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Semiclassical Spectra of Gauge Fields

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We study the asymptotic behavior of the eigenvalues of the Schrödinger operator with a vector potential on a compact manifold, as Planck's constant tends to zero. We obtain estimates in terms of periodic trajectories of Wong's flow which are uniform in the "charge" parameter.

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1. INTRODUCTION

Let $P \to M$ be a compact principal G bundle, over a Riemannian manifold M. A gauge field on M is defined by a connection on P. Choose a bi-invariant metric on G; then, since the base M has a fixed Riemannian

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metric, there is a bijective correspondence between connections on $P \to M$ and G-invariant metrics on P which make $P \to M$ into a Riemannian submersion, and that induce on every fiber the same metric as the one induced by G. This correspondence goes as follows: a metric and a connection correspond to each other iff for every $p \in P$ the horizontal space is orthogonal to the fiber, and at each point the differential of the projection is an isometry when restricted to the horizontal subspace. Given a connection on P, one obtains, for every irreducible representation π_λ of G, a connection $\nabla_\lambda: C^\infty(M, E_\lambda) \to C^\infty(M, T^*M \otimes E_\lambda)$ on sections of the associated vector bundle $E_\lambda \to M$. Here $\lambda \in \mathfrak{t}^*$ is the highest weight of the representation π_λ, having chosen a maximal torus $T \subset G$ and an ordering of the roots. The bundle $T^*M \times E_\lambda$ has a product connection which we also denote ∇_λ. If we compose these two operators and follow the result with the map γ: $T^*M \otimes T^*M \to \mathbb{R}$ defined by the Riemannian metric on M, we obtain a Laplace–Beltrami operator,

$$H_\lambda^0 = -\gamma \circ \nabla_\lambda \circ \nabla_\lambda : C^\infty(M, E_\lambda) \to C^\infty(M, E_\lambda). \tag{1}$$

This is the quantum Hamiltonian for a particle with configuration space M, charge λ, and subject to the gauge field defined by the connection on P. The semiclassical analysis we pursue, following previous work including [12, 16, 18, 26, 27], is of the following nature. (For the scalar potential case see for example [2, 17, 24, 25].) Given $V \in C^\infty(M)$, set

$$H_\lambda = \hbar^2 H_\lambda^0 + V, \qquad \hbar = |\lambda + \delta|^{-1}. \tag{2}$$

Here δ is half the sum of the positive roots. We are interested in the spectral behavior of H_λ as $|\lambda| \to \infty$ in a Weyl chamber. Motivation to take $\hbar^{-1} \to \infty$ and $|\lambda| \to \infty$ at the same rate is discussed in [27, 28]. The particular choice $\hbar = |\lambda + \delta|^{-1}$ is related to the identity $-\pi_\lambda(\Delta_G) = |\lambda + \delta|^2 - |\delta|^2$, where Δ_G is the Laplace operator on G,

$$\pi_\lambda(-\Delta_G + |\delta|^2) = \hbar^{-2} I. \tag{3}$$

As in [16, 27] the spectral behavior of H_λ as $|\lambda| \to \infty$ can be analyzed in terms of the joint spectrum of commuting operators on P, as follows. Let Δ_P^G denote the action on $C^\infty(P)$ derived from Δ_G via the G-action on P. Let $\mathcal{D}_\lambda$ denote the subspace of $C^\infty(P)$ on which G acts like copies of π_λ. Then $\Delta_G^P|_{\mathcal{D}_\lambda} = -|\lambda + \delta|^2 + |\delta|^2 I$. Now the representation theory of G implies that

$$\mathcal{D}_\lambda \cong \text{sum of } d_\lambda \text{ copies of } C^\infty(M, E_\lambda), \tag{4}$$

where d_λ is the dimension of V_λ, the representation space of π_λ. Furthermore, if we set

$$L = \Delta + V_1(x)\, \Delta_G^P - |\delta|^2\, V(x), \tag{5}$$

$$A = -\Delta_G^P + |\delta|^2 \tag{6}$$

where Δ is the Laplace operator on P and $V_1 = V - 1$, then the operators A and L commute, both leave $\mathscr{D}_\lambda$ invariant, and we have

$$A = |\lambda + \delta|^2 = \hbar^{-2} \qquad \text{on } \mathscr{D}_\lambda, \tag{7}$$

and, under the identification (4),

$$-L|_{\mathscr{D}_\lambda} \cong \text{sum of } d_\lambda \text{ copies of } \hbar^{-2} H_\lambda. \tag{8}$$

If $V > 1$, then L is elliptic on P. Adding a constant to a general V can accomplish this, so we will assume this has been arranged. Equations (7) and (8) show that our semiclassical problem can be formulated as a joint eigenvalue problem for the operators A and L.

In [27], (5)–(8) was used to write the trace of $f(H_\lambda)$, with $f \in \mathscr{S}(\mathbb{R})$, in the form

$$\operatorname{Tr} f(H_\lambda) = d_\lambda^{-1} \operatorname{Tr}[f(-A^{-1}L)|_{\mathscr{D}_\lambda}], \tag{9}$$

and the right-hand side of (9) was analyzed as follows. For a suitable class of operators K on $C^\infty(P)$, with Schwartz kernel $k(p, q)$, the *G-trace* of K is defined as the following distribution on G:

$$\operatorname{Tr}_G K(g) = \int_P k(p \cdot g, p)\, dV_p. \tag{10}$$

One can show (cf. Section 3.1) that if K commutes with the G-action on $C^\infty(P)$, $\operatorname{Tr}_G K$ is a central distribution on G, and if $\chi_\lambda \in C^\infty(G)$ is the character of π_λ, then

$$\langle \operatorname{Tr}_G K, \chi_\lambda \rangle = d_\lambda^{-1} \operatorname{Tr}(K|_{\mathscr{D}_\lambda}). \tag{11}$$

Thus if we define $\beta(\lambda + \delta)$ by

$$d_\lambda \beta(\lambda + \delta) = \langle \operatorname{Tr}_G f(-A^{-1}L), \chi_\lambda \rangle, \tag{12}$$

then (9) becomes

$$d_\lambda^{-1} \operatorname{Tr} f(H_\lambda) = \beta(\lambda + \delta). \tag{13}$$

One of the main results of [27] is that, for f a Schwartz function on the line, the G-trace of $f(-A^{-1}L)$ is a distribution conormal to $\{e\}$, where $e \in G$ is the identity element. More precisely,

$$\operatorname{Tr}_G f(-A^{-1}L) \in I^{m + d/4}(G; T_e^*G), \qquad m = \dim M, \qquad d = \dim G. \tag{14}$$

Thus (13) holds with

$$\beta \in S^m(\mathfrak{t}^*), \tag{15}$$

and this leads to a complete asymptotic expansion of $d_\lambda^{-1} \operatorname{Tr}_G f(H_\lambda)$ as $\lambda \to \infty$ in a Weyl chamber.

The asymptotic analysis of (13) involves looking at a part of the spectrum of H_λ of fixed width as $|\lambda| \to \infty$. A finer analysis involves shrinking the spectral width as $|\lambda| \to \infty$; Hörmander's classic paper [19] does such an analysis (in the setting $G = \{e\}$) with spectral width of the order of $\hbar$. It is our aim here to obtain a result of similar sharpness, incorporating also ideas developed by Duistermaat and Guillemin [5]. Results on this spectral width were obtained in [13–16] as $|\lambda| \to \infty$ along a *ray* within a Weyl chamber. The results here amalgamate those of [16, 27], in simultaneously looking at this narrow spectral band and doing so as $\lambda \to \infty$ uniformly in a cone contained in a Weyl chamber.

Specifically, we will analyze the asymptotic behavior of

$$\operatorname{Tr} f(\hbar^{-1} H_\lambda^{-1/2}(H_\lambda - c)), \qquad \hbar = |\lambda + \delta|^{-1}, \tag{16}$$

as $|\lambda| \to \infty$, for a given $c \in \mathbb{R}$. This is a measure of the distribution of the spectrum of H_λ about c. In view of the discussion above, (16) is equal to

$$d_\lambda^{-1} \operatorname{Tr}[f(Q)|_{\mathscr{D}_\lambda}] = \langle \operatorname{Tr}_G f(Q), \chi_\lambda \rangle, \tag{17}$$

where

$$Q = (-L)^{-1/2} (-L - cA) \in OPS^1(P) \tag{18}$$

is self-adjoint. We will analyze this for f such that $\hat{f} \in C_0^\infty(\mathbb{R})$. We will also make geometric assumptions implying that Q has simple characteristics, and that the Hamilton vector field of its symbol on the characteristic manifold is nowhere radial; thus Q is an operator of real principal type.

The analysis of (17) will be carried out in three steps. First we show that, under suitable assumptions, $f(Q)$ is a Fourier integral operator; we compute its canonical relation and its symbol. Next we show that in good cases the G-trace of an FIO is a Lagrangian distribution on G, and apply this to $\operatorname{Tr}_G f(Q)$. Next we must explore the asymptotic behavior of the Fourier coefficients of the G-trace, i.e., of

$$\tau(\lambda) = \langle \operatorname{Tr}_G f(Q), \chi_\lambda \rangle, \tag{19}$$

as $|\lambda| \to \infty$.

In the analysis of (19), one can use the Weyl integration formula and character formula to write

$$\langle v, \chi_\lambda \rangle = |W|^{-1} \sum_{w, w' \in W} (\det ww')\, \hat{\mu}(w(\lambda + \delta) - w'(\delta)), \tag{20}$$

where W is the Weyl group, of cardinality $|W|$, provided ν is central and μ is the restriction of ν to the maximal torus T, with Fourier transform $\hat{\mu}$. Now, the distribution $\nu = \mathrm{Tr}_G f(Q)$ is too singular for its restriction to T to exist in the simple sense of restricting a continuous function. Worse, the device of composing Fourier integral distributions, under clean intersection hypotheses on their Lagrangians, which serves us so well up through the construction of $\mathrm{Tr}_G f(Q)$, definitely tends to *break down* at the step of restricting to T. In particular, (20) does not generally work, with $\nu = \mathrm{Tr}_G f(Q)$. However, we can write instead

$$\begin{aligned}\langle \nu, \chi_\lambda \rangle &= |\lambda + \delta|^s \langle \nu_s, \chi_\lambda \rangle \\ &= |\lambda + \delta|^s \, |W|^{-1} \sum_{w, w' \in W} (\det ww')\, \hat{\mu}_s(w(\lambda + \delta) - w'(\delta)),\end{aligned} \tag{21}$$

where $\nu_s = (\delta^2 - \Delta_G)^{-s/2} \nu$, s is chosen so large that ν_s is continuous on G, and $\mu_s = \nu_s|_T$. Thus μ_s is well defined, but it might be a more complicated object than a Lagrangian distribution on T.

We obtain information on $\mathrm{Tr}_G f(Q)$, and hence on (19), in terms of the flow induced by the principal symbol of $L^{1/2}$ on the Poisson manifold

$$\mathscr{W} = T^*P/G \tag{22}$$

(the Wong bundle, see [9, 29, 33, 34]), on the energy level $\sigma_L = c^2$. Thus our major result is a kind of Poisson formula, where the phase space is the Poisson manifold $\mathscr{W}$. Results on the geometry of the Wong flow, and its influence on $\mathrm{Tr}_G f(Q)$, are given in Section 5. The nature of the singularities is governed by the periodic orbits on the energy surface $\sigma_L = c^2$, with periods contained in the support of $\hat{f}$; the singular support of $\mathrm{Tr}_G f(Q)$ consists of elements $g \in G$ which move the initial point to the final point of an orbit in T^*P projecting over a periodic trajectory in $\mathscr{W}$. We note here the (initially) surprising result that, when G has rank $\geqslant 2$, isolated periodic orbits are not the rule; rather periodic orbits tend to come in families. This is treated in Proposition 5.7. Depending on the geometry of the Wong flow, $\hat{\mu}_s$ in (21) might have a simple asymptotic expansion derivable by the stationary phase method, or it might have a "nonclassical" asymptotic behavior as $|\lambda| \to \infty$.

Our main conclusions on the asymptotic behavior of $\tau(\lambda)$, defined by (19), are given in Section 6. In Theorem 6.1 we describe the behavior of $\tau(\lambda)$ when $\hat{f}$ is supported on an interval $(-T, T)$ containing no nontrivial periods of the Wong flow. In that case we show that $\tau(\lambda) = d_\lambda a(\lambda)$ with $a(\lambda) \in S^{m-1}(\mathfrak{t}^*)$, having leading term $a_0(\lambda)$ equal to $\hat{f}(0)$ times the Liouville volume of a natural geometrical object; here $m = \dim M = \dim P - \dim G$. Theorem 6.3 deals with situations where other periods of the Wong flow lead to Lagrangian singularities in the restriction of $\mathrm{Tr}_G f(Q)$ to T, possibly with a microlocal cutoff applied. In Section 6.3 we present a family of examples, involving particularly $G = U(2)$, illustrating some types of classical and non-classical asymptotics alluded to in the preceding paragraph.

We will also consider a generalization of (2),

$$H_\lambda = \hbar^2 H_\lambda^0 + i\hbar \pi_\lambda(X) + V, \tag{23}$$

where X is a section of the bundle $\mathfrak{g}_{ad} = P \times_{ad} \mathfrak{g}$ over M. The extra term $i\hbar\pi_\lambda(X)$ arises from what is called a Higgs field. Modifications necessary to treat this case will be discussed in Section 6.

2. Functions of Operators of Real Principal Type

We begin by establishing some notation. For every smooth manifold X, we will consider on the cotangent bundle of X, T^*X, the symplectic form

$$\omega = -d\theta, \tag{24}$$

where θ is the tautological one-form on T^*X. (In the notation of classical mechanics, $\theta = \sum p_j\, dq_j$ while $\omega = \sum dq_j \wedge dp_j$.) If $\Lambda \subset T^*X\backslash 0$ is a closed conic Lagrangian submanifold, we will use Hörmander's notation $I^m(X, \Lambda)$ for the spaces of Lagrangian distributions associated with Λ, see [20, Chap. XXV]. We will also use the following standard notation and terminology: A canonical relation $\mathscr{C}$ from the cotangent bundle of a manifold Y to the cotangent bundle of a manifold X is a submanifold, $\mathscr{C} \subset T^*X\backslash 0 \times T^*Y\backslash 0$, such that

$$\mathscr{C}' \stackrel{\text{def}}{=} \{(x, \xi; y, \eta) \in T^*X \times T^*Y; (x, \xi; y, -\eta) \in \mathscr{C}\} \tag{25}$$

is a Lagrangian submanifold. We will also use the following notation: if $\bar{x} = (x, \xi) \in T^*X$, we let $\bar{x}' = (x, -\xi)$. If $\mathscr{C}$ is a closed conic canonical relation from $T^*Y\backslash 0$ to $T^*X\backslash 0$, a Fourier integral operator associated with $\mathscr{C}$ is an operator $F: C_0^\infty(Y) \to C^\infty(X)$ whose Schwartz kernel belongs to one of the spaces $I^m(X \times Y; \mathscr{C}')$.

