

## TEICHMÜLLER GEODESICS AND ENDS OF HYPERBOLIC 3-MANIFOLDS

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### 1. INTRODUCTION

THE THEORY of the structure and deformations of hyperbolic 3-manifolds depends in an essential way on a good understanding of the geometry of incompressible maps of surfaces into these manifolds. Consider, for example, a hyperbolic 3-manifold  $N$  homeomorphic to  $S \times \mathbf{R}$  where  $S$  is a closed surface of genus  $g > 1$ . Thurston and Bonahon showed how to fill the convex hull of  $N$  with “pleated surfaces” homotopic to the obvious map  $S \rightarrow S \times \{0\}$  (see Sections 2.3, 2.4), and these surfaces have induced metrics which determine points in the Teichmüller space  $\mathcal{T}(S)$  of conformal (or hyperbolic) structures on  $S$ . It has been conjectured that the locus of these points is related in an approximate way to a geodesic in  $\mathcal{T}(S)$ , and this is known to be true for a class of examples arising from hyperbolic structures on surface bundles over a circle (see [8]). This kind of information has implications concerning the geometry of  $N$ , which will be discussed more fully in a forthcoming paper ([25]).

From a more differential-geometric point of view, one can consider, for any metric  $\sigma$  on  $S$ , a map  $f_\sigma : S \rightarrow N$  of least “energy” (see Section 3) in the above homotopy class. This least energy is then some function  $\mathcal{E}(\sigma)$ , and  $f_\sigma$  is a *harmonic map*. One can then ask about the locus of points  $[\sigma]$  in  $\mathcal{T}(S)$  where  $\mathcal{E}$  is bounded above by a given constant, and its relationship to the above set of metrics induced from pleated surfaces.

We give here a proof of the following result in this direction, where the crucial restrictive hypothesis we need to make is a positive lower bound on the injectivity radius  $\text{inj}_N(x)$  for all  $x \in N$ .

**THEOREM A.** *Let  $N = \mathbf{H}^3/\Gamma$  be a hyperbolic 3-manifold,  $S$  a closed surface of genus at least 2, and  $[f: S \rightarrow N]$  a  $\pi_1$ -injective homotopy class of maps. Suppose that there is some constant  $\varepsilon_0 > 0$  so that  $\text{inj}_N(x) \geq \varepsilon_0$  for all  $x \in N$ .*

*Then there is a Teichmüller geodesic segment, ray, or line  $L$  in the Teichmüller space  $\mathcal{T}(S)$  and constants  $A, B$  depending only on  $\chi(S)$  and  $\varepsilon_0$  such that*

1. *Every Riemann surface on  $L$  can be mapped into  $N$  by a map in  $[f]$  with energy at most  $A$ .*
2. *Every pleated surface  $g: S \rightarrow N$  homotopic to  $f$  determines an induced hyperbolic metric on  $S$  that lies in a  $B$ -neighborhood of  $L$ .*

The main application of this result is to answer affirmatively Thurston’s “ending lamination conjecture”, for the special case of hyperbolic manifolds (with incompressible boundary) admitting a positive lower bound on injectivity radius. This theorem, together with some extensions of results of Cannon and Thurston on the structure of limit sets of surface groups, will appear in [25].

The ending lamination conjecture states, roughly, that a hyperbolic manifold is determined uniquely by the asymptotic geometry of its ends. Let us state here one version of this, which is proven in [25].

Let  $e$  be a simply degenerate end of  $N$  (see Section 2.4), corresponding to an incompressible boundary component  $S$  of the compact manifold homotopy-equivalent to  $N$ . The *ending lamination* of  $e$  is an element of  $GL(S)$  (see Section 2.1) which is obtained as a limit of any sequence of simple closed curves in  $S$  whose geodesic representatives in  $N$  are eventually contained in any neighborhood of  $e$ . The connection of this to Theorem A is, essentially, that the ending lamination determines an “endpoint at infinity” for the ray  $L$ , which in turn prescribes the internal geometry of  $N$ . Thus we have, for example:

**AN ENDING LAMINATION THEOREM ([25]).** *Let  $N_1$  and  $N_2$  be two homeomorphic hyperbolic 3-manifolds, homotopy-equivalent to a compact manifold with incompressible boundary, and for which  $\text{inj}_{N_i}(x) \geq \varepsilon_0 > 0$  for all  $x \in N_i$ . If all ends of  $N_i$  are simply degenerate and corresponding ends have the same ending laminations, then the two manifolds are isometric.*

(We omit for simplicity the statement of the case when geometrically finite ends occur).

