&\quad=-(-1)^{n(\gamma)}\varphi_+^\lambda(\gamma)
\end{aligned}$$

for the case $\varepsilon(w)=-1$, and similarly,

$$\begin{aligned}
\varepsilon(w_0)\,\delta(w_2)\,\varphi_+^{\lambda,\varnothing}(w_1 w_2)&=(-1)^{n(n-2)/8+|\gamma/|}\cdot(-1)^{n(\gamma)+|\gamma/|}\varphi_+^\lambda(\gamma)\\
&=(-1)^{(n-2)/4+n(\gamma)}\varphi_+^\lambda(\gamma)
\end{aligned}$$

for the case $\varepsilon(w)=+1$. In either case, $\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w)$ will vanish unless $\lambda=\gamma$. Under these circumstances, (7.8) implies that the Ith summand in (8.8) is

$$\begin{aligned}
&i^{(n-\ell(\lambda)+3)/2}(-1)^{n(\lambda)}\sqrt{z_\lambda} &&\text{if}\quad n=0 \bmod 4\\
&(-i)^{(\ell(\lambda)-2)/2}(-1)^{n(\lambda)}\sqrt{z_\lambda} &&\text{if}\quad n=2 \bmod 4.
\end{aligned}$$

Since each summand is independent of I, the claimed formulas follow.

Now consider $\lambda\in DP^+$. In these cases, we have

$$\begin{aligned}
&E_\pm^{\lambda,\lambda}S^{\lambda,\lambda}T_\pm^\lambda w(w_0\otimes v_0\otimes v_1\otimes v_2)\\
&\quad=\tfrac{1}{2}\delta(w_2)\,\varepsilon(w_0'u)\,w_0'u\otimes[x(x\pm y)\,x^{l_1}y^{l_2}v_0\otimes Sw_2v_2\otimes w_1v_1\\
&\qquad\pm y(x\pm y)\,x^{l_1}y^{l_2}v_0\otimes w_2v_2\otimes Sw_1v_1],
\end{aligned}$$

using "+" throughout for $n=0 \bmod 4$ and "−" throughout for $n=2 \bmod 4$. From this it is clear that we must have $\varepsilon(w_1 w_2)=+1$ (and hence $l_1=l_2 \bmod 2$) to obtain a nonzero trace. This forces $\varepsilon(w)=+1$ for $n=0 \bmod 4$ and $\varepsilon(w)=-1$ for $n=2 \bmod 4$. By the same reasoning used above, we may therefore assume that w is the canonical representative of some class $(2\gamma, \varnothing)$ or $(\varnothing, 2\gamma)$, according to whether $n=0 \bmod 4$ or

$n = 2 \bmod 4$. We leave the reader to evaluate the traces of $x(x \pm y)\, x^{l_1} y^{l_2}$ and $y(x \pm y)\, x^{l_1} y^{l_2}$, and thereby obtain the following analogue of (8.8):

$$\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w) = \begin{cases} \sum_{w_0} i^{n/4}\delta(w_2)\,\varepsilon(w_0)\,\Delta^{\varepsilon}\varphi^{\lambda,\varnothing}(w_1 w_2) & \text{if} \quad n = 0 \bmod 4 \\ \sum_{w_0} i^{(n-6)/4}\varepsilon\delta(w_2)\,\varepsilon(w_0)\,\Delta^{\varepsilon}\varphi^{\lambda,\varnothing}(w_1 w_2) & \text{if} \quad n = 2 \bmod 4. \end{cases}$$

Here one needs to make use of the fact that the trace of $U^{\lambda}w$ on $\mathbf{C}\tilde{W}^{n/2} \otimes \mathbf{C}^2 \otimes V \otimes V$ is twice the value of $\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w)$. The remainder of the proof proceeds as usual. Apply (7.9) to deduce that $\Delta^{\varepsilon\delta}\varphi^{\lambda,\lambda}(w)$ will vanish unless $\lambda = \gamma$, and then use Lemma 8.2 to evaluate each of the terms appearing in the above sums. ∎

The following result is a corollary of Theorems 3.2, 8.1, 8.3, and 8.4; it summarizes the overall structure of the irreducible representations for the factor set $[-1, -1, +1]$.

