

## Renegotiation and Symmetry in Repeated Games\*

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The paper derives a theory of renegotiation-proofness in symmetric repeated games based on a notion of "equal bargaining power." According to *consistent bargaining equilibrium* a player can mount a credible objection to a continuation equilibrium in which he receives a particular expected present discounted value, if there are other self-enforcing agreements that never give any player such a low continuation value after any history. The definition does not imply strongly symmetric solutions. But under modest assumptions, consistent bargaining equilibria of infinitely repeated games with perfect monitoring *are* strongly symmetric. Such solutions have an unusually elementary characterization. *Journal of Economic Literature* Classification Numbers: C7, C72. © 1993 Academic Press, Inc.

### 1. INTRODUCTION

In the context of repeated games, the term "renegotiation" refers to the ongoing *bargaining* that takes place among the players as the supergame unfolds. But considerations of bargaining power are curiously absent from the formulations of renegotiation-proof equilibrium proposed in the

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literature. The leading theories assume that conflicts of interest are resolved by a unanimity rule<sup>1</sup>: an equilibrium plan in progress is overthrown in favor of a credible alternative if and only if the change has unanimous approval. While this Pareto criterion is less than satisfying, there is no commonly accepted theory of bargaining power in repeated games to take its place.

Even in static bargaining problems, it is hard to argue compellingly for a particular prediction regarding the division of surplus. If the problem is symmetric, things are easier: although the solution remains logically indeterminate, a symmetric division of the surplus suggests itself as a reasonable guess.<sup>2</sup> This raises the possibility that one could make progress in symmetric repeated games by incorporating the heroic assumption of "equal bargaining power" into a theory of renegotiation. That is the exercise we attempt here. The resulting solution concept is again best viewed as an educated guess, one that we feel is a better bench mark than the unanimity criterion can provide.

In this spirit, we consider symmetric discounted repeated games and begin by remarking that a natural interpretation of equal bargaining power is (among other things) that the players will receive symmetric payoffs in the equilibrium negotiated at the beginning of the game, as long as the Pareto frontier of the set of credible supergame equilibria includes a symmetric element. It is tempting to extend this reasoning to say that the symmetry of the subgame in which players find themselves after any history (possibly including deviations from equilibrium play) suggests that the continuation equilibrium in the subgame ought to be symmetric. We argue that there should be no such presumption: *even in the subgame* it may be in the interests of the worst-off player not to insist on symmetry.

The line of reasoning that supports this assertion is an elaboration of the approach to renegotiation theory<sup>3</sup> taken by Pearce [17]. The approach is most easily understood by thinking first of the simple case of a symmetric

<sup>1</sup> This is true, for example, in the independent work of Bernheim and Ray [11] and Farrell and Maskin [13] that opened the literature, and in the related papers by Benoit and Krishna [8], Bernheim, Peleg, and Whinston [10], and van Damme [21], as well as in the alternative approaches of Asheim [7] and Pearce [17]. The frameworks presented by Greenberg [15] and Bergin and MacLeod [9] do not specify any one criterion.

<sup>2</sup> Here too there is ample room for disagreement, especially if the opportunity set is not convex. For example, if the possible payoff pairs in a two-person bargaining situation are (5, 5), (4, 8), and (8, 4), perhaps (5, 5) is not a convincing prediction. In its favor, one could make the following (inconclusive) case. If player 1 is pressing for (8, 4), for instance, player 2 can demand (5, 5), arguing that player 1 cannot expect player 2 to find 4 acceptable if player 1 will not agree to 5.

<sup>3</sup> One could also modify other definitions of renegotiation-proof equilibrium (see Footnote 1 above) by replacing the Pareto criterion with an assumption of equal bargaining power; we work here with the approach that we find the most persuasive.

two-person repeated game all of whose (subgame perfect) equilibria are symmetric. The supergame equilibrium value set can be regarded as a subset of  $\mathbf{R}$ , say  $V$ , with maximum  $\bar{v}$  and minimum  $\underline{v}$ . An equilibrium that achieves  $\bar{v}$  may be supported by the "threat" that after certain histories (for example, if someone is observed to deviate, or, with imperfect monitoring, an unfavorable signal arises) the value of the "continuation equilibrium" in the ensuing subgame will be  $\underline{v}$ . Although the threat is subgame perfect, it is not credible in another sense: it seems plausible that the players could convince one another to abandon the worst equilibrium, on the grounds that it constitutes an unnecessarily harsh punishment. While players understand that in order to enjoy mutual cooperation they must accept different continuation payoffs after different histories, they will not accept a "punishment" if there exists another continuation equilibrium that never needs to use such a severe punishment. In other words, because players care about the future rather than the past, they ask themselves not whether a certain punishment was needed to deter deviations earlier in the game, but whether the punishment is inescapable in the sense that in the future any equilibrium must inevitably rely on punishments at least as harsh.

Consider now the more general case of a symmetric game in which some supergame equilibrium values are asymmetric. How should players exploit the equal bargaining power associated with the symmetry of their roles? Not, we contend, by insisting on symmetric payoffs in every subgame: in some cases this would result in unnecessarily low payoffs for both players. Rather, a supergame continuation value pair  $(a, b)$  with  $a < b$  should be acceptable to player 1 as long as there is no other subgame perfect equilibrium in which, in all subgames, each player receives at least some value  $c > a$ . It might happen, for example, that all "strongly symmetric equilibria" (those that treat players symmetrically in every subgame) must use continuation value threats of  $(2, 2)$  or worse, whereas the asymmetric threats  $(3, 5)$  and  $(5, 3)$  sustain a variety of equilibria whose continuation values never drop below 3 for either player. (It is easy to find examples of this sort; a simple one is provided below.) Consequently, if a deviation by player 1 is followed by a path with value  $(3, 5)$ , player 1 accepts the asymmetry because in a symmetric regime, punishments of value 2 would be unavoidable. Formally, we say that a subgame perfect equilibrium  $\sigma$  is a *consistent bargaining equilibrium* (CBE) if the infimum of the values of continuation equilibria (taken over all subgames and players) of  $\sigma$  is at least as great as the corresponding infimum for any other subgame perfect equilibrium. In the context of symmetric games, this specializes the definition of renegotiation-proofness given by Pearce [17].

While it is intriguing that symmetric bargaining power need not lead to symmetric punishments, it would be tremendously helpful when analyzing

a particular game to know that one could restrict attention to strongly symmetric profiles. Abreu [1] showed in an oligopolistic model that in the traditional perfect monitoring setting without renegotiation, optimization within the class of strongly symmetric profiles yields easily described equilibria with vivid properties. But he further showed that more severe punishments can usually be achieved outside the strongly symmetric class and that the structure of the optimal punishment tends, unfortunately, to be complex. One of our principal goals is to provide conditions under which renegotiation and equal bargaining power *imply* strong symmetry and to explore the properties of strongly symmetric CBEs.

For any finite symmetric game  $G$  satisfying standard regularity assumptions and having an equilibrium in pure strategies, the associated infinitely repeated discounted game  $G^\infty(\delta)$  has a CBE. While much of the paper restricts attention to games with perfect monitoring, the definitions apply equally to imperfect monitoring models. For the latter, we can show that severest CBE punishments are often *not* strongly symmetric. By contrast, their counterparts in perfect monitoring models usually *are* strongly symmetric.

It is by now well understood that the crucial step in determining what kind of behavior can be supported in a particular supergame is to compute the worst credible threats that are available to the players. Theorem 2 of Section 2 provides an unexpectedly simple characterization of the worst CBE punishment in the repeated game with discount factor  $\delta$ : it is just the maximized value of a function  $f$  defined in an elementary way using the payoff function of the one-shot game  $G$ . Specifically, for any symmetric strategy profile  $x$  in  $G$ ,  $\delta f(x)$  is the difference between the payoff when  $x$  is played, and  $(1 - \delta)$  times any player's *best response* payoffs against the same profile. If  $G$  is the well-known linear Cournot model, for example, this result produces *closed-form expressions* (for any value of the discount factor and any number of firms) for maximally collusive punishment values. This degree of tractability suggests that one can readily derive the theory's implications in a variety of applied areas.

The worst credible punishments in the class of games whose CBE solutions we characterize have a "stick-and-carrot" structure similar to that established by Abreu [1] for the standard theory without renegotiation. They have two "phases," the first serving to give the players low payoffs for a number of periods, and the second following the equilibrium path of the best strongly symmetric CBE (which without loss of generality can be taken to be *stationary*).

The assumptions required to generate these results are nontrivial restrictions on the component game. Nonetheless, many games of economic interest satisfy the assumptions. As an example, Appendix A presents a class of games including Cournot oligopolies with convex cost functions

and a family of demand functions containing the linear and constant elasticity demands as special cases. Readers interested in seeing the theory applied to price-setting oligopolies and to multimarket collusion (Bernheim and Whinston [12]) may consult Abreu, Pearce, and Stacchetti [6].