2.1. $f(Q)$ *as a Fourier integral operator*

Throughout this section, X will be a compact manifold. Let $Q \in OPS^1(X)$ be a self-adjoint operator whose (real) principal symbol has non-radial simple characteristics. Specifically, this means that if $q \in C^\infty(T^*X\backslash\{0\})$ is the principal symbol of Q, then zero is a regular value of q, and its Hamilton vector field H_q is nowhere radial on $\Sigma = q^{-1}(0)$. If $f: \mathbb{R} \to \mathbb{R}$ is bounded and continuous, $f(Q)$ is a bounded operator on $L^2(X)$ given by the spectral theorem. In this section we will show that in good cases $f(Q)$ is an FIO. The result of principal use for this paper is the following, already advertised in [2]:

PROPOSITION 2.1. *If $\hat{f} \in C_0^\infty(\mathbb{R})$, then*

$$f(Q) \in I^{-1/2}(X, X; \Lambda_f'), \tag{26}$$

where

$$\begin{aligned}&(x, y; \xi, \eta) \in \Lambda_f \\ &\quad \Leftrightarrow q(x, \xi) = 0 \text{ and } \exists t \in \operatorname{supp}(\hat{f}) \text{ such that } \phi_t(x, \xi) = (y, \eta),\end{aligned} \tag{27}$$

where $\{\phi_t\}$ denotes the Hamilton flow of q.

The set Λ_f, as defined by (27), is a closed, immersed canonical relation with boundary. However, the Schwartz kernel of $f(Q)$ will be microlocally supported in the interior of Λ. With some care in defining symbols, the standard theory extends to such operators, see below.

While we will not make direct use of it in this paper, we also note the following result.

PROPOSITION 2.2. *If $f \in S^m_{1,0}(\mathbb{R})$, and $\hat{f}$ has compact support, then*

$$f(Q) \in I^{-1/2, m+1/2}_{1,0}(X, X; \Delta', \Lambda'). \tag{28}$$

Here Δ' is the graph of the identity canonical transformation on $T^*X \setminus \{0\}$, Λ' is defined by (27), and the class in (28) is the class of Fourier integral operators associated with a pair of cleanly intersecting Lagrangians, as studied in [22, 11]. The proof we give here parallels arguments for the case when Q is elliptic given by Taylor [30, 31], and Colin de Verdière [4]. We begin with the identity

$$f(Q) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(t)\, e^{itQ}\, dt. \tag{29}$$

Considering the group of Fourier integral operators e^{itQ}, we see that if the support of $\hat{f}$ is in $(-\varepsilon, \varepsilon)$, ε small, then for a given $u \in \mathcal{D}'(X)$, $f(Q)u$ moves the wave front set of u by a small amount. This enables us to localize the analysis and reduce the problem to the model case $Q = D_1 = -i\partial/\partial x_1$ on $\mathbb{R}^n$, $n = \dim(X)$.

LEMMA 2.3. *Propositions* (2.1) *and* (2.2) *hold for the operator $Q = D_1 = i\partial/\partial x_1$ acting on distributions on $\mathbb{R}^n$.*

Proof. Since on the Fourier transform side D_1 is the operator of multiplication by ξ_1, the Schwartz kernel K_f of $f(D_1)$ is

$$K_f(x, y) = \int e^{i[(x_1 - y_1)\xi_1 + (x' - y')\xi']} f(\xi_1)\, d\xi, \tag{30}$$

where we are splitting the variables $x = (x_1, x')$ and $\xi = (\xi_1, \xi')$. This expression does not exhibit K_f as an oscillatory integral of the standard type, because $f(\xi_1)$ is not a symbol (as a function of (ξ_1, ξ')). If we do the $d\xi_1$ integral, we obtain

$$K_f(x, y) = \int e^{i(x' - y')\cdot\xi'} \hat{f}(y_1 - x_1)\, d\xi'. \tag{31}$$

Equation (26) follows immediately from this. Now (30) is of the form (2.1) in [6], namely of the form

$$k(x, y) = \int e^{i[(x_1 - y_1 - s)\xi_1 + (x' - y')\xi' + s\sigma]} a(x, y, s, \xi, \sigma)\, d\sigma\, ds\, \delta\xi, \tag{32}$$

with

$$a(x, y, s, \xi, \sigma) = f(\sigma). \tag{33}$$

Generally, (32) defines $k \in I^{p,l}(\mathbb{R}^n, \mathbb{R}^n; \Delta', \Lambda_1')$ provided $a \in S^{p',l'}$ with $p' = p + \frac{1}{2}$ and $l' = l - \frac{1}{2}$, which means that a satisfies estimates of the form

$$|D_\xi^\alpha D_\sigma^\beta D_{x,y,s}^\gamma a| \leqslant C_{\alpha,\beta,\gamma} \langle \xi \rangle^{p' - |\alpha|} \langle \sigma \rangle^{l' - |\beta|}. \tag{34}$$

These estimates are certainly satisfied by (33) if f satisfies the assumptions in Proposition 2.2 (with $l' = m$, $p' = 0$), which finishes the proof. ∎

We recall a few additional facts on the distribution k defined by (32), with (34). It turns out that

$$k \in I^{p+l}(\mathbb{R}^n, \mathbb{R}^n; \Delta' \backslash \Gamma'), \tag{35}$$

microlocally near $\Delta' \backslash \Gamma'$, where $\Gamma = \Delta \cap \Lambda_1$, and

$$k \in I^p(\mathbb{R}^n, \mathbb{R}^n; \Lambda_1' \backslash \Gamma') \tag{36}$$

microlocally near $\Lambda' \backslash \Gamma'$. Moreover, according to [1, 6, 7],

$$I^{p,l} \circ I^{p',l'} \subset I^{p + p' + 1/2, l + l' - 1/2} \tag{3.7}$$

while each $A \in I^{p,l}$ defines a bounded operator

$$A\colon H^s_{\text{comp}} \to H^{s+s_0}_{\text{loc}} \tag{38}$$

provided $\max(p+\frac{1}{2}, p+l) \leqslant -s_0$. Our definition of order of $f(Q)$ is consistent with that of [6], except for an apparent misprint: in [6] the authors write $p' = p - n/2 - \frac{1}{2}$, which seems inconsistent with (35) and (36).

Let us now prove the propositions. The operator Q of real principal type is known to be microlocally conjugate to D_1. More precisely, for $WF(u)$ in a sufficiently small conic subset Γ of $T^*X\backslash\{0\}$, there exists a Fourier integral operator, V, elliptic on a neighborhood of Γ, such that

$$Ve^{itQ}u = e^{itD_1}Vu \qquad \text{mod } C^\infty \tag{39}$$

for $t \in (-\varepsilon, \varepsilon)$. Therefore, for such u, we have

$$Vf(Q)u = f(D_1)\, Vu \qquad \text{mod } C^\infty. \tag{40}$$

Writing a general $u \in \mathscr{D}'(X)$ as a finite sum of distributions with small wave front sets and using lemma (2.3), we see that there is ε small enough so that (26) and (28) hold for all functions f with $\operatorname{supp} \hat{f} \subset (-\varepsilon, \varepsilon)$. Using a partition of unity on $\mathbb{R}$ to write a general compactly supported $\hat{f}$ as a finite sum of terms each supported in a small interval, we can write

$$f(Q) = \sum_{j=1}^{N} e^{it_jQ} f_j(Q), \qquad \operatorname{supp} \hat{f}_j \subset (-\varepsilon, \varepsilon), \tag{41}$$

which leads to the proof of the propositions. We end by noting that the special case of Proposition 2.2 where $f(Q) = Q^\lambda$, $\operatorname{Re} \lambda = m$, is given in [1].

We now turn to a description of the half-density part of the symbol of the operator $f(Q)$, when $\hat{f} \in C_0^\infty$.

2.2. *Symbolic Calculus*

When describing the symbol of $f(Q)$, it is necessary to be more precise about the immersed canonical relation Λ_f associated with $f(Q)$. We pause to describe this in more general terms, for future reference.

Let X be an n-dimensional manifold, and $\tilde{X} = T^*X\backslash\{0\}$. First of all, if $\Theta \subset \tilde{X}$ is a conic Lagrangian submanifold which is not necessarily closed, denote by $I^m(X, \Theta)$ the space of distributions which are microlocally supported in the *interior* of Θ and satisfy the standard estimates defining the Hörmander spaces $I^m(X; \Theta)$ in case Θ is closed. Now let Λ be an n-dimensional manifold together with a free action of the multiplicative group $\mathbb{R}^+$. We will say that Λ is conic, and, more generally, a subset of Λ will be called conic iff it is invariant under the $\mathbb{R}^+$ action. We also give ourselves a smooth map,

$$\Phi: \Lambda \to \tilde{X}, \tag{42}$$

and assume that (a) it is a Lagrangian immersion, (b) it intertwines the given $\mathbb{R}^+$ action on Λ with fiber multiplication on $\tilde{X}$, and (c) it has clean self-intersections.

DEFINITION 2.4. A distribution $u \in \mathscr{D}'(X)$ will be said to be in $I^m(X; \Lambda, \Phi)$ iff there is a finite collection of open conic subsets of Λ, $\{U_j\}$, such that (i) for each j the restriction of Φ to U_j is an embedding, and (ii) there are distributions $u_j \in I^m(X; \Phi(U_j))$ such that

$$u = \sum_j u_j. \tag{43}$$

The condition on the self-intersection of Φ to be clean enables us to define the symbol of $u \in I^m(X; \Lambda, \Phi)$ as a half-density on Λ with values on a version of the Maslov line bundle. This is based on the following lemma.

LEMMA 2.5. *Let $U_1, U_2 \subset \Lambda$ be open conic subsets such that the restriction of Φ to each of them is an embedding. Assume that $u_j, u_j' \in I^{\cdot}(X; \Phi(U_j))$ are such that $u_1 + u_2 = u_1' + u_2'$. Then $u_j = u_j'$ modulo smooth functions, $j = 1, 2$, provided $\Phi(U_1) \cap \Phi(U_2)$ has positive codimension in $\Phi(U_1)$.*

Proof. Letting $v_j = u_j - u_j'$, we suppose $v_1 + v_2 = 0$. Then the principal symbol of v_1 vanishes on $\Phi(U_1) \backslash \Phi(U_2)$, hence on all of $\Phi(U_1)$ by continuity, since by assumption v_j have classical symbols. Similarly, the principal symbol of v_2 vanishes on all of $\Phi(U_2)$. By induction, the complete symbols of v_j also vanish. ∎

Having set up these general definitions, let us go back to the operators $f(Q)$ of the beginning of this section. Let $\Sigma = q^{-1}(0) \subset \tilde{X}$,

$$\Lambda = \Sigma \times \mathbb{R}, \tag{44}$$

and

$$\Phi\colon \tilde{X} \times \mathbb{R} \to \tilde{X} \times \tilde{X} \tag{45}$$

be defined by $\Phi(x, \xi; t) = (x, \xi; \phi_t(x, \xi)')$. Then, by Proposition 2.1, for each f with $\hat{f} \in C_0^\infty(\mathbb{R})$, $f(Q) \in I^{-1/2}(X \times X; \Lambda, \Phi)$. We now describe the symbol of this operator.

LEMMA. 2.6. *The half-density part of the symbol of $f(Q)$ is*

$$\hat{f}(t)\, |\sigma|^{1/2} \otimes |dt|^{1/2}, \tag{46}$$

where σ denotes the Liouville measure on Σ.

Proof. Appealing once again to the microlocal normal form for operators of real principal type, it is enough to prove the lemma for the model operator $Q = D_1$ on $\mathbb{R}^n$. Again, the result is trivial in the model case. ∎

2.3. *A Condition for H_q to be Non-radial*

We now study when the operator we are primarily interested in, namely $Q = (-L)^{-1/2}(-L - cA)$ of Section 1, has non-radial simple characteristics. It is immediate that this is the case iff the principal symbol $p(x, \xi)$ of the operator $-L - cA$ has a nowhere-radial Hamilton vector field, H_p. We will give a useful criterion for this condition to hold.

Note that $L + cA = L_0 + [V(x) - c]\,\Delta_G^P + (\text{lower order})$, where $L_0 = \Delta - \Delta_G^P$ is doubly characteristic on the conormal space to the horizontal lifts of TM to TP, while Δ_G^P is doubly characteristic on $\mathscr{H}^*P$, the conormal bundle to the fibers of $P \to M$. Thus the principal symbol $p(x, \xi)$ of $-L - cA$ has the following form, which we will study. Consider now a symbol $p(x, \xi)$ which is a homogeneous polynomial of degree 2 in ξ, and of the form

$$p(x, \xi) = a(x, \xi) + b(x)\, c(x, \xi). \tag{47}$$

We assume

$$0 \text{ is a regular value of } b, \tag{48}$$

which of course implies that $\Sigma = b^{-1}(0)$ is a smooth manifold. For $-L - cA$, (48) amounts to the assumption that c is a regular value of V. We also suppose

$$a(x, \xi) \geqslant 0, \qquad c(x, \xi) \geqslant 0. \tag{49}$$

We will make the following further hypothesis, satisfied by $-L - cA$, on the nature of $a(x, \xi)$ and $c(x, \xi)$. Namely, we suppose that at each $x \in P$, T_x^*P splits as $V_{1x} \oplus V_{2x}$; write $\xi = (\xi', \xi'')$ in this splitting. We suppose $a(x, \xi)$ is a positive definite quadratic form in ξ', and $c(x, \xi'')$ a positive definite quadratic form in ξ''. We call this "hypothesis S."

Under hypotheses (47)–(49) and S, we have that p is elliptic where $b(x) > 0$. For $b(x) < 0$, $p(x, \xi)$ is a non-degenerate quadratic form in ξ. Thus $d_\xi p$ can vanish (with $\xi \neq 0$) only over Σ. We will establish the following.

Proposition 2.7. *Under hypothesis (47)–(49) and S, $p(x, \xi)$ has simple characteristics, on which H_p is nowhere radial, provided*

$$\operatorname{Char}(a(x, \xi)) \cap N^*\Sigma \setminus \{0\} = \varnothing. \tag{50}$$

Proof. We only have to check whether $d_x p(x, \xi)$ can be proportional to the canonical one form $\alpha = \sum_j \xi_j \, dx_j$, for some (x, ξ) in the characteristic set of p with $x \in \Sigma$, i.e., in case

$$(x, \xi) \in Z = \{(x, \xi) \in \text{Char } a \colon x \in \Sigma\}. \tag{51}$$

Note that Z is the zero set of $d_\xi p$. Now

$$d_x p(x, \xi) = c(x, \xi) \, db(x) \tag{52}$$

on Z. Hence if dp and α are parallel at $(x, \xi) \in Z$, it must be that $(x, \xi) \in N^*\Sigma$. Under hypothesis (50), there are not any such (x, ξ), and conversely. ∎

Note that this proposition applies to $p(x, \xi) = \xi_1^2 + x_1 \xi_2^2$, but not to $\xi_1^2 + x_2 \xi_2^2$.