COROLLARY 8.5. *The irreducible spin representations of* $W_n([-1, -1, +1])$ *can be indexed so that the submodules of* $\Theta \otimes \Phi^{\lambda,\mu}$ *are labelled by the unordered pair* $\{\lambda, \mu\}$, *where* $\lambda, \mu \in DP$. *In the following table,* $n_{\lambda,\mu}$ *denotes the number of modules indexed by* $\{\lambda, \mu\}$, $m_{\lambda,\mu}$ *denotes their multiplicity in* $\Theta \otimes \Phi^{\lambda,\mu}$, *and* $o_{\lambda,\mu}$ *denotes the size of their* L_n*-orbit.*

$n_{\lambda,\mu}$	$o_{\lambda,\mu}$	$m_{\lambda,\mu}$	
1	1	2	*if* $\lambda = \mu$ *and* $\ell(\lambda)$ *is even*
4	1	1	*if* $\lambda = \mu$ *and* $\ell(\lambda)$ *is odd*
2	2	1	*if* $n - \ell(\lambda) - \ell(\mu)$ *is even and* $\lambda \neq \mu$
1	4	1	*if* $n - \ell(\lambda) - \ell(\mu)$ *is odd.*

9. THE FACTOR SETS $[\pm 1, \pm 1, -1]$

The orthogonal group O_n has a double cover Pin(n) that can be represented as a subgroup of the multiplicative group of the Clifford algebra $\mathscr{C}_n$. It follows that one may obtain a projective representation of any subgroup of O_n (e.g., a reflection group) by suitably restricting any linear representation of $\mathscr{C}_n$. The representations of Weyl groups that arise in this fashion were first investigated in a series of papers by Morris (e.g., see [Mo2]).

To describe the representations of W_n that this technique produces, define an algebra homomorphism $\mathbf{C}W_n^\gamma \to \mathscr{C}_n$ for the factor set $\gamma = [-1, -1, -1]$ by setting

$$\sigma_j \mapsto \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}), \qquad \tau \mapsto \xi_1.$$

The fact that this does generate an algebra homomorphism is an immediate consequence of the defining relations (1.2). We may thus obtain a projective representation of W_n for the factor set $[-1, -1, -1]$ by composing $\mathbf{C}W_n^\gamma \to \mathscr{C}_n$ with any linear representation $\mathscr{C}_n \to \mathrm{End}(V)$.

If n is even, the algebra $\mathscr{C}_n$ is simple and thus isomorphic to the matrix algebra $M(2^{n/2})$. For odd n, we have $\mathscr{C}_n \cong M(2^{(n-1)/2}) \otimes M(2^{(n-1)/2})$. (See [St1] or [ABS] for an explicit isomorphism.) It follows that $\mathscr{C}_n$ has one irreducible representation for even n, and two such representations for odd n. Since $\mathbf{C}W_n^\gamma \to \mathscr{C}_n$ is surjective, the above construction yields one irreducible spin representation of $W_n([-1, -1, -1])$ for even n and two for odd n. We will denote these representations generically by Ψ; in situations where we need to emphasize that there are two choices for odd n, we will write $\Psi_\pm$.

Let ψ denote the character of Ψ.

THEOREM 9.1. *For $\alpha \in OP$ and $\beta \in EP$, we have*

$$\psi(\alpha, \beta) = \begin{cases} (-1)^{(n-\ell(\alpha))/2 + \ell(\beta)} 2^{(\ell(\alpha)+\ell(\beta))/2} & \text{if } n \text{ is even} \\ (-1)^{(n-\ell(\alpha))/2} 2^{(\ell(\alpha)+\ell(\beta)-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

The only other nonzero values of $\psi(\alpha, \beta)$ occur when n is odd, $\alpha = \varnothing$, and β is arbitrary. In these cases, we have $\psi_\pm(\varnothing, \beta) = \pm i^{(n-1)/2}(-1)^{n(\beta)} 2^{(\ell(\beta)-1)/2}$.