The brief treatment of imperfect monitoring models provided by Section 3 establishes that strong symmetry is *not* a general implication of the definition of a consistent bargaining equilibrium. Fudenberg, Levine, and Maskin [14] identify<sup>4</sup> a large class of games with unobservable actions in which first-best outcomes in the supergame can be approached asymptotically in equilibrium as  $\delta$  approaches 1. By Proposition 7 of Pearce [17] this is true even if one restricts attention to consistent bargaining equilibria. But we show that imposing strong symmetry leads to inefficiencies that do not vanish asymptotically. Thus, in imperfect monitoring models with patient players, consistent bargaining equilibria will usually violate strong symmetry. Section 4 concludes the paper.

#### *Limitations of Consistent Bargaining Equilibrium*

The purpose of this subsection is to place CBE in a particular context and to emphasize how cautiously it should be interpreted.

One way of summarizing the problem of renegotiation is as follows. For the purpose of creating incentives, a group of players may find it useful to make threats, embedded in the negotiated equilibrium plan, about the value of continuation payoffs in some eventuality (a history ending with a certain deviation, for example). If the contingency in question actually arises, players may be tempted to abandon the plan, perhaps because it is unpleasant for all concerned, or perhaps because it is intolerably asymmetric. In principle, it is easy to agree that the interests of the group “*ex ante*” and “*ex post*” may conflict radically. But modeling this formally requires a precise expression of “the interests of the group.” We would like to say that a cooperative agreement  $\sigma$  is *renegotiation-proof* if after each history  $h$  there is no subgame perfect equilibrium all of whose continuation values are strictly preferred by the group to  $\sigma|_h$ . What theory results depends on the binary relation one uses to represent group preferences.<sup>5</sup>

One candidate that occurs naturally to economists is the Pareto criterion: action stream  $x$  is preferred to  $y$  if each individual prefers  $x$  to  $y$ . We find the Pareto rule unattractive in this context; it gives a single player undue veto power over departures from a verbal agreement. It is unclear whether the group can credibly commit to respecting such a verbal agreement *ex post*. More plausible to us is the hypothesis that the bargaining power of each player, derived from the structure of the game, will play a

<sup>4</sup> See also the references cited in Section 3 below.

<sup>5</sup> For a more detailed discussion along these lines, see Abreu and Pearce [4].

primary role in determining the division of surplus, both at the beginning of the game and following any history. An obvious difficulty, and a severe one, is that there is no authoritative bargaining theory for the division of surplus in repeated games. Our strategy in this paper is to restrict attention to symmetric games, where equal division can be used as a natural benchmark.

Even in symmetric environments, one may judge the consistent bargaining solution to be unduly biased toward equal shares.<sup>6</sup> It is hard to argue convincingly one way or the other, and CBE certainly cannot be viewed as a definitive solution concept for symmetric repeated games. But we find it a more persuasive construction than the solution concepts previously available.

A further cautionary remark is in order. The CBE concept implicitly compares the utilities of different players. Whether this is an advantage or disadvantage of the theory (relative to other solution concepts that do not make reference to interpersonal comparisons) depends on one's views concerning the likelihood that interpersonal comparisons play a role in negotiations. The solution concept is intended to apply only to situations in which players feel that their circumstances are the same as one another's, not just in the ordinal description of the game, but also in the utility consequences of physical outcomes.

## 2. PERFECT MONITORING

The class of games considered in this section are symmetric repeated games with *perfect monitoring*.

### 2.1. Preliminaries

The *stage game* is denoted  $G = (S_1, \dots, S_n; \Pi_1, \dots, \Pi_n)$ , where  $N = \{1, \dots, n\}$  is the set of players,  $S_i$  is the set of pure strategies for player  $i$ ,  $S = S_1 \times \dots \times S_n$ , and  $\Pi_i: S \rightarrow \mathbf{R}$  is player  $i$ 's payoff function. The stage game is symmetric in that

- (i)  $S_i = S_1$  for all  $i$ .
- (ii) For each permutation  $\tau$  of  $\{1, \dots, n\}$ ,  $\Pi_i(s_{\tau(1)}, \dots, s_{\tau(n)}) = \Pi_{\tau(i)}(s_1, \dots, s_n)$  for all  $s \in S$  and all  $i$ .

In addition we assume the following:

- (A1)  $S_i$  is compact.
- (A2)  $\Pi_i$  is continuous.

<sup>6</sup> To the extent that this is true, the asymmetric results of Section 3 are particularly striking.

The associated *repeated game* is denoted  $G^\infty(\delta)$ , where  $\delta \in (0, 1)$  is the discount factor. We refer to  $n$ -tuples  $(z_1, \dots, z_n)$  by the corresponding unsubscripted symbol  $z$ . Also  $z_{-i} = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$ . Each player  $i$  chooses an action  $s_i(t) \in S_i$  in every period  $t = 1, 2, \dots$ . The perfect monitoring assumption is that  $s_i(t)$  may depend on the entire history of *all* players' previous choices  $s(1), \dots, s(t-1)$ . A repeated game (pure) strategy for player  $i$  is denoted  $\sigma_i$ , and  $\sigma$  denotes a strategy *profile*. Throughout we confine attention to pure strategies. Also,

- $\mathbf{s} = \{s(t)\}_{t=1}^\infty$ , where  $s(t) = (s_1(t), \dots, s_n(t)) \in S$ , denotes a *path*.
- $v_i(\mathbf{s}) = ((1 - \delta)/\delta) \sum_{t=1}^\infty \delta^t \Pi_i(s(t))$  is the *average* discounted payoff to  $i$  from the path  $\mathbf{s}$ . Note that first-period payoffs are discounted.
- $\hat{v}_i(\sigma)$  denotes the (average discounted) payoff to  $i$  from the strategy profile  $\sigma$ .
- $v_i(\mathbf{s}; t) = ((1 - \delta)/\delta) \sum_{\tau=0}^\infty \delta^{\tau+1} \Pi_i(s(t + \tau))$  is the payoff to  $i$  along the path  $\mathbf{s}$  from period  $t$  onward, discounted to the beginning of period  $t$ .

Let  $H = \bigcup_{t=0}^\infty S^t$  be the set of all histories, where  $S^0 = \{\emptyset\}$  and  $\emptyset$  denotes the null history. For all  $h \in H$ ,  $\sigma|_h$  denotes the strategy profile induced by  $\sigma$  on the subgame following the history  $h$ . By convention  $\sigma|_\emptyset = \sigma$ . We are interested in a subset of the set of *subgame perfect equilibria* (Selten [19, 20]).

**DEFINITION.** For each strategy profile  $\sigma$ ,  $C(\sigma) = \{\hat{v}(\sigma|_h) | h \in H\}$  is the set of “continuation values” of  $\sigma$ , including the value of  $\sigma$  itself, and  $l(\sigma) = \inf\{\min\{w_1, \dots, w_n\} | (w_1, \dots, w_n) \in C(\sigma)\}$ .

Let  $e = (1, \dots, 1) \in \mathbf{R}^n$ , and for any  $y \in S_1$ , let  $y \cdot e$  denote the  $n$ -tuple  $(y, \dots, y)$ . We adopt the convention that  $\mathbf{x} = \{x(t) \cdot e\}_{t=1}^\infty$ ,  $x(t) \in S_1$ , denotes a *symmetric* path while  $\mathbf{s} = \{s(t)\}_{t=1}^\infty$ ,  $s(t) \in S$ , is, as previously defined, an arbitrary, possibly asymmetric, path in  $S$ . Let

$$\begin{aligned} \bar{\Pi}_i(s) &= \max\{\Pi_i(s'_i, s_{-i}) | s'_i \in S_i\} \\ \pi(x) &= \Pi_1(x, \dots, x), \\ \bar{\pi}(x) &= \bar{\Pi}_1(x, \dots, x). \end{aligned}$$

That is,  $\bar{\Pi}_i(s)$  is player  $i$ 's best response payoff (in the stage game) when other players play according to  $s_{-i}$ ;  $\pi(x)$  is a player's payoff at the symmetric action choice  $x$ ; and  $\bar{\pi}(x)$  is a player's best response payoff when all other players choose the action  $x$ .

A path  $\mathbf{s}$  is an equilibrium path if it can be “supported” by a (credible) punishment.

DEFINITION. A path  $\mathbf{s}$  is supportable by  $w \in \mathbf{R}$  if for each  $i = 1, \dots, n$  and each  $t = 1, 2, \dots$ ,

$$(1 - \delta)(\bar{\Pi}_i(\mathbf{s}(t)) - \Pi_i(\mathbf{s}(t))) \leq \delta(v_i(\mathbf{s}; t + 1) - w).$$

For a symmetric  $\mathbf{x}$  these conditions reduce to

$$(1 - \delta)(\bar{\pi}(x(t)) - \pi(x(t))) \leq \delta(v_1(\mathbf{x}; t + 1) - w) \quad \text{for all } t.$$

A strategy profile of  $G^\delta$  may be viewed as a rule specifying an initial path and punishments for any deviation from the initial path or from a previously prescribed punishment. Let  $\hat{\sigma}(\mathbf{s}, \mathbf{x})$  denote the *simple profile* (see Appendix B for a definition) with initial path  $\mathbf{s}$  and a single symmetric punishment  $\mathbf{x}$ . Any single player deviation from an ongoing path ( $\mathbf{s}$  or  $\mathbf{x}$ ) is responded to simply by (re)starting  $\mathbf{x}$ . From Abreu [2] we have the following.