COROLLARY 2.8. *The operator Q given by* (18), *with L and A given by* (5), (6), *is of real principal type as long as c is a regular value of $V \in C^\infty(M)$.*

3. THE G-TRACE

In [26, 27], the G-trace was defined for the action of a compact Lie group G on a principal bundle $P \to M$ for a class of pseudodifferential operators on P, and the G-trace was analyzed as a pseudodifferential operator on G. The operators in question were the $A \in OPS^m(P)$ with complete symbol vanishing to infinite order on the conormal bundle $\mathscr{H}^*P$ to the fibers of $P \to M$. Here we extend this analysis to a class of Fourier integral operators.

3.1. *Generalities*

We begin with the definition of the G-trace, which we present here in a more general context than that indicated above. Namely, let $\{U(g) \colon g \in G\}$ be a unitary representation of a Lie group G on a Hilbert space $\mathscr{H}$, and B a bounded operator on $\mathscr{H}$. Roughly, the G-trace of B is that function on G defined by the formula

$$\text{Tr}_G \, B(g) = \text{Tr}[U(g) \circ B]. \tag{53}$$

This is clearly well defined if B is of trace class, but we want to consider other cases, in which (53) leads to a distribution on G. Thus (53) is a formal description for an object whose precise definition is the following: for $v \in C_0^\infty(G)$,

$$\langle v, \text{Tr}_G \, B \rangle = \text{Tr}[U(v) \circ B]. \tag{54}$$

The condition required on B for this to make sense is that the map $C_0^\infty(G) \to \mathscr{L}(H)$ given by $v \mapsto U(v)B$ be a continuous map from $C_0^\infty(G)$ into the Banach space of trace class operators on $\mathscr{H}$. Note that, if $w \in C_0^\infty(G)$, we have

$$\langle (\mathrm{Tr}_G B) * v, w \rangle = \mathrm{Tr}\, U(w * \check{v}) B, \tag{55}$$

where $\check{v}(g) = v(g^{-1})$.

We are primarily interested in the case where B commutes with all the $U(g)$. In that case we have:

PROPOSITION 3.1. *Assume that, $\forall g \in G$, B and $U(g)$ commute. Then $\mathrm{Tr}_G B$ is central, that is*

$$\forall g, g_1 \in G, \qquad \mathrm{Tr}_G B(g^{-1} g_1 g) = \mathrm{Tr}_G B(g_1). \tag{56}$$

Furthermore, if G is compact, B is trace class, $\{\pi_\lambda\}$ denotes the (equivalence classes of f) irreducible unitary representations of G and $\{\chi_\lambda\}$ their characters, then

$$\mathrm{Tr}_G B(g) = \sum_{\pi_\lambda} d_\lambda^{-1} (\mathrm{Tr}\, B|_{\mathscr{D}_\lambda})\, \chi_\lambda(g), \tag{57}$$

where $\mathscr{D}_\lambda$ denotes the maximal subspace of $\mathscr{H}$ on which G acts by copies of π_λ, and d_λ is the dimension of π_λ.

Proof. The first statement follows from the following calculation. In all generality,

$$\mathrm{Tr}_G B(g g_1) = \mathrm{Tr}\, U(g)\, U(g_1) B = \mathrm{Tr}\, U(g_1)\, B U(g). \tag{58}$$

Now if U and B commute, this is equal to $\mathrm{Tr}\, U(g_1)\, U(g) B$, which is to say, to $\mathrm{Tr}_G B(g_1 g)$, and (56) follows. For the proof of the second part, recall that the orthogonal projector, $P_\lambda \colon \mathscr{H} \to \mathscr{D}_\lambda$, is equal to

$$P_\lambda = d_\lambda \int_G \overline{\chi_\lambda(g)}\, U(g)\, dg. \tag{59}$$

Thus

$$\mathrm{Tr}[B|_{\mathscr{D}_\lambda}] = d_\lambda \int_G \mathrm{Tr}_G B(g)\, \overline{\chi_\lambda(g)}\, dg. \tag{60}$$

Since $\mathrm{Tr}_G B$ is central, (57) follows from (60). ∎

Having given a general description of the G-trace, we consider some examples. Our first example is not of direct relevance to the main theme of this paper, but it indicates one of several other contexts in which one could study the G-trace. Namely, U could be an irreducible unitary representation of a (noncompact) semi-simple Lie group G. Then, as is known [32], $U(v)$ itself is trace class for any $v \in C_0^\infty(G)$, so $\operatorname{Tr}_G B$ is defined for any bounded operator B on $\mathscr{H}$. In this case, the distributional trace of U is known to be in $L^1_{\mathrm{loc}}(G)$, so $U(v)$ is trace class for any $v \in L^\infty_{\mathrm{comp}}(G)$. From (55) it follows that, for any bounded B on $\mathscr{H}$, convolution with $\operatorname{Tr}_G B$ (on the left) maps $L^2_{\mathrm{comp}}(G) \to L^2_{\mathrm{loc}}(G)$.

Cases of greatest interest to us at present involve those in which U arises from a (right) action of G as a group of isometries of a Riemannian manifold X; U acts on $L^2(X)$ as

$$U(g)\, f(x) = f(x \cdot g), \qquad x \in X, \qquad g \in G. \tag{61}$$

If $R_g\colon X \to X$ is the map $R_g(x) = x \cdot g$, then we are assuming that R_e is the identity and that $R_{g_1} \circ R_{g_2} = R_{g_2 g_1}$ for all $g_1, g_2 \in G$, so that $g \mapsto U(g)$ is a group homomorphism. We suppose that B has a (distributional) kernel $b(x, y)$, so that

$$Bf(x) = \int_X b(x, y)\, f(y)\, dV(y). \tag{62}$$

Then, formally, $\operatorname{Tr}_G B$ is given by

$$\operatorname{Tr}_G B(g) = \int_X b(x \cdot g, x)\, dV(x), \tag{63}$$

or, more precisely

$$\langle \operatorname{Tr}_G B, v \rangle = \int_X \int_G v(g)\, b(x \cdot g, x)\, dg\, dV(x). \tag{64}$$

The condition that B commute with U is

$$\forall x, y \in X, \qquad g \in G, \qquad b(x, y) = b(x \cdot g, y \cdot g). \tag{65}$$

For such representations U, $U(v)$ is of trace class for every $v \in C_0^\infty(G)$ provided X is compact and $U(v)$ has a smooth distributional kernel, which happens if G acts transitively on X. In such a case, $\operatorname{Tr}_G B$ is well defined for all B bounded on $L^2(X)$, indeed for any $B\colon L^2(X) \to \mathscr{D}'(X)$. This case is not disjoint from the case of irreducible representations U of semi-simple G, since principal series representations arise in the form considered in the last paragraph. In such a case as the principal series, we would want to consider a generalization, to G-actions on X not preserving a volume element, in which case square roots of Jacobians appear in (61) and (63).

In case $X = P$ is a principal bundle, the transitivity condition mentioned above does not hold. Restrictions on B are required to assure that $U(v)B$ is of trace class for every $v \in C^\infty(G)$. We consider this in the next section.

3.2. *Microlocal Construction of* $\mathrm{Tr}_G\, B$

Let us now investigate the construction of the G- trace of an operator from the microlocal point of view. We place ourselves in the following setting, somewhat more general than the one have just considered. Let X be a compact manifold, endowed with a smooth positive density, dV, and assume that G acts on X on the right,

$$\begin{aligned} &X \times G \to X \\ &(x, g) \to x \cdot g, \end{aligned} \tag{66}$$

with $x \cdot e = x$ and $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$, and preserving the density dV. Then G is unitarily represented in $\mathscr{H} = L^2(X, dV)$, by (61). We will think of the representation as a single operator $\mathscr{U}$ from $C^\infty(X)$ to $C^\infty(X \times G)$, by setting

$$\forall f \in C^\infty(X), \qquad \mathscr{U}(f)(x, g) = f(x \cdot g). \tag{67}$$

The Schwarz kernel of $\mathscr{U}$ is a distribution on $(X \times G) \times X$; it is clearly a delta function along the graph of the action. More precisely, let

$$\mathscr{G} = \{(x, g, y) \mid y = x \cdot g\} \subset X \times G \times X. \tag{68}$$

Then the Schwartz kernel of $\mathscr{U}$ is a delta distribution along $\mathscr{G}$, and in particular is conormal with respect to $\mathscr{G}$. We can write the Schwartz kernel of $\mathscr{U}$ symbolically as follows:

$$\mathscr{U}(x, g, y) = \delta(y - x \cdot g). \tag{69}$$

For future reference, we now describe the conormal bundle of $\mathscr{G}$. In order to do this, we need to (i) lift the action of G to the punctured cotangent bundle of X, $\tilde{X} = T^*X \backslash 0$, and (ii) introduce the moment map of the lifted action. We refer to [10] for details of what follows. To avoid introducing cumbersome notation, given $g \in G$ we will denote by the same letter the diffeomorphism of X defined by g and the action. This diffeomorphism has a natural lift to $\tilde{X}$ defined by the recepie

$$\forall x \in X, \qquad \xi \in T_x^*X \qquad (x, \xi) \cdot g = (x \cdot g, d(g)_x^{*\,-1}(\xi)). \tag{70}$$

The lifted diffeomorphism is symplectic, and if we restrict the lifted action to a one-parameter subgroup of G, the resulting flow is in fact Hamiltonian.

The moment map is a way to describe the corresponding Hamiltonians, all at once. More precisely, let $\mathfrak{g}$ be the Lie algebra of G, and, for each $A \in \mathfrak{g}^*$, let $A^\#$ denote the vector field on X defined by A and the action of G:

$$A^\#_x = \frac{d}{dt} x \cdot (\exp tA)|_{t=0}. \tag{71}$$

Then the moment map referred to above is the map $\Phi\colon \tilde{X} \to \mathfrak{g}^*$ defined by the identity

$$\forall A \in \mathfrak{g}, \qquad (x, \xi) \in \tilde{X}, \qquad \langle \Phi(x, \xi), A \rangle = \langle A^\#_x, \xi \rangle; \tag{72}$$

we cite a couple of its properties.

LEMMA 3.2. (a) $\forall A \in \mathfrak{g}$ *the one-parameter subgroup of symplectomorphisms of* $\tilde{X}$ *which is the lifting of* $x \mapsto x \cdot \exp(tA)$ *has the function*

$$(x, \xi) \mapsto \langle \Phi(x, \xi), A \rangle$$

for Hamiltonian, and (b) *for all* (x, ξ) *and* g,

$$\Phi((x, \xi) \cdot g) = \mathrm{Ad}^*_g \, \Phi(x, \xi). \tag{73}$$

Here Ad^*_g is the transpose of the adjoint representation $\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$ given by $A \mapsto d/dt[g(\exp tA)\, g^{-1}]|_{t=0}$. We do not get that Φ is equivariant with respect to the standard co-adjoint representation $g \mapsto \mathrm{Ad}^*_{g^{-1}}$ because we are working with a right action. The proof of (73) is an easy exercise; for the proof of (a) see [10, Eq. (29.2), p. 221].

As mentioned above, the moment map enters into the description of the conormal bundle to the graph $\mathscr{G}$ of the action:

LEMMA 3.3. *Identify the contangent bundle of the group,* T^*G, *with* $G \times \mathfrak{g}^*$ *using left translations. Then the conormal bundle of* $\mathscr{G}$ *is equal to*

$$N^*\mathscr{G} = \{(x, \xi;\, g, \mathrm{Ad}^*_g \, \Phi(x, \xi);\, (x, -\xi) \cdot g);\, g \in G,\, (x, \xi) \in \tilde{X}\}. \tag{74}$$

Let us now look at the construction of the G-trace of an operator on X, not necessarily commuting with the action of G. We begin with smoothing operators. Let $b \in C^\infty(X \times X)$, and denote by B the corresponding smoothing operator which of course is of trace class since X is compact.

LEMMA 3.4.

$$\mathrm{Tr}_G \, B(g) = \int_X b(x \cdot g, x)\, dV(x). \tag{75}$$

Proof. The Schwartz kernel of the composition $U(g)B$ is the smooth function

$$U(g)\, B(x, y) = b(x \cdot g, y), \tag{76}$$

as one can easily check. As it is well known that the trace of a smoothing operator on a compact manifold is obtained by integrating its Schwartz kernel along the diagonal, (75) follows. ∎

Let us denote by $K: C^\infty(X \times X) \to C^\infty(G)$ the operator defined by (75), that is, $K(b) = \mathrm{Tr}_G\, B$.

LEMMA 3.5. *As a distribution on $G \times X \times X$, the Schwartz kernel of K is obtained from that of $\mathscr{U}$ by the following permutation of the variables:*

$$K(g, x, y) = \mathscr{U}(y, g, x). \tag{77}$$

Hence the Schwartz kernel of K is a Lagrangian distribution in the space $I^{-d/4}(G \times X \times X; \mathscr{C}')$ where d is the dimension of G and $\mathscr{C}$ the canonical relation

$$\mathscr{C} = \{(g, \mathrm{Ad}_g^*\, \Phi(x, \xi); (x, \xi) \cdot g; x, -\xi);\ g \in G, (x, \xi) \in \tilde{X}\}. \tag{78}$$

Proof. One has

$$\mathrm{Tr}_G\, B(g) = \int_X dV(x) \int_X b(y, x)\, \mathscr{U}(g, x, y)\, dV(y)$$

$$= \int_{X \times X} \mathscr{U}(g, y, x)\, b(x, y)\, dV(x)\, dV(y). \tag{79}$$

Interchanging the variables and "priming" in $T^*(X \times X)$ transforms $N^*\mathscr{G}$ into (78). ∎

We are now ready to discuss the problem of constructing the G-trace of more general operators on X. Notice that $\mathscr{C}$ is *not* contained in $(T^*G - \{0\}) \times (T^*(X \times X) - \{0\})$, which reflects the fact that K does *not* extend to all of $\mathscr{D}'(X \times X)$. The problem arises from the points in $\Phi^{-1}(0)$; microlocally away from this set K is a regular Fourier integral operator. More precisely, we have:

COROLLARY 3.6. *Let $\Lambda \subset \tilde{X} \times \tilde{X}$ be a closed Lagrangian with the property that its projection into the second factor has empty intersection with $\Phi^{-1}(0)$. Then K has a continuous extension to*

$$K: I^m(X \times X; \Lambda) \to \mathscr{D}'(G), \tag{80}$$

that is, $\forall b \in I^m(X \times X; \Lambda)$, the G-trace of the corresponding operator B is a well-defined distribution $\mathrm{Tr}_G\, B \in \mathscr{D}'(G)$.

Of course much more is true; by the general composition theorem for Fourier integral operators, if suitable clean intersection hypotheses are satisfied, given $b \in I'(X \times X, \Lambda)$ with Λ as above, $\mathrm{Tr}_G\, B$ is in some space $I^{m'}(G; \Gamma)$ with $\Gamma \subset T^*G$ an immersed Lagrangian. Before discussing the clean intersection condition, let us see what Γ should be.