One can interpret the ending lamination, at least in certain cases, as encoding information about the topological structure of the action of a Kleinian group on its limit set. In particular, the ending lamination theorem and the techniques of its proof also yield

**A TOPOLOGICAL RIGIDITY THEOREM ([25]).** *Let  $\Gamma_1$  and  $\Gamma_2$  be Kleinian groups isomorphic to a closed surface group  $\pi_1(S)$ , and suppose  $\text{inj}_{N_i}(x) \geq \varepsilon_0$  for all  $x \in N_i = \mathbf{H}^3/\Gamma_i$ .*

*If the actions of  $\Gamma_1$  and  $\Gamma_2$  on the Riemann sphere are conjugate by a homeomorphism  $\Phi$  whose restriction to the domains of discontinuity is conformal, then  $\Phi$  is in fact a Möbius transformation.*

Note that this is an extension, in this particular case, of a deep theorem of Sullivan [31], which gives the same conclusion but requires as an additional hypothesis that  $\Phi$  be globally quasi-conformal. Sullivan’s theorem is in fact used in the proof of the above theorems – our techniques are used to supply the quasi-conformality.

*Remarks.* A few words are in order on the significance of the restrictive hypotheses of the theorem.

The assumption of  $\pi_1$ -injectivity of  $[f]$  is made in order to apply the geometric tameness results of Thurston and Bonahon [32, 4] (see Section 2.4). In the general case of interest, when a boundary component of the compact core of  $N$  may be compressible, it is not known that  $N$  is geometrically tame. Canary [6] has made progress on this question, and in particular has shown that if  $N$  is known to be homeomorphic to the interior of a compact manifold with boundary, then  $N$  is geometrically tame. One can speculate that, in these cases, it should be possible to prove an analogue of Theorem A.

The second serious restriction is the lower bound on injectivity radius. Our methods depend heavily on this, and in particular on the compactness of certain spaces of hyperbolic manifolds, which holds because of the lower bound (see Section 4). In general, manifolds can develop “thin parts” and cusps, and a direct extension of our methods fails. Indeed, it is not even clear in that case what the precise form of the theorem should be. The presence of thin parts in the three-manifold leads to pleated surfaces which also decompose into thick and thin parts, and any successful extension of the ideas of this paper should involve relative versions of the various estimates (Lemma 5.2 in particular), which hold on subsurfaces.

Although there is hope for this to work, there seem to be serious combinatorial and geometric complications.

A final comment on the method is that the role played by harmonic maps in the proof is, in fact, minor. Most of the technical lemmas are about pleated surfaces, and it should be possible to construct a similar proof from which harmonic maps are entirely absent. It seems unclear at this point whether the harmonic map techniques or the pleated surface techniques offer greater hope for approaching the general case.

**Plan of the paper**

The proof is rooted in the profuse but slippery analogies between Teichmüller space and hyperbolic space. Given the initial data of a  $\pi_1$ -injective homotopy class of maps  $[f: S \rightarrow N]$ , let  $\mathcal{P} \subset \mathcal{T}(S)$  denote the set of hyperbolic metrics on  $S$  induced by pleated surfaces (see Section 2 for definitions). The main step in proving Theorem A will turn out to be showing that  $\mathcal{P}$  is *quasi-convex*, in the sense that any geodesic with endpoints in  $\mathcal{P}$  lies in a uniformly bounded neighborhood of  $\mathcal{P}$ . To do this, we shall exhibit  $\mathcal{P}$  (or rather a closely related set) as the image of a *contracting projection*, in analogy with the closest-point projection onto a convex set in  $\mathbf{H}^n$ .