(P) The simple profile  $\hat{\sigma}(\mathbf{s}, \mathbf{x})$  is a subgame perfect equilibrium if and only if the paths  $\mathbf{s}$  and  $\mathbf{x}$  are supportable by  $v_1(\mathbf{x})$ .

Henceforth, we will typically refer to subgame perfect equilibrium simply as equilibrium. We assume that

(A3)'  $G^\delta$  has a (subgame perfect) equilibrium.

A simple sufficient condition on  $G$  which guarantees (A3)' is

(A3)  $G$  has a Nash equilibrium in pure strategies.

## 2.2. Bargaining Power and Renegotiation

Our main definition is motivated as follows. Players do not care about symmetry per se. Rather, a player exploits his bargaining power by refusing to accept a continuation payoff, say  $w$ , unless *all* (subgame perfect) equilibria rely on punishments at least as harsh as  $w$ .

DEFINITION. An equilibrium  $\sigma$  is a CBE if for any equilibrium  $\gamma$ ,  $l(\gamma) \leq l(\sigma)$ .

If  $\sigma$  is a CBE, it is impossible for any player  $i$  to object, following some history  $h$ , that his continuation payoff  $\bar{v}_i(\sigma|h)$  is intolerably low (and to demand renegotiation of the agreement). Punishments of at least this severity are an inevitable part of any self-enforcing implicit agreement. Conversely, we interpret equal bargaining power to mean that a player may demand  $l(\sigma)$  after *any* history  $h$ .

THEOREM 1 (Existence). Under (A1), (A2), and (A3)', a consistent bargaining equilibrium exists.

*Proof.* See Appendix B.



Recall that a repeated game strategy for player  $i$  is a sequence of functions  $\sigma_i = (\sigma_i(1), \sigma_i(2), \dots)$  where  $\sigma_i(t): S^{t-1} \rightarrow S_i$ .

DEFINITION. A strategy profile  $\sigma$  is strongly symmetric if for all  $i \in N$ , (i)  $\sigma_i(1) = \sigma_1(1)$ , and (ii)  $\sigma_i(t)(h) = \sigma_1(t)(h)$  for all  $t \geq 2$  and  $h \in S^{t-1}$ .

Does the equal bargaining power assumption imply equal treatment in the sense of identical behavior after all contingencies? Not necessarily, as the following simple example shows. In this version of the Prisoner's Dilemma, no CBE is strongly symmetric.

EXAMPLE. Consider the two-player stage game

	$U$	$D$
$u$	20, 20	5, 30
$d$	30, 5	7, 7

Set  $\delta = 1/2$ . Consider a strongly symmetric equilibrium  $\sigma$  which involves play of  $(u, U)$  after some history. Given that the one-period gain from cheating at  $(u, U)$  is 10, it follows that  $(1 - \delta)10 \leq \delta(20 - l(\sigma))$ . Hence  $l(\sigma) \leq 10$ . If  $\sigma$  has players use  $(d, D)$  always, then obviously  $l(\sigma) = 7$ . But there exist equilibria which are not strongly symmetric, with all continuation values higher than 10. Let  $s^1 = (d, U)$  and  $s^2 = (u, D)$ , and consider the following strategy profile  $\hat{\sigma}$ . Start with  $s^1$ . If players use  $s^1$  in period  $t$ ,  $s^2$  is to be played in  $(t + 1)$ , and vice versa. If row deviates in  $t$ ,  $s^2$  is to be played in  $(t + 1)$ , and if column deviates in  $t$ ,  $s^1$  is played in  $t + 1$ . It may be checked directly that  $\hat{\sigma}$  is an equilibrium. Furthermore,  $l(\hat{\sigma}) = (5 + 30\delta)/(1 + \delta) = 40/3 > 10 \geq l(\sigma)$  for any strongly symmetric equilibrium  $\sigma$ . Thus, in this game it is in the players' self-interest to permit themselves to be treated asymmetrically.

### 2.3. Characterizations

While the preceding example demonstrates that strong symmetry cannot be guaranteed a priori, it is analytically an extremely attractive restriction and leads to tractable characterizations. The next assumption plays a central role in establishing that strongly symmetric CBE's exist.

(A4) For all  $s \in S$  there exists  $x \in S_1$  such that

$$(i) \quad \pi(x) \geq \frac{1}{n} \sum_i \Pi_i(s)$$

$$(ii) \quad \bar{\pi}(x) - \pi(x) \leq \frac{1}{n} \sum_i [\bar{\Pi}_i(s) - \Pi_i(s)].$$

This assumption requires that for any action profile there exists a symmetric action profile which yields higher average payoffs and lower average gains from cheating. That is, stage game asymmetries do not, of themselves, increase average payoffs or reduce average temptations to cheat.

**LEMMA 1 (Smoothing).** *Given Assumption 4, for any path  $\mathbf{s}$  supportable by  $w \in \mathbf{R}$ , there exists a symmetric path  $\mathbf{x}$  supportable by  $w$  such that*

$$v_1(\mathbf{x}; t) \geq \frac{1}{n} \sum_{i=1}^n v_i(\mathbf{s}; t) \quad \text{for } t = 1, 2, \dots$$

*Proof.* Consider  $\mathbf{s} = \{s(t)\}_{t=1}^{\infty}$  supportable by  $w$ . Then for all  $i$ ,

$$(1 - \delta)(\bar{\Pi}_i(s(t)) - \Pi_i(s(t))) \leq \delta(v_i(\mathbf{s}; t + 1) - w).$$

Hence,

$$(1 - \delta) \frac{1}{n} \sum_i [\bar{\Pi}_i(s(t)) - \Pi_i(s(t))] \leq \delta \left( \frac{1}{n} \sum_i v_i(\mathbf{s}; t + 1) - w \right).$$

By (A4) there exists a symmetric path  $\mathbf{x} = \{x(t) \cdot e\}_{t=1}^{\infty}$  such that

$$\pi(x(t)) \geq \frac{1}{n} \sum_i \Pi_i(s(t)) \quad \text{and} \quad \bar{\pi}(x(t)) - \pi(x(t)) \leq \frac{1}{n} \sum_i [\bar{\Pi}_i(s(t)) - \Pi_i(s(t))].$$

Hence,  $v_1(\mathbf{x}; t) \geq (1/n) \sum_i v_i(\mathbf{s}, t)$ , and  $(1 - \delta)(\bar{\pi}(x(t)) - \pi(x(t))) \leq \delta(v_1(\mathbf{x}; t + 1) - w)$  for all  $t$ . Q.E.D.

The next assumption appears in the proof of Theorem 2. It is used there to resolve an integer problem (time is discrete) and should be viewed more as a convenience than as an essential component of the basic argument. See the discussion following the proof. Note that (A3) and (A4) *imply* that  $G$  has a symmetric Nash equilibrium. Denote by  $x^{cn}$  the symmetric equilibrium which yields the highest payoff. By (A1) and (A2),  $x^{cn}$  is well defined.

(A5) For any  $z \in S_1$  and  $a$  such that  $\pi(x^{cn}) \leq a \leq \pi(z)$ , there exists  $y \in S_1$  such that  $a = \pi(y)$ , and  $\bar{\pi}(y) - \pi(y) \leq \bar{\pi}(z) - \pi(z)$ .

While (A4) and (A5) are nontrivial assumptions, they are both satisfied in all the examples we consider in Appendix A and in Abreu, Pearce, and Stacchetti [6].

**DEFINITION.** If  $\sigma$  and  $\gamma$  are CBE's, it follows that  $l(\sigma) = l(\gamma)$ . Let  $\underline{l} = l(\sigma)$  for any CBE  $\sigma$ .

The key result of this section is Theorem 2. If one sets aside questions of “openness,” the argument is roughly the following. Symmetric CBE paths exist,<sup>7</sup> because by Lemma 1 any asymmetric CBE path can be “averaged” across players while preserving incentive compatibility. It is easy to show that among the best symmetric CBE paths there is at least one that is stationary. For any action  $x$ , let  $f(x)$  be the greatest value that would support the stationary symmetric path on which action  $x$  is always chosen. Such a punishment would be just sufficient to deter a deviation; therefore,  $\pi(x) = (1 - \delta)\bar{\pi}(x) + \delta f(x)$ , that is,  $f(x) = (1/\delta)(\pi(x) - (1 - \delta)\bar{\pi}(x))$ . Let  $z$  be the action chosen on some stationary symmetric CBE path. We know that  $\underline{r} \leq f(z)$ . Let  $x^*$  maximize  $f$ . Consider a path  $\mathbf{x}$  in which  $x^{cn}$  is played for  $t$  periods, followed by  $x^*$  forever. Except for integer problems (the proof resolves these using (A5)),  $t$  can be chosen so that the path has value  $f(x^*)$ . Moreover, the value of the path is weakly increasing, so all its continuation values are at least  $f(x^*)$ . There are no profitable deviations from the simple profile  $\hat{\sigma}(\mathbf{x}, \mathbf{x})$ , therefore by the definition of  $\underline{r}$  we have  $\underline{r} \geq f(x^*)$ . Thus,  $\underline{r} \leq f(z) \leq f(x^*) \leq \underline{r}$ . This characterizes  $\underline{r}$  as a simple function of  $\delta$  and the data of the component game.