LEMMA 3.7. *Let $b \in I'(X \times X, \Lambda)$, where Λ satisfies the assumptions of Corollary* 3.6. *Then $WF\, \mathrm{Tr}_G\, B \subset \Gamma$, where*

$$\Gamma = \{(g, \gamma) \mid \exists \bar{x} \in \tilde{X} \text{ such that } (\bar{x} \cdot g, \bar{x}') \in \Lambda \text{ and } \gamma = \mathrm{Ad}_g^*\, \Phi(\bar{x})\}. \tag{81}$$

(*We have denoted points in $\tilde{X}$ with an overbar to distinguish them from points in X.*) *Hence $g \in G$ is in the projection of Γ iff*

$$N^*\{(x \cdot g, x); x \in X\} \cap \Lambda \neq \varnothing. \tag{82}$$

Proof. That the wave-front set of $\mathrm{Tr}_G\, B$ is contained in (81) follows from (78) and the calculus of wave-front sets. The second statement follows from the fact that

$$N^*\{(x \cdot g, x); x \in X\} = \{((x, \xi) \cdot g, (x, -\xi)); (x, \xi) \in \tilde{X}\}. \quad \blacksquare \tag{83}$$

The set Γ in general will not be connected, and in general $K(b)$ will be a Lagrangian distribution whose order may vary from one connected component of Γ to another.

Let us now discuss the clean intersection condition that ensures that $\mathrm{Tr}_G\, B$, $b \in I'(X \times X, \Lambda)$, is a Lagrangian distribution. Let

$$\mathscr{F} = \{(g, \bar{x}) \in G \times \tilde{X}; (\bar{x} \cdot g, \bar{x}') \in \Lambda\}. \tag{84}$$

Two bits of notation: for every $v \in T_{\bar{x}}\tilde{X}$, let $v' \in T_{\bar{x}'}\tilde{X}$ be the image of v under the differential of the diffeomorphism $\bar{y} \mapsto \bar{y}'$, and we continue to identify $TG \cong G \times \mathfrak{g}$ using left translations.

THEOREM 3.8. *Let Λ satisfy the hypothesis of Corollary* 3.6, *and assume furthermore that*

(a) *$\mathscr{F}$ is a submanifold of $G \times \tilde{X}$, and*

(b) *At every $(g, \bar{x}) \in \mathscr{F}$, the tangent space to $\mathscr{F}$ is equal to the set of all $(A, v) \in \mathfrak{g} \times T_{\bar{x}}\tilde{X}$ such that*

$$(dg_{\bar{x}}(v) + A^\sharp_{\bar{x} \cdot g}, v') \in T_{(\bar{x} \cdot g, \bar{x}')}\Lambda. \tag{85}$$

Then $\forall b \in I^m(X \times X; \Lambda)$ *one has* $\mathrm{Tr}_G B \in I^s(G, \Gamma)$, *where* $s = m - 3(\dim G)/4 + (\dim \mathscr{F})/2$.

Proof. The clean intersection condition ensuring that $\mathrm{Tr}_G B$ is a Lagrangian distribution [20], is that the following should be a clean intersection diagram:

$$\begin{array}{ccc} \mathscr{F}^{\#} & \longrightarrow & \mathscr{C} \\ \downarrow & & \downarrow \\ T^*G \times \Lambda & \hookrightarrow & T^*G \times \tilde{X} \times \tilde{X}. \end{array} \tag{86}$$

Here $\mathscr{F}^{\#} \subset T^*G \times \tilde{X} \times \tilde{X}$ is the set of all $(g, \mathrm{Ad}_g^* \Phi(\bar{x}); \bar{x} \cdot g; \bar{x}')$ such that $(\bar{x} \cdot g, \bar{x}') \in \Lambda$. Now the map

$$\begin{aligned} &G \times \tilde{X} \to T^*G \times \tilde{X} \times \tilde{X} \\ &(g, \bar{x}) \mapsto (g, \mathrm{Ad}_g^* \Phi(\bar{x}); \bar{x} \cdot g; \bar{x}') \end{aligned} \tag{87}$$

is an embedding whose image is $\mathscr{C}$, and it induces a diffeomorphism of sets, $\mathscr{F}^{\#} \cong \mathscr{F}$. Thus $\mathscr{F}^{\#}$ is a submanifold iff $\mathscr{F}$ is, and (85) is simply the translation under (87) of the condition that (86) be clean. For the calculation of the order, note that the excess of the diagram (86) is equal to $\dim \mathscr{F} - \dim G$. ∎

The case we are immediately concerned with is when the action of G on $\tilde{X}$ is free; we will assume this is the case in the remainder of this section. This of course includes the case where $X = P$ is a principal G bundle, but it is somewhat more general (think of the action of S^1 on the punctured cotangent bundle of the two sphere by rotations around the z axis). We will keep the notation $\tilde{X} = T^*X - \{0\}$. Let Λ satisfy the assumption of Corollary 3.6, and let

$$\mathscr{I} = \{(\bar{x} \cdot g, x'); \bar{x} \in \tilde{X}, g \in G\}. \tag{88}$$

Notice that, because the action of G on $\tilde{X}$ is free, $\mathscr{I}$ is a smooth submanifold of $\tilde{X} \times \tilde{X}$, in fact diffeomorphic to $G \times \tilde{X}$ via the map $(g, \bar{x}) \mapsto (\bar{x} \cdot g, \bar{x}')$.

Recall that, if (M, ω) is a symplectic manifold, a submanifold $\Sigma \subset M$ is called *co-isotropic* iff $\forall x \in \Sigma$ the tangent space $T_x\Sigma$ contains its symplectic orthogonal. Then there is a smooth foliation (called the null foliation) of Σ with the property that $\forall x \in \Sigma$ the tangent space to the leaf through x is $T_x\Sigma^{\perp}$. With this terminology, we can re-state the clean intersection condition of the previous theorem as follows:

PROPOSITION 3.9. *$\mathscr{I}$ is a co-isotropic submanifold of the symplectic manifold $\tilde{X} \times \tilde{X}$, and the map*

$$
\begin{aligned}
F: \mathscr{I} &\to G \times g^* \cong^L T^*G \\
(\bar{x} \cdot g, \bar{x'}) &\mapsto (g, \mathrm{Ad}_g^* \Phi(\bar{x})),
\end{aligned}
\tag{89}
$$

where $G \times \mathfrak{g}^* \cong^L T^*G$ is the trivialization using left translations, is a submersion whose fibers are finite unions of leaves of the null foliation of $\mathscr{I}$. Moreover, the clean intersection condition [(a) *and* (b)] *of Theorem* 3.8 *holds iff $\mathscr{I}$ and Λ intersect cleanly, in which case the restriction of F to $\mathscr{I} \cap \Lambda$ is a map locally of constant rank whose image is the immersed Lagrangian Γ of* (81).

Proof. The tangent space to $\mathscr{I}$ at a point $(\bar{x} \cdot g, \bar{x}')$ is the set of all vectors of the form

$$
(dg_{\bar{x}}(v) + A^{\sharp}_{\bar{x} \cdot g}, v'), \qquad v \in T_{\bar{x}}\tilde{X}, \qquad A \in \mathfrak{g}. \tag{90}
$$

It follows easily that the symplectic orthogonal of this space consists of those vectors of the form $(dg_{\bar{x}}(w), w')$, where $w \in T_{\bar{x}}\tilde{X}$ satisfies

$$
\forall A \in \mathfrak{g}, \qquad \omega_{\bar{x}}(A^{\sharp}_{\bar{x}}, w) = 0. \tag{91}
$$

Such vectors are of the form (90), so $\mathscr{I}$ is co-isotropic. Moreover, the general theory of the moment map implies that (91) is equivalent to $d\Phi_{\bar{x}}(w) = 0$. For every $\alpha \in \mathfrak{g}^*$, define the Kostant–Kirillov skew-symmetric bilinear form Ω_α on $\mathfrak{g}$ by

$$
\forall A, B \in \mathfrak{g}, \qquad \Omega_\alpha(A, B) = \langle \alpha, [A, B] \rangle. \tag{92}
$$

Then a calculation shows that the differential of F is given by

$$
\begin{aligned}
&dF_{(\bar{x} \cdot g, \bar{x}')}(dg_{\bar{x}}(v) + A^{\sharp}_{\bar{x} \cdot g}, v')(\cdot) \\
&\quad = (A, \Omega_{\mathrm{Ad}_g^*(\bar{x})}(A, \cdot) + \mathrm{Ad}_g^*(d\Phi_{\bar{x}}(v))(\cdot)).
\end{aligned}
\tag{93}
$$

Hence the kernel of dF is at every point the symplectic orthogonal of $T\mathscr{I}$. Formula (93) shows that F is a submersion iff Φ is, and this follows from the assumption that the action of G on $\tilde{X}$ is free (it is known that the kernel of the transpose of $d\Phi_{\bar{x}}$ is the Lie algebra of the isotropy subgroup of $\bar{x}$). This proves the first part of the proposition.

To prove the second part, note that the embedding $(g, \bar{x}) \mapsto (\bar{x} \cdot g, \bar{x})$ from $G \times \tilde{X}$ to $\tilde{X} \times \tilde{X}$, which parametrizes $\mathscr{I}$, induces a diffeomorphism of sets $\pi: \mathscr{F} \to \mathscr{I} \cap \Lambda$. Thus $\mathscr{I} \cap \Lambda$ is a manifold iff $\mathscr{F}$ is. The condition on the tangent spaces is easily verified using the differential of π. To prove the last statement, we need a formula for the symplectic form on $G \times \mathfrak{g}^* \cong^L T^*G$.

Pick $(g, \alpha) \in G \times \mathfrak{g}^*$, and identify $T_g G \cong \mathfrak{g}$ using left translations. Then the symplectic form can be shown to be

$$\begin{aligned}(\omega_{T^*G})_{(g,\alpha)} &((A_1, \beta_1), (A_2, \beta_2)) \\ &= \langle \beta_2, A_1 \rangle - \langle \beta_1, A_2 \rangle + \Omega_\alpha(A_1, A_2). \end{aligned} \tag{94}$$

If we were working with *left* actions, we would have that the pull-back by F of this form is the restriction to $\mathscr{I}$ of the symplectic form on $T^*X \times T^*X$. However, since we are working with a right action, $F^*\Omega = (\omega)_{T^*X \times T^*X}|_{\mathscr{I}}$, where Ω is the two form on $G \times \mathfrak{g}^*$ obtained from (94) by changing the sign of the last term in the right-hand side. Thus the last statement follows from general facts of reduction of Lagrangian submanifolds with respect to co-isotropic submanifolds. ∎

3.3. *Restricting the G-trace to a Cone*

Assume the action of G on $X = P$ is free, so that $P \to M$ is a principal G-bundle. In the applications to particles in gauge fields, we will make dynamic assumptions on the Wong flow implying that the clean intersection condition of Theorem 3.8 holds if we replace T^*P by an open set of the form $\Phi^{-1}(U)$, with $U \subset \mathfrak{g}^*$ an Ad-invariant conic open set. As we will now see, this is enough to get a hold of $\mathrm{Tr}_G\, B|_{\mathscr{D}_\lambda}$, for λ in an invariant conic open set U_0 with $\overline{U_0} \subset U$. Recall that an element $\mu \in \mathfrak{g}^*$ is called regular iff its isotropy subgroup is a maximal torus.

THEOREM 3.10. *Let $U \subset \mathfrak{g}^*$ be an invariant, conic open set containing only regular elements, and let U_0 a smaller invariant conic open set whose closure is contained in U. Then there is a 0-order, self-adjoint, G-equivariant operator, $\mathscr{P}$, on $\mathscr{D}'(P)$, with the following three properties:* (i) *$\mathscr{P}$ is a pseudodifferential operator microlocally away from the conormal bundle to the fibers of $P \to M$,* (ii) *$\mathscr{P}$ is microlocally supported in $\Phi^{-1}(U)$, and* (iii) *for every integral element $\mu \in U_0$, $\mathscr{P}|_{\mathscr{D}_\mu}$ is the identity.*

Proof. By standard results, established in Chapter XII, Section 6 of [30], there exists a bi-invariant $\mathscr{P}_0 \in OPS^0(G)$ microlocally supported in U and equal to the identity on the linear span in $C^\infty(G)$ of all matrix elements of representations π_μ, $\mu \in U_0$. If we denote by π_P the natural action of G on $L^2(P)$, then $\mathscr{P} = \pi_P(\mathscr{P}_0)$ is the desired operator. ∎

If we now apply Theorem 3.6 to an operator of the form $B \circ \mathscr{P}$, we obtain the following result:

COROLLARY 3.11. *Let $\Lambda \subset \tilde{X} \times \tilde{X}$ ($\tilde{X} = T^*P \backslash \{0\}$) be a closed homogeneous canonical relation satisfying the hypotheses of Corollary 3.6. Let U*

and U_0 be as in Theorem 3.10. Assume furthermore that the clean intersection condition of Theorem 3.8 is met, with $\tilde{X}$ replaced by $\Phi^{-1}(U)$. Then, $\forall b \in I^m(P \times P, \Lambda')$ such that the associated operator B commutes with G, there exists a central distribution $\nu \in I^s(G, \Gamma_U)$ having the same Fourier coefficients as $\mathrm{Tr}_G B$ on every integral $\mu \in U_0$, namely, $\mathrm{Tr}\, B|_{\mathscr{D}_\mu}$. Here $s = m - 3(\dim G)/4 + (\dim \mathscr{F})/2$, and

$$\Gamma_U = \{(g, \gamma);\ \exists \bar{x} \in \Phi^{-1}(U) \text{ such that } (\bar{x} \cdot g, \bar{x}') \in \Lambda \text{ and } \gamma = \mathrm{Ad}_G^* \Phi(\bar{x})\}. \tag{95}$$

Of course, $\nu = \mathrm{Tr}_G(B \circ \mathscr{P})$.

4. Fourier Analysis of the G-Trace

4.1. *Generalities*

Let $\nu \in \mathscr{D}'(G)$ be a central Lagrangian distribution. Keeping the notation of Section 3, let $\{\pi_\lambda\}$ be the set of irreducible unitary representations of G and $\{\chi_\lambda\}$ the corresponding set of characters. Here we assume that we have chosen a maximal torus, $T \subset G$, and an ordering of the roots, and λ belongs to the highest weight lattice (intersected with a Weyl chamber) in $\mathfrak{t}^*$. As any central distribution, ν can be written in the form

$$\nu = \sum_\lambda \langle \nu, \chi_\lambda \rangle\, \chi_\lambda. \tag{96}$$

In this section we discuss how the asymptotic behavior of $\langle \nu, \chi_\lambda \rangle$ as $|\lambda| \to \infty$ is governed by the microlocal picture of ν. Our primary interest is when ν is the G-trace of an operator commuting with G. We can often analyze the behavior of its Fourier coefficients in the interior of the positive Weyl chamber.

