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A linear representation of the mapping class group $\mathcal{M}$ and the theory of winding numbers

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Abstract

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This paper describes a linear representation Φ of the mapping class group $\mathcal{M}$, of an orientable surface S with one boundary component. The representation Φ extends the symplectic representation, and is defined for surfaces of arbitrary genus $g>1$. The main tools used to define Φ are crossed homomorphisms $e_X : \mathcal{M} \mapsto H^1(S;\mathbb{Z})$ which are defined using nonvanishing vector fields X on S, and the theory of winding numbers of curves on surfaces described by Chillingworth in [1, 2]. These crossed homomorphisms were essentially described by Morita in [6]. A geometric interpretation of Φ is then given. If T_1S denotes the unit tangent bundle of S, then Φ records the action of $\mathcal{M}$ on $H_1(T_1S;\mathbb{Z})$. The kernel of Φ is then characterized using knowledge of the crossed homomorphisms e_X. If matrix entries are taken modulo $2g-2$, the representation Φ factors through the mapping class group $\bar{\mathcal{M}}$ of a closed orientable surface of genus $g>1$. Thus Φ induces representations of $\bar{\Phi}_n$ of $\bar{\mathcal{M}}$ for any $n \mid 2g-2$. The $\bar{\Phi}_n$ were discovered by Sipe in [7, 8], and it is noted that her characterization of the image of $\bar{\Phi}_n$ carries over to the integer valued case. The structure found in characterizing ker Φ is then used to study ker $\bar{\Phi}_n$. In particular, it is shown that a quotient of ker $\bar{\Phi}_n$ is a semidirect product for each even n dividing $2g-2$.

Keywords: Crossed homomorphism, winding numbers, mapping class group.

AMS (MOS) Subj. Class.: Primary 57M99; secondary 55M25, 57N07.

Introduction

This paper describes a linear representation Φ of the mapping class group $\mathcal{M}$ of an orientable surface S with one boundary component. Let $H_1 = H_1(S;\mathbb{Z})$. Then $\mathcal{M}$ acts on H_1 and this action is given by the well-known symplectic representation

$$\rho : \mathcal{M} \mapsto \mathrm{Sp}(2g;\mathbb{Z}),$$

where g is the genus of S. The Torelli group, $\mathcal{I}$, is the subgroup of $\mathcal{M}$ acting trivially on H_1; in other words, $\mathcal{I} = \ker \rho$. The representation Φ extends ρ, and is defined for surfaces of genus $g > 1$. The main tools used to define Φ are crossed homomorphisms $e_x : \mathcal{M} \mapsto H^1(S; \mathbb{Z})$ which are defined using nonvanishing vector fields X on S, and winding numbers of curves on surfaces described by Chillingworth in [1, 2]. Given a nonvanishing, smooth vector field X on S, and a curve γ on S, the winding number of γ with respect to X can be intuitively defined as the number of times the tangent vector to γ rotates with respect to the X-vector as γ is traversed once positively. The notion of winding numbers is used, in turn, to define crossed homomorphisms $e_x : \mathcal{M} \mapsto H^1(S; \mathbb{Z})$. Intuitively, for $f \in \mathcal{M}$, the class $e_x(f) \in H^1(S; \mathbb{Z})$ measures how f changes the winding numbers of homology classes with respect to X. The fact that the e_x are crossed homomorphisms is precisely what allows for the definition of representations Φ_x of $\mathcal{M}$. It should be noted that, even though $e_x(f)$ is well defined on homology classes, the winding numbers of homologous curves are not necessarily equal. Hence the action of $\mathcal{I}$ on winding numbers is not trivial.

The use of vector fields in the definition of Φ_x suggests a geometric interpretation. Let T_1S be the unit tangent bundle of S, and $\tilde{H}_1 = H_1(T_1S; \mathbb{Z})$. Given any $f \in \mathcal{M}$, its derivative Df acts on $\tilde{H}_1$, and one would expect a representation of $\mathcal{M}$ in terms of this action. The difficulty in finding such a representation arises in choosing a basis for $\tilde{H}_1$, and then calculating the action of Df on this basis. The vector field X will be used to lift curves on S to curves in T_1S, and the crossed homomorphism e_x facilitates the necessary calculations. Theorem 2.2 then shows that, for any $f \in \mathcal{M}$, $\Phi_x(f)$ gives the action of Df on $\tilde{H}_1$ with respect to a particular basis. A different choice of vector field corresponds to a change of basis for $\tilde{H}_1$. Thus the representations Φ_x are all conjugate, and are considered a single representation Φ.

Since Φ extends the symplectic representation ρ, it is immediate that $\ker \Phi \subset \mathcal{I}$. Moreover, the fact that the action of $\mathcal{I}$ on winding numbers is nontrivial implies that $\ker \Phi$ is a proper subgroup of $\mathcal{I}$. In order to understand $\ker \Phi$, then, it is necessary to study $\Phi|_{\mathcal{I}}$. The representation Φ is calculated on an infinite set of generators for $\mathcal{I}$, giving rise to a characterization of $\ker \Phi$.

Let $\bar{S}$ be the closed surface obtained from S by attaching a 2-disc along its boundary, and $\bar{\mathcal{M}}$ its mapping class group. Let $\bar{\rho} : \bar{\mathcal{M}} \mapsto \mathrm{Sp}(2g; \mathbb{Z})$ denote the corresponding symplectic representation, and $\bar{\mathcal{I}} = \ker \bar{\rho}$. The groups $\mathcal{M}$ and $\bar{\mathcal{M}}$ are closely related. In particular, if $T_1\bar{S}$ is the unit tangent bundle of $\bar{S}$, then Johnson showed in [5] that the following sequence is exact:

$$1 \to \pi_1(T_1\bar{S}) \to \mathcal{M} \xrightarrow{\text{pr}} \bar{\mathcal{M}} \to 1. \tag{1.1}$$

Moreover, the subgroup $\pi_1(T_1\bar{S})$ of $\mathcal{M}$ is actually a subgroup of $\mathcal{I}$, giving the exact sequence

$$1 \to \pi_1(T_1\bar{S}) \to \mathcal{I} \to \bar{\mathcal{I}} \to 1. \tag{1.2}$$

A natural question arises. Does the representation Φ factor through $\bar{\mathcal{M}}$? The calculations of $\Phi|_{\mathcal{I}}$ show that Φ factors through $\bar{\mathcal{M}}$ when the matrices $\Phi(f)$ are taken with entries in $\mathbb{Z}_n$ rather than $\mathbb{Z}$ for any $n \mid 2g-2$ (where g is the genus of S). Let Φ_n denote the representation Φ with matrix entries taken in $\mathbb{Z}_n$. For $n \mid 2g-2$, then, the representations Φ_n induce representations $\bar{\Phi}_n$ of $\bar{\mathcal{M}}$. Sipe discovered the representations $\bar{\Phi}_n$ in [7], and characterized the image of $\bar{\Phi}_n$ in [8]. It is noted that Sipe's characterization of im $\bar{\Phi}_n$ carries over to im Φ. Following the notation in [7, 8], let $N_{g,n}$ be the congruence subgroup of level n in $\mathrm{Sp}(2g; \mathbb{Z})$, and $G_{g,n} = \ker \bar{\Phi}_n$. Sipe shows that $\bar{\rho}(G_{g,n}) = N_{g,n}$. The characterization of ker Φ is used to describe $\bar{\mathscr{C}}_{g,n} = G_{g,n} \cap \bar{\mathscr{I}}$, obtaining the exact sequence

$$1 \to \bar{\mathscr{C}}_{g,n} \to G_{g,n} \to N_{g,n} \to 1. \tag{1.3}$$

Moreover, it is shown that, for n even and dividing $2g-2$, a quotient of $G_{g,n} = \ker \bar{\Phi}_n$ is a semidirect product. It is interesting to note that in [8] Sipe shows that im $\bar{\Phi}_n$ is a semidirect product for n odd, and that a quotient of ker $\bar{\Phi}_n$ is a semidirect product for n even.