**THEOREM 2.** *Let  $f(x) = (1/\delta)(\pi(x) - (1 - \delta)\bar{\pi}(x))$ . Then  $\underline{r} = \max_{x \in S_1} f(x)$ .*

*Proof.* From the definition of a CBE it is clear that any CBE path is supportable by  $\underline{r}$ . Hence by Theorem 1 and Lemma 2 there exists a symmetric path  $\mathbf{z} = \{z(t) \cdot e\}_{t=1}^\infty$  which is supportable by  $\underline{r}$ . Let  $\tilde{z} \in \text{cl}\{z(t)\}_{t=1}^\infty$  (where “cl” is “closure”) satisfy  $\pi(\tilde{z}) \geq \pi(z(t))$  for all  $t$ . Then  $\pi(\tilde{z}) \geq v_1(\mathbf{z}; t + 1)$ ,  $t = 1, 2, \dots$ . By the continuity of  $\pi$  and  $\bar{\pi}$  it follows that  $(1 - \delta)(\bar{\pi}(\tilde{z}) - \pi(\tilde{z})) \leq \delta(\pi(\tilde{z}) - \underline{r})$ . That is,  $\underline{r} \leq (1/\delta)(\pi(\tilde{z}) - (1 - \delta)\bar{\pi}(\tilde{z})) \leq \max_x f(x)$ . Let  $x^*$  be any solution to the problem  $\max f$ , and define  $v_1^* = \pi(x^*)$ . Observe that  $f$  may be rewritten as  $f(x) = \pi(x) - ((1 - \delta)/\delta)(\bar{\pi}(x) - \pi(x))$ . Hence

$$\pi(x^{cn}) = f(x^{cn}) \leq f(x^*) = \pi(x^*) - \frac{(1 - \delta)}{\delta} (\bar{\pi}(x^*) - \pi(x^*)) \leq v_1^*.$$

Assume that  $f(x^*) > f(x^{cn})$ . Since  $x^{cn}$  is a payoff maximal symmetric Nash equilibrium (NE), this implies that  $x^*$  is not an NE. Hence  $v_1^* > f(x^*)$ . Also,  $\pi(x^*) > \pi(x^{cn})$ . Let

$$T = \max \left\{ \tau \left| \frac{(1 - \delta)}{\delta} (\delta + \delta^2 + \dots + \delta^\tau) \pi(x^{cn}) + \delta^\tau v_1^* \geq f(x^*) \right. \right\}.$$

<sup>7</sup> We refer to a path  $\mathbf{s}$  as a CBE path if there exists a CBE  $\gamma$  with outcome  $\mathbf{s}$ .

By the continuity of  $\pi$  and (A5) there exists  $y$  such that

$$\frac{(1-\delta)}{\delta} [(\delta + \dots + \delta^T) \pi(x^{cn}) + \delta^{T+1} \pi(y)] + \delta^{T+1} v_1^* = f(x^*),$$

and  $\bar{\pi}(y) - \pi(y) \leq \bar{\pi}(x^*) - \pi(x^*)$ . Let  $\underline{x} = \{\underline{x}(t) \cdot e\}_{t=1}^\infty$  be the symmetric path where  $\underline{x}(t) = x^{cn}$ ,  $t = 1, \dots, T$ ;  $\underline{x}(T+1) = y$ ;  $\underline{x}(t) = x^*$ ,  $t = T+2, T+3, \dots$ . Then  $v_1(\underline{x}) = f(x^*)$  and  $v_i(\underline{x}; t+1) > f(x^*)$  all  $t = 1, 2, \dots$ . Also from above,  $(1-\delta)(\bar{\pi}(x^*) - \pi(x^*)) = \delta(\pi(x^*) - f(x^*))$ . Since  $\bar{\pi}(x^{cn}) = \pi(x^{cn})$ , and  $\bar{\pi}(y) - \pi(y) \leq \bar{\pi}(x^*) - \pi(x^*)$ , it follows that  $\underline{x}$  is supportable by  $v_1(\underline{x}) = f(x^*)$ , and hence by (P),  $\hat{\sigma}(\underline{x}, \underline{x})$  is an equilibrium. Since  $l(\hat{\sigma}(\underline{x}, \underline{x})) = f(x^*)$ ,  $r \geq f(x^*)$ . Combine this with the earlier inequality  $r \leq \max f$  to complete the proof for the case  $f(x^*) > f(x^{cn})$ . The case where  $x^{cn}$  maximizes  $f$  is trivial. Then  $T = \infty$  and  $\underline{x}$  is the constant symmetric path  $x^{cn}$  forever. Q.E.D.

It is clear from the proof that (A5) is used only to solve an integer problem. If  $T$  as defined above exceeds 0 then (A5) may be dropped if public randomization is allowed. In period  $T+1$ , players play  $x^{cn}$  with some probability  $\alpha$  and  $x^*$  with complementary probability. Thereafter they play  $x^*$  forever. Our results need to be modified somewhat if  $T=0$  and (A5) is violated.

Before we comment on this rather striking formula it is useful to have some further results. Let  $\underline{x}$  be as defined in the proof above. First note that the strongly symmetric simple profile  $\hat{\sigma}(\underline{x}, \underline{x})$  is a CBE and yields the payoff  $r$ . The following lemma is elementary.

LEMMA 2. *A path  $\mathbf{s}$  is a CBE path if and only if*

- (i)  $\mathbf{s}$  is supportable by  $r$ , and
- (ii)  $v_i(\mathbf{s}; t) \geq r$  for  $i = 1, \dots, n$  and  $t = 1, 2, \dots$ .

*Under these conditions the simple profile  $\hat{\sigma}(\mathbf{s}, \underline{x})$  is a CBE.*

*Proof.* See Appendix B.

An immediate consequence of Lemmas 1 and 2 is

THEOREM 3. *For any CBE  $\gamma$  there exists a strongly symmetric CBE  $\sigma$  such that  $\hat{v}_1(\sigma) \geq (1/n) \sum_i \hat{v}_i(\gamma)$ .*

Thus, under our assumptions players' self-interest does not force them to accept asymmetries (recall the example and the earlier discussion). The minimum payoff  $l(\sigma)$  is not improved by permitting asymmetric treatment. Our principle of equal bargaining power therefore implies that players may, without loss, insist on symmetric payoffs both prior to and following

a deviation. We therefore restrict attention to  $R = \{\bar{v}_1(\sigma) \mid \sigma \text{ is a strongly symmetric CBE}\}$ . It is straightforward to show that  $R$  is a compact set.

LEMMA 3.  $R$  is compact.

*Proof.* See Appendix B.

Two numbers of special interest are  $\underline{r} = \min R (=l(\sigma) \text{ for any CBE } \sigma)$ , and  $\bar{r} = \max R$ . The former is the worst credible punishment payoff, and the latter the best, or most “collusive,” payoff. It is remarkable that both these numbers, which emerge from a potentially complex intertemporal incentive compatibility problem, may be expressed in terms of explicit, trivially computable formulae.

DEFINITION. Let  $x^*$  satisfy  $x^* \in \arg \max f$  and  $\pi(x^*) \geq \pi(y)$  for all  $y \in \arg \max f$ . This notation is useful in characterizing  $\bar{r}$ .

THEOREM 4.  $\bar{r} = \pi(x^*)$ .

*Proof.* Let  $y$  be a symmetric CBE path with payoff  $\bar{r}$ . By Lemma 2,  $y$  is supportable by  $\underline{r}$ . Let  $\tilde{y} \in \text{cl}\{y(t)\}$  satisfy  $\pi(\tilde{y}) \geq \pi(y(t))$  for all  $t$ . By continuity of  $\pi$  and  $\bar{\pi}$  (this step is analogous to the proof of Theorem 2),  $(1 - \delta)(\bar{\pi}(\tilde{y}) - \pi(\tilde{y})) \leq \delta(\pi(\tilde{y}) - \underline{r})$ . Hence,  $\underline{r} \leq (1/\delta)(\pi(\tilde{y}) - (1 - \delta)\bar{\pi}(\tilde{y}))$ . It follows from Theorem 2 that  $\tilde{y}$  maximizes  $f$ . Thus,  $\pi(x^*) \geq \pi(\tilde{y})$ , and  $v_1(x^*) \geq \pi(\tilde{y}) \geq v_1(y) = \bar{r}$ . Since  $x^*$  is supportable by  $\underline{r}$ , it follows from Lemma 2 that  $x^*$  is a CBE path. Therefore,  $\bar{r} = v_1(x^*) = \pi(x^*)$  as required. Q.E.D.