We begin by making the following general remarks. Assume ν is any central distribution in $H^s(G)$ with $s > d/2$, $d = \dim(G)$. Then $\nu \in C(G)$; let $\nu^\sharp$ denote the restriction of ν to T. By the Weyl integration formula,

$$\langle \nu, \chi_\lambda \rangle = |W|^{-1} \langle \nu^\sharp, |D|^2 \chi_\lambda \rangle_T. \tag{97}$$

Here D is the Weyl denominator,

$$D(g) = \sum_{w \in W} (\det w)\, e_{w(\delta)}(g), \tag{98}$$

where $e_\lambda \colon T \to S^1$ is the character of T with differential $2\pi\lambda$, W is the Weyl group, and δ is half the sum of the positive roots. If we introduce now in (97) the Weyl character formula,

$$\forall g \in T, \qquad \chi_\lambda(g) = D(g)^{-1} \sum_{w \in W} (\det w)\, e_{w(\lambda+\delta)}(g), \tag{99}$$

we get

$$\langle v, \chi_\lambda \rangle_G = |W|^{-1} \left\langle D \cdot v^\sharp, \sum_{w \in W} (\det w)\, e_{w(\lambda+\delta)}(\theta) \right\rangle_T. \tag{100}$$

Introducing the definition of D, this easily becomes

$$|W|^{-1} \sum_{w, w' \in W} (\det ww')\, \hat{\mu}(w(\lambda+\delta) - w'(\delta)), \tag{101}$$

where $\hat{\mu}$ denotes the Fourier transform of $\mu = v^\sharp$. In many cases, the asymptotic behavior of $\hat{\mu}(w(\lambda+\delta) - w'(\delta))$ as $|\lambda| \to \infty$ can be studied via the method of stationary phase. One expects that if v is a Lagrangian distribution, this behavior is governed by the symbol of v.

Notice that the restriction that v be represented by a continuous function can be lifted as follows. Let Δ denote the Laplace operator on G associated to a bi-invariant metric with total volume one. Then any Ad-invariant distribution v has the property that

$$\pi_\lambda((|\delta|^2 - \Delta)^{s/2} v) = |\lambda + \delta|^s\, \pi_\lambda(v), \tag{102}$$

for all $s \in \mathbb{R}$. If $v \in H^\sigma(G)$, set $v_1 = (|\delta|^2 - \Delta)^{s/2} v$ choosing $s < \sigma - d/2$. Then $v_1 \in C(G)$, and we can apply to it the analysis of the previous paragraph. Since, by (102)

$$\langle v, \chi_\lambda \rangle = |\lambda + \delta|^{-s} \langle v_1, \chi_\lambda \rangle, \tag{103}$$

asymptotic information on the Fourier coefficients of v_1 translates into information on the Fourier coefficients of v.

4.2. *Fourier Analysis of Central Conormal Distributions*

To obtain uniform information on the Fourier coefficients of v, it is natural to try to restrict v to the maximal torus, T. Now the restriction operator,

$$\rho: C_0^\infty(G) \to C_0^\infty(T) \tag{104}$$

is *not* a regular Fourier integral operator because its Schwartz kernel is a Lagrangian distribution with respect to

$$\mathscr{C} = \{(x, \xi_0; x, \xi) : x \in T, \text{ and } \xi_0 = \xi|_{T_x T}\}, \tag{105}$$

and $\mathscr{C} \subset T^*T \times T^*G$ contains covectors of the form $(x, 0; x, \xi)$ with $p \neq 0$. Thus it is the conormal bundle of T that contains the "bad" directions that prevent ρ from having an extension to $\mathscr{D}'(G)$. On the other hand, if $\Gamma \subset T^*G \setminus \{0\}$ is a closed conic Lagrangian satisfying

$$\Gamma \cap N^*T = \emptyset, \tag{106}$$

then ρ has a continuous extension to $I^m(G; \Gamma)$, and, in fact, provided the standard clean intersection condition is met, it maps this class into some $I^{m'}(T; \mathscr{C} \circ \Gamma)$.

There are simple cases where condition (106) is violated; for example take Γ the conormal space to the identity element! This is a very important case for us, as $\mathrm{Tr}_G\, f(Q)$ has a big singularity at e whenever $\hat{f}(0) \neq 0$. In this section we show how to get around this problem. Our main tools will be the formulas (100)–(101), supplemented by (102)–(103). More generally, in this section we take v to be a central conormal distribution on G, with wave front set in the conormal bundle to a smooth Ad-invariant submanifold $X \subset G$. We will assume that X intersects T cleanly. Recall that this means that (a) the intersection $Z = T \cap X$ is a manifold, and that (b) $\forall x \in Z$, $T_x Z = T_x T \cap T_x X$. The *excess* of the intersection is defined to be the non-negative integer

$$e = \dim G + \dim Z - [\dim T + \dim X], \tag{107}$$

so that a clean intersection is transverse iff its excess is zero.

PROPOSITION 4.1. *Assume X intersects T cleanly, with excess e, and let $r = \dim T$, $d = \dim G$. Let k denote the codimension of Z in T. Then, provided*

$$m < -d/4 + (k - e)/2, \tag{108}$$

the restriction operator ρ has a continuous extension to

$$\rho\colon I^m(G; N^*X) \to I^{m'}(T; N^*Z), \tag{109}$$

where

$$m' = m + (d - r)/4 + e/2. \tag{110}$$

Proof. This follows from the characterizations of conormal distributions with classical symbols in terms of their asymptotic behavior as the singular set is approached. The condition (108) implies the restriction to T is integrable. ∎

We now show how Proposition 4.1 together with (100)–(103) lead to an explicit analysis of the case $X = \{e\}$, in effect giving an alternative derivation of the result from [30] which led to the deduction of (15) in the Introduction from (13)–(14).

PROPOSITION 4.2. *If* $v \in I^{\mu + d/4}(G; T_e^*\backslash 0)$ *is central, then* $\langle v, \chi_\lambda \rangle$ *has the form*

$$\langle v, \chi_\lambda \rangle = d_\lambda q(\lambda + \delta), \tag{111}$$

with a Weyl-invariant $q \in S^\mu(\mathfrak{t}^*)$.

Proof. Using (102)–(103), we can assume without loss of generality that $\mu + d - r < 0$, so Proposition 4.1 implies the restriction $v^\sharp$ of v to T exists and

$$v^\sharp \in I^{\mu + r/4 + d - r}(T, T_e^* T \backslash 0). \tag{112}$$

Now (100) implies

$$\begin{aligned} \langle v, \chi_\lambda \rangle &= |W|^{-1} \sum_{w \in W} (\det w)\, \hat{\sigma}(w(\lambda + \delta)) \\ &= r(\lambda + \delta) \end{aligned} \tag{113}$$

where

$$\sigma = D \cdot v^\sharp \in I^{\mu + r/4 + (d - r)/2}. \tag{114}$$

The extra smoothness of σ over $v^\sharp$ is due to the fact that D vanishes to order $(d - r)/2$ at e, by the formula

$$D(g) = e_\delta(g) \prod_{\alpha \in R^+} (1 - e_{-\alpha}(g)), \qquad g \in T, \tag{115}$$

where R^+ denotes the set of positive roots of $\mathfrak{g}$. It follows that

$$\hat{\sigma} \in S^{\mu + (d - r)/2}(\mathfrak{t}^*). \tag{116}$$

Furthermore, we have

$$r(w(\lambda + \delta)) = (\det w)\, r(\lambda + \delta), \tag{117}$$

a property in common with the dimension formula

$$d_\lambda = \prod_{\alpha \in R^+} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}. \tag{118}$$

In particular, $r(\lambda + \delta) = 0$ for $\lambda + \delta$ in the walls of a Weyl chamber, so the quotient $r(\lambda + \delta)/d_\lambda = q(\lambda + \delta)$ is smooth, hence a symbol, of order μ. This proves (111). ∎

The case $X=\{e\}$ is a special case of the situation where X is a single conjugacy class; we say a little more about this here. Thus pick $g_0 \in T$, and let $X=\{gg_0 g^{-1}; g \in G\}$ be the conjugacy class of g_0. We are interested in restricting distributions on G conormal to X to the maximal torus T, and so we must examine the intersection $X \cap T$. We will need several facts about roots of compact Lie groups, that we now recall. Let R denote the set of real roots, and R^+ the set of positive roots, and for each $\alpha \in R$, let e_α be the character of T satisfying

$$\forall H \in \mathfrak{t}, \qquad e_\alpha(\exp(H)) = e^{2\pi i \alpha(H)}. \tag{119}$$

Furthermore, for each $\alpha \in R^+$, let $M_\alpha \leqslant \mathfrak{g}$ denote the real α-isotypical summand of the adjoint action of T on $\mathfrak{g}$. Then the Lie algebra of the centralizer $Z(g_0)$ of g_0 is known to equal

$$LZ(g_0) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_{g_0}} M_\alpha, \tag{120}$$

where

$$R_{g_0} = \{\alpha \in R^+; g_0 \in \ker e_\alpha\}, \tag{121}$$

see [3, Proposition V.2.3]. We summarize what we need in the following:

PROPOSITION 4.3. *If $X \subset G$ is a conjugacy class, containing $g_0 \in T$,*

$$X \cap T = \bigcup_{w \in W} w(g_0), \tag{122}$$

where W is the Weyl group. Moreover, the intersection is always clean, with excess equal to

$$\varepsilon(g_0) = 2 \cdot \sharp R_{g_0}. \tag{123}$$

Proof. The first statement is well known; see [3, Lemma IV.2.5]. We now verify the clean intersection condition. After left translation to the identity by g_0^{-1}, what we must show is that

$$\mathfrak{t} \cap \{\mathrm{Ad}_{g_0^{-1}}(A) - A; A \in \mathfrak{g}\} = 0. \tag{124}$$

Let $V = \mathrm{Ad}_{g_0^{-1}}(A) - A$, assume $V \in \mathfrak{t}$, and let $H \in \mathfrak{t}$. Then

$$\begin{aligned} 0 = [H, V] &= [H, \mathrm{Ad}_{g_0^{-1}} A] - [H, A] \\ &= \mathrm{Ad}_{g_0^{-1}}[\mathrm{Ad}_{g_0} H, A] - [H, A] \\ &= \mathrm{Ad}_{g_0^{-1}}[H, A] - [H, A]. \end{aligned} \tag{125}$$

Now $B \in \mathfrak{g}$ is such that $\mathrm{Ad}_{g_0^{-1}}(B) = B$ iff B is in the Lie algebra of $Z(g_0)$, the centralizer of g_0. It follows that

$$\forall H \in \mathfrak{t}, \qquad [H, A] \in \mathfrak{t} \oplus \bigoplus_{\alpha \in R_{g_0}} M_\alpha, \tag{126}$$

but since the M_α are the isotypical summands of the Ad_T action on $\mathfrak{g}$, this implies that A itself is in the right-hand side of (126). By the previous remark, this means that $\mathrm{Ad}_{g_0^{-1}} A = A$, which implies $V = 0$, and cleanness follows. Finally, (123) follows from (120) and the fact that for each α the dimension of M_α is two. ∎

In terms of the Weyl denominator, one has:

COROLLARY 4.4. *The excess of the intersection* (122) *equals* $\varepsilon(g_0) = 2j$, *where* j *is the order of vanishing of* D *at* g_0 (*normalized so that* $j = 0$ *if* $D(g_0) \neq 0$).

Proof. This follows from the denominator formula (115), together with (123). ∎

If X is a conjugacy class, and $v \in I^m(G; N^*X)$, the formula (113) for $\langle v, \chi_\lambda \rangle$ continues to hold, where $\sigma = D \cdot v^\#$ is a sum of conormal distributions associated to an orbit of the Weyl group. Thus $\hat{\sigma}$ is a sum of terms, each of which is a product of a symbol and an oscillatory factor. This is illustrated by the following simple example; take $v = v_X$ homogeneous measure on X:

$$\langle v_X, f \rangle = \int_G f(g^{-1} g_0 g)\, dg. \tag{127}$$

In this case,

$$\langle v_X, \chi_\lambda \rangle = \chi_\lambda(g_0) \tag{128}$$

is a sum of oscillatory terms by Weyl's formula (99).

The case of N^*X where X is a conjugacy class other than $\{e\}$ actually does not arise so frequently as the conic Lagrangian of $\mathrm{Tr}_G f(Q)$, as we will see in Section 5.2, Proposition 5.7. In fact, for $g_1 \in G$ to belong to the singular support of $\mathrm{Tr}_G f(Q)$, we need g_1 to take the initial point $\bar{x}$ to the final point in an integral curve of H_q in $T^*P \backslash 0 = \tilde{X}$ projecting over a closed orbit of the Wong flow, as explained in Section 5. The corresponding point in $\Gamma = WF\, \mathrm{Tr}_G f(Q)$ is $(g_1, \Phi(\bar{x}))$. We will see that inverse images under Φ of coadjoint orbits in $\mathfrak{g}^*$ lie over symplectic leaves of the Wong bundle. Under the hypotheses of Proposition 5.7, it will follow that the conic

Lagrangian manifold Γ in $T^*G\backslash 0$ contains, as an open subset, the conormal bundle to a hypersurface Y in G, swept out by an $(r-1)$-parameter family of conjugacy classes, of maximal dimension.

There remains the question of what the entire connected component(s) of Γ containing N^*Y look like. One possibility is that it continues to be the conormal bundle of a smooth, Ad invariant, hypersurface in G, which happens if the closure $\bar{Y}$ is smooth. That $\bar{Y}$ may or may not be smooth is illustrated by cases of products $P=G\times M$, with $G=SU(3)$, for example. See Section 6.3 for some examples illustrating what $\mathrm{Tr}_G\, f(Q)$ and its restriction to T might look like, and how their Fourier coefficients might behave.

5. The G-trace of $f(Q)$

5.1. *Clean-Intersection Criteria*

From now on we take $X=P$ a principal fiber bundle, and look more carefully at the sort of canonical relations Λ that arise when taking functions of operators of real principal type. Let Q be a first-order, self-adjoint pseudodifferential operator on X of real principal type commuting with G, and let q denote its principal symbol. Thus $q\colon \tilde{X}\to \mathbb{R}$ is a smooth function which (i) is G-invariant, (ii) is positive-homogeneous of degree 1, (iii) zero is a regular value of q, and (iv) the Hamilton vector field of q, H_q, is not radial at any point of $\Sigma=q^{-1}(0)$. We will also find it necessary to assume the following:

(H1) Nowhere on Σ is the Hamilton vector field of q colinear with a vector of the form $A^\sharp_{\bar{x}}$, $A\in \mathfrak{g}^*$, and

(H2) The intersection $\Sigma\cap\Phi^{-1}(0)$ is empty.

We note that hypotheses (H1) and (H2) are satisfied when Q is of the form (18), with L and A given by (5)–(6). The situation for (H2) is simple; $\Phi^{-1}(0)$ is the nonormal bundle $\mathscr{H}^*P$ to the fibers of $P\to M$, and the requirement that this be disjoint from Char Q clearly holds in the case (18).