The paper is organized as follows. Section 1 is preliminary in nature, describing the tools needed to define Φ. Section 2 provides the main treatment of the representation Φ, and Theorem 2.2 gives the desired geometric interpretation of Φ_x. Knowledge of the crossed homomorphisms e_x is used to calculate Φ on generators of $\mathscr{I}$; thereby obtaining a characterization of ker Φ. The calculations are then furthered to more general diffeomorphisms in $\mathscr{I}$. The relationship between the representations $\bar{\Phi}_n$ of $\bar{\mathcal{M}}$ discovered by Sipe and the representation Φ of $\mathcal{M}$ is studied in Section 3. In particular, the results of Section 2 are used to reveal more of the structure of the groups $G_{g,n} = \ker \bar{\Phi}_n$.

Remark. The representation Φ was actually found in a somewhat different context from that described in this paper. Squier, in [9], describes a method of defining two-parameter representations of Artin groups. The group $\mathcal{M}$ admits a presentation with "Artin group" and two extra relations (see [10]). The question of when these two extra relations are satisfied was asked. This happens precisely when the specialization $t = -s^{-1}$ is made, where (t, s) are the parameters in Squier's work. In that case, it turned out that the remaining parameter occured only trivially, and the resulting representation was Φ. From this point of view, Theorem 2.2 is surprising indeed.

1. Preliminaries and notation

In this section the necessary background information and definitions are given. Mapping class groups are discussed in Section 1.1, with an emphasis on the Torelli group. Section 1.2 deals with the unit tangent bundle of a surface with one boundary component and conventions about vector fields. Winding numbers are the topic of Section 1.3. Here the crossed homomorphisms e_x are defined, and some of their properties discussed. This material, although it may be the least familiar to the reader, is the most important in what is to come. Here is a summary of the notation introduced so far:

- $\bar{S}$ a closed orientable surface of genus $g > 1$,
- $S = \bar{S} - 2$-disc,
- $\mathcal{M}$, $\bar{\mathcal{M}}$ the respective mapping class groups,
- $H_1 = H_1(S; \mathbb{Z}) = H_1(\bar{S}; \mathbb{Z})$; $H^1 = H^1(S; \mathbb{Z}) = H^1(\bar{S}; \mathbb{Z})$,
- ρ, $\bar{\rho}$ the respective symplectic representations, with kernels $\mathcal{I}$, $\bar{\mathcal{I}}$,
- T_1S, $T_1\bar{S}$ the respective unit tangent bundles,
- $\tilde{H}_1 = H_1(T_1S; \mathbb{Z})$,
- X an arbitrary, smooth, nonvanishing vector field on S,
- e_x the crossed homomorphism defined by X,
- Φ_x the representation defined using e_x,
- Φ the representation given by any vector field,
- Φ_n the representation Φ with matrix entries taken in $\mathbb{Z}_n$,
- $\bar{\Phi}_n$ the induced representations of $\bar{\mathcal{M}}$,
- $N_{g,n}$ the congruence subgroup of level n in $\mathrm{Sp}(2g; \mathbb{Z})$,
- $G_{g,n} = \ker \bar{\Phi}_n$; $\mathcal{C}_{g,n} = G_{g,n} \cap \bar{\mathcal{I}}$.

Other notation will be introduced; however, this reference may be helpful.

1.1.

Given S, $\bar{S}$ as above, the curves $\{\alpha_i, \beta_i\}_{i=1,\dots,g}$, pictured in Fig. 1(a) will be called the standard generators for both $\pi_1(S, *)$ and $\pi_1(\bar{S}, *)$. A collection of simple closed curves $\{a_i, b_i\}_{i=1,\dots,g}$ will be called a symplectic basis for H_1 if $\langle a_i, a_j \rangle = 0 = \langle b_i, b_j \rangle$, and $\langle a_i, b_j \rangle = \delta_{ij}$, where $\langle\, ,\rangle$ denotes the intersection pairing in H_1. The projections $\{a_i, b_i\}$ of the curves $\{\alpha_i, \beta_i\}$ form a symplectic basis.

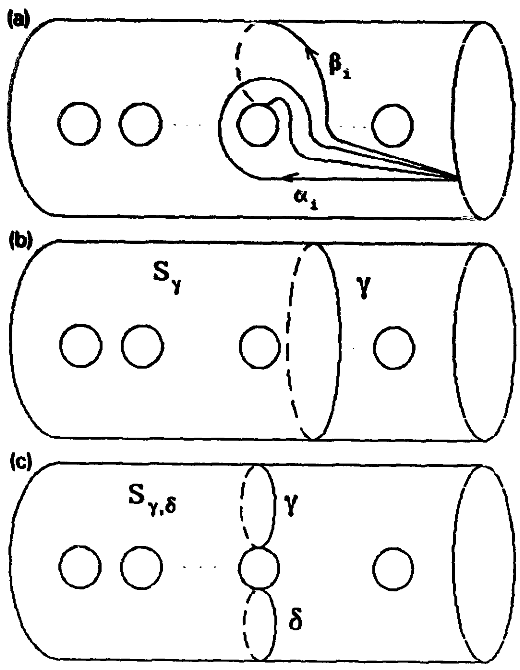

Fig. 1.

Recall that the representation Φ, which will be described, extends the symplectic representation; therefore, the kernel of Φ is contained in $\mathcal{I}$. In order to understand ker Φ, it is necessary to know more about the structure of the group $\mathcal{I}$. Toward this end, two special types of curves on S are distinguished which will give rise to important classes of maps in $\mathcal{I}$.

Definition 1.1.1. A simple closed curve γ on S is a *bounding simple closed curve* (BSCC) if it bounds a subsurface $S_\gamma \subset S$. The genus of S_γ is the genus $g(\gamma)$ of γ (see Fig. 1(b)).

Definition 1.1.2. A pair of disjoint, homologous, simple closed curves (γ, δ) is called a *bounding pair* (BP) if γ is not homologous to 0. Then the pair (γ, δ) bound a subsurface $S_{\gamma,\delta}$, and the genus of the bounding pair is the genus of $S_{\gamma,\delta}$ (see Fig. 1(c)).

By conventions, the subsurface S_γ (respectively $S_{\gamma,\delta}$) is chosen to be the subsurface which γ (respectively (γ, δ)) bounds that does not contain the boundary of S. The curves just defined give rise to the following diffeomorphisms in $\mathcal{I}$.

Definition 1.1.3. A Dehn twist about a BSCC γ is called a *BSCC map* and denoted T_γ.

Definition 1.1.4. A *bounding pair map* (BP map) is comprised of opposite twists $T_\gamma T_\delta^{-1}$ about a BP (γ, δ). The genus of a BP map is the genus of the associated BP.

Johnson showed in [3] that, for $g \geq 3$, $\mathcal{I}$ is actually generated by BP maps of genus one. To derive the corresponding result for $\bar{\mathcal{I}}$, $g \geq 3$, one notes that $\bar{\mathcal{I}}$ is a quotient of $\mathcal{I}$ (see exact sequences (1.1), (1.2)). Thus $\bar{\mathcal{I}}$ is also generated by BP maps of genus one, for $g \geq 3$. Moreover, in the process of proving the exactness of sequence (1.2), Johnson shows that $\pi_1(T_1\bar{S}) \subset \mathcal{I}$ is generated by BP maps of maximal genus; i.e., of genus $g-1$. This fact will be used in determining how to obtain representations of $\bar{\mathcal{M}}$ from Φ.

The case $g = 2$ is a bit different. In this case, Johnson showed that $\mathcal{I}$ is generated by BP maps of genus one together with BSCC maps. Note that when $g = 2$, any BP map is of maximal genus, thus $\pi_1(T_1\bar{S})$ is the group of BP maps. The subgroup of $\mathcal{I}$ (respectively $\bar{\mathcal{I}}$) generated by all BSCC maps will be denoted by $\mathcal{K}$ (respectively $\bar{\mathcal{K}}$). It will be shown that $\mathcal{K} \subset \ker \Phi$; therefore, BP maps of genus one will be of primary importance when calculating $\Phi|_{\mathcal{I}}$ for all $g \geq 2$.

