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Outer Automorphisms of Upper Triangular Matrices

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The outer automorphism group of the upper triangular matrices over the field of two elements is calculated. A. J. Weir (*Proc. Amer. Math. Soc.* **6** (1955), 454–464) performed a similar calculation for fields of odd characteristic, and we borrow the term extremal automorphism from his work. The results have implications in the study of stable splittings: the classifying space of U_n has three dominant summands when $n = 4$ and only one dominant summand when $n \geq 5$, in the sense of G. Nishida (Stable homotopy type of classifying spaces of finite groups, preprint (1986)).

Let U_n denote the subgroup of upper triangular matrices in $GL_n(\mathbb{F}_2)$. The outer automorphism group of U_n is generated by the obvious symmetry, perhaps called a flip or an anti-transpose, the central automorphisms and the extremal automorphisms. The central automorphisms lie in the kernel of the map $\mathrm{Out}(U_n) \to \mathrm{Out}(U_n/\text{center})$. The extremal automorphisms are described later. The term is borrowed from the work of A. J. Weir [3].

G. Nishida [2] has shown that the idempotents of the semisimple quotient of the group ring $\mathbb{F}_2[\mathrm{Out}(U_n)]$ lift to idempotents in the ring of 2-local stable self-maps of the classifying space BU_n, and correspond to dominant summands, which are those not detected on any proper subgroups. This yields summands of the cohomology ring $H^*(U_n; \mathbb{F}_2)$ as a module over the Steenrod algebra. The results of this paper imply that BU_4 has three dominant summands (although two are isomorphic) and BU_n has only one dominant summand for $n \geq 5$.

Explicitly, we have:

THEOREM. *The outer automorphism groups of U_n are*

(1) $\mathrm{Out}(U_3) = \mathrm{Out}(D_8) \cong \mathbb{Z}/2$

(2) $\mathrm{Out}(U_4) \cong \Sigma_3 \times \mathbb{Z}/2$

(3) $\mathrm{Out}(U_n) \cong (\mathbb{Z}/2)^{n-1} \times_T \mathbb{Z}/2$ *for* $n \geq 5$.

Remark. In part (3), the notation signifies a semi-direct product.

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The split $\mathbb{Z}/2$ is the anti-transpose, and the normal subgroup $(\mathbb{Z}/2)^{n-1}$ is generated by $n-3$ central automorphisms and two extremal automorphisms.

COROLLARY. *The semisimple quotient of* $\mathbb{F}_2[\mathrm{Out}(U_n)]$ *is trivial unless* $n=4$, *in which case the quotient is* $\mathbb{Z}/2 \times M_2(\mathbb{F}_2)$. *Thus* BU_4 *has three dominant summands, and the two corresponding to the Steinberg idempotents of* $M_2(\mathbb{F}_2)$ *are isomorphic.*

Proof of the Theorem. U_n has $n-1$ generators, the "off diagonal" matrices $I_n + e_{i,i+1}$. The automorphisms are determined by the action on these generators. There is an important automorphism $\sigma: U_n \to U_n$ which is a flip or anti-transpose $I_n + e_{i,i+1} \to I_n + e_{n-i, n+1-i}$. This is the only non-trivial element of $\mathrm{Out}(D_8)$, and is a split quotient of every $\mathrm{Out}(U_n)$.

The center of U_n is a single copy of $\mathbb{Z}/2$, which must be fixed by all automorphisms. Thus we have a map $\mathrm{Out}(U_n) \to \mathrm{Out}(U_n/\text{center})$. Then the center of U_n/center must be preserved as a subgroup, and so on, and we obtain a map

$$\mathrm{Out}(U_n) \to \mathrm{Out}((\mathbb{Z}/2)^{n-1}) \cong GL_{n-1}(\mathbb{F}_2).$$