Proof. Take the elements $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$ as a basis of $\mathscr{C}_n$, where $I = \{i_1 < \cdots < i_k\}$ ranges over the subsets of $\{1, ..., n\}$. For $w \in \mathbf{C}W_n^\gamma$, let $\xi_I(w)$ denote the coefficient of ξ_I in the $\mathscr{C}_n$-image of w. By Proposition 3.1 of [St1] (a description of the irreducible character(s) of $\mathscr{C}_n$), we have

$$\begin{aligned} \psi(w) &= 2^{n/2}\xi_\varnothing(w) && \text{if } n \text{ is even} \\ \psi_\pm(w) &= 2^{(n-1)/2}\xi_\varnothing(w) \pm (2i)^{(n-1)/2}\zeta(w) && \text{if } n \text{ is odd,} \end{aligned} \tag{9.1}$$

where $\zeta = \xi_1 \cdots \xi_n$ denotes the basis element corresponding to $I = \{1, ..., n\}$.

To evaluate $\xi_\varnothing(w)$, first consider the case in which w is a positive, canonical k-cycle. If the S_n-image of w is $(j+1, j+2, ..., j+k)$, then the $\mathscr{C}_n$-image of w is

$$2^{-(k-1)/2}(\xi_{j+1} - \xi_{j+2}) \cdots (\xi_{j+k-1} - \xi_{j+k}),$$

so there is no constant term unless k is odd. In that case, only one of the 2^{k-1} terms that arise in the expansion of the above expression has a nonzero constant term. This single term is of the form $(-\xi_{j+2})(\xi_{j+2})(-\xi_{j+4})(\xi_{j+4})\cdots$, so we conclude that $\xi_{\varnothing}(w)=(-1/2)^{(k-1)/2}$.

To evaluate the constant term for a negative cycle, first note that the $\mathscr{C}_n$-image of τ_j is $(-1)^{j-1}\xi_j$. This follows by induction on j and the fact that $(\xi_2-\xi_1)\,\xi_1(\xi_2-\xi_1)=-2\xi_2$. Therefore, the $\mathscr{C}_n$-image of a negative, canonical k-cycle w will be of the form

$$(-1)^{k+j-1}\,2^{-(k-1)/2}(\xi_{j+1}-\xi_{j+2})\cdots(\xi_{j+k-1}-\xi_{j+k})\,\xi_{j+k}, \tag{9.2}$$

so there will be no constant term unless k is even. In that case, it is easy to see that $\xi_{\varnothing}(w)=(-1)^{j+k/2-1}\,(1/2)^{(k-1)/2}$.

For the general case, assume that w is the canonical representative of some class (α,β), and let $w=w_1\cdots w_l$ be its defining factorization as a product of canonical cycles. Note that $\xi_{\varnothing}(w)=\xi_{\varnothing}(w_1)\cdots\xi_{\varnothing}(w_l)$. We may therefore assume $\alpha\in OP$ and $\beta\in EP$ since the above analysis shows that $\xi_{\varnothing}(w)$ would otherwise be zero. Under these circumstances, the ith negative cycle of w (the one of length β_i) includes the element $\tau_{j+\beta_i}$ as part of its defining factorization, where $j=|\alpha|+\beta_1+\cdots\beta_{i-1}$. However, since $\alpha\in OP$ and $\beta\in EP$, it follows that $j=\ell(\alpha)=n \bmod 2$, and so we have

$$\begin{aligned}\xi_{\varnothing}(w)&=\prod_{i=1}^{\ell(\alpha)}(-1/2)^{(\alpha_i-1)/2}\prod_{i=1}^{\ell(\beta)}(-1)^{n+\beta_i/2-1}(1/2)^{(\beta_i-1)/2}\\&=(-1)^{(n-1)\ell(\beta)}(-1)^{(n-\ell(\alpha))/2}(1/2)^{(n-\ell(\alpha)-\ell(\beta))/2}.\end{aligned}$$