1.2.

Some properties of the unit tangent bundle, which will be needed later, are reviewed in this paragraph. Let S have some Riemannian structure, inducing a norm $\| \ \|_p$

on each fiber TS_p of the tangent bundle TS. Then T_1S is the collection of all tangent vectors of unit length, i.e.,

$$T_1S=\bigcup_{p\in S}\{v\in TS_p \mid \|v\|_p=1\}.$$

The unit tangent bundle $T_1\bar{S}$ of $\bar{S}$ is defined similarly. It is well known that T_1S is the trivial bundle, or $T_1S=S\times S^1$; hence, $\tilde{\pi}_1=\pi_1(T_1S)=\pi_1(S)\times \mathbf{Z}$ and $\tilde{H}_1=H_1\times\mathbf{Z}=\mathbf{Z}^{2g+1}$. At times it will be useful to think of H_1 as a subgroup of $\tilde{H}_1$ in this manner. In terms of exact sequences, that above description gives

$$1\to\mathbf{Z}\to\tilde{H}_1\to H_1\to 1, \tag{1.2.1}$$

where the kernel is generated by the fiber class $[z]\in\tilde{H}_1$. In the case of the closed surface, $T_1\bar{S}$ is no longer trivial. The first homology of $T_1\bar{S}$, however, is easily described by $\bar{H}_1=H_1(T_1\bar{S};\mathbf{Z})=\mathbf{Z}^{2g}\times\mathbf{Z}_{2g-2}$. In other words, the fiber class $[z]\in\bar{H}_1$ has order $2g-2$ (for details, see [7, 8], for instance).

Since the tangent bundle TS of S is the trivial bundle, S admits nonvanishing, smooth vector fields. If X is a nonvanishing smooth vector field on S, it induces a smooth global section of T_1S by the formula

$$X(p)=X(p)/\|X(p)\|_p. \tag{1.2.2}$$

(Both the vector field and the section will be denoted by X.) In what is to follow, a vector field X on S will mean a global smooth section of T_1S. A formula similar to (1.2.2) can be used to define the diffeomorphism Df of T_1S induced by $f\in\mathcal{M}$. The derivative Df of f induces a diffeomorphism of T_1S, also denoted by Df, defined by

$$Df(p,v)=(f(p),\ Df(v)/\|Df(v)\|_{f(p)}). \tag{1.2.3}$$

Via differentiation, then, an action of $\mathcal{M}$ on $\tilde{H}_1$ is obtained, and this action will be calculated. One anticipatory remark is in order before discussing winding numbers. Since $\tilde{H}_1=\mathbf{Z}^{2g+1}$, the action of Df on $\tilde{H}_1$ will be given by a $(2g+1)\times(2g+1)$ matrix with integer coefficients. Moreover, Df will always preserve the fiber class $[z]\in\tilde{H}_1$. If $\tilde{a}_i$, $\tilde{b}_i$ are lifts of a_i, b_i to $\tilde{H}_1$, then the matrix of Df with respect to the basis $\{z,\tilde{a}_1,\dots,\tilde{a}_g,\tilde{b}_1,\dots,\tilde{b}_g\}$ of $\tilde{H}_1$ will have the vector $(1,0,\dots,0)$ as its first column.

1.3.

In this section, the definition of winding numbers of curves on surfaces is given, as well as a review of some of their properties. An interpretation of winding numbers in terms of intersections in T_1S is also given. This definition is particularly useful for the calculations required in the proof of Theorem 2.2. The crossed homomorphism e_x is then defined, and some of its properties outlined. In particular, the behavior of the e_x under composition in $\mathcal{M}$ is calculated. This formula will suggest a way to define representations Φ_x of $\mathcal{M}$. First, Remark 2 on p. 221 of [1] is reproduced.

Let γ be a regular closed curve on S which is parameterized by arc length (i.e., $\|\gamma'(t)\|_{\gamma(t)}=1$ for all t). Then $T_1S|_\gamma$ is a torus T^2. Let X be a vector field on S and $X(\gamma)=X|_\gamma$, then it is easy to see that $X(\gamma)$ is a curve on T^2. Choose an orientation on T_1S. Let α be the homotopy class of $X(\gamma)$ and β the homotopy class corresponding to a fiber with induced orientation; then, $\pi_1(T^2)=\langle\alpha,\beta\,|\,\alpha\beta\alpha^{-1}\beta^{-1}=1\rangle$. Since γ is parameterized by arc length, the curve $\bar\gamma=(\gamma(t),\gamma'(t))$ lies on T^2, and $\bar\gamma=\beta^m\alpha\in\pi_1(T^2)$ for some integer m. Chillingworth defines $\omega_X(\gamma)=m$. Note that the sign of m depends on a choice of orientation for T_1S.

Hodgson suggested an alternate description of this definition in terms of intersections in the homology of T_1S. If X is a smooth global section of T_1S, then $X(S)$ is a surface in T_1S, i.e., $X(S)\in H_2(T_1S,\partial T_1S;\mathbb{Z})$. By Poincaré duality, $X(S)\in\tilde H^1$, and the value of $X(S)$ on any class $[\bar\gamma]\in\tilde H_1$ is given by $\langle X(S),\bar\gamma\rangle$. Choose an orientation for T_1S so that $\langle X(S),z\rangle=1$, where $[z]$ is the fiber class with induced orientation. Calculating $X(S)$ on the generators α,β of $\pi_1(T^2)$ gives

$$\langle X(S),\beta\rangle=\langle X(S),z\rangle=1, \tag{1.3.1}$$

and

$$\langle X(S),\alpha\rangle=\langle X(S),X(\gamma)\rangle=0. \tag{1.3.2}$$

Hence $\langle X(S),\bar\gamma\rangle=\langle X(S),\beta^m\alpha\rangle=m$, and the following definition can be made:

$$\omega_X(\gamma)=\langle X(S),\bar\gamma\rangle.$$

This definition, with careful choices of orientations, coincides with the one given by Chillingworth.

This definition of $\omega_X(\gamma)$ is only for curves on the surface S; however, Chillingworth shows that the notion of winding number can be extended to the case of the closed surface if values are taken in $\mathbb{Z}_{2g-2}$ rather than $\mathbb{Z}$. This fact will be useful in determining how the representations Φ_X of $\mathcal{M}$ can induce representations of $\bar{\mathcal{M}}$.

Two properties of winding numbers are used in defining the crossed homomorphisms $e_X:\mathcal{M}\mapsto H^1$. First, given two vector fields X_1, X_2 on S, there is a well-defined element $e_{1,2}\in H^1$ given by

$$e_{1,2}[\gamma]=\omega_{X_1}(\gamma)-\omega_{X_2}(\gamma) \tag{1.3.3}$$

for any $[\gamma]\in H_1$. This is well defined on $H_1\subset\tilde H_1$ since $e_{1,2}$ vanishes on the fiber class. Secondly, note that $\omega_{DfX}(f\gamma)=\omega_X(\gamma)$.

Definition 1.3.1. Let X be a vector field on S. Then $e_X:\mathcal{M}\mapsto H^1$ is defined by

$$e_X(f)[\gamma]=\omega_X(f\gamma)-\omega_X(\gamma)$$

for any $f\in\mathcal{M}$, $\gamma\in H_1$.