For $n \geqslant 5$, the image of this map is just a $\mathbb{Z}/2$ generated by the flip σ, but for $n=4$ the map $\mathrm{Out}(U_4) \to GL_3(\mathbb{F}_2)$ has image isomorphic to $GL_2(\mathbb{F}_2) \cong \Sigma_3$. In U_4, the normalizer of the $\mathbb{Z}/2$ subgroup generated by either $I_4 + e_{1,2}$ or $I_4 + e_{3,4}$ is isomorphic to $D_8 \times \mathbb{Z}/2$, but the normalizer of the subgroup $\langle I_4 + e_{2,3} \rangle$ is $(\mathbb{Z}/2)^4$. Perhaps the best interpretation of this image $GL_2(\mathbb{F}_2)$ is as linear maps of the $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup of $H^2((\mathbb{Z}/2)^3; \mathbb{F}_2)$ generated by the two K-invariants for the central extension:

$$1 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to U_4/\text{center} \to \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1.$$

The elements in the kernel of the map $\mathrm{Out}(U_n) \to \mathrm{Out}(U_n/\text{center})$ are referred to as central automorphisms, which constitute an elementary abelian subgroup $(\mathbb{Z}/2)^{n-3}$. The generators are automorphisms φ_i that twist with the central element

$$\varphi_i(I_n + e_{i,i+1}) = I_n + e_{i,i+1} + e_{n-1,n}$$

and

$$\varphi_i(I_n + e_{j,j+1}) = I_n + e_{j,j+1} \qquad \text{for} \quad i \neq j.$$

Note that for $i=1$ or $i=n-1$ the automorphisms φ_1 and φ_{n-1} are inner; for example φ_1 is conjugation by $I_n + e_{2,n}$.

Let $\varphi: U_n \to U_n$ be any automorphism, and let us consider the image of the element $I_n + e_{1,2}$. Note that the commutator $[I_n + e_{1,2}, I_n + e_{2,n}] = I_n + e_{1,n}$ is nontrivial, and also the $(\mathbb{Z}/2)^3$ subgroup $\langle I_n + e_{1,n-1}, I_n + e_{2,n}, I_n + e_{1,n} \rangle$ is preserved. This implies that either $\varphi(I_n + e_{1,2}) = (I_n + e_{1,2}) \cdot \varepsilon$ or that $\varphi(I_n + e_{1,2}) = (I_n + e_{n-1,n}) \cdot \varepsilon'$. In the latter situation, $\sigma \circ \varphi(I_n + e_{1,2}) = (I_n + e_{1,2}) \cdot \varepsilon$. By composing with an inner automorphism we can assume that $\varepsilon \in U_{n-1}$, that is, the expression or "word" ε contains no elements or "letters" of the first row of U_n. In fact, since $I_n + e_{1,2}$ is of order 2, it is clear that ε contains no elements of the second row, so $\varepsilon \in U_{n-2}$.

Assume momentarily that $\varepsilon \in U_{n-3}$ (no elements from the third row). Note that the subgroup consisting of the first row of U_n has several properties: it is elementary abelian, normal, and equals the closure under inner automorphisms of the first element $I_n + e_{1,2}$. These properties will be preserved by any automorphism φ. Consider the conjugation of $(I_n + e_{1,2}) \cdot \varepsilon$ given by

$$(I_n + e_{2,3})(I_n + e_{1,2})(\varepsilon)(I_n + e_{2,3}) = (I_n + e_{1,3})(I_n + e_{1,2}) \cdot \varepsilon.$$

Then the image of the first row subgroup contains the elements $I_n + e_{1,3}$, and thus by continued conjugation, the rest of the row $I_n + e_{1,k}$ with $k > 2$. This image should be an abelian group, so ε must commute with $I_n + e_{1,k}$ for $k > 2$. But then $\varepsilon = 1$ and so φ (or perhaps $f_{\text{inner}} \circ \sigma \circ \varphi$) fixes the matrix $I_n + e_{1,2}$.