To evaluate $\zeta(w)$, observe that the $\mathscr{C}_n$-image of w is a product of $n-\ell(\alpha)$ linear terms, whereas ζ is a product of n such terms. Hence, in order for $\zeta(w)$ to be nonzero, we must have $\alpha=\varnothing$ (and thus $\beta\notin EP$, assuming n is odd). It is easy to see that the coefficient of $\xi_{j+1}\cdots\xi_{j+k}$ in (9.2) is $(-1)^{k+j-1}\,(1/2)^{(k-1)/2}$, so we have

$$\zeta(w)=\prod_{i=1}^{\ell(\beta)}(-1)^{\beta_1+\cdots+\beta_i-1}\,(1/2)^{(\beta_i-1)/2}. \tag{9.3}$$

For odd n, we have $\beta_1+\cdots+\beta_i-1=\beta_{i+1}+\beta_{i+2}+\cdots \bmod 2$, so the above expression simplifies to $\zeta(w)=(-1)^{n(\beta)}\,(1/2)^{(n-\ell(\beta))/2}$. The claimed results now follow from (9.1) and our formulas for $\xi_{\varnothing}(w)$ and $\zeta(w)$. ∎

The tensor products $\Psi\otimes X^{\lambda,\mu}$ and $\Psi\otimes\Phi^{\lambda,\mu}$ permit us to easily create a large supply of projective representations for the factor sets $[-1,-1,-1]$ and $[+1,-1,-1]$, respectively. The following results show that all irreducible representations for these factor sets are of this form.

THEOREM 9.2. *The irreducible spin representations of $W_n([-1, -1, -1])$ are $\Psi \otimes X^{\lambda, \varnothing}$ (for n even) and $\Psi_\pm \otimes X^{\lambda, \varnothing}$ (for n odd), where λ ranges over the partitions of n.*

Proof. By separating a partition into its odd and even parts, it is easy to see that for any cycle type μ, there is a corresponding W_n-class of the form (OP, EP) whose S_n-image is of type μ. Since Theorem 9.1 shows that $\psi(\alpha, \beta)$ is nonzero for all classes of the type (OP, EP), it follows that for even n, the characters $\psi\chi^{\lambda, \varnothing}$ are linearly independent. For odd n, Theorem 9.1 shows that $(\psi_+ - \psi_-)(\varnothing, \beta)$ is also nonzero for all partitions β of n, so the characters $\psi_\pm \chi^{\lambda, \varnothing}$ are likewise linearly independent.

Conversely, Theorem 2.1 implies that the split classes for the factor set $[-1, -1, -1]$ are indexed by (OP, EP), along with $(\varnothing, P)$ for odd n. Thus, the characters listed above form a basis for the space spanned by spin characters, and so we need only to prove that these characters are irreducible. For this, it suffices to show that $\psi\chi^{\lambda, \varnothing}$ is of norm 1 (for even n) and that $(\psi_+ + \psi_-)\chi^{\lambda, \varnothing}$ is of norm $\sqrt{2}$ for odd n. If (α, β) indexes a W_n-class such that $\alpha \cap \beta = \varnothing$, then the order $z_{\alpha, \beta}$ of the W_n-centralizer common to this class is $2^{\ell(\alpha) + \ell(\beta)} z_{\alpha \cup \beta}$. Therefore, for the case of even n, Theorem 9.1 implies

$$\begin{aligned} \|\psi\chi^{\lambda, \varnothing}\|^2 &= \sum_{\substack{\alpha \in OP \\ \beta \in EP}} 2^{\ell(\alpha) + \ell(\beta)} z_{\alpha, \beta}^{-1} |\chi^\lambda(\alpha \cup \beta)|^2 \\ &= \sum_\mu z_\mu^{-1} |\chi^\lambda(\mu)|^2 = \|\chi^\lambda\|_{S_n}^2 = 1. \end{aligned}$$