Here, and for the remainder of the paper, $f\gamma$ denotes $f_*\gamma$ where f_* is the automorphism of $\pi_1(S)$ induced by f. The fact that $e_X(f)\in H^1$ follows from the fact that $\omega_x(f\gamma)=\omega_{Df^{-1}x}(\gamma)$ and the previous discussion. The most important property of the e_x is the composition law:

$$\begin{aligned} e_x(fh)[\gamma] &= \omega_x(fh\gamma)-\omega_x(\gamma) \\ &= \omega_x(fh\gamma)-\omega_x(h\gamma)+\omega_x(h\gamma)-\omega_x(\gamma) \\ &= e_x(f)[h\gamma]+e_x(h)[\gamma]. \end{aligned} \tag{1.3.4}$$

Or, more simply stated:

$$e_x(fh)[\gamma]=e_x(f)[h\gamma]+e_x(h)[\gamma], \tag{1.3.4}$$

for all $f, h \in \mathcal{M}$ and all $[\gamma]\in H_1$. Now consider elements $u\in H^1$ as vectors in $\mathbb{Z}^{2g}$ by

$$u=[u(a_1),\ldots,u(a_g),u(b_1),\ldots,u(b_g)]$$

where $\{a_i, b_i\}$ is a symplectic basis for H_1. Consider the action of $\mathcal{M}$ on H^1 in terms of vectors $u\in H^1$ and the symplectic representation ρ. The reader can convince himself that the following formula is true:

$$\begin{aligned} u[h\gamma] &= [u(a_1),\ldots,u(b_g)]\cdot(\rho(h)\cdot[\gamma]) \\ &= ([u(a_1),\ldots,u(b_g)]\cdot\rho(h))\cdot[\gamma] \\ &= (u\cdot\rho(h))\cdot[\gamma], \end{aligned} \tag{1.3.5}$$

where all products are usual matrix multiplication and $[\gamma]\in H_1$ is considered to be a vector. Now use (1.3.5) to rewrite (1.3.4) as

$$e_x(fh)=e_x(f)\cdot\rho(h)+e_x(h). \tag{1.3.6}$$

This formula indicates how to use the crossed homomorphism e_x to define a representation Φ_x of $\mathcal{M}$. Before doing so, however, some remarks about $e_x|_{\mathcal{I}}$ are in order. An immediate consequence of (1.3.6) is that the $e_x|_{\mathcal{I}}$ are homomorphisms since $\rho(h)$ is the identity matrix for $h\in\mathcal{I}$. The following property of the homomorphisms $e_x|_{\mathcal{I}}$ was proven by Johnson in [4]. His argument is repeated here.

Lemma 1.3.2. *The homormorphism $e_x|_{\mathcal{I}}$ is independent of the choice of vector field X.*

Proof. Let X_1, X_2 be vector fields on S, and $f\in\mathcal{I}$. Then

$$\begin{aligned} [e_{x_1}(f)-e_{x_2}(f)][\gamma] &= \omega_{x_1}(f\gamma)-\omega_{x_1}(\gamma)-[\omega_{x_2}(f\gamma)-\omega_{x_2}(\gamma)] \\ &= \omega_{x_1}(f\gamma)-\omega_{x_2}(f\gamma)-[\omega_{x_1}(\gamma)-\omega_{x_2}(\gamma)] \\ &= e_{1,2}[f\gamma]-e_{1,2}[\gamma] \\ &= e_{1,2}[\gamma]-e_{1,2}[\gamma]=0. \end{aligned} \tag{1.3.7}$$

The last equality holds since $f\in\mathcal{I}$. □

One reason to consider $e_x|_{\mathcal{I}}$ is that $\ker\Phi_x\subset\mathcal{I}$. It will turn out that changing the choice of vector field X gives rise to conjugate representations. Hence $\ker\Phi_x$ is independent of the choice of vector field X. Lemma 1.3.2 is just another way of stating this fact.

2. The representation $\Phi : \mathcal{M} \mapsto \widetilde{\mathrm{Sp}}_x$

In this section the representation Φ_x is defined using the crossed homomorphism e_x, and the dependence of Φ_x on the vector field X is studied. The definition of winding numbers is used to show that Φ_x calculates the action of $\mathcal{M}$ on $\tilde{H}_1$. As a result of this geometric interpretation, the image of $\mathcal{M}$ under Φ_x is denoted by $\widetilde{\mathrm{Sp}}_x$. The representation $\Phi|_{\mathcal{I}}$ is then studied. In particular, the image of $\mathcal{I}$ under Φ is characterized along with ker Φ.

Definition 2.1. Given a vector field X, the representation $\Phi_x : \mathcal{M} \mapsto \widetilde{\mathrm{Sp}}_x$ is given by

$$\Phi_x(f) = \begin{pmatrix} 1 & e_x(f) \\ 0 & \rho(f) \end{pmatrix} \tag{2.1}$$

where $\rho(f)$ is the image of $f \in \mathcal{M}$ under the symplectic representation and $e_x(f)$ is considered to be a vector.

To see that Φ_x is indeed a representation, formula (1.3.6) is used in the following calculation.

$$\Phi_x(f)\Phi_x(h) = \begin{pmatrix} 1 & e_x(f) \\ 0 & \rho(f) \end{pmatrix} \begin{pmatrix} 1 & e_x(h) \\ 0 & \rho(h) \end{pmatrix} = \begin{pmatrix} 1 & e_x(f) \cdot \rho(h) + e_x(h) \\ 0 & \rho(f)\rho(h) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e_x(fh) \\ 0 & \rho(fh) \end{pmatrix} = \Phi_x(fh). \tag{2.2}$$

The following theorem gives the desired geometric interpretation of the representation Φ_x.

Theorem 2.2. *Let X be a vector field on S. Then $\Phi_x(f)$ calculates the action of Df on $\tilde{H}_1$ with respect to the basis $\{z, \tilde{a}_i, \tilde{b}_i\}$, where z is the fiber class and*

$$\tilde{a}_i = \bar{a}_i - \omega_x(a_i) \cdot z; \; \tilde{b}_i = \bar{b}_i - \omega_x(b_i) \cdot z.$$

Proof. The calculation shall be performed for $\tilde{a}_1$, all other cases are completely analogous. Suppose $\rho(f)[a_1] = (\sum_{i=1}^{g} m_i \cdot a_i + n_i \cdot b_i)$ in H_1 and consider $Df(\tilde{a}_1)$:

$$Df(\tilde{a}_1) = Df(\bar{a}_1 - \omega_x(a_1) \cdot z) = Df(\bar{a}_1) - \omega_x(a_1) \cdot z = \overline{fa_1} - \omega_x(a_1) \cdot z \tag{2.3}$$

since $Df(z) = z$ and $Df(\bar{\alpha}) = \overline{f\alpha}$ for any regular closed curve on S. It is necessary, then, to compute $\overline{fa_1}$ in terms of the basis $\{z, \tilde{a}_i, \tilde{b}_i\}$ for $\tilde{H}_1$. Note that, since $\overline{fa_1}$ and $(\sum_{i=1}^{g} m_i \cdot \bar{a}_i + n_i \cdot \bar{b}_i)$ project to the same homology class in H_1,

$$\overline{fa_1} - \sum_{i=1}^{g} m_i \cdot \bar{a}_i + n_i \cdot \bar{b}_i = n \cdot z \tag{2.4}$$

for some integer n. To calculate n, intersect both sides of equation (2.4) with $X(S) \in \tilde{H}^1$. By the definition of winding numbers, the equality

$$\left\langle X(S), \overline{fa_1} - \sum_{i=1}^{g} m_i \cdot \bar{a}_i + n_i \cdot \bar{b}_i \right\rangle = \langle X(S), nz \rangle$$

becomes

$$\omega_X(fa_1) - \sum_{i=1}^{g} m_i \cdot \omega_X(a_i) + n_i \cdot \omega_X(b_i) = n. \tag{2.5}$$