To establish (H1), we can replace q by $p(x,\xi)$, having properties (47)–(49) and S, as set out in Section 2.3. Note that $A^\sharp_{\bar{x}}=H_\Psi$, where, for $\bar{x}=(x,\xi)\in T^*P\backslash 0$, $\Psi(x,\xi)=\langle\Phi(x,\xi),A\rangle=\langle A^\sharp_x,\xi\rangle$, as in (72). We want to show that dp and $d\Psi$ are not colinear at any (x,ξ) in Char p. Now, with respect to the splitting $\xi=(\xi',\xi'')$ arising from the connection on P, $\Psi(x,\xi)$ is a linear form in ξ'' alone. Thus, at a point of colinearity, we must have $\xi'=0$, or equivalently, $(x,\xi)\in\mathrm{Char}\, a$. If also $(x,\xi)\in\mathrm{Char}\, p$, then $b(x)=0$, so $d_\xi p=0$ at such a point. But $d_\xi\Psi$ is nowhere zero, so colinearity is impossible, granted that p has simple characteristics.

For an open $J \subset \mathbb{R}$, consider

$$\Lambda_J = \{(\bar{x}, \bar{y}') \in \Sigma \times \Sigma; \exists t \in J \text{ such that } \bar{y} = \phi_t(\bar{x})\}, \tag{129}$$

where $\{\phi_t\}$ is the Hamilton flow of q. Although in general this in only an *immersed* Lagrangian, it is embedded if J is small enough, and the arguments of Section 2.2 show that it is enough to consider that case. Note that, by (H2), Corollary 3.6 applies to this canonical relation, and

$$\mathscr{I} \cap \Lambda_J = \{(\bar{x}, \bar{y}') \in \Sigma \times \Sigma; \exists t \in J, g \in G, \text{ such that } \bar{y} = \bar{x} \cdot g = \phi_t(\bar{x})\}. \tag{130}$$

This shows that the right setting to understand this problem is in terms of the Wong bundle,

$$\mathscr{W} = T^*P/G. \tag{131}$$

This is a smooth manifold, which serves as the phase space for the Wong equations of motion for a particle in a background gauge field, [23, 29, 33, 34]. Since ϕ_t commutes with the G-action, there is a smooth flow ψ_t on $\mathscr{W}$ such that the natural projection π: $T^*P \to \mathscr{W}$ intertwines ϕ_t and ψ_t. Then

$$(\bar{x}, \bar{y}') \in \mathscr{I} \cap \Lambda_J \qquad \Leftrightarrow \qquad \exists t \in J \text{ such that } \pi(\bar{y}) = \psi_t(\pi(\bar{x})). \tag{132}$$

In other words, it is the periodic trajectories of ψ_t with periods in J that produce the singularities of the G-trace of $f(Q)$, where f is a smooth function with $\operatorname{supp} \hat{f} \subset J$. Our goal in this section is to interpret the clean intersection condition and study the singularities of $\operatorname{Tr}_G f(Q)$ purely in terms of the geometry of $\{\psi_t\}$.

We begin by recalling a few facts about the Wong bundle, $\mathscr{W}$. First of all, $\mathscr{W}$ is a Poisson manifold, since it is the quotient of a symplectic manifold by a free Hamiltonian action. The symplectic leaves of $\mathscr{W}$ are known to be the submanifolds of the form

$$\Phi^{-1}(\mathcal{O})/G \subset \mathscr{W}, \tag{133}$$

where $\mathcal{O} \subset \mathfrak{g}^*$ is a co-adjoint orbit of G. Moreover, if $\tilde{q}$ denotes the function on $\mathscr{W}$ whose pull-back to $\tilde{X}$ is equal to q, then $\{\psi_t\}$ is the Hamiltonian flow of $\tilde{q}$. We are interested in the flow $\{\psi_t\}$ restricted to the image of the characteristic set Σ in $\mathscr{W}$, that is, in

$$\mathscr{Y} = \Sigma/G = \tilde{q}^{-1}(0). \tag{134}$$

Note that the multiplicative group $\mathbb{R}^+$ acts everywhere, and commutes with the flows. It is convenient to break this homogeneity in the following way. Let a be the function on T^*P which is the square of the norm of the vertical component of a covector. Thus a is the pull-back via Φ of the square of the Ad-invariant norm on $\mathfrak{g}^*$. It follows that $\{q, a\} = 0$, by the G-invariance of q. Let $\tilde{a}$ be the function on $\mathscr{W}$ defined by a, and let $\mathscr{Y}_1 = \mathscr{Y} \cap \tilde{a}^{-1}(1)$. By (H1), this is a submanifold of $\mathscr{W}$, which is the union of symplectic leaves, and is invariant under the flow ψ_t.

Our immediate goal is to prove the following proposition:

PROPOSITION 5.1. *The clean intersection condition is satisfied by Λ_J iff the following is a clean-intersection diagram,*

$$\begin{array}{ccc} \mathscr{P}_1 & \longrightarrow & \mathscr{Y}_1 \times J \\ \big\downarrow & & \big\downarrow \\ \mathscr{Y}_1 & \xrightarrow{\Delta} & \mathscr{W} \times \mathscr{W}, \end{array} \tag{135}$$

where the arrow on the right is $(y, t) \mapsto (y, \psi_t(y)')$, Δ is the diagonal embedding, and

$$\mathscr{P}_1 = \{(y, T) \in \mathscr{Y}_1 \times J;\ y = \psi_T(y)\}. \tag{136}$$

Proof. By Proposition 3.9, the condition on Λ_J is that the diagram

$$\begin{array}{ccc} \mathscr{F}_0 & \longrightarrow & \Sigma \times J \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathscr{I} & \longrightarrow & \tilde{X} \times \tilde{X} \end{array} \tag{137}$$

be a clean-intersection diagram, where $f(\bar{x}, t) = (\phi_t(\bar{x}), \bar{x})$ parametrizes Λ_J,

$$\mathscr{F}_0 = \{(\bar{x}, t) \in \Sigma \times J;\ \exists g \in G \text{ such that } \phi_t(\bar{x}) = \bar{x} \cdot g\}, \tag{138}$$

and $\varphi(\bar{x}, t) = (\bar{x}, \phi_t(\bar{x})')$. First we indicate how the cleanness of this diagram is equivalent to the cleanness of the homogeneous version of (135) that is, of

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & \mathscr{Y} \times J \\ \big\downarrow & & \big\downarrow \\ \mathscr{Y} & \longrightarrow & \mathscr{W} \times \mathscr{W}, \end{array} \tag{139}$$

where $\mathscr{P}$ is defined as in (136), with $\mathscr{Y}_1$ replaced by $\mathscr{Y}$. Now diagram (137) fibers over diagram (139). More precisely, this is a particular instance of the following situation: one has a fibration $\Pi: X \to Y$ and two submanifolds, $A, B \subset Y$. The assertion is that A and B intersect cleanly iff $\Pi^{-1}(A)$ and $\Pi^{-1}(B)$ do. We leave the reader to convince herself that this is so. Thus the clean intersection condition is equivalent to the cleanness of (139). A similar argument shows that this is equivalent to the cleanness of (135). ∎

5.2. *Some Geometry of the Wong Flow*

Now we look in more detail at the geometry of the closed trajectories of $\{\psi_t\}$. We first look at the period $T=0$.

LEMMA 5.2. *Zero is an isolated point in the period spectrum of the restriction of* $\{\psi_t\}$ *to* $\mathscr{Y}$, *and* A_J *satisfies the clean intersection condition if* J *is a small enough neighborhood of zero.*

Proof. The condition of non-radiality of Ξ_q together with (H2), imply that the flow on $\mathscr{Y}_1$ does not have any fixed points, and so by compactness its period spectrum is bounded away from zero. Cleanness is left to the reader to check; it follows from the assumption (H1). ∎

We now turn to the geometry of the nontrivial periodic orbits of the Wong flow. As we will see, the main difference with the generic Hamilton flow on a symplectic manifold is that *non-degenerate trajectories generically arise in families.* We will use the following notation: for every $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$ (where the last inclusion is defined by the bi-invariant metric on G), $\mathscr{W}_\lambda \subset \mathscr{W}$ is the symplectic leaf

$$\mathscr{W}_\lambda = \Phi^{-1}(\mathcal{O}_\lambda)/G, \tag{140}$$

where $\mathcal{O}_\lambda$ is the co-adjoint orbit of G through λ. One can easily check that $\mathscr{W}_\lambda$ can be naturally identified with

$$\mathscr{W}_\lambda \cong \Phi^{-1}(\lambda)/G_\lambda, \tag{141}$$

where G_λ is the isotropy subgroup of λ.

To study the geometry of periodic trajectories of the Wong flow, we need to study how the symplectic leaves (of maximal dimension) of $\mathscr{W}$ are sewn together. Let $U \subset \mathfrak{t}^* \subset \mathfrak{k}^*$ be the interior of the positive Weyl chamber, and let

$$Y = \Phi^{-1}(U). \tag{142}$$

Note that the isotropy subgroup of every $\mu \in U$ is the maximal torus, T, and thus Y is invariant under T.

LEMMA 5.3. *Y is a symplectic submanifold of T^*P. Moreover, the maximal torus T acts on Y in a Hamiltonian fashion, with moment map Φ_T making the following diagram commutative:*

$$\begin{array}{ccc} Y & \xrightarrow{\Phi_T} & \mathfrak{t}^* \\ \downarrow & & \downarrow \\ T^*P & \xrightarrow{\Phi} & \mathfrak{g}^*. \end{array} \tag{143}$$

Proof. The first statement is an easy consequence of Theorem 26.7 of [10]; the second follows trivially. ∎

It follows from the previous considerations that the symplectic leaves $\mathscr{W}_\mu$ with $\mu \in U$ are (as symplectic manifolds!) the reduced manifolds of Y under the action of T at points $\mu \in U$:

$$\mathscr{W}_\mu \cong \Phi_T^{-1}(\mu)/T. \tag{144}$$

(Actually, one should check that the symplectic forms agree, but this is easy to see.) This fact will help us get a symplectic normal form for a neighborhood of $\mathscr{W}_\lambda$. We learned the following argument from Eugene Lerman [21], where he uses it to get a "one-line proof" of the Duistermaat–Heckman formula. Pick $\lambda \in V$, and let A be any connection on the principal T bundle

$$\Phi^{-1}(\lambda) - \mathscr{W}_\lambda. \tag{145}$$

For every $\mu \in U$, let $X_\mu = \Phi_T^{-1}(\mu)$.

PROPOSITION 5.4. *There exists an open conic neighborhood of λ, $V \subset U$, and a T-equivariant diffeomorphism, φ, making the following diagram commutative:*

$$\begin{array}{ccc} X_\lambda \times V & \xrightarrow{\varphi} & \Phi_T^{-1}(V) \\ \downarrow & & \downarrow{\scriptstyle \Phi_T} \\ V & \longrightarrow & \mathfrak{t}^*. \end{array} \tag{146}$$

Moreover, the pull-back under φ of the symplectic form is equal to

$$p_1^*\omega_\lambda + d\langle \pi_2, A \rangle, \tag{147}$$

where p_1 is the composition of the projections $X_\lambda \times V \to X_\lambda \to \mathscr{W}_\lambda$, ω_λ is the symplectic form on $\mathscr{W}_\lambda$, A is the $\mathfrak{t}$-valued connection form, and π_2 the projection onto the second factor, $X_\lambda \times V \to V \subset \mathfrak{t}^*$.

Proof. One can show directly that (147) is a symplectic form on $X_\lambda \times V$, and that $X_\lambda \times \{\lambda\}$ is a coisotropic submanifold. In fact, the restriction of (147) to $X_\lambda \times \{\lambda\} \cong X_\lambda$ is the same as the restriction of the symplectic form on Y to X_λ. Now invoke the equivariant version of the coisotropic embedding theorem [8] (the uniqueness part) to conclude the existence of φ with the desired properties. ∎

The following is an immediate consequence of this result. Let

$$\mathscr{W}_V = \bigcup_{\mu \in V} \mathscr{W}_\mu. \tag{148}$$

It is clear that $\mathscr{W}_V$ is an open subset of $\mathscr{W}$.

COROLLARY 5.5. *The trivialization* (146) *induces a diffeomorphism*

$$\mathscr{W}_V \cong \mathscr{W}_\lambda \times V, \tag{149}$$

mapping $\mathscr{W}_\mu$ onto $\mathscr{W}_\lambda \times \{\mu\}$. Moreover, this is an isomorphism of Poisson manifolds if the Poisson structure on the right-hand side is defined by the family of two forms $\{\omega_\mu ; \mu \in V\}$ on $\mathscr{W}_\lambda$ given by

$$\omega_\mu = \omega_\lambda + \langle \lambda - \mu, F_A \rangle, \tag{150}$$

where F_A is the curvature of the connection A.

Our main application of these results is the following. Let $\gamma \subset \mathscr{Y}_1$ be a periodic trajectory of our flow. It is entirely contained in some symplectic leaf; thus there is a unique $\lambda \in \mathfrak{t}^*$ such that $|\lambda| = 1$ and

$$\gamma \subset \mathscr{Y}_1 \cap \mathscr{W}_\lambda. \tag{151}$$

Note that $\mathscr{Y}_1 \cap \mathscr{W}_\lambda = \mathscr{W}_\lambda \cap \tilde{q}^{-1}(0)$. We will assume that γ is non-degenerate, in the following sense:

DEFINITION 5.6. γ will be said to be non-degenerate, iff the following two conditions are satisfied:

(N.1) As a trajectory of q on the symplectic manifold $\mathscr{W}_\lambda$, γ is non-degenerate, and

(N.2) λ belongs to the interior of a Weyl chamber.

As we will now show, these assumptions imply that γ belongs to a family of closed trajectories. We will use the model for the symplectic leaves provided by Corollary 5.5.

PROPOSITION 5.7. *Let γ be a non-degenerate trajectory of q on the Poisson manifold $\mathscr{W}$, lying in the symplectic leaf $\mathscr{W}_\lambda$ and with energy $q=0$. Then there exists a conic open neighborhood of λ in the interior of a Weyl chamber, V, and smooth maps $F\colon V \to \mathscr{W}_V$ and $T\colon V \to \mathbb{R}$, such thar $\forall \mu \in V$*

$$\psi_{T(\mu)}(F(\mu)) = F(\mu), \tag{152}$$

and $F(\mu)$ has the property

$$F(\mu) \in \mathscr{W}_\mu \cap \{q=0\}. \tag{153}$$

Proof. Choose a connection A on the principal T bundle $X_\lambda \to \mathscr{W}_\lambda$, and let V be as in Corollary 5.5. We can identify the flow of q on $\mathscr{W}_\mu$, $\mu \in V$ with the flow of a Hamiltonian q_μ on $\mathscr{W}_\lambda$ with the symplectic structure (150). The Hamiltonian q_μ is of course q pulled-back by the diffeomorphism identifying the symplectic leaves $\mathscr{W}_\mu$ and $\mathscr{W}_\lambda$, and depends smoothly on μ. With this notation, $q_\lambda = q$. Let $Z = \mathscr{W}_\lambda \cap \{q=0\}$. By the assumption (H2) and the compactness of P, Z itself is compact. Let $\mathscr{U}$ be a tubular neighborhood of Z in $\mathscr{W}_\lambda$, with projection $\pi\colon \mathscr{U} \to Z$. By shrinking V if necessary, we may assume without loss of generality that $\forall \mu \in V$

$$Z_\mu := \mathscr{W}_\lambda \cap \{q_\mu = 0\} \subset \mathscr{U}. \tag{154}$$

Moreover, by the compectness of Z we may assume that the inclusion (154) is a *section* of $\pi\colon \mathscr{U} \to Z$. By this we mean that the restriction of π to Z_μ is a diffeomorphism. For each $\mu \in V$, let Ξ_μ be the vector field on Z obtained by the following procedure: take the Hamilton vector field of q_μ (with respect to the symplectic form (150)) on the energy surface $\{q_\mu = 0\}$, and project it via $\pi\colon \mathscr{U} \to Z$ to a vector field Ξ_μ on Z. Obviously, the flow of Ξ_μ is smoothly conjugate to the Hamilton flow of q on $\mathscr{W}_\mu \cap \{q=0\}$, and the vector fields Ξ_μ depend smoothly on μ with Ξ_λ equal to the Hamilton vector field of q restricted to Z.

