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LETTER TO THE EDITOR

Discussion of "Generalized beam theory applied to shear stiffness", *Int. J. Solids Structures*, Vol. 27, No. 15, pp. 1955–1967 (1991)

Dr Renton develops his expressions for the shear contribution to beam deflections by an energy argument based on the reasonable premise that the reactions at a fixed support should do no work. This approach would certainly seem to be preferable to the largely *ad hoc* methods used by earlier authors. However, an alternative interpretation is to regard the theory of beams as the beginning of an asymptotic expansion of the solution of a fully three-dimensional elasticity problem in terms of a small parameter, ε, defined as the ratio between a representative dimension in the beam cross-section and the beam length. In this context, the bending deflection term—about which there is no disagreement—would be the first term of the expansion and the shear deflection might be defined as the second term, which is generally two orders higher in ε.

For the thin rectangular cantilever, $0 < x < a$, $-b < y < b$, built-in at $x = a$, the appropriate three-dimensional problem is defined by the boundary conditions

$$u_x = u_y = 0,\ x = a,\ -b < y < b;\quad \sigma_{xx} = 0,\ x = 0,\ -b < y < b;\quad \int_{-b}^{b} \sigma_{xy}\, \mathrm{d}y = F,\ x = 0 \tag{1–3}$$

where the displacement boundary conditions at the built-in end are imposed in the "strong" or pointwise sense.

The usual polynomial elasticity solution of this problem involves a distortion of initially plane sections due to shear and hence can only satisfy these conditions in a weak sense—either in terms of the displacement and/or displacement gradient at one or more discrete points or in terms of an average displacement.

To the best of the present author's knowledge, the exact problem—which will involve a self-equilibrated "Saint-Venant" corrective stress field near the built-in end—has never been solved. However, some properties of this corrective field can be determined without completing a full solution. For example, by applying Betti's reciprocal theorem to the approximate solution, using a state of simple bending as auxiliary solution, it can be shown that, if the distorted end section is restored to a plane by a self-equilibrated distribution of normal stress, σ_{xx}, the location of the resulting plane corresponds to the 'integral' boundary condition

$$\int_{-b}^{b} u_x(a, y)\, y\, \mathrm{d}y = 0. \tag{4}$$

Unfortunately, the boundary condition on the component u_y cannot be dealt with as simply, but it is interesting to note that the use of eqn (4) in the approximate solution, coupled with the related integral conditions

$$\int_{-b}^{b} u_x(a, y)\, \mathrm{d}y = 0,\quad \int_{-b}^{b} u_y(a, y)\, \mathrm{d}y = 0 \tag{5, 6}$$

gives very nearly the same result as Renton's energy argument. Since Renton's argument is derived from considerations of strain energy in the approximate solution itself, we should not be surprised to find that his result is recovered exactly if (6) is replaced by the end condition

$$\int_{-b}^{b} u_x(a, y)(b^2 - y^2)\,dy = 0. \tag{7}$$

It is important to remark that, though the corrective stress field at the built-in end is localized in the Saint-Venant sense, the corresponding corrective displacement field is not necessarily localized, since the rigid body motion of the region beyond the end zone will generally be affected by the precise end conditions applied. Furthermore, it seems reasonable to expect that the extra constraint (e.g. on the strain component e_{yy}) implied by the strong end conditions (1) above would result in a stiffer restraint than that predicted by the elementary beam theory. Somewhat similar effects are obtained when a cantilever is subjected to torsion and the end-plane is restrained from warping, in which case the rotation of the free end can be significantly reduced (Timoshenko and Goodier, 1970).

It is perhaps instructive to consider the simpler problem in which the rectangular cantilever beam is loaded only by a bending moment at the free end—i.e. in which the conditions (2, 3) at $x = 0$ are replaced by

$$\sigma_{xy} = 0, \quad \int_{-b}^{b} \sigma_{xx}\, y\,dy = M. \tag{8}$$

In this case, the elementary solution predicts no distortion of plane sections and there is no shear force, but Poisson's ratio effects ensure that there is a non-zero value of e_{yy}, which must be constrained by a local corrective field at the built-in end. This local field will itself account for some strain energy and, as a result, the rotation of the applied moment, M, will be less than that predicted by the elementary theory. Furthermore, this reduction in rotation must be concentrated in the end zone, so that its effect on the beam displacement is seen principally as a rigid body rotation, which in turn will cause the end deflection to be less than that predicted by the elementary theory by an amount which is proportional to the length of the beam. This term is of the same order in the supposed asymptotic expansion as the shear deflection term, notwithstanding the fact that in the present problem there is no shear force to produce such a deflection.

The purpose of this perhaps rather laboured discussion is to show that "Saint-Venant" type end effects in beam problems produce corrections to the beam deflections that are of the same order in an asymptotic expansion of the exact three-dimensional solution as those due to legitimate shear deflection effects, even though the latter are generally significantly larger, as demonstrated by the relatively minor differences between previously published estimates listed in Renton's Introduction. Thus, the attempt to place shear deflections on a legitimate footing by asymptotic expansion seems doomed to failure.

Shear deflection estimates also introduce other paradoxical effects into the beam theory which deserve further discussion. For example, we might define a concentrated moment, M, applied at the point x on a beam as the limit as $\delta x \to 0$ of a pair of equal and opposite concentrated transverse forces of magnitude $M/\delta x$ at the points x and $x + \delta x$ respectively. However, the infinitesimal region of beam between x and $x + \delta x$ will experience a shear deflection which tends to a constant rather than zero as $\delta x \to 0$, suggesting that a concentrated moment should be associated with a step in transverse deflection. On the other hand, no such deflection would be expected if the concentrated moment were regarded as due to two equal and opposite horizontal forces acting (say) at the top and bottom of the beam. This suggests that a consistent second-order beam theory would need to encompass higher order information about the method of load application (and hence about the local stress state in the beam), beyond a mere statement of force resultants.

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