Combining (2.4), (2.5), and using the definition of $\{\tilde{a}_i, \tilde{b}_i\}$, gives

$$\begin{aligned} \overline{fa_1} &= \sum_{i=1}^{g} m_i \cdot \bar{a}_i + n_i \cdot \bar{b}_i - \sum_{i=1}^{g} (m_i \cdot \omega_X(a_i)) \cdot z + (n_i \cdot \omega(b_i)) \cdot z + \omega_X(fa_1) \cdot z \\ &= \sum_{i=1}^{g} m_i \cdot (\bar{a}_i - \omega_X(a_i) \cdot z) + n_i \cdot (\bar{b}_i - \omega_X(b_i) \cdot z) + \omega_X(fa_1) \cdot z \\ &= \sum_{i=1}^{g} m_i \cdot \tilde{a}_i + n_i \cdot \tilde{b}_i + \omega_X(fa_1) \cdot z. \end{aligned} \tag{2.6}$$

With this calculation, formula (2.3) becomes

$$\begin{aligned} Df(\tilde{a}_1) &= \sum_{i=1}^{g} m_i \cdot \tilde{a}_i + n_i \cdot \tilde{b}_i + [\omega_X(fa_1) - \omega_X(a_1)] \cdot z \\ &= \sum_{i=1}^{g} m_i \cdot \tilde{a}_i + n_i \cdot \tilde{b}_i + e_X(f)[a_1] \cdot z = \Phi_X(f) \cdot [\tilde{a}_1]. \end{aligned} \tag{2.7}$$

In a similar manner it is shown that $Df(\tilde{\gamma}) = \Phi_X(f) \cdot [\tilde{\gamma}]$ for all $[\tilde{\gamma}] \in \tilde{H}_1$. □

An immediate consequence of Theorem 2.2 is the following

Corollary 2.3. *If X_1, X_2 are vector fields on S, then the representations Φ_{X_1}, Φ_{X_2} are conjugate.*

Proof. By the theorem, both Φ_{X_1} and Φ_{X_2} describe the action of $\mathcal{M}$ on $\tilde{H}_1$ with respect to the appropriate bases; therefore, they are conjugate. □

Since the representations Φ_X are all conjugate, they are considered a single representation Φ. Note, however, that the group $\widetilde{\mathrm{Sp}}_X$ depends on the choice of X.

From Definition 2.1 it is easy to see that the representation Φ extends the symplectic representation as promised. In order to describe $\ker \Phi$, it is necessary to consider $\Phi|_{\mathcal{I}}$. For $f \in \mathcal{I}$, definition (2.1) gives

$$\Phi_X(f) = \begin{pmatrix} 1 & e_X(f) \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{2.8}$$

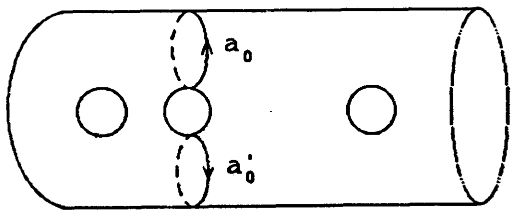

Fig. 2.

Here Id is the $2g \times 2g$ identity matrix. Recall that the homomorphism $e_x|_{\mathcal{I}}$ is independent of the choice of vector field; hence, the following results and calculations are independent of the choice of vector field. The first task is to calculate the matrix $\Phi_x(T_{a_0}T_{a_0'}^{-1})$, where (a_0, a_0') is the BP of genus one pictured in Fig. 2. If $f_0 = T_{a_0}T_{a_0'}^{-1}$, then $f_0(\alpha_i)$ is conjugate to α_i for $i \neq 2$; likewise, $f_0(\beta_i)$ is conjugate to β_i for all i. Since winding numbers are well defined on conjugacy classes, it suffices to consider $e_x(f_0)[\alpha_2]$ in the calculation of $e_x(f_0)$. The formula for calculation of winding numbers given in [1] yields $e_x(f_0)[\alpha_2] = -2$. Thus, considering $e_x(f_0)$ as a vector gives

$$e_x(f_0) = (0, -2, 0, \dots, 0). \tag{2.9}$$

This calculation can be used to calculate $e_x(T_\gamma T_\delta^{-1})$ for any BP (γ, δ) of genus one. Since such maps generate $\mathcal{I}$ for $g \geq 3$, the desired characterization of im $\Phi|_{\mathcal{I}}$ and ker Φ will follow.

Proposition 2.4. *Let (γ, δ) be a BP of genus one, with γ oriented so that $S_{\gamma,\delta}$ lies to its left. Then*

$$e_x(T_\gamma T_\delta^{-1}) = 2(\langle\gamma, a_1\rangle, \dots, \langle\gamma, a_g\rangle, \langle\gamma, b_1\rangle, \dots, \langle\gamma, b_g\rangle).$$

Proof. For $(\gamma, \delta) = (a_0, a_0')$, equation (2.9) proves the proposition. Now let (γ, δ) be any BP of genus one, and let $f \in \mathcal{M}$ be such that $fa_0 = \gamma$, $fa_0' = \delta$. Then $T_\gamma T_\delta^{-1} = fT_{a_0}T_{a_0'}^{-1}f^{-1}$, and

$$\begin{aligned}\Phi_x(T_\gamma T_\delta^{-1}) &= \begin{pmatrix}1 & e_x(f)\\ 0 & \rho(f)\end{pmatrix}\begin{pmatrix}1 & e_x(f_0)\\ 0 & \text{Id}\end{pmatrix}\begin{pmatrix}1 & e_x(f^{-1})\\ 0 & \rho(f^{-1})\end{pmatrix}\\ &= \begin{pmatrix}1 & e_x(f^{-1}) + e_x(f)\cdot\rho(f^{-1}) + e_x(f_0)\cdot\rho(f^{-1})\\ 0 & \text{Id}\end{pmatrix}.\end{aligned} \tag{2.10}$$

But $0 = e_x(ff^{-1}) = e_x(f)\cdot\rho(f^{-1}) + e_x(f^{-1})$, so (2.10) becomes

$$\Phi_x(T_\gamma T_\delta^{-1}) = \begin{pmatrix}1 & e_x(f_0)\cdot\rho(f^{-1})\\ 0 & \text{Id}\end{pmatrix}.$$

Once it is shown that $e_x(f_0)\cdot\rho(f^{-1}) = 2(\langle\gamma, a_1\rangle, \dots, \langle\gamma, b_g\rangle)$, the proof will be complete. The calculation preceding the proposition showed that

$$e_x(f_0) = 2(\langle a_0, a_1\rangle, \dots, \langle a_0, b_g\rangle) = (e_x(f_0)[a_1], \dots, e_x(f_0)[b_g]);$$

therefore

$$\begin{aligned} e_x(f_0)\cdot\rho(f^{-1}) &= (e_x(f_0)[f^{-1}a_1],\ldots,e_x(f_0)[f^{-1}b_g]) \\ &= 2(\langle a_0, f^{-1}a_1\rangle,\ldots,\langle a_0, f^{-1}b_g\rangle) \\ &= 2(\langle fa_0, a_1\rangle,\ldots,\langle fa_0, b_g\rangle). \end{aligned} \tag{2.11}$$

By the choice of f, however, $fa_0=\gamma$ and (2.11) becomes

$$e_x(f_0)\cdot\rho(f^{-1}) = 2(\langle\gamma, a_1\rangle,\ldots,\langle\gamma, b_g\rangle) \tag{2.12}$$

as desired. □

Proposition 2.4 facilitates the calculation of $e_x(f)$ for any $f=T_{\gamma_1}T_{\delta_1}^{-1}\cdots T_{\gamma_n}T_{\delta_n}^{-1}\in \mathcal{I}$, where (γ_i,δ_i) are BPs of genus one.

Corollary 2.5. *If $f=T_{\gamma_1}T_{\delta_1}^{-1}\cdots T_{\gamma_n}T_{\delta_n}^{-1}\in\mathcal{I}$ where each (γ_i,δ_i) is a BP of genus one, then*

$$\begin{aligned} e_x(f) &= e_x(T_{\gamma_1}T_{\delta_1}^{-1}\cdots T_{\gamma_n}T_{\delta_n}^{-1}) \\ &= 2\left(\left\langle\sum_{i=1}^{n}\gamma_i, a_1\right\rangle,\ldots,\left\langle\sum_{i=1}^{n}\gamma_i, b_g\right\rangle\right). \end{aligned} \tag{2.13}$$

Proof. This is a direct consequence of Proposition 2.4 and the fact that $e_x|_{\mathcal{I}}$ is a homomorphism. □

Corollary 2.5 can now be used to characterize ker Φ.