The main arguments of the proof start in Section 6, where we define the projection  $\Pi$ , and relate it to the energy function  $\mathcal{E}: \mathcal{T}(S) \rightarrow \mathbf{R}_+$ , which assigns to  $\sigma \in \mathcal{T}(S)$  the energy of the unique map  $f_\sigma \sim f$  which is harmonic with respect to  $\sigma$ . Roughly speaking,  $\Pi(\sigma)$  may be thought of as an approximation to the pullback by  $f_\sigma$  of the metric of  $N$ . We show in Proposition 6.2 that  $\log \mathcal{E}(\sigma)$  is approximately the translation distance,  $d(\sigma, \Pi(\sigma))$ . We then define

$$\mathcal{L}_a = \{[\sigma] \in \mathcal{T}(S) : d(\sigma, \Pi(\sigma)) \leq a\}$$

which, for suitable  $a > 0$ , approximates both the bounded-energy locus  $\mathcal{H}_A = \{[\sigma] : \mathcal{E}(\sigma) \leq A\}$ , and the locus  $\mathcal{P}$  of pleated surface metrics.

In Section 7 we prove the main result about  $\Pi$ , Theorem 7.1, which states that  $\Pi$  is “contracting”, in the sense that arbitrarily large balls sufficiently far from  $\mathcal{L}_a$  are mapped by  $\Pi$  to sets of bounded diameter. This allows us to show, in Theorem 8.1, that in fact,  $\mathcal{L}_a$  is quasi-convex. The rest of the proof of Theorem A, which is given at the end of Section 8, is a fairly brief topological argument, utilizing the geometric tameness results of Thurston and Bonahon.

Section 5 contains some technical lemmas regarding pleated surfaces, among which Lemma 5.2 is the most important. This lemma controls the placement of pleated surfaces in  $N$  in terms of the intersection number of their pleating loci. It is also one of the places in the paper where the most crucial use is made of the injectivity radius condition.

The basis for most of the estimates in the paper is provided by the discussion in Section 4 of geometric limits and compactness of spaces of hyperbolic manifolds (with appropriate injectivity radius bounds).

Sections 2 and 3 are essentially expository. Section 2 reviews some basic definitions and theorems from the theory of surfaces and hyperbolic 3-manifolds. Section 3 introduces harmonic maps, and states Theorems 3.2 and 3.3 from [24], which provide the link between harmonic maps and pleated surfaces.

**2. NOTATION AND BASIC FACTS**

The following notation will be used throughout: if  $A$  and  $B$  are subsets of a metric space  $X$  we define  $d_X(A, B)$  to be the *infimum* of  $d_X(a, b)$  over  $a \in A, b \in B$ . We will use this notation

when  $X$  is a hyperbolic manifold  $N$ , or Teichmüller space  $\mathcal{T}(S)$ . In the latter case we will tend to omit the subscript.

### 2.1 Laminations

In this paper,  $S$  will always be a closed surface of genus  $g > 1$ . Let  $\mathcal{S}(S)$  denote the set of homotopy classes of simple closed curves on  $S$ . If a hyperbolic metric  $\sigma$  on  $S$  is chosen we may represent each class in  $\mathcal{S}(S)$  by a (unique) geodesic representative, and then include  $\mathcal{S}(S)$  in the set  $GL(S, \sigma)$  of *geodesic laminations*, namely all closed sets in  $S$  foliated by  $\sigma$ -geodesics (see [32, 7]). This set carries a natural (albeit non-Hausdorff) topology of geometric convergence.

Denote by  $\mathcal{ML}(S, \sigma)$  the set of *transversely measured* geodesic laminations—a transverse measure assigns positive Borel measures to arcs that intersect the lamination transversely, and these measures are taken to each other under isotopies that preserve the leaves of the lamination.  $\mathcal{ML}(S, \sigma)$  is a much nicer space—under the weak topology induced by the measures (and with respect to which the measure-forgetting map from  $\mathcal{ML}(S, \sigma)$  to  $GL(S, \sigma)$  is continuous),  $\mathcal{ML}(S, \sigma)$  is homeomorphic to  $\mathbf{R}^{6g-6}$ .