To summarize,  $\underline{r} = \max f$ ,  $\bar{r} = \pi(x^*)$ , and  $x^*$ , the constant path  $x^*$  forever, is the most collusive symmetric CBE path.

Let  $x^m$  denote any maximizer of  $\pi$ . The proofs of the next two results assume that  $S_1$  is an interval and use the following regularity assumptions:

(A6) The functions  $\pi$  and  $\bar{\pi}$  are continuously differentiable.

(A7) The points  $x^{cn}$ ,  $x^m$  lie in the interior of  $S_1$ , and  $\bar{\pi}'(x^{cn})$ ,  $\bar{\pi}'(x^m) \neq 0$ . Furthermore, for all  $x \notin \text{int } S_1$ ,  $\pi(x^{cn}) > \pi(x)$ .

THEOREM 5.  $\underline{r} > \pi(x^{cn})$ , and  $\bar{r} < \pi(x^m)$  for all  $\delta \in (0, 1)$ .

*Proof.* By theorem 2,  $\underline{r} = \max f = \max(\pi(x) - ((1 - \delta)/\delta)[\bar{\pi}(x) - \pi(x)])$ . Hence, by (A6) and (A7),  $\arg \max f \subset \text{int } S_1$  and  $f'(z) = 0$  for all  $z \in \arg \max f$ . Obviously,  $\underline{r} \geq \pi(x^{cn})$ . Now observe that if

$$\underline{r} = \pi(x^{cn}) = \pi(x^{cn}) - \frac{(1 - \delta)}{\delta} [\bar{\pi}(x^{cn}) - \pi(x^{cn})] = \frac{1}{\delta} (\pi(x^{cn}) - (1 - \delta)\bar{\pi}(x^{cn})),$$

then  $x^{cn} \in \arg \max f$  and  $f'(x^{cn}) = 0$ . Noting that  $\bar{\pi}'(x^{cn}) = \pi'(x^{cn})$  we have  $\bar{\pi}'(x^{cn}) = 0$ , contradicting (A7). Finally, suppose that  $\bar{r} = \pi(x^*) = \pi(x^m)$ . Then  $x^*$  maximizes  $\pi$  and  $\pi'(x^*) = 0$ . By assumption  $\bar{\pi}'(x^*) \neq 0$ , which contradicts the requirement  $f'(x^*) = 0$ . Q.E.D.

This should be contrasted with the usual theory without renegotiation, where the first inequality of Theorem 5 (with  $\underline{r}$  replaced by  $\underline{y}$ , the minimum of the equilibrium value set) is reversed, and the second inequality (with  $\bar{r}$  replaced by  $\bar{v}$ , the symmetric maximum of the equilibrium value set) holds with equality for sufficiently high  $\delta$ .

**THEOREM 6.** *As functions of  $\delta$ ,  $\underline{r}(\delta)$  and  $\bar{r}(\delta)$  are strictly increasing. Furthermore,  $\lim_{\delta \rightarrow 1} \underline{r}(\delta) = \lim_{\delta \rightarrow 1} \bar{r}(\delta) = \pi(x^m)$ .*

*Proof.* Let  $f(x; \delta) = \pi(x) - ((1 - \delta)/\delta)[\bar{\pi}(x) - \pi(x)]$ . By Theorem 2,  $\underline{r}(\delta) = \max f(x; \delta)$ . Consider  $\delta_1 < \delta_2$  and let  $x_i$  maximize  $f(x; \delta_i)$ . By Theorem 5,  $x_1$  does not define a symmetric NE and  $\bar{\pi}(x_1) > \pi(x_1)$ . Since  $(1 - \delta_1)/\delta_1 > (1 - \delta_2)/\delta_2$ ,

$$\underline{r}(\delta_2) = \max f(x; \delta_2) \geq f(x_1; \delta_2) > f(x_1; \delta_1) = \underline{r}(\delta_1).$$

Hence,  $\underline{r}(\delta)$  is strictly increasing in  $\delta$ .

From the first order conditions of the problem  $\max f(x; \delta_i)$ , it is clear that  $x_1 \neq x_2$ . Hence,  $f(x_1; \delta_1) > f(x_2; \delta_1)$  and  $f(x_2; \delta_2) > f(x_1; \delta_2)$ . These inequalities may be seen to imply  $\pi(x_2) > \pi(x_1)$ . Hence,  $\bar{r}(\delta)$  is strictly increasing in  $\delta$ . Finally, observe that  $\pi(x^m) \geq \pi(x) - ((1 - \delta)/\delta)(\bar{\pi}(x) - \pi(x))$  for all  $x$  and  $\delta$ , and  $\lim_{\delta \rightarrow 1} [\pi(x^m) - ((1 - \delta)/\delta)(\bar{\pi}(x^m) - \pi(x^m))] = \pi(x^m)$ .

Q.E.D.

The limit results of Theorem 6 are in the spirit of Proposition 3 of Pearce [17]. Our additional structure yields the new result that  $\bar{r}(\delta)$  is strictly monotonic.

Recall the punishment path  $\mathbf{x}$  constructed in the proof of Theorem 2 and the interpretation in Theorem 4 of  $x^*$  forever as the most collusive CBE path. Then  $\mathbf{x}$  has two phases: an initial phase of low payoffs (the stick) followed by a phase of the highest (renegotiation proof) payoffs available (the carrot). This is analogous to the symmetric stick-and-carrot punishments of Abreu [1].

Under an additional assumption this structure may be expressed more crisply: phase I consists of exactly one period.

(A8) If  $(1 - \delta)\pi(x^{cn}) + \delta\bar{r} > \underline{r}$ , there exists  $y \in S_1$  such that  $(1 - \delta)\pi(y) + \delta\bar{r} = \underline{r}$  and  $\bar{\pi}(y) \leq \bar{\pi}(x^{cn})$ .

This assumption is, for instance, satisfied in symmetric Cournot supergames with constant marginal cost.

DEFINITION. For all  $x_1, x_2 \in S_1$ , let  $\xi(x_1, x_2)$  denote the symmetric path in which  $x_1$  is played in the first period and  $x_2$  in all subsequent periods.

THEOREM 7. *There exists  $\underline{x}_1 \in S_1$  such that  $\hat{\sigma}(\xi(\underline{x}_1, x^*), \xi(\underline{x}_1, x^*))$  is a CBE and  $v_1(\xi(\underline{x}_1, x^*)) = \underline{r}$ .*

*Proof.* If the hypothesis of (A8) is false, then in the proof of Theorem 2,  $T=0$  and we may set  $\underline{x}_1 = a$ , as defined there. If not, let  $y$  be as in (A8) and set  $\underline{x}_1 = y$ . Then  $(1 - \delta) \bar{\pi}(\underline{x}_1) + \delta \underline{r} \leq (1 - \delta) \pi(x^{cn}) + \delta \underline{r} \leq \underline{r} = (1 - \delta) \pi(\underline{x}_1) + \delta \bar{r}$ , where the second inequality follows from  $\underline{r} \geq \pi(x^{cn})$ . Hence  $(1 - \delta)(\bar{\pi}(\underline{x}_1) - \pi(\underline{x}_1)) \leq \delta(\bar{r} - \underline{r})$ . Therefore,  $\mathbf{z} = \xi(\underline{x}_1, x^*)$  is supportable by  $\underline{r}$ , and since  $v_1(\mathbf{z}) = \underline{r}$ , by (P),  $\hat{\sigma}(\mathbf{z}, \mathbf{z})$  is an equilibrium. Also,  $v_1(\mathbf{z}; t) = \bar{r}$ ,  $t = 2, 3, \dots$ . Thus  $\hat{\sigma}(\mathbf{z}, \mathbf{z})$  is a CBE, and the proof is complete.

Q.E.D.

Theorems 2 and 4 emphasize how easily the best and worst renegotiation proof *payoffs* may be computed. The simple stick-and-carrot structure of the associated *strategies* is given a sharp expression in Theorem 7.

### 3. IMPERFECT MONITORING

In repeated games with perfect monitoring, some histories of play discriminate sharply among the participants: perhaps one player has deviated from cooperative behavior, while all others have conformed to some agreement. Evidence distinguishing one player from another tends to be less conclusive in models in which publicly observed signals are only stochastically related to players' private decisions. This suggests that there is if anything less reason to treat players asymmetrically after certain histories in imperfect monitoring models than in supergames with perfect monitoring. We show that on the contrary, consistent bargaining equilibria under imperfect monitoring will commonly violate strong symmetry, unlike their counterparts in perfect monitoring environments. In other words, sometimes players find it in their interest to submit gracefully to asymmetric treatment.

The results presented below are chosen with the intention of conveying as succinctly as possible the idea that asymmetric continuation payoffs arise quite naturally despite the presumption of equal bargaining power. No attempt is made to describe the specific structure of optimally collusive equilibria.