Now assume that $\varphi(I_n + e_{1,2}) = (I_n + e_{1,2}) \cdot \varepsilon$ with $\varepsilon \in U_{n-2}$, and let $I_n + e_{3,k}$ be the first element of the third row of U_n that appears in the expression for ε. Write $\varepsilon = (I_n + e_{3,k}) \cdot \varepsilon'$, and again consider the conjugation by $I_n + e_{2,3}$:

$$\begin{aligned}
&(I_n + e_{2,3})(I_n + e_{1,2})\, \varepsilon (I_n + e_{2,3}) \\
&\quad = (I_n + e_{1,2})(I_n + e_{1,3})(I_n + e_{2,k})(I_n + e_{3,k})(I_n + e_{2,3})\, \varepsilon'(I_n + e_{2,3}) \\
&\quad = (I_n + e_{1,2})(I_n + e_{3,k})\, \varepsilon'(I_n + e_{1,3})(I_n + e_{2,k})(I_n + e_{1,k}) \cdot \lambda,
\end{aligned}$$

where λ is an element of the abelian subgroup generated by $I_n + e_{ij}$ and $I_n + e_{2,j}$ for $j > k$.

Now conjugate $\varphi(I_n + e_{1,2})$ by $I_n + e_{k,n}$; if $k < n$

$$\begin{aligned}
&(I_n + e_{k,n})(I_n + e_{1,2})(I_n + e_{3,k})\, \varepsilon'(I_n + e_{k,n}) \\
&\quad = (I_n + e_{1,2})(I_n + e_{3,k})(I_n + e_{3,n})\, \varepsilon' \lambda',
\end{aligned}$$

where λ' is in the subgroup of the last column of U_n generated by $I_n + e_{j,n}$ for $4 \leq j \leq k-1$. Thus the image under φ of the first row subgroup contains both $(I_n + e_{1,3})(I_n + e_{2,k})(I_n + e_{1,k})\, \lambda$ and $(I_n + e_{3,n})\, \lambda'$. But the commutator of these two elements is $I_n + e_{1,n}$, which contradicts the fact that this subgroup should be abelian.

Unless $k = n$, we see that $\varepsilon = 1$ and $I_n + e_{1,2}$ is fixed by φ. When $k = n$, we find a type of automorphism referred to as an extremal automorphism (for a similar definition, see [3]):

$$\varphi_e(I_n + e_{1,2}) = (I_n + e_{1,2})(I_n + e_{3,n})$$
$$\varphi_e(I_n + e_{1,3}) = (I_n + e_{1,3})(I_n + e_{2,n})(I_n + e_{1,n})$$
$$\varphi_e(I_n + e_{i,j}) = I_n + e_{i,j} \qquad \text{otherwise,}$$

There are only two extremal automorphisms, the φ_e above and $\sigma\varphi_e\sigma$. These generate a $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup of $\mathrm{Out}(U_n)$.

So, if necessary by composing with an extremal automorphism, we can assume our transformed automorphism φ fixes $I_n + e_{1,2}$. Then the first row is preserved as a group, and by composing with further inner automorphisms, we can obtain a map φ fixing the top row element-wise.

The set of those automorphisms which fix both the top row and the quotient U_{n-1} can be shown to be the cohomology group $H^1(U_{n-1}; (\mathbb{Z}/2)^{n-1})$ with twisted coefficients [1]. This yields $(\mathbb{Z}/2)^{n-2} \subset \mathrm{Out}(U_n)$ generated by the central automorphisms and the extremal automorphisms $\sigma\varphi_e\sigma$.

Now I claim that any automorphism of U_n which acts as the identify on the first row will also act as the identify on the quotient U_{n-1}. We may inductively assume that $\mathrm{Out}(U_{n-1}) \simeq (\mathbb{Z}/2)^{n-2} \times_T \mathbb{Z}/2$, generated by $n-4$ central automorphisms, two extremal automorphisms, and the flip σ. Showing that none of these extends to an automorphism of U_n fixing the top row element-wise follows from simple commutativity relations with $I_n + e_{1,2}$ and $I_n + e_{1,n-2}$. This completes the proof.

References

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