The choice of metric is not important—there is a canonical homeomorphism between  $GL(S, \sigma)$  and  $GL(S, \sigma')$ , and between  $\mathcal{ML}(S, \sigma)$  and  $\mathcal{ML}(S, \sigma')$ , for any  $\sigma'$ , which restricts to the identity on  $\mathcal{S}(S)$ . Thus we will suppress the metric, writing  $GL(S)$  and  $\mathcal{ML}(S)$ . (See also [14, 19, 26]). We also define the projectivized space  $\mathcal{PML}(S) = \mathcal{ML}(S)/\mathbf{R}_+$ , formed by identifying proportional measures.  $\mathcal{PML}(S)$  is in fact a sphere, but it is only important to us that it is compact.

For a simple closed geodesic a measure is given by a single positive number, called a weight. The set  $\mathbf{RS}(S)$  of weighted homotopy classes is dense in  $\mathcal{ML}(S)$ , and some basic geometric notions generalize from closed curves to measured laminations.

Given a hyperbolic metric  $\sigma$  and  $\gamma \in \mathcal{S}(S)$  denote by  $l_\sigma(\gamma)$  the length of the geodesic representative of  $\gamma$ . This extends (by scaling) to  $\mathbf{RS}(S)$ , and from there to a continuous function on  $\mathcal{ML}(S)$ .  $l_\sigma(\lambda)$  can also be described directly as the mass of the measure defined locally as the product of the transverse measure with length along the leaves of  $\lambda$ . In addition, this function is continuous in  $\sigma$  (taken as an element of the Teichmüller space of  $S$ —see the next section).

The *geometric intersection number*  $i(\cdot, \cdot)$  of two closed geodesics is the cardinality of their set of transverse intersections. More topologically, this is the minimum of the cardinalities of intersection of any homotopic representatives. This function also extends to a continuous symmetric function on  $\mathcal{ML}(S) \times \mathcal{ML}(S)$ , which is homogeneous under scaling of the measures. If  $i(\lambda, \mu) = 0$ , then any two components of their supports are either disjoint or equal. This means in particular that the union of their supports forms a lamination, which we denote  $\lambda \cup \mu$  (we note that one may sum the measures of  $\lambda$  and  $\mu$  to obtain a measure on  $\lambda \cup \mu$ , but this will not be important for us).

Finally, recall Ahlfors and Beurling's notion of *extremal length*  $E_\sigma(\gamma)$  for  $\gamma \in \mathcal{S}(S)$  (see, for example, the books by Strebel [30] and Gardiner [12]). This is the quantity  $\sup_{\sigma'} l_{\sigma'}^2(\gamma)$ , where the supremum is over metrics  $\sigma'$  of area 1, which are conformally equivalent to  $\sigma$ . Here  $l_{\sigma'}$  is the infimum of  $\sigma'$ -lengths of representatives of  $\gamma$  (note that there is not necessarily a unique geodesic). It is useful to know that this supremum is actually realized by a metric which is Euclidean except on a finite set of singularities, and which is foliated (away from the singularities) by geodesic curves which are representatives of  $\gamma$ . Hence in this metric  $S$  is obtained by identifications on the boundary of a Euclidean cylinder whose core is homotopic to  $\gamma$ , and  $E_\sigma(\gamma)$  is the reciprocal of the modulus of this cylinder.

Extremal length extends to a continuous function on  $\mathcal{ML}(S)$  (see Kerckhoff [17]), which scales according to  $E_\sigma(c\lambda) = c^2 E_\sigma(\lambda)$ . The dependence on  $\sigma$  is again continuous.