The model is a repeated partnership. We set out the model and notation below, emphasizing only those aspects which do not overlap with Section 2. It is now assumed that player  $i$  selects an action in period  $t$  from a finite set  $S_i$ . His choice  $s_i(t)$  is *unobservable* to  $j \neq i$  but the realization of

a random variable  $\theta(t)$  is publicly observed at the end of period  $t$ . The signal  $\theta$  can take one of  $m$  values  $\theta_1, \dots, \theta_m$ , and  $p_i(s)$  denotes the probability of signal  $\theta_i$  given the action profile  $s \in S$ . We assume that  $\theta$  has *constant support*:  $p_i(s) > 0$  for all  $i = 1, \dots, m$  and  $s \in S$ . Player  $i$ 's realized payoff in period  $t$  depends on his own action and on the realization of the signal. His payoff  $\Pi_i(s(t))$  in the component game is the expectation of his realized payoff  $\pi_i(s_i(t), \theta(t))$ . Thus a player cares about the unobserved actions of others only insofar as these determine the distribution of the payoff relevant signal. The component game is symmetric in that  $S_i = S_1$  for all  $i$ , and  $\Pi_{\tau(i)}(s_1, s_2, \dots, s_n) = \Pi_i(s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(n)})$  for any permutation  $\tau$  of  $\{1, \dots, n\}$ .

A strategy  $\sigma_i$  for player  $i$  in the repeated game is a sequence of measurable functions  $\sigma_i(1), \sigma_i(2), \dots$ , where  $\sigma_i(1) \in S_1$  and for  $t \geq 2$ ,  $\sigma_i(t): (S_i \times \Theta)^{t-1} \rightarrow S_i$ , where  $\Theta = \{\theta_1, \dots, \theta_m\}$ . Let  $v(\sigma)$  denote the vector of average (expected) present discounted payoffs when the strategy profile  $\sigma$  is used. Define  $p(s) = (p_1(s), \dots, p_m(s))$  and  $P^i(s) = \{p(s'_i, s_{-i}) \mid s'_i \neq s_i, s'_i \in S_i\}$ . Denote by  $G^\infty(\delta)$  the repeated game with discount factor  $\delta$ . Fudenberg, Levine, and Maskin [14] have introduced the following condition.<sup>8</sup>

**DEFINITION.** The stage game satisfies the *pairwise full rank condition* if the collection of vectors  $\{p(s)\} \cup P^i(s) \cup P^j(s)$  is linearly independent for all  $s \in S$ , and  $i \neq j$ .

We will assume that

(M1) The stage game has a symmetric Nash equilibrium in pure strategies.

(M2) If  $x \cdot e \in \arg \max \sum_i \Pi_i(s)$ , then  $\bar{\pi}(x) > \pi(x)$ .

(M3) The stage game satisfies the pairwise full rank condition.<sup>9</sup>

We invoke (M1) only for convenience and (M2) guarantees that the stage game is non-trivial in that efficient behavior is never a Nash equilibrium. Finally (M3) implies that the signal contains information which permits deviations by different players to be distinguished.

We first record a simple corollary of a folk theorem of Fudenberg, Levine, and Maskin [14] which implies that a symmetric first-best payoff

<sup>8</sup> We adapt their conditions and results in an obvious way to conform with our assumption of pure strategies. They work with the more general class of *public* strategies.

<sup>9</sup> It might seem at first sight that the pairwise full rank condition is inconsistent with symmetry. This is emphatically not the case. To construct a simple example, let  $\xi = (\xi_1, \dots, \xi_n)$  be a multi-dimensional random variable where the  $\xi_i$ 's are independent, and the probability distribution of each  $\xi_i$  depends only on  $s_i$ . Of course, in a non-trivial example  $\pi_i$  will depend on all the  $\xi_j$ 's.



can be approached in equilibrium as  $\delta$  tends to 1. Related results are given by Matsushima [16] and, in a static setting, by Williams and Radner [22].

The pairwise full rank condition is required for the result below. Let  $\pi^* = (1/n) \max_s \sum_i \Pi_i(s)$ , and  $V(\delta) = \{v(\sigma) | \sigma \text{ is a (perfect Bayesian) equilibrium}^{10} \text{ of } G^\infty(\delta)\}$ .

**PROPOSITION 1** (Fudenberg, Levine, and Maskin, 1989). *For all  $\varepsilon > 0$  there exists  $\underline{\delta}$  such that for all  $\delta \geq \underline{\delta}$  there exists  $u \in V(\delta)$  with  $|\pi^* \cdot e - u| \leq \varepsilon$ .*

The preceding theorem is easily translated to a statement about consistent bargaining equilibria by appealing to a limiting characterization of renegotiation-proof equilibria.

**PROPOSITION 2** (Pearce, 1987). *For all  $\delta', u \in V(\delta')$  and  $\varepsilon > 0$ , there exists  $\underline{\delta}$  such that for all  $\delta \geq \underline{\delta}$  there exists an equilibrium  $\sigma$  of  $G^\infty(\delta)$  such that for all  $i$ ,  $\inf_h v_i(\sigma|h) \geq u_i - \varepsilon$ .*

The next result on consistent bargaining equilibria is an immediate corollary of the preceding propositions.

**COROLLARY.** *For all  $\varepsilon > 0$  there exists  $\underline{\delta}$  such that for all  $\delta > \underline{\delta}$  there exists a consistent bargaining equilibrium  $\bar{\sigma}$  with  $l(\bar{\sigma}) \geq \pi^* - \varepsilon$ .*

We now show that restricting attention to *strongly symmetric* equilibria bounds payoffs away from efficiency uniformly in  $\delta$ . Our main result then follows immediately. Theorem 8 is closely related to the work of Radner, Myerson, and Maskin [18] and various subsequent papers (see, for example, Abreu, Milgrom, and Pearce [3] and Fudenberg, Levine, and Maskin [14]).

**THEOREM 8.** *There exists  $\Delta > 0$  such that for all  $\delta$  and for all strongly symmetric equilibria  $\sigma$  of  $G^\infty(\delta)$ ,  $v_1(\sigma) \leq \pi^* - \Delta$ .*

*Proof.* See Appendix B.

It is now easy to argue that equal bargaining power need not imply strong symmetry; in many supergames *none* of the consistent bargaining equilibria is strongly symmetric. In the notation of Theorem 8,  $l(\sigma) \leq \pi^* - \Delta$  for every strongly symmetric equilibrium  $\sigma$  of  $G^\infty(\delta)$  (regardless of the values of  $\delta$ ). Setting  $\varepsilon = \Delta/2$  in the Corollary to Propositions 1 and 2, we find a  $\underline{\delta} < 1$  such that for all  $\delta \geq \underline{\delta}$  there exists an equilibrium  $\bar{\sigma}$  with  $l(\bar{\sigma}) \geq \pi^* - \Delta/2$ . That is,  $l(\bar{\sigma}) > l(\sigma)$  for any strongly symmetric equilibrium  $\sigma$ , therefore no CBE can be strongly symmetric. This result for patient players is recorded in Theorem 9.

<sup>10</sup> See Fudenberg, Levine, and Maskin [14]. Note that given our constant support assumption, there is no essential distinction between Nash and perfect equilibrium.

**THEOREM 9.** *Under (M1), (M2), and (M3) there exists  $\bar{\delta} < 1$  such that for all  $\delta \geq \bar{\delta}$  there exists no strongly symmetric consistent bargaining equilibrium.*

The benefit of treating players asymmetrically after certain histories is easily explained. Players' incentives to cooperate depend upon their payoffs varying with the realizations of the random signal. In a strongly symmetric regime this means that surplus is systematically thrown away; with a finite signal space (or whenever the relevant likelihood ratios are bounded above) there is consequently an inescapable efficiency loss. Relaxing symmetry introduces the possibility of passing surplus from player to player instead of destroying it. Thus, rather than punish both players in a two-person game whenever certain signals arise, one may reward one player at the expense of the player whose "record" is less favorable.

#### 4. CONCLUSION

This paper suggests a particular approach to the problem of renegotiation in symmetric repeated games. The theory developed is based on the idea that players will tolerate asymmetries in continuation payoffs precisely to the extent that even the worst-off player in any subgame finds this to be in his interest. In games with imperfect monitoring, our solution concept often leads to asymmetric continuation payoffs, despite the equal bargaining power of the players. But we give conditions under which solutions of games with perfect monitoring are strongly symmetric. Under these conditions we provide simple formulae for the computation of the most collusive credible equilibrium and the value of the severest credible punishment. In contrast to the traditional theory without renegotiation, severest punishment paths here take an almost naively intuitive form: following a number of periods of "Cournot–Nash reversion," play returns to (constrained) maximal collusion. A variety of oligopolistic models satisfy the required conditions. Cournot oligopolies with modest restrictions are in this class; the linear Cournot supergame is solved fully in closed form, for all discount factors and any number of firms in Appendix A.

We explained briefly in the Introduction why we find some bargaining theory along the lines suggested in this paper more plausible than the Pareto criterion, which is totally insensitive to bargaining considerations. The Pareto criterion has some appeal as a *necessary* condition for the absence of renegotiation, but is entirely inadequate as a sufficient condition. We also expressed our reservations about our consistent bargaining solution.