The remainder of the proof is standard. Pick a base point $w \in \gamma$, and a cross-section $C \subset Z$ of the flow of q on Z containing γ. Condition (N.1) on γ is precisely that w is a non-degenerate fixed point of the return map R_γ: $C \to C$ of the flow of q. For μ sufficiently close to λ, C is still a cross-section for the flow of Ξ_μ, and the associated return maps R_μ depend smoothly on μ. Thus by Lefschetz' theorem there is a smooth map $f\colon V \to C$ such that $f(\lambda) = w$ and $\forall \mu \in V$ $R_\mu(f(\mu)) = f(\mu)$,. But this means precisely that the trajectory of Ξ_μ through $f(\mu)$ is periodic. Since the entire system is homogeneous with respect to the action of $\mathbb{R}^+$, one can take V to be conic. ∎

Remarks. We will make use later of the following remarks:

(R.1) By (153), under the isomorphism (149) the mapping F is of the form

$$F(\mu) = (F_0(\mu), \mu), \tag{155}$$

for some smooth map $F_0 \to \mathscr{W}_\lambda$.

(R.2) If we differentiate the relation (152), we obtain that for all tangent vectors $\delta\mu \in T_\mu V \cong \mathfrak{t}^*$

$$(I - d(\psi^\mu_{T(\mu)}))\, d(F_0)(\delta u) = dT_\mu(\delta u)\, \Xi^\mu, \tag{156}$$

where ψ^μ is the flow of q_μ on $(\mathscr{W}_\lambda, \omega_\mu)$, and Ξ^μ is its infinitesimal generator.

5.3. *The Singularities of* $\operatorname{Tr}_G f(Q)$

The previous considerations on the geometry of the Wong flow have the following implications on the singularities of $\operatorname{Tr}_G f(Q)$. We keep the notation of Section 5.1, in particular the manifold Λ_J is defined by (129), and the Fourier transform of f is assumed to be included in the interval $J \subset \mathbb{R}$.

First we look at the singularity at the identity. The following is an easy consequence of Lemma 5.2:

THEOREM 5.8. *If the only period in J is zero,*

$$\operatorname{Tr}_G f(Q) \in I^{n-3d/4-1}(G; T_e^*G - \{0\}),$$

where $n = \dim X$ and $d = \dim G$.

Proof. We just will indicate the calculation of the order. The excess of the original diagram (86) is $e = 2n - 1 - d$, because the manifold $\mathscr{F}^\#$ is in this case diffeomorphic to Σ. Since the order of K is $-d/4$ and that of $f(Q)$ is $-1/2$, the order of the G-trace is $-d/4 - 1/2 + n - (d+1)/2$. ∎

Next we consider the singularity created by non-degenerate periodic trajectories. Consider a non-degenerate periodic trajectory γ of the Wong flow, in the sense of Definition 5.6. Let V, F, T be as in Proposition 5.7. We will now show that the clean intersection condition is satisfied, assuming that all the periodic trajectories of the Wong flow on $\mathscr{Y}$ with periods in J are among the ones produced by Proposition 5.7. Since we are working with the non-homogeneous version of $\mathscr{Y}$, etc., let $V_1 = \{\mu \in V;\ |\mu| = 1\}$. Denote by U the open subset of $\mathfrak{g}^*$ consisting of all the vectors whose Ad_G^* orbit intersects V. Define

$$\mathscr{I}^U = \{(\bar{x} \cdot g, \bar{x}');\ \bar{x} \in \Phi^{-1}(U),\ g \in G\} \tag{157}$$

and

$$\Lambda_J^U = \{(\bar{x}, \bar{y}') \in \Lambda_J;\ \bar{y} \in \Phi^{-1}(U)\}. \tag{158}$$

THEOREM 5.9. *Let γ be a non-degenerate periodic trajectory of the Wong flow, and let V, F, T be given by Proposition 5.7. Assume furthermore that*

$$\mathscr{P}_1 \cap [\mathscr{W}_V \times \mathbb{R}] = \{(\psi_t(F(\mu)), T(\mu));\ \mu \in V_1,\ t \in [0, T(\mu)]\}. \tag{159}$$

Pick $V_0 \subset \mathfrak{g}^*$ open, conic and invariant with closure contained in V. Then there is a central Lagrangian distribution, $v \in I^{-1/4 - d/4 + r/2}(G, \Gamma_U)$, having the same Fourier coefficients as $\mathrm{Tr}_G\, f(Q)$ at every integral element $\mu \in V_0$. Here

$$\begin{aligned}\Gamma_U = \{&(g, \mu);\ \exists \bar{x} \in \Phi^{-1}(U),\ t \in J \text{ such that } \bar{x} = \phi_t(\bar{x}) \cdot g \\ &\text{and } \mu = \mathrm{Ad}_g^* \,\Phi(\bar{x})\}.\end{aligned} \tag{160}$$

Proof. We first prove that the assumptions imply that $\mathscr{I}^U$ and Λ_J^U intersect cleanly. The condition that (135) be a clean-intersection diagram means that (a) $\mathscr{P}_1$ should be a submanifold of $\mathscr{Y}_1 \times J$, and (b) at every $(y, T) \in \mathscr{P}_1$ the tangent space at P should equal the set of all $(v, \tau) \in T_y\mathscr{Y}_1 \times \mathbb{R}$ such that

$$v = \tau \Xi_y + d(\psi_T)_y\,(v), \tag{161}$$

where Ξ is the infinitesimal generator of ψ_t. Regarding (a), note that F is an embedding, by remark (R.1). Thus (159) shows that $\mathscr{P}_1$ is diffeomorphic to the graph of a map, and hence it is a manifold. Let us now consider its tangent space at a point parametrized by μ and t. It consists of all vectors of the form

$$(\sigma\Xi + d(\psi_t)\, dF(\delta u),\ dT(\delta\mu)) \tag{162}$$

with $\sigma \in \mathbb{R}$ and $\delta\mu \in \mathfrak{t}^*$. What we must check is that every vector (v, τ) satisfying (161) is of the form (162). First localize: let us work on $\mathscr{Y} \cap \mathscr{W}_V \subset \mathscr{W}_\lambda \times V$. Decompose $v = (v_1, v_2)$, where v_1 is tangent to $\mathscr{W}_\lambda$ and $v_2 \in T_\mu V = \mathfrak{t}^*$. As we will show, we can take $\delta\mu = v_2$. What we must show now is that

$$v_1 - d(\psi_t^\mu)\, d(F_0)_\mu\,(v_2) \tag{163}$$

is a multiple of Ξ^μ. There are two steps in the proof of this fact.

Claim 1. $v_1 - d(F_0)(v_2)$ is tangent to $q_\mu^{-1}(0)$. To prove this just compute $d(q_\mu)$ applied to this vector, using the identity

$$0 = d(q_\mu(v_2)) + d_\mu(q)(v_2), \tag{164}$$

which follows by differentiating the identity $q_\mu(F_0(\mu)) = 0$.

Claim 2. The result, ζ, of applying $[I - d(\psi_T^\mu)]$ to (163) is a multiple of Ξ^μ. Indeed, by (161),

$$\zeta = \tau \Xi^\mu - (I - d(\psi_T^\mu))\, dF_0(v_2), \tag{165}$$

which, by remark (R.2), is a multiple of Ξ^μ.

To finish the proof, note that Claim 2 and the non-degeneracy of the trajectory corresponding to μ imply that (163) is a multiple of Ξ^μ. ∎

The considerations made in the previous proof have the following consequence, which is an answer to the question of how the period function T changes:

COROLLARY 5.10. *In the setting of the previous propositions, assume that for given $\mu \in V$ and $\delta\mu \in \mathfrak{t}^*$, $d_\mu(q)(\delta\mu) \neq 0$. Then*

$$dT_\mu(\delta\mu) = \frac{\omega_\mu(v, d(\psi^\mu_{T(\mu)})(v))}{d_\mu(q)(\delta\mu)}. \tag{166}$$

6. APPLICATIONS TO PARTICLES IN GAUGE FIELDS

6.1. *Asymptotic Expansions*

Here we will state the main results on spectral asymptotics which follow from the machinery we have developed. As stated in the introduction, our goal is to analyze the behavior of

$$\tau(\lambda) = \operatorname{Tr} f(\hbar^{-1} H_\lambda^{-1/2}(H_\lambda - c)) = \langle \operatorname{Tr}_G f(Q), \chi_\lambda \rangle, \tag{167}$$

where

$$Q = (-L)^{-1/2}\,(-L - cA) \tag{168}$$

and f is a Schwartz function on the line such that $\hat{f} \in C_0^\infty(\mathbb{R})$. Note that, by the results from Section 2.3, this is an operator of real principal type, as long as c is regular of $V \in C^\infty(M)$.

THEOREM 6.1. *Suppose $\hat{f}$ is supported on an interval $(-T, T)$ which does not contain any nontrivial periods of the Wong flow on $\mathscr{Y}$. Then*

$$\tau(\lambda) = d_\lambda a(\lambda) \text{ with } a(\lambda) \in S^{n-d-1}(\mathfrak{t}^*). \tag{169}$$

Here n (resp. d) is the dimension of P (resp. G). Furthermore, the leading term in the classical expansion of $a(\lambda)$ is

$$a_0(\lambda) = \hat{f}(0) \operatorname{Vol}[\mathscr{W}_\lambda \cap \tilde{q}^{-1}(0)], \tag{170}$$

where $\mathscr{W}_\lambda$ is the symplectic leaf of the Wong bundle corresponding to λ, and Vol *stands for Liouville measure.*

Next we analyze the contribution to the asymptotic expansion of $\tau(\lambda)$ arising from periodic trajectories. Let γ be a non-degenerate periodic trajectory of the Wong flow lying in the symplectic leaf $\mathscr{W}_{\lambda_0}$, and assume that the support of $\hat{f}$ is such that the singularities of $\operatorname{Tr}_G f(Q)$ in an invariant open cone containing λ_0 are those arising from the periodic trajectories branching off γ. Precisely, assume that condition (159) holds. Then Theorem (5.9) ensures that $\operatorname{Tr}_G f(Q)$ is a Lagrangian distribution on G microlocally in a smaller cone. To obtain from this asymptotic information on $\tau(\lambda)$, we will make a generic assumption ensuring that we can restrict the distribution of Theorem (5.9) to T:

DEFINITION 6.2. Let T be the period of γ. We will say that γ has *regular holonomy* iff given $\bar{x} \in T^*P$ above γ, every $g \in G$ such that $\phi_T(\bar{x}) \cdot \bar{x} = \bar{x}$ is a regular element, meaning that g is *not* in more than one maximal torus.

THEOREM 6.3. *Assume the condition* (159) *of Theorem* 5.9, *and assume furthermore that γ has regular holonomy. Then, after perhaps shrinking V to a smaller open cone containing λ_0, for every $V_0 \subset \mathfrak{t}^*$ an open cone with closure contained in V there exists a Lagrangian distribution v on T having the same Fourier coefficients on V_0 as* $\operatorname{Tr}_G f(Q)$. *Indeed,*

$$v \in I^{(r-1)/4}(T; \Theta), \tag{171}$$

where r is the dimension of T and

$$\Theta = \{(g, \mu) \in T \times \mathfrak{t}^*;\ \exists \bar{x} \in \Psi^{-1}(V) \text{ and } t \in J \text{ such that } \bar{x} = \phi_t(\bar{x}) \text{ and } \mu = \Phi(\bar{x})|_{\mathfrak{t}^*}\}. \tag{172}$$

Proof. We will first check that

$$\Gamma_U \cap N^*T = \varnothing. \tag{173}$$

Here Γ_U is as in (160). Assume $(g, \mu) \in \Gamma_U$, where $g \in T$. Thus there is a $t \in J$ and an $\bar{x} \in \Phi^{-1}(U)$ such that $\bar{x} = \phi_t(\bar{x} \cdot g)$. If we recall that U is the saturation of V by Ad* orbits, we see that there exists $h \in G$ such that

$$\alpha \stackrel{\text{def}}{=} \mathrm{Ad}_h^* \, \Phi(\bar{x}) = \Phi(\bar{x} \cdot h) \tag{174}$$

is in V. Thus $\bar{x} \cdot h \in Y$, where Y is the symplectic section of (142). Now $\bar{x} \cdot h = \phi_t(\bar{x} \cdot gh)$, and, since Y is invariant under ϕ, it follows that $\bar{x} \cdot gh \in Y$. Hence

$$\beta \stackrel{\text{def}}{=} \Phi(\bar{x} \cdot gh) = \mathrm{Ad}_h^* \circ \mathrm{Ad}_g^* \, \Phi(\bar{x}) \tag{175}$$

is in the interior of the positive Weyl chamber. By (174), we have

$$\begin{aligned} \beta &= \mathrm{Ad}_h^* \, \mathrm{Ad}_g^* \, \mathrm{Ad}_{h_{-1}}^*(\alpha) \\ &= \mathrm{Ad}_{h_{-1}gh}^*(\alpha). \end{aligned} \tag{176}$$

By Lemma IV.2.5 of [3], this implies that α and β are in the same orbit of the Weyl group, but, since they both lie in the interior of the positive Weyl chamber, necessarily $\alpha = \beta$. By (176), $h^{-1}gh \in T$. Now since the set of regular points is open, by shrinking V if necessary we can assume that g is a regular element. Hence h must be in the normalizer of T (for otherwise g would lie in the two distinct maximal tori T and hTh^{-1}). Hence $\mathrm{Ad}_g^* \, \Phi(\bar{x}) = \mathrm{Ad}_{h^{-1}}^*(\beta)$ is in $\mathfrak{t}^*$, and is non-zero. This proves (173).