Definition 2.6. The Chillingworth subgroup $\mathscr{C}\subset\mathcal{I}$ is defined by

$$\mathscr{C}=\left\{f=T_{\gamma_1}T_{\delta_1}^{-1}\cdots T_{\gamma_n}T_{\delta_n}^{-1}\,|\,\text{each }(\gamma_i,\delta_i)\text{ a BP of genus one, and}\left(\sum_{i=1}^{n}2\gamma_i\right)=0\text{ in }H_1\right\}.$$

Define the subgroups $\mathscr{C}_{g,n}$ similarly, the requirement being that $(\sum_{i=1}^{n}2\gamma_i)=0$ in $H_1(S;\mathbb{Z}_n)$ rather than H_1. The subgroups $\mathscr{C}_{g,n}$ will be useful in studying the structure of the groups $G_{g,n}$ of [7, 8].

In order to characterize ker Φ, Corollary 2.5 will be used together with the following fact about the group $\mathscr{K}\subset\mathcal{I}$ generated by Dehn twists on BSCCs. Chillingworth shows in [2] that, if γ is a SCC, the following formula is satisfied:

$$\omega_x(T_\gamma\beta)=\omega_x(\beta)+\langle\gamma,\beta\rangle\cdot\omega_x(\gamma), \tag{2.14}$$

for any curve β on S. if γ is a BSCC, then $\langle\gamma,\beta\rangle=0$, and formula (2.14) shows that $T_\gamma\in\ker\Phi$. Since $\mathscr{K}$ is generated by such maps, $\mathscr{K}\subset\ker\Phi$.

Corollary 2.7. *For $g \geqslant 3$, $\mathscr{C} = \ker \Phi$.*

Proof. Recalling that $\mathscr{I}$ is generated by BP maps of genus one, this is a direct consequence of Corollary 2.5 and Definition 2.6. □

Remark. The case $g = 2$ is special. For the purposes of this paper, it suffices to say that it can be shown that $\mathscr{K} = \ker \Phi$.

Corollary 2.5 can also be used to characterize im $\Phi|_{\mathscr{I}}$.

Proposition 2.8. *For $g \geqslant 2$, the matrix $\begin{pmatrix}1 & v\\ 0 & \mathrm{Id}\end{pmatrix}$ is in the image of $\mathscr{I}$ under Φ if and only if $v \in (2\mathbb{Z})^{2g}$.*

Proof. Corollary 2.5 implies necessity for the subgroup of $\mathscr{I}$ generated by BP maps of genus one. For $g \geqslant 3$, this implies necessity for $\mathscr{I}$. For $g = 2$, Corollary 2.5, together with the remarks preceding Corollary 2.7, imply necessity. Conversely, with careful choices of BP maps of genus one, it is easy to see that generators of $(2\mathbb{Z})^{2g}$ are contained in $\mathrm{im}(e_x|_{\mathscr{I}})$. Since $e_x|_{\mathscr{I}}$ is a homomorphism, the proof is complete. □

In order to see how Φ can induce representations of $\bar{\mathcal{M}}$, the image of $\pi_1(T_1\bar{S}) \subset \mathscr{I}$ under Φ must be studied. Recall that $\pi_1(T_1\bar{S})$ is generated by BP maps of maximal genus; hence it is necessary to calculate Φ on BP maps of arbitrary genus.

Proposition 2.9. *Let $T_\gamma T_\delta^{-1}$ be a BP map of genus g'. Then*

$$e_x(T_\gamma T_\delta^{-1}) = 2g'(\langle\gamma, a_1\rangle, \ldots, \langle\gamma, b_g\rangle)$$

where γ is oriented so that $S_{\gamma,\delta}$ lies on its left.

Proof. The idea is to rewrite $T_\gamma T_\delta^{-1}$ in terms of BP maps of genus one, and use Corollary 2.5 for the calculation. Let $\varepsilon_1, \ldots, \varepsilon_{g'-1}$ be the curves pictured in Fig. 3 with the given orientations. Then

$$T_\gamma T_\delta^{-1} = (T_\gamma T_{\varepsilon_1}^{-1})(T_{\varepsilon_1} T_{\varepsilon_2}^{-1}) \cdots (T_{\varepsilon_{g'-1}} T_\delta^{-1}),$$

and, using Corollary 2.5,

$$e_x(T_\gamma T_\delta^{-1}) = 2\left(\left\langle\left[\gamma + \sum_{i=1}^{g'-1} \varepsilon_i\right], a_1\right\rangle, \ldots, \left\langle\left[\gamma + \sum_{i=1}^{g'-1} \varepsilon_i\right], b_g\right\rangle\right). \qquad (2.15)$$

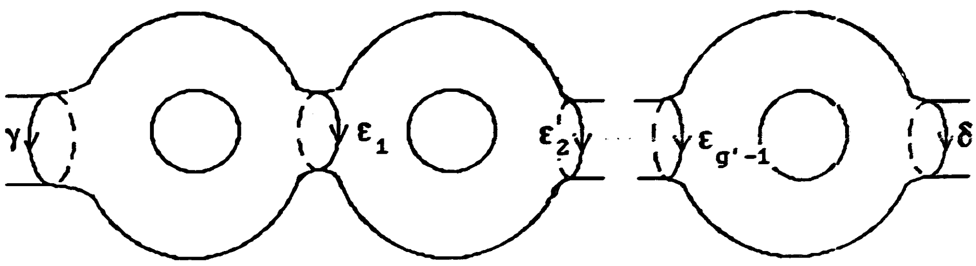

Fig. 3.

However, γ is homologous to ε_i for each i and $[\gamma+\sum_{i=1}^{g'-1}\varepsilon_i]=g'\cdot[\gamma]$. Equation (2.15) becomes

$$e_X(T_\gamma T_\delta^{-1})=2g'(\langle\gamma,a_1\rangle,\dots,\langle\gamma,b_g\rangle), \tag{2.16}$$

proving the proposition. □

Note that if (γ,δ) is a BP of maximal genus, then $g'=g-1$. Proposition 2.9, together with the fact that $\pi_1(T_1\bar{S})$ is generated by BP maps of maximal genus, imply

Corollary 2.10. *For $g\geq 2$, the matrix $\left(\begin{smallmatrix}1 & v\\ 0 & \mathrm{id}\end{smallmatrix}\right)$ is in the image of $\pi_1(T_1\bar{S})$ under Φ if and only if $v\in((2g-2)\mathbb{Z})^{2g}$.*

Recall that Φ_n denotes Φ with matrix entries taken in $\mathbb{Z}_n$. Then Corollary 2.10 shows that $\pi_1(T_1\bar{S})\subset\ker\Phi_n$ for $n\,|\,2g-2$. Hence the Φ_n induce representations $\bar{\Phi}_n$ of $\bar{\mathcal{M}}$ for $n\,|\,2g-2$. The representations $\bar{\Phi}_n$ of $\bar{\mathcal{M}}$ were discovered by Sipe, and are the topic of the next section.

One final remark is in order. If $\mathcal{M}_*$ is the mapping class group of a once-punctured surface, then there is the exact sequence

$$1\to\mathbb{Z}\to\mathcal{M}\to\mathcal{M}_*\to 1, \tag{2.17}$$

where the kernel is the infinite cyclic subgroup generated by a Dehn twist about the boundary of S. Since the representation Φ is trivial on BSCC maps, it follows that Φ induces a representation of $\mathcal{M}_*$, also denoted by Φ. All the results of this section carry over to the case of the mapping class group of a once-punctured surface.