The case for odd n follows similarly. ∎

THEOREM 9.3. *For even n, the irreducible spin representations of $W_n([+1, -1, -1])$ are $\Psi \otimes \Phi^{\lambda, \varnothing}$ $(\lambda \in DP^+)$ and $\Psi \otimes \Phi_\pm^{\lambda, \varnothing}$ $(\lambda \in DP^-)$. For odd n, the irreducible spin representations are $\Psi_\pm \otimes \Phi^{\lambda, \varnothing}$ $(\lambda \in DP^+)$ and $\Psi_\pm \otimes \Phi_\pm^{\lambda, \varnothing}$ $(\lambda \in DP^-)$.*

Proof. Following the previous argument, recall that the spin characters φ^λ of $\tilde{S}_n$ are supported on the classes indexed by OP and DP^-. Note that there exist $\tilde{W}_n$-classes of the form (OP, EP) (and of the form $(\varnothing, P)$ for odd n) whose $\tilde{S}_n$-image is of each of these types. Since Theorem 9.1 shows that ψ is nonzero on each of these classes, we may deduce that for even n (resp., odd n), the characters $\psi\varphi^{\lambda, \varnothing}$ and $\psi\varphi_\pm^{\lambda, \varnothing}$ (resp., $\psi_\pm \varphi^{\lambda, \varnothing}$ and $\psi_\pm \gamma_\pm^{\lambda, \varnothing}$) are linearly independent.

Conversely, Theorem 2.1 shows that the split classes for the factor set $[+1, -1, -1]$ are indexed by $(OP, \varnothing)$ and the $\varepsilon = -1$ portion of (DOP, DEP), along with $(\varnothing, OP)$ and $(\varnothing, DP^-)$ for odd n. Since there is an obvious one-to-one correspondence between DP^- and the $\varepsilon = -1$

portion of (DOP, DEP), it follows that the number of characters we have constructed is correct; i.e., they span the entire space of spin characters for this factor set. To complete the proof, we only need to verify that these characters are irreducible. For example, assuming that n is even and $\lambda \in DP^-$, Theorem 9.1 implies

$$\|\psi(\varphi_+^{\lambda,\varnothing} + \varphi_-^{\lambda,\varnothing})\|^2 = 4 \sum_{\alpha \in OP} z_\alpha^{-1} |\varphi_+^\lambda(\alpha)|^2 = \|\varphi_+^\lambda + \varphi_-^\lambda\|_{S_n}^2 = 2,$$

so $\psi\varphi_+^{\lambda,\varnothing}$ and $\psi\varphi_-^{\lambda,\varnothing}$ must both be irreducible. The other cases can be treated similarly. ∎

Since $\varepsilon\chi^{\lambda,\varnothing} = \chi^{\lambda',\varnothing}$, it follows that $\Psi \otimes X^{\lambda,\varnothing}$ is self-associate with respect to ε if and only if $\lambda \in SC$. Similarly, $\Psi \otimes \Phi^{\lambda,\varnothing}$ is self-associate with respect to ε if and only if $\lambda \in DP^+$. In either case, the ε-associator is clearly of the form $1 \otimes S$, where S denotes the ε-associator for X^λ (as in Section 6) or Φ^λ (as in Section 8). We therefore have

$$\begin{aligned} \Delta^\varepsilon(\psi\chi^{\lambda,\varnothing}) &= \psi\Delta^\varepsilon\chi^{\lambda,\varnothing} \qquad \text{if} \quad \lambda \in SC \\ \Delta^\varepsilon(\psi\varphi^{\lambda,\varnothing}) &= \psi\Delta^\varepsilon\varphi^{\lambda,\varnothing} \qquad \text{if} \quad \lambda \in DP^+. \end{aligned}$$

Since we already have explicit formulas for $\Delta^\varepsilon\chi^{\lambda,\varnothing}$ and $\Delta^\varepsilon\varphi^{\lambda,\varnothing}$ (cf. Theorems 6.2 and 8.1), we now have explicit formulas for these difference characters too.