We would have to prove next that the intersection of Γ_U with the canonical relation underlying the restriction operator from G to Y is clean. This would finish the proof of the theorem, thanks to Theorem 5.9. However, we can simply note that what we have proved is that we can replace G by T in Theorem 5.9. ∎

The leading order term in the asymptotic behavior of the Fourier coefficients of v along rays has been determined in [16]; it is the usual term in the trace formula associated to a periodic trajectory, in this case a trajectory of the Wong flow. Theorem 6.2 shows that in non-degenerate cases this estimate is uniform in cones.

6.2. *Higgs Fields*

As a generalization of the family of operators given by (2), we also analyze contributions to a gauge field Hamiltonian due to a "Higgs field." Thus we consider

$$H_\lambda = \hbar^2 H_\lambda^0 + i\hbar \pi_\lambda(X) + V, \tag{177}$$

where X is a section of $\mathfrak{g}_{\mathrm{Ad}} = P \times_{\mathrm{Ad}} \mathfrak{g}$. In this case, X gives rise to a vector field on P, tangent to the fibers of $P \to M$, such that

$$Y|_{\mathscr{D}_\lambda} \cong d_\lambda \text{ copies of } \pi_\lambda(X). \tag{178}$$

Thus, in analogy with (8), we can say that

$$-L + iA^{1/2} Y|_{\mathscr{D}_\lambda} \cong d_\lambda \text{ copies of } \hbar^{-2} H_\lambda. \tag{179}$$

We could therefore produce an analogue of (168) with $-L$ replaced by $-L + iA^{1/2}Y$, but a technical problem arises in the analysis of this operator, because $A^{1/2}$ is not a pseudodifferential operator on P; its "symbol" is singular on the subset of $T^*P \backslash \{0\}$ conormal to the fibers of $P \to M$. This technical problem can be overcome by the device of adding one more variable.

Thus we work on $P \times S^1$, and we let $\partial_\theta = \partial/\partial\theta$ on $C^\infty(S^1)$. We will make a partial replacement of $A^{1/2}$ by $D_\theta = (1/i)\,\partial_\theta$. With α denoting a small parameter and K a positive constant, set

$$\mathscr{L} = \Delta + (\tilde{V} - 1)\,\Delta_G^P - |\delta|^2 \tilde{V} + \alpha\,\partial_\theta Y + K\alpha^2\,\partial_\theta^2, \tag{180}$$

where

$$\tilde{V} = V - 1. \tag{181}$$

and, similarly as before, we assume without loss of generality that $V > K + 1$, so $\tilde{V} > 1$. Now we set

$$\mathscr{D}_{\lambda,k} = \{u \in C^\infty = \{u \in C^\infty(P \times S^1);\, G \text{ acts like } \pi_\lambda \text{ and } D_\theta = k\}. \tag{182}$$

Then

$$-\mathscr{L}|_{\mathscr{D}_{\lambda,k}} \cong d_\lambda \text{ copies of } \hbar^{-2} H_\lambda, \tag{183}$$

provided

$$\hbar = |\lambda + \delta|^{-1} = (\alpha k)^{-1}. \tag{184}$$

The differential operator $\mathscr{L}$ is strongly elliptic on $P \times S^1$, and $-\mathscr{L}$ is positive definite, provided K is taken to be sufficiently large. The operators $\mathscr{L}$, A, and D_θ all commute. Now, in place of (179), we can use the fact that

$$(-\mathscr{L})^{-1/2}\,(-\mathscr{L} - cA)|_{\mathscr{D}_{\lambda,k}} \cong d_\lambda \text{ copies of } \hbar^{-1} H_\lambda^{-1/2}(H_\lambda - c), \tag{185}$$

granted (184). Thus we are led to analyze

$$\mathrm{Tr}_{G \times S^1} f(Q), \tag{186}$$

where

$$Q = (-\mathscr{L})^{-1/2} (-\mathscr{L} - cA) \tag{187}$$

is a first-order, self-adjoint pseudodifferentail operator on $P \times S^1$ of real principal type. The analysis of this is done in the same spirit as in the case when there is not a Higgs field.

6.3. *Examples*

We illustrate some of the phenomena dealt with in the analysis of this paper with a simple family of examples. This family contains cases when the clean-intersection condition for restriction of $\mathrm{Tr}_G f(Q)$ to T is violated, and suggests further sorts of analytical problems to tackle in future work.

Consider product bundles

$$P = M \times G, \tag{188}$$

with the trivial connection. The corresponding metric on P is a product metric, so

$$\Delta_P = \Delta_M + \Delta_G^P. \tag{189}$$

We will take $V = 1$, so that we are in the set-up of the introduction with

$$L = \Delta_P, \qquad A = -\Delta_G^P + |\delta|^2. \tag{190}$$

Hence

$$\begin{aligned} L + ca &= \Delta_M - (c-1)\, \Delta_G^P + |\delta|^2 \\ &= \Delta_M - a\, \Delta_G^P + c\, |\delta|^2, \end{aligned} \tag{191}$$

where we have set $a = c - 1$. This operator is elliptic if $a < 0$, degenerate if $a = 0$, and of principal type for $a > 0$. Our object of study is the G-trace of $f(Q)$, where

$$Q = (-L)^{-1/2} (-L - cA). \tag{192}$$

As before, this will shed some light on the asymptotic behavior of

$$\mathrm{Tr}\, f(\hbar^{-1} H_\lambda^{-1/2} (H_\lambda - c)) = d_\lambda^{-1}\, \mathrm{Tr}\, f(Q)|_{\mathscr{D}_\lambda} \tag{193}$$

for large $|\lambda|$. The left-hand side of (193), a measure of the number of eigenvalues of H_λ near c, is expressed in terms of a measure of the number of eigenvalues of $-L$ close to those of cA. Recall that we take $\hat{f} \in C_0^\infty(\mathbb{R})$. For $c < 1$, $a < 0$ and Q is elliptic; then $f(Q)$ is a smoothing operator and (193) is rapidly decreasing as $\lambda \to \infty$. The interesting case is $c > 1$, so $a > 0$, and we concentrate on this. By scaling the metric on G, we may as well suppose that $a = 1$.

Thus the geometry here is controlled by the null bicharacteristic flow associated to $\Delta_M - \Delta_G^P$. Write $(x, \xi) \in T^*M$, $(g, \gamma) \in T^*G \cong G \times \mathfrak{g}^*$. Then

$$\Sigma = \operatorname{Char} Q = \{(x, \xi; g, \gamma); |\xi|_x = |\gamma|\}. \tag{194}$$

Note that for the WGS-bundle we have the identification $T^*P/G \cong T^*M \times \mathfrak{g}^*$, and

$$Y = \Sigma/G = \{(x, \xi, \gamma) \in T^*M \times \mathfrak{g}^*; |\xi|_x = |\gamma|\}. \tag{195}$$

The Hamiltonian flow on $T^*(M \times G)$ is of the form

$$\operatorname{Geo}_t^G(g, \gamma) = (\sigma_\gamma(t)\, g, \gamma) \tag{196}$$

where $\sigma_\gamma(t)$ is the one parameter group $\sigma_\gamma(t) = \exp tX$, $x \in \mathfrak{g}$ corresponding to $\gamma \in \mathfrak{g}^*$ under the isomorphism provided by the bi-invariant metric on G. Then the criterion that the Wong flow has a periodic orbit, of period T, is that there exist $(x, \xi; g, \gamma) \in \Sigma$ and $g_1 \in G$ such that

$$\operatorname{Geo}_T^M(x, \xi) = (x, \xi), \qquad \operatorname{Geo}_{-T}^G(g, \gamma) \cdot g_1 = (g, \gamma). \tag{197}$$

The first condition is that M has a periodic geodesic of length T; the last condition here is equivalent to

$$g_1 = \sigma_\gamma(-T)^{-1}. \tag{198}$$

The set of such g_1's makes up the singular support of $\kappa = \operatorname{Tr}_G f(Q)$, provided $T \in \operatorname{supp} \hat{f}$.

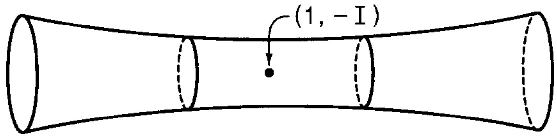
Thus the singular support of κ consists of a union of images $\Sigma(T_j)$ under $\exp\colon \mathfrak{g} \to G$ of spheres $S(T_j)$ of radius T_j (centered at 0) in $\mathfrak{g}$, where $\{T_j\}$ is the set of periods of geodesics on M (assuming this set is discrete). The wave front set of the singularity of κ lying over $\Sigma(T_j)$ is the flow-out of $T_e^*G \setminus 0$ under the time $-T_j$ geodesic flow. In case exp has nonsingular derivative on $S(T_j)$, this is the conormal bundle of the smooth manifold $\Sigma(T_j)$. The set where exp is singular is described as follows. Identifying TG with $G \times \mathfrak{g}$ by left translations, we have

$$d\exp(X)\, Y = \Xi(\operatorname{ad} X)\, Y, \qquad \Xi(a) = (e^a - 1)/a. \tag{199}$$

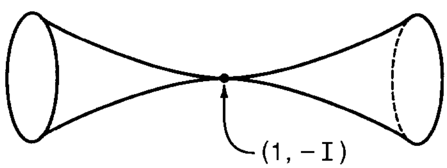
Thus exp is singular at $X \in \mathfrak{g}$ provided ad X has eigenvalues of the form $\lambda = 2\pi i m$, m a non-zero integer.

We consider some specific groups. First, if $G = SU(2)$, then with an appropriate normalization of the metric on $\mathfrak{g}$, $\exp\colon \mathfrak{g} \to G$ is singular on spheres of radii $R_j = \pi j$, $j = 1, 2, \ldots$ with image $\pm I$. Thus, if one of these numbers is a period T for the geodesic flow on M, one gets a contribution to $\mathrm{Tr}_G\, f(Q)$ belonging to $I^*(G, T_g^*G\backslash 0)$, with $g = I$ or $-I$. Proposition 4.5 applies if $g = I$ and an obvious variant applies if $g = -I$.

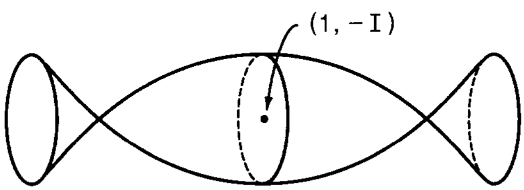
Further phenomena arise if we consider $G = U(2)$, or its double cover $U(1) \times SU(2)$, with Lie algebra $\mathfrak{g} = \mathbb{R} \oplus su(2)$; take $Y = (y, X) \in \mathfrak{g}$. If $y^2 + |X|^2 = 1$ and $|y|$ is small, so that we are considering a small neighborhood of the "equator" in the sphere $S^3 \subset \mathbb{R}^4 \cong \mathfrak{g}$, its image in $U(1) \times SU(2)$ under $\exp tY$ for t slightly smaller than π looks like this:



where a cross section diffeomorphic to S^2 is drawn as a circle. At $t = \pi$, the image has a cusp singularity:

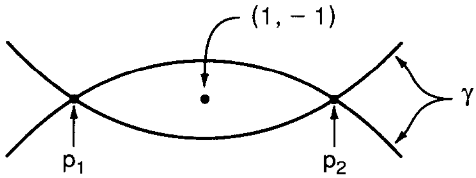


For t slightly larger than π, the image looks like this, with two conic singularities:



Thus if the geodesic flow on M has a period $T > \pi$, one gets a contribution μ to $\mathrm{Tr}_G\, f(Q)$ consisting of a Lagrangian distribution associated to $N^*\Sigma\backslash 0$ defined in the obvious way over the conic points, so that fiber over a regular point is a union of two rays, while the fiber over a conic point is a union of two cones.

The way the maximal torus T^2 sits in $U(t) \times SU(2)$, its normal bundle does not intersect $N^*\Sigma\backslash 0$, so the contribution μ has a well defined restriction $\mu^\# \in \mathscr{D}'(T^2)$; we need not use the construction (102)–(103). The singular support of $\mu^\#$, $\Sigma \cap T^2$, is a union γ of two arcs surrounding $(1, -1) \in T^2$:



However, the wave-front set of $\mu^{\#}$ is generally not just $N^*\gamma\backslash 0$, the union of the normal bundles of the two smooth arcs. Rather, the union of two quadrants in $T^*_{p_j}T^2$, lying over the intersection points p_1, p_2 is also typically contained in $WF(\mu^{\#})$. Thus locally near each p_j, $\mu^{\#}$ is a distribution associated to two transversally intersecting Lagrangians, of the sort studied in [22, 11].

These last statements can be proved by considering the following model situation. For $z=(y,x)=(y,x_1,x_2,x_3)\in\mathbb{R}^4$, consider $v=\delta(|x|-|y|)/y$. Let $v^{\#}$ be the restriction of v to the (y,x_1)-plane, so

$$v^{\#}=\delta(|x_1|-|y|)/y=\delta(x_1-y)/y+\delta(x_1+y)/y, \tag{200}$$

interpreted in the principal value sense. Note that $\hat{v}^{\#}(\eta,\xi_1)$ is piecewise constant on $\mathbb{R}^2$, constant on each of the 4 quadrants in $\mathbb{R}^2$ separated by $\eta=\pm\xi_1$, and vanishing on two of these quadrants due to cancellation. The distribution $\mu^{\#}$ on T^2 is a curvy version of this, up to a pseudodifferential operator factor. Note that in the Weyl integral formula (101) is a factor $D(g)$ which vanishes at p_j. Multiplying the model $v^{\#}$ by a linear factor annihilates the extra wave front set of $y=x_1=0$; in the Fourier transform representation this amounts to applying a derivative to the piecewise constant $\hat{v}^{\#}$. If a pseudodifferential operator is applied to $v^{\#}$, this "accidental" annihilation effect does not occur in general, though the order of the "extra" singularity is lowered.

In the curvy situation on $T^2\subset U(1)\times SU(2)$, the Fourier transform $\hat{\mu}^{\#}$ may have a more complicated behavior than that of the model $\hat{v}^{\#}$. We have no complete analysis of it to describe here, though effecting such an analysis is an example of an interesting class of problems arising in the study of Fourier integral distributions associated to transversally intersecting Lagrangians, on which one can hope to obtain progress. We merely note here that $\hat{\mu}^{\#}$ has a classical asymptotic behavior in all directions in $\mathbb{R}^2$ except four, corresponding to the locus of intersection of these Lagrangians, where the behavior of the Fourier transform will be more subtle.

APPENDIX: Notation Index

Symbol	Equation where defined
H_λ^0	(1)
H_λ	(2), (23)
Tr_G	(10), (54)
$f(Q)$	(17)
$\tau(\lambda)$	(19), (167)
$\mathscr{W}$	(22), (131)
A_f	(27)
Φ	(72)
Γ	(81)
$\mathscr{F}$	(84)
$\mathscr{I}$	(88)
Γ_U	(95), (160)
e	(107)
A_j	(129)
$\tilde{q}$	(134)
$\mathscr{Y}$	(134)
$\mathscr{W}_\lambda$	(140)
$\mathscr{W}_V$	(148)
$\mathscr{I}^U$	(157)
A_J^U	(158)

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