3. The representations $\bar{\Phi}_n$ and the groups $G_{g,n}$

The context in which Sipe discovered the representations $\bar{\Phi}_n$ is described in this section, and her characterization of im $\bar{\Phi}_n$ is mentioned (for details see [8]). This characterization extends to one of $\widetilde{\mathrm{Sp}}_{X_0}$, with the appropriate choice of vector field X_0 (recall that $\widetilde{\mathrm{Sp}}_X$ depends on the choice of vector field). The results of Section 2 are then applied to gain further insight into the structure of the groups $G_{g,n}$.

Sipe discovered the representations $\bar{\Phi}_n$ while calculating the action of $\bar{\mathcal{M}}$ on nth roots of the canonical bundle of the surface $\bar{S}$. Some of the results found in [7, 8], are reviewed here. First, by a Chern class argument, the canonical bundle admits nth roots if and only if $n\,|\,2g-2$; therefore, it will be assumed that $n\,|\,2g-2$, where g is the genus of $\bar{S}$. Sipe shows that there is a one-to-one correspondence between nth roots of the canonical bundle and the set

$$\Psi_n=\{\xi\in H^1(T_1\bar{S};\mathbb{Z}_n)\,|\,\xi(z)=-1\}.$$

This topological description of nth roots is used in what follows. In order to calculate the action of $\bar{\mathcal{M}}$ on nth roots, then, it suffices to know the action of $\bar{\mathcal{M}}$ on $H_1(T_1\bar{S};\mathbb{Z}_n)$. Sipe was able to calculate this action, obtaining the representations $\bar{\Phi}_n$. Recall that

Φ_n denotes the representation Φ of $\mathcal{M}$ with matrix entries taken in $\mathbf{Z}_n$. Furthermore, for $n|2g-2$, the representation Φ_n induces a representation of $\bar{\mathcal{M}}$. The induced representation coincides with $\bar{\Phi}_n$ since both give the action of $\bar{\mathcal{M}}$ on $H_1(T_1\bar{S};\mathbf{Z}_n)$. Thus Sipe's results give information about Φ_n, and it is natural to ask which results carry over to the representation Φ. Sipe's characterization of im $\bar{\Phi}_n$, for example, immediately carries over to the integer-valued case.

With respect to the basis $\{z, \bar{a}_i+z, \bar{b}_i+z\}$ for $\tilde{H}_1$, the following characterization of the image of $\bar{\Phi}_n$ was given:

Theorem 3.1 (Sipe). *The matrix* $\left(\begin{smallmatrix}1 & v\\ 0 & B\end{smallmatrix}\right)$ *is contained in the image of* $\bar{\Phi}_n$ *if and only if*

$$\mathrm{Diag}(B^tNB)-v\in(2\mathbf{Z}_n)^{2g},$$

where $B\in\mathrm{Sp}(2g;\mathbf{Z}_n)$ and $N=\left(\begin{smallmatrix}0 & \mathrm{Id}\\ 0 & 0\end{smallmatrix}\right)$.

Here Diag(A) is the $2g$ vector whose ith component is a_{ii} and B^t is the transpose of B.

Let X_0 be the vector field such that $\omega_{X_0}(\alpha_i)=-1$, $\omega_{X_0}(\beta_i)=-1$ for all i. Then, by Theorem 2.2, Φ_{X_0} calculates the action of $\mathcal{M}$ on $\tilde{H}_1$ with respect to the basis $\{z, \bar{a}_i+z, \bar{b}_i+z\}$.

Corollary 3.2. *The matrix* $\left(\begin{smallmatrix}1 & v\\ 0 & B\end{smallmatrix}\right)$ *is in* $\widetilde{\mathrm{Sp}}_{X_0}$ *if and only if*

$$\mathrm{Diag}(B^tNB)-v\in(2\mathbf{Z})^{2g}.$$

Proof. This is an immediate consequence of Theorem 2.2 and a direct generalization of Sipe's arguments in [8] to the integer-valued case. □

Note that Proposition 2.8 is the equivalent of Lemma 1 in [8], and that the results of Section 2 reveal more of the geometric structure of $\Phi|_{\mathcal{I}}$. Although Sipe's arguments regarding im $\bar{\Phi}_n$ immediately generalize to the integer-valued case, her discussion of ker $\bar{\Phi}_n = G_{g,n}$ does not.

Let $N_{g,n}\subset\mathrm{Sp}(2g,\mathbf{Z})$ be the congruence subgroup of level n. Sipe shows that $\bar{\rho}(G_{g,n})=N_{g,n}$, where $\bar{\rho}$ is the symplectic representation of $\bar{\mathcal{M}}$. This result does not apply to the integer-valued case since ker $\Phi\subset\mathcal{I}$. In fact, the results of Section 2 can be used to further describe the structure of $G_{g,n}$. The fact that $\bar{\rho}(G_{g,n})=N_{g,n}$ characterizes the symplectic part of $G_{g,n}$, while the results in Section 2 can be used to characterize $G_{g,n}\cap\bar{\mathcal{I}}$. Recall the exact sequence (1.1) and Definition 2.6 of the groups $\mathcal{C}_{g,n}$.

Definition 3.3. Let $\bar{\mathcal{C}}_{g,n}$ be the image of $\mathcal{C}_{g,n}$ in $\bar{\mathcal{M}}$, and let $\bar{\mathcal{C}}$ be the image of the Chillingworth subgroup in $\bar{\mathcal{M}}$.

An immediate consequence of the definitions is the following

Proposition 3.4. $G_{g,n}\cap\bar{\mathcal{I}}=\bar{\mathcal{C}}_{g,n}$; *hence the following sequence is exact:*

$$1\to\bar{\mathcal{C}}_{g,n}\to G_{g,n}\xrightarrow{\bar{\rho}}N_{g,n}\to 1. \qquad (3.1)$$

Proof. Definition 3.3, Corollary 2.5, and the fact that Φ_n induces $\bar{\Phi}_n$. □

The characterization of $G_{g,n} \cap \bar{\mathcal{I}}$ is virtually a direct consequence of properties of $\Phi|_{\mathcal{I}}$ discussed in Section 2. With a little work, however, even more of the structure of $G_{g,n}$ is revealed.

Note that $\bar{\mathscr{C}}_{g,2g-2} \subset G_{g,n}$ for all $n|2g-2$. In other words, $\bar{\mathscr{C}}_{g,2g-2}$ is the subgroup of $\bar{\mathcal{I}}$ which acts trivially on nth roots of the canonical bundle for all $n|2g-2$. It will be shown that, for n even, the group $G_{g,n}/\bar{\mathscr{C}}_{g,2g-2}$ is a semidirect product. The remainder of the discussion, then, will be restricted to the case where n is even and $n|2g-2$. The strategy will be to obtain a semidirect product structure on quotients of certain subgroups in $\mathcal{M}$ using results from Section 2, and then show that these quotients are isomorphic to $G_{g,n}/\bar{\mathscr{C}}_{g,2g-2}$. If $\mathscr{G}_{g,n} \subset \mathcal{M}$ denotes ker Φ_n, then $\mathrm{pr}(\mathscr{G}_{g,n}) = G_{g,n}$ where pr is as in sequence (1.1). Moreover, as a result of sequences (1.1), (1.2), and Sipe's characterization of $\bar{\rho}(G_{g,n})$, it follows that $\rho(\mathscr{G}_{g,n}) = N_{g,n}$. Definition 2.6 and Corollary 2.5 imply that $\mathscr{C}_{g,n} = \mathscr{G}_{g,n} \cap \mathcal{I}$, giving rise to the exact sequence

$$1 \to \mathscr{C}_{g,n} \to \mathscr{G}_{g,n} \xrightarrow{\rho} N_{g,n} \to 1. \tag{3.2}$$

Since $\mathscr{C} \subset \mathscr{C}_{g,n}$ for all n, and $\mathscr{C} \subset \mathcal{I}$, the sequence (3.2) modulo $\mathscr{C}$ becomes

$$1 \to \mathscr{C}_{g,n}/\mathscr{C} \to \mathscr{G}_{g,n}/\mathscr{C} \xrightarrow{\rho} N_{g,n} \to 1. \tag{3.3}$$