Next consider the $\varepsilon\delta$-associates. Since the classes of the form (OP, EP) all lie in the $\varepsilon\delta = +1$ portion of W_n, Theorem 9.1 implies that $\varepsilon\delta \otimes \Psi \cong \Psi$ for even n. Also, since the classes of the form $(\varnothing, P)$ lie in the $\varepsilon\delta = -1$ portion of W_n (for odd n), we have $\varepsilon\delta \otimes \Psi_+ \cong \Psi_-$. Hence, $\Psi \otimes X^{\lambda,\varnothing}$ and $\Psi \otimes \Phi^{\lambda,\varnothing}$ are self-associate with respect to $\varepsilon\delta$ if and only if n is even. In that case, ζ anticommutes with each generator ξ_i, and hence, with the $\mathscr{C}_n$-images of both σ_j and τ_j. Since $\zeta^2 = (-1)^{n/2}$, it follows that (the representing matrix for) $i^{n/2}\zeta$ is the $\varepsilon\delta$-associator of Ψ. From this we may conclude that $i^{n/2}\zeta \otimes 1$ is the $\varepsilon\delta$-associator for both $\Psi \otimes X^{\lambda,\varnothing}$ and $\Psi \otimes \Phi^{\lambda,\varnothing}$, and hence

$$\Delta^{\varepsilon\delta}(\psi\chi^{\lambda,\varnothing}) = \chi^{\lambda,\varnothing}\Delta^{\varepsilon\delta}\psi \qquad \text{and} \qquad \Delta^{\varepsilon\delta}(\psi\varphi^{\lambda,\varnothing}) = \varphi^{\lambda,\varnothing}\Delta^{\varepsilon\delta}\psi.$$

PROPOSITION 9.4. *For even n, we have $\Delta^{\varepsilon\delta}\psi(w) = 0$ unless w belongs to a W_n-class of the form $(\varnothing, \beta)$. In that case,*

$$\Delta^{\varepsilon\delta}\psi(\varnothing, \beta) = (-i)^{n/2}(-1)^{n(\beta) - \ell(\beta)} 2^{\ell(\beta)/2}.$$

Proof. For any $\xi \in \mathscr{C}_n$, the coefficient of $\xi_\varnothing$ in $i^{n/2}\zeta\xi$ is the same as the coefficient of ζ in $(-i)^{n/2}\xi$. Since $\xi_\varnothing$ is the only basis element with nonzero trace in any representation of $\mathscr{C}_n$ (cf. (9.1) and [St1, Sect. 3]), it follows that

$$\Delta^{\varepsilon\delta}\psi(w) = (-2i)^{n/2}\,\zeta(w).$$

In the proof of Theorem 9.1, we showed that $\zeta(w)=0$ unless w belongs to a W_n-class of the form $(\varnothing, \beta)$. Assuming w is the canonical representative of such a class, (9.3) implies

$$\zeta(w) = (-1)^{n(\beta)-\ell(\beta)}(1/2)^{(n-\ell(\beta))/2},$$

using the fact that $\beta_1+\cdots+\beta_i=\beta_{i+1}+\beta_{i+2}+\cdots$ mod 2 for even n. The claimed formula for $\Delta^{\varepsilon\delta}\psi$ now follows. ∎