It is clear that no theory yet articulated is entirely compelling. Why do we bother, then, to work out the details of a *particular* theory? Our reasons are roughly as follows:

(i) To illustrate precisely how a theory taking some kind of bargaining power into account can be integrated conceptually with Pearce's approach to renegotiation-proofness in repeated games.

(ii) To demonstrate that carrying out the exercise in (i) can yield a nonempty solution concept that allows a degree of cooperation, while limiting its extent and form.

(iii) To show that the importance of variations in the relative shares of the players is, ironically, greater under imperfect monitoring (where the evidence discriminates less sharply between players).

Adoption of an alternative bargaining theory can surely be expected to alter the details (and to destroy some of the simplicity) of our analysis. But, as Abreu and Pearce [4] argue, in many cases, exercise (i) above can still be carried out. Furthermore, we think that the features in (ii) and (iii) are broadly representative of what one would find by exploring more complicated alternatives.

#### APPENDIX A

The purpose of this appendix is to suggest by example that the assumptions of Section 2 will be satisfied in a range of natural repeated economic models and to demonstrate the tractability of the solution concept. Further examples are given in Abreu, Pearce, and Stacchetti [6].

We consider here a class of quantity-setting oligopolistic supergames of the sort studied in Abreu [1]. Identical firms produce a homogeneous product at constant marginal cost  $c > 0$ . The industry inverse demand function is denoted  $p$ . Then  $\Pi_i(s_1, \dots, s_n) = (p(\sum s_j) - c)s_i$ , where  $s_i$  is the output of firm  $i$ .

Under reasonable assumptions this model fits into the framework above. These assumptions are:

(Q1)  $p: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  is continuous and strictly decreasing. Also,  $\lim_{z \rightarrow 0} p(z) > c$ , and  $\lim_{z \rightarrow \infty} p(z) = 0$ .

(Q1) implies that there exists  $\bar{M}(\delta)$  such that  $-\Pi_1(\bar{M}(\delta), 0, \dots, 0) > (\delta/(1-\delta)) \sup_z \Pi_1(z, 0, \dots, 0)$ . The loss to a firm from producing an output of  $\bar{M}(\delta)$  or more cannot be recouped by any possible future gain. Thus w.l.o.g. we may restrict firms to output choices in the interval  $[0, \bar{M}(\delta)]$ .

(Q2)  $S_i = [0, \bar{M}(\delta)]$ ,  $i = 1, \dots, n$ .

(Q3)  $G = (S_1, \dots, S_n; \Pi_1, \dots, \Pi_n)$  has a symmetric pure strategy Nash equilibrium.

(Q1) to (Q3) imply (A1) to (A3) of Section 2. They also imply (A4), since  $\pi(1/n \sum_i s_i) = (1/n) \sum_i \Pi_i(s_i)$ , and by Lemma 21 of Abreu [1],  $\bar{\Pi}_i$  is a convex function. If in addition we assume that  $\pi$  is concave, (A5) holds.

(Q4)  $\pi$  is concave.

Sufficient conditions (on primitives) for (Q4) are that the demand function is linear, or has constant elasticity greater than unity. Corresponding to (A6) we now have

(Q5)  $\pi$  and  $\bar{\pi}$  are continuously differentiable.

If the inverse demand function is differentiable then, of course, so is  $\pi$ ; since  $\bar{\pi}$  is convex it follows that it is differentiable almost everywhere.

(Q1) to (Q3) imply  $\pi(x^{cn}) > 0$ . Also  $\pi(0) = 0$  and  $\pi(\bar{M}(\delta)) < 0$ . Together with (Q5), Corollary 3 (reproduced below) and Lemma 21 of Abreu [1], (A7) is implied.

COROLLARY 3 (Abreu, 1986). *Let  $x_2 > x_1 \geq 0$ . Then  $\bar{\pi}(x_1) = \bar{\pi}(x_2) = 0$ , or  $\bar{\pi}(x_1) > \bar{\pi}(x_2)$ .*

Finally (A8) follows from Corollary 3 (above) and the fact that if  $(1 - \delta) \pi(x^{cn}) + \delta v_2 > v_1 \geq 0$ , we may always choose  $y \geq x^{cn}$  such that  $(1 - \delta) \pi(y) + \delta v_2 = v_1$ . Thus, the symmetric oligopolistic quantity-setting supergame with assumptions (Q1) to (Q5) satisfies all the assumptions of Section 2. The relevant picture is Fig. 1.

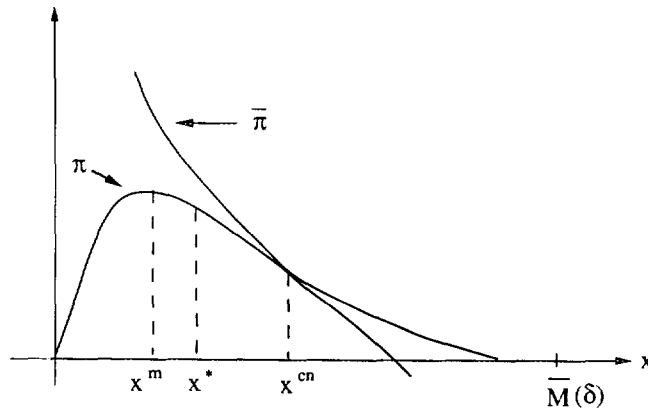


FIGURE 1

For illustrative purposes we compute a linear example. The inverse demand function is

$$p(z) = \begin{cases} \alpha - \beta z & z \leq \alpha/\beta \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha, \beta > 0$  and  $\alpha > c$ . Then

$$\pi(x) = (\alpha - c - n\beta x)x \text{ when } x \leq \frac{\alpha}{n\beta} \quad \text{and} \quad \pi(x) = -cx \text{ otherwise, and}$$

$$\bar{\pi}(x) = \frac{1}{4\beta} (\alpha - c - (n-1)\beta x)^2 \text{ when } x \leq \frac{\alpha - c}{(n-1)\beta} \quad \text{and} \quad \text{zero otherwise.}$$

By definition,  $f(x) = (1/\delta)(\pi(x) - (1-\delta)\bar{\pi}(x))$ , and by Theorem 2,  $r = \max f(x)$ . This problem has a unique maximum at

$$x^* = \frac{\alpha - c}{\beta} \left[ \frac{(n+1) - \delta(n-1)}{4n + (1+\delta)(n-1)^2} \right].$$

Hence

$$r = \frac{(\alpha - c)^2}{\beta} \left[ \frac{1}{4n + (1-\delta)(n-1)^2} \right].$$

By Theorem 4,  $\bar{r} = \pi(x^*)$ . That is,

$$\bar{r} = \frac{(\alpha - c)^2}{\beta} \left[ \frac{(n+1)^2 - \delta^2(n-1)^2}{(4n + (1-\delta)(n-1)^2)^2} \right].$$

It follows directly that in agreement with Theorem 6,

$$\lim_{\delta \rightarrow 1} r(\delta) = \lim_{\delta \rightarrow 1} \bar{r}(\delta) = \frac{(\alpha - c)^2}{4n\beta} = \pi(x^m).$$

### APPENDIX B

We provide here proofs omitted in the text. The following definition and result from Abreu [2] will be useful.

DEFINITION. Let  $s^i, i = 0, 1, \dots, n$ , be paths in  $S$ . The simple (strategy) profile  $\tilde{\sigma}(s^0, s^1, \dots, s^n)$  specifies

- (i) play according to  $s^0$  until some player deviates singly from  $s^0$ ,

(ii) for any  $j \in N$ , play  $\mathbf{s}^j$  if the  $j$ th player deviates singly from  $\mathbf{s}^i$ ,  $i = 0, 1, \dots, n$ , where  $\mathbf{s}^i$  is an ongoing previously specified path. Continue with  $\mathbf{s}^i$  if no deviations occur or if two or more players deviate simultaneously.

(P') The simple profile  $\tilde{\sigma}(\mathbf{s}^0, \mathbf{s}^1, \dots, \mathbf{s}^n)$  is a subgame perfect equilibrium if and only if  $(1 - \delta)(\bar{\Pi}_j(\mathbf{s}^i(t)) - \Pi_j(\mathbf{s}^i(t))) \leq \delta(v_j(\mathbf{s}^i; t + 1) - v_j(\mathbf{s}^i))$  for all  $j = 1, \dots, n$ ,  $i = 0, 1, \dots, n$  and  $t = 1, 2, \dots$

*Proof* (Theorem 1). Let  $\underline{r} = \sup\{l(\sigma) \mid \sigma \text{ is an equilibrium}\}$ . By (A1), (A2), and (A3)'  $\underline{r}$  is well defined. We complete the proof by exhibiting an equilibrium  $\sigma$  with  $l(\sigma) = \underline{r}$ . Let  $\{\sigma^n\}_{n=1}^\infty$  be a sequence of equilibria such that  $(\underline{r} - l(\sigma^n)) \leq 1/\eta$ . For each  $\sigma^n$  there exists a history  $h$  and a player  $i$  such that  $0 \leq \underline{r} - \bar{v}_i(\sigma^n|_h) \leq 1/\eta$ . Since  $\sigma^n$  is an equilibrium so also is  $\sigma^n|_h$ . Furthermore, by symmetry there exists an analogous equilibrium in which player  $i$  and player 1 are interchanged. That is, w.l.o.g. we may (in addition) assume that  $0 \leq (\underline{r} - v_1(\mathbf{s}^{0n})) \leq 1/\eta$ , where  $\mathbf{s}^{0n}$  is the initial path of  $\sigma^n$ .