Recall that $\mathscr{C} = \ker \Phi$; therefore, the groups $\mathscr{C}_{g,n}/\mathscr{C}$, $\mathscr{G}_{g,n}/\mathscr{C}$ can be thought of as subgroups of the matrix group $\widetilde{\mathrm{Sp}}_{x_0}$. Explicit use of Corollary 3.2, the choice of vector field X_0, and the fact that n is even, gives the following descriptions of the groups $\mathscr{C}_{g,n}/\mathscr{C}$, $\mathscr{G}_{g,n}/\mathscr{C}$:

$$\mathscr{C}_{g,n}/\mathscr{C} = \Phi_{x_0}(\mathscr{C}_{g,n}) = \left\{ \begin{pmatrix} 1 & v \\ 0 & \mathrm{Id} \end{pmatrix} \,\middle|\, v \in (n\mathbb{Z})^{2g} \right\}, \tag{3.4}$$

and

$$\mathscr{G}_{g,n}/\mathscr{C} = \Phi_{x_0}(\mathscr{G}_{g,n}) = \left\{ \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix} \,\middle|\, v \in (n\mathbb{Z})^{2g},\ B \in N_{g,n} \right\}. \tag{3.5}$$

Note that since n is even, $B \in N_{g,n}$ implies $B \equiv (\mathrm{Id}) \bmod 2$, and

$$\mathrm{Diag}(B^t N B) \equiv \mathrm{Diag}((\mathrm{Id})^t N(\mathrm{Id})) \equiv (0, \ldots, 0) \bmod 2. \tag{3.6}$$

Hence, if $\left(\begin{smallmatrix} 1 & v \\ 0 & B \end{smallmatrix}\right)$ satisfies conditions (3.5), then it satisfies the conditions of Corollary 3.2. This remark also implies that the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right)$ is in $\Phi_{x_0}(\mathscr{G}_{g,n})$ for n even and all $B \in N_{g,n}$. Thus any matrix $\left(\begin{smallmatrix} 1 & v \\ 0 & B \end{smallmatrix}\right) \in \Phi_{x_0}(\mathscr{G}_{g,n})$ can be written uniquely as a product $\left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & v \\ 0 & \mathrm{Id} \end{smallmatrix}\right)$, where both $\left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & v \\ 0 & \mathrm{Id} \end{smallmatrix}\right)$ are in $\Phi_{x_0}(\mathscr{G}_{g,n})$. In other words, there is a semidirect product structure on $\mathscr{G}_{g,n}/\mathscr{C}$. Under the obvious identification of $\mathscr{C}_{g,n}/\mathscr{C}$ with $(n\mathbb{Z})^{2g}$ the previous discussion gives the isomorophism

$$\mathscr{G}_{g,n}/\mathscr{C} \cong (n\mathbb{Z})^{2g} \rtimes N_{g,n}. \tag{3.7}$$

Now that the desired semidirect product structure has been obtained on quotients of subgroups of $\mathcal{M}$, it remains to see how this structure projects to subgroups in $\bar{\mathcal{M}}$. Recall that the representation Φ induces representations of $\bar{\mathcal{M}}$ when matrix entries are taken modulo $2g-2$. Moreover, $\mathscr{C}_{g,2g-2}/\mathscr{C} \subset \mathscr{C}_{g,n}/\mathscr{C}$ for $n|2g-2$. Using the identification of $\mathscr{C}_{g,n}/\mathscr{C}$ with $(n\mathbf{Z})^{2g}$, it is clear that

$$(\mathscr{C}_{g,n}/\mathscr{C})/(\mathscr{C}_{g,2g-2}/\mathscr{C}) \cong (n\mathbf{Z})^{2g}/((2g-2)\mathbf{Z})^{2g} \cong (\mathbf{Z}_k)^{2g}, \tag{3.8}$$

where $k=(2g-2)/n$. Now consider sequence (3.3) modulo the subgoup $\mathscr{C}_{g,2g-2}/\mathscr{C}$, obtaining

$$1 \to (\mathscr{C}_{g,n}/\mathscr{C})/(\mathscr{C}_{g,2g-2}/\mathscr{C}) \to \mathscr{G}_{g,n}/\mathscr{C}_{g,2g-2} \to N_{g,n} \to 1, \tag{3.9}$$

which becomes, after the appropriate identifications,

$$1 \to (\mathbf{Z}_k)^{2g} \to \mathscr{G}_{g,n}/\mathscr{C}_{g,2g-2} \to N_{g,n} \to 1. \tag{3.10}$$

Consider $\mathscr{G}_{g,n}/\mathscr{C}$ to be the semidirect product $(n\mathbf{Z})^{2g} \rtimes N_{g,n}$, and note that moding out by $((2g-2)\mathbf{Z})^{2g}$ only affects the $(n\mathbf{Z})^{2g}$ factor. Hence $\mathscr{G}_{g,n}/\mathscr{C}_{g,2g-2}$ is a semidirect product as well. More explicitly,

$$\mathscr{G}_{g,n}/\mathscr{C}_{g,2g-2} \cong (\mathbf{Z}_k)^{2g} \rtimes N_{g,n}. \tag{3.11}$$

This semidirect product structure on subgroups in $\mathcal{M}$ actually projects to a semidirect product structure on subgroups in $\bar{\mathcal{M}}$.

Proposition 3.5. *For n even and $n|2g-2$, $G_{g,n}/\bar{\mathscr{C}}_{g,2g-2} \cong (\mathbf{Z}_k)^{2g} \rtimes N_{g,n}$ where $k=(2g-2)/n$.*

Proof. By equation (3.11), all that needs to be shown is that

$$G_{g,n}/\bar{\mathscr{C}}_{g,2g-2} \cong \mathscr{G}_{g,n}/\mathscr{C}_{g,2g-2}. \tag{3.12}$$

Since $\ker(\mathrm{pr}) = \pi_1(T_1\bar{S}) \subset \mathscr{C}_{g,2g-2}$, the proposition is a consequence of the definitions of $\mathscr{G}_{g,n}$ and $\bar{\mathscr{C}}_{g,2g-2}$. Explicitly,

$$\mathscr{G}_{g,n}/\mathscr{C}_{g,2g-2} \cong (\mathscr{G}_{g,n}/\pi_1(T_1\bar{S}))/(\mathscr{C}_{g,2g-2}/\pi_1(T_1\bar{S})) = G_{g,n}/\bar{\mathscr{C}}_{g,2g-2}. \qquad \square$$

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References

[1] D.R.J. Chillingworth, Winding numbers on surfaces I, Math. Ann. 196 (1972) 218-249.
[2] D.R.J. Chillingworth, Winding numbers on surfaces II, Math. Ann. 199 (1972) 131-153.
[3] D. Johnson, Homeomorphisms of a surface which act trivially on homology, Proc. Amer. Math. Soc. 75 (1979) 119-125.
[4] D. Johnson, An Abelian quotient of the mapping class group $\mathcal{I}_g$, Math. Ann. 249 (1970) 225-242.
[5] D. Johnson, The structure of the Torelli group I: A finite set of generators for $\mathcal{I}$, Ann. of Math. 118 (1983) 423-442.
[6] S. Morita, Families of Jacobian manifolds and characteristic classes of surface bundles II, Proc. Japan Acad. Ser. A 61 (1985) 112-115.
[7] P. Sipe, Roots of the canonical bundle of the universal Teichmüller curve and certain subgroups of the mapping class group, Math. Ann. 260 (1982) 67-92.
[8] P. Sipe, Some finite quotients of the mapping class group of a surface, Proc. Amer. Math. Soc. 97 (1986) 515-524.
[9] C. Squier, Matrix representations of Artin groups, Preprint.
[10] B. Wajnryb, A simple presentation for the mapping class group of an orientable surface, Israel J. Math. 45 (1983) 157-174.