For the character δ, we have $\delta\otimes(\Psi\otimes X^{\lambda,\varnothing})\cong(\varepsilon\delta\otimes\Psi)\otimes(\varepsilon\otimes X^{\lambda,\varnothing})$, and similarly for $\Psi\otimes\Phi^{\lambda,\varnothing}$. Therefore, the only representations that are self-associate with respect to δ are those that are already self-associate with respect to both ε and $\varepsilon\delta$. For $\Psi\otimes X^{\lambda,\varnothing}$ this requires $\lambda\in SC$ and n even; for $\Psi\otimes\Phi^{\lambda,\varnothing}$ this requires $\lambda\in DP^+$ and n even. Since the associators $1\otimes S$ and $i^{n/2}\zeta\otimes 1$ obviously commute, we may use $i^{n/2}\zeta\otimes S$ as the δ-associator, and so we have

$$\Delta^{\delta}(\psi\chi^{\lambda,\varnothing}) = \Delta^{\varepsilon\delta}\psi\cdot\Delta^{\varepsilon}\chi^{\lambda,\varnothing} \qquad \text{if} \quad \lambda\in SC \text{ and } n \text{ is even}$$

$$\Delta^{\delta}(\psi\varphi^{\lambda,\varnothing}) = \Delta^{\varepsilon\delta}\psi\cdot\Delta^{\varepsilon}\varphi^{\lambda,\varnothing} \qquad \text{if} \quad \lambda\in DP^+ \text{ and } n \text{ is even.}$$

Since we have now constructed all of the associators and difference characters for the factor sets $[-1,-1,-1]$ and $[+1,-1,-1]$, we may now obtain the corresponding results for the factor sets $[-1,+1,-1]$ and $[+1,+1,-1]$ as a corollary of Theorem 3.2. The following results summarize the overall structure of these representations.

COROLLARY 9.5. *The irreducible spin representations of $W_n([-1,+1,-1])$ can be labeled by $\mathbf{Z}_2$-orbits of the form $\{\lambda,\lambda'\}$, where λ ranges over the partitions of n. The index $\{\lambda,\lambda'\}$ labels submodules of $\Theta\otimes\Psi\otimes X^{\lambda,\varnothing}$. In the following table, n_λ denotes the number of modules indexed by $\{\lambda,\lambda'\}$, m_λ denotes their multiplicity in $\Theta\otimes\Psi\otimes X^{\lambda,\varnothing}$, and o_λ denotes the size of the L_n-orbit of $\Psi\otimes X^{\lambda,\varnothing}$.*

n_λ	o_λ	m_λ	
1	1	2	*if* $\lambda\in SC$ *and n is even*
2	2	1	*if* $\lambda\in SC$ *and n is odd, or* $\lambda\notin SC$ *and n is even*
1	4	1	*if* $\lambda\notin SC$ *and n is odd.*

COROLLARY 9.6. *The irreducible spin representations of* $W_n([+1, +1, -1])$ *can be labeled by partitions of n in DP, so that* λ *indexes submodules of* $\Theta \otimes \Psi \otimes \Phi^{\lambda, \varnothing}$. *In the following table,* n_λ *denotes the number of modules indexed by* λ, m_λ *denotes their multiplicity in* $\Theta \otimes \Psi \otimes \Phi^{\lambda, \varnothing}$, *and* o_λ *denotes the size of the* L_n*-orbit of* $\Psi \otimes \Phi^{\lambda, \varnothing}$.

n_λ	o_λ	m_λ	
1	1	2	*if* $\lambda \in DP^+$ *and n is even*
2	2	1	*if* $\lambda \in DP^+$ *and n is odd, or* $\lambda \in DP^-$ *and n is even*
1	4	1	*if* $\lambda \in DP^-$ *and n is odd.*

APPENDIX: THE WEYL GROUP $W(D_n)$

Assume $n \geqslant 4$ and define $s_0 = ts_1t$, so that the reflections $s_0, s_1, ..., s_{n-1}$ generate $W(D_n)$, the Weyl group of the root system D_n. If we fix a particular factor set $\alpha = [\varepsilon_1, \varepsilon_2, \varepsilon_3]$ for W_n and define $\sigma_0 = \tau\sigma_1\tau$, then the subalgebra of $\mathbf{C}W_n^\alpha$ generated by $\sigma_0, ..., \sigma_{n-1}$ will be a twisted group algebra for $W(D_n)$. To classify the algebras that arise in this fashion, recall that the Coxeter presentation for $W(D_n)$ consists of the usual relations for $s_1, ..., s_{n-1}$ (as Coxeter generators for S_n), along with