The rest of the proof mimics Proposition 2 of Abreu [2]. We endow  $\Omega = S^\infty$  with the product topology. By (A1) and (A2)  $v: \Omega \rightarrow \mathbf{R}^n$  is continuous, and by Tychonoff's theorem  $\Omega$  is compact. We may w.l.o.g. take  $\{\mathbf{s}^{0n}\}$  to be a convergent sequence. Let  $\mathbf{s}^0 = \lim \mathbf{s}^{0n}$ . By definition,  $v_i(\mathbf{s}^{0n}; t) \geq l(\sigma^n)$  for all  $i, t$ . Hence we have  $v_i(\mathbf{s}^0; t) \geq \underline{r}$  for all  $i, t$ . Also  $v_1(\mathbf{s}^0) = \underline{r}$ . Let  $\mathbf{s}^i$  be obtained from  $\mathbf{s}^0$  by interchanging the roles of players 1 and  $i$ . That is, writing  $\mathbf{s}^k = \{(s_1^k(t), \dots, s_n^k(t))\}_{t=1}^\infty$ ,  $k = 0, 1, \dots, n$ , we have  $s_j^k(t) = s_j^0(t)$ ,  $j \neq 1, i$ ,  $s_1^i(t) = s_1^0(t)$ , and  $s_i^i(t) = s_i^0(t)$ . Clearly  $v_i(\mathbf{s}^i) = \underline{r}$  and  $v_j(\mathbf{s}^i; t) \geq \underline{r}$  for all  $i, j, t$ . We now argue that  $\sigma(\mathbf{s}^0, \mathbf{s}^1, \dots, \mathbf{s}^n)$  is an equilibrium. Suppose not. Then by (P'),  $\bar{\Pi}_j(\mathbf{s}^0(t)) - \Pi_j(\mathbf{s}^0(t)) > (\delta/(1 - \delta))(v_j(\mathbf{s}^0; t + 1) - \underline{r})$  for some  $j, t$ . Since  $\mathbf{s}^{0n} \rightarrow \mathbf{s}^0$  and  $l(\sigma^n) \rightarrow \underline{r}$ , by continuity for  $\eta$  large enough  $\bar{\Pi}_j(\mathbf{s}^{0n}(t)) - \Pi_j(\mathbf{s}^{0n}(t)) > (\delta/(1 - \delta))(v_j(\mathbf{s}^{0n}; t + 1) - l(\sigma^n))$ . But then by (P),  $\sigma^n$  is not an equilibrium, a contradiction. Hence  $\tilde{\sigma}(\mathbf{s}^0, \mathbf{s}^1, \dots, \mathbf{s}^n)$  is an equilibrium. Since its minimum continuation value is  $\underline{r}$ , it is a CBE. Q.E.D.

*Proof* (Lemma 2) (Necessity). Let  $\sigma$  be a CBE with initial path  $\mathbf{s}$ . By the definition of a CBE,  $l(\sigma) = \underline{r}$ . This establishes (ii). To establish (i) simply note that  $\mathbf{s}$  must be supportable by  $l(\sigma)$ .

(Sufficiency). By (P),  $\hat{\sigma}(\mathbf{s}, \underline{\mathbf{x}})$  is an equilibrium. Its continuation values are  $\{v(\mathbf{a}; t) \mid \mathbf{a} = \mathbf{s}, \underline{\mathbf{x}} \text{ and } t = 1, 2, \dots\}$ . Hence,  $l(\hat{\sigma}(\mathbf{s}, \underline{\mathbf{x}})) \geq \underline{r}$ , and  $\hat{\sigma}(\mathbf{s}, \underline{\mathbf{x}})$  is a CBE. Q.E.D.

*Proof* (Lemma 3). From (A1) and (A2),  $R$  is bounded. Consider a sequence of symmetric CBE paths  $\{\mathbf{x}^n\}$  such that  $\lim v_1(\mathbf{x}^n) = a$ . We need to show that  $a \in R$ . Endow  $\Omega = S^\infty$  with the product topology. By Tychonoff's theorem,  $\Omega$  is compact. Assume w.l.o.g. that  $\mathbf{x}^n \rightarrow \mathbf{x}$ . By

continuity,  $v_i(\mathbf{x}; t) = \lim v_i(\mathbf{x}^n; t) \geq \underline{r}$  for all  $i$  and  $t \geq 1$ . By (P') and the definition of a CBE,  $\mathbf{x}^n$  is supportable by  $\underline{r}$ . Hence by Lemma 2  $\mathbf{x}$  is a CBE path with  $v_1(\mathbf{x}) = a$ . Q.E.D.

*Proof* (Theorem 8). Fix  $\delta$ . Let  $\sigma$  be a payoff maximal strongly symmetric (perfect Bayesian) equilibrium. By Corollary 2 of Abreu, Pearce, and Stacchetti [5], such a  $\sigma$  exists. Let  $\bar{v} = v_1(\sigma)$ . As argued in that paper we may w.l.o.g. take players' behavior in  $\sigma$  to be a function only of publicly observed outcomes and not of the history of their own past choices. Let  $\sigma|_\theta$  denote the behavior induced by  $\sigma$  after a first period outcome  $\theta$  (which by assumption is independent of players' first period choices). Since  $\sigma$  is a perfect Bayesian equilibrium, so also is  $\sigma|_\theta$ . Hence,  $\underline{v} \leq w_\theta \leq \bar{v}$ , where  $w_\theta = v_1(\sigma|_\theta)$  and  $\underline{v}$  is the worst strongly symmetric equilibrium payoff. Let  $\alpha_\theta \in (0, 1)$  be defined by  $w_\theta = \bar{v} - \alpha_\theta(\bar{v} - \underline{v})$ .

Let  $\sigma_1(1) = x$  be the action played in the first period. Then  $\bar{v} = (1 - \delta)\pi(x) + \delta \sum_\theta p_\theta w_\theta$ , where  $p_\theta$  is the probability of the outcome  $\theta$  when all players use action  $x$ . Suppose  $x \cdot e \notin A = \arg \max \sum_i \Pi_i(s)$ . Then  $\pi(x) < \pi^*$  and  $\bar{v} \leq (1 - \delta)\pi(x) + \delta \bar{v}$ . Therefore  $\bar{v} \leq \pi(x) < \pi^*$ , and  $\bar{v} \leq \pi^* - \Delta_1$ , where  $\Delta_1 = \min\{\pi^* - \pi(x) | x \cdot e \in S \setminus A\} > 0$  (recall that  $S_1$  is a finite set.) Now suppose  $x \cdot e \in A$ . Then by (M2) there exists  $s_1 \neq x$  such that  $\Pi_1(s_1, x, \dots, x) > \Pi_1(x, \dots, x)$ . Let  $g > 0$  be the difference between these payoffs. Let  $q_\theta$  be the probability of outcome  $\theta$  when the action profile is  $(s_1, x, \dots, x)$ . Since  $\sigma$  is an equilibrium, it follows that

$$g \leq \frac{\delta}{1 - \delta} \sum_\theta (p_\theta - q_\theta)(\bar{v} - \alpha_\theta(\bar{v} - \underline{v})) = \frac{\delta}{1 - \delta} (Q - P)(\bar{v} - \underline{v}),$$

where  $P = \sum \alpha_\theta p_\theta$  and  $Q = \sum \alpha_\theta q_\theta$ . Let  $m = (Q - P)/P$ . Then  $(1 - \delta)(g/m) \leq \delta(\bar{v} - \underline{v})P$ . By the constant support assumption,  $P > 0$ . Together with the finite outcome, finite action assumption it follows that  $m$  is bounded above ( $(m + 1)$  is a likelihood ratio) by some finite  $\bar{m}$ , independent of  $\alpha_\theta$ ,  $x \cdot e \in A$ , and the profitable deviation  $s_1$ . But  $\bar{v} = (1 - \delta)\pi^* + \delta \sum p_\theta[\bar{v} - \alpha_\theta(\bar{v} - \underline{v})]$ . Therefore,  $(1 - \delta)\bar{v} = (1 - \delta)\pi^* - \delta P(\bar{v} - \underline{v}) \leq (1 - \delta)\pi^* - (1 - \delta)(g/m)$ . Hence  $\bar{v} \leq \pi^* - (g/m)$ . Let  $\Delta_2 = g/\bar{m} > 0$ . Set  $\Delta = \min\{\Delta_1, \Delta_2\}$  to complete the proof. Q.E.D.

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