$$s_0^2 = 1, \qquad (s_0 s_i)^2 = 1 \ (i \neq 2), \qquad (s_0 s_2)^3 = 1.$$

By comparison, as a consequence of (1.2), we have

$$(\sigma_0\sigma_1)^2 = (\tau\sigma_1)^4 = \varepsilon_3, \qquad (\sigma_0\sigma_i)^2 = \varepsilon_1 \ (i > 2), \qquad (\sigma_0\sigma_2)^3 = \pm 1.$$

By substituting $\sigma_0 \to \pm\sigma_0$ (if necessary), we may insist that $(\sigma_0\sigma_2)^3 = 1$. Since these relations are independent of ε_2, we conclude that there are four distinct factor sets for $W(D_n)$ that arise as restrictions of W_n-factor sets.

As representatives for these four factor sets, we may choose $[+1, +1, +1]$, $[-1, +1, +1]$, $[-1, -1, -1]$, and $[+1, -1, -1]$. The corresponding W_n-representations were constructed in Sections 5, 7, and 9; namely, $X^{\lambda,\mu}$, $\Phi^{\lambda,\mu}$, $\Psi \otimes X^{\lambda,\varnothing}$, and $\Psi \otimes \Phi^{\lambda,\varnothing}$. Of course, the pairs of δ-associate representations in this list will be isomorphic when restricted to $W(D_n)$ (or a suitable double cover), and the representations that are self-associate with respect to δ will split into two irreducible representations; namely the eigenspaces of the δ-associators constructed in Sections 6, 8, and 9. The self-associate cases are summarized in Table II.

The Schur multiplier of $W(D_n)$ is $\mathbf{Z}_2^2$ for $n \geqslant 5$, and $\mathbf{Z}_2^3$ for $n = 4$ [IY, H]. It follows that in the case $n \geqslant 5$, all of the factor sets for $W(D_n)$ are restric-

TABLE II

Factor set	Representation	Self-associate case	Difference character
$[+1, +1, +1]$	$X^{\lambda,\mu}$	$\lambda=\mu$	$\Delta^{\delta}\chi^{\lambda,\lambda}$
$[-1, +1, +1]$	$\Phi^{\lambda,\mu}$	$\lambda=\mu$	$\Delta^{\delta}\varphi^{\lambda,\lambda}$
$[-1, -1, -1]$	$\Psi\otimes X^{\lambda,\varnothing}$	$\lambda\in SC$, n even	$\Delta^{\varepsilon\delta}\psi\Delta^{\varepsilon}\chi^{\lambda,\varnothing}$
$[+1, -1, -1]$	$\Psi\otimes\Phi^{\lambda,\varnothing}$	$\lambda\in DP^{+}$, n even	$\Delta^{\varepsilon\delta}\psi\Delta^{\varepsilon}\varphi^{\lambda,\varnothing}$

tions of W_n-factor sets, and so all projective representations of $W(D_n)$ are of the form described above. If $n=4$, the group of diagram automorphisms for $W(D_n)$ permit arbitrary permutation of the generators s_0, s_1, and s_3. By applying the same permutations to $\sigma_0, \sigma_1, \sigma_3$, we thus obtain the eight twisted groups algebras, corresponding to the eight choices

$$(\sigma_0\sigma_1)^2=\pm 1, \qquad (\sigma_1\sigma_3)^2=\pm 1, \qquad (\sigma_0\sigma_3)^2=\pm 1.$$

Since the factor sets $[+1, +1, +1]$ and $[-1, -1, -1]$ are invariant under these permutations, it follows that the projective representations of $W(D_4)$ corresponding to the four new factor sets can be obtained by applying the two 3-cycles of $\{\sigma_0, \sigma_1, \sigma_3\}$ to the representations for the factor sets $[-1, +1, +1]$ and $[+1, -1, -1]$.

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