We remark that the comparison between extremal and hyperbolic lengths on a given Riemann surface is not completely straightforward (see [21]). The inequality

$$E_\sigma(\lambda) \geq \frac{l_\sigma^2(\lambda)}{2\pi|\chi(S)|}$$

follows immediately from the definitions and the Gauss–Bonnet theorem, but a bound in the other direction for  $E_\sigma$  of a closed curve is, in the worst case, exponential in  $l_\sigma$ . When both are small (for a closed curve), they are related approximately linearly. We will use the following comparison, which is natural because it is scale invariant, but does not hold uniformly in  $\mathcal{T}(S)$ . A proof, which is just a compactness argument, appears in [23, §8].

LEMMA 2.1. *Given a closed surface  $S$  and  $\varepsilon_0 > 0$  there exists  $c_0$  such that, for any hyperbolic metric  $\sigma$  on  $S$  with injectivity radii at least  $\varepsilon_0$  at any point,*

$$c_0 \leq \frac{l_\sigma^2(\lambda)}{E_\sigma(\lambda)} \leq 2\pi|\chi(S)|,$$

for any  $\lambda \in \mathcal{ML}(S)$ .

### 2.2 Teichmüller space

Good general references for this material may be found in the books by Abikoff [1] and Gardiner [13]. The Teichmüller space  $\mathcal{T}(S)$  of a surface is the space of conformal structures (alternately hyperbolic structures) on  $S$ , where two structures are considered equivalent if there is a conformal map (alternately isometry) between them isotopic to the identity.  $\mathcal{T}(S)$  carries a natural topology which is induced by a variety of possible natural metrics. We will work with the *Teichmüller metric*, which assigns to  $[\sigma], [\tau] \in \mathcal{T}(S)$  the distance  $d_T(\sigma, \tau) = \frac{1}{2} \log K$ , where  $K$  is the smallest quasi-conformal distortion of a homeomorphism from  $(S, \sigma)$  to  $(S, \tau)$  that is isotopic to the identity (note that  $d_T$  is symmetric, which may not be obvious from the definition). For our purposes it will suffice to know that

THEOREM 2.2 (Kerckhoff [17]) *For any two points  $[\sigma], [\tau] \in \mathcal{T}(S)$ ,*

$$K = \sup_\gamma \frac{E_\tau(\gamma)}{E_\sigma(\gamma)},$$

where  $E$  denotes extremal length as in the previous section, and  $\gamma$  ranges over the measured laminations in  $S$ .

$\mathcal{T}(S)$  is homeomorphic to  $\mathbf{R}^{6g-6}$  where  $g > 1$  is the genus of  $S$  (and in fact has a natural smooth and even a complex structure). The metric is not non-positively curved as was once thought (see [22]), but it has some qualitative properties of a non-positively curved space. In particular, between every two points is a unique shortest path, called a Teichmüller geodesic, which extends uniquely to an infinite geodesic. We will also need to know that  $\mathcal{T}(S)$  is complete, and that the following holds:

LEMMA 2.3. *Let  $K \subset \mathcal{T}(S)$  be a compact set and  $\{L_i\}$  a sequence of Teichmüller geodesics such that  $L_i \cap K \neq \emptyset$ . Then there exists a subsequence of  $\{L_i\}$  that converges on compact sets to a geodesic  $L$ , whose length (possibly infinite) is  $l(L) = \limsup l(L_i)$ .*

This is in fact true for any complete, locally compact geodesic metric space, by an application of Ascoli’s theorem (see e.g. [9]). In our case it follows immediately from the

above claims about geodesics, or from the stronger fact that one can define an exponential map for  $\mathcal{F}(S)$  based at any point, which is a diffeomorphism.

### 2.3 Pleated surfaces and realizability

Let  $N$  denote a complete hyperbolic three-manifold. A *pleated surface* (or *pleated map*) is a map  $g: S \rightarrow N$  together with a hyperbolic metric  $\rho$  on  $S$ , such that  $g$  is *path-isometric* with respect to  $\rho$  (takes rectifiable paths to paths of equal length) and which maps at least one geodesic segment through any  $p \in S$  to a geodesic segment in  $N$  (see [33, 32, 7]).

It follows from this definition that there is a geodesic lamination  $\lambda$  on  $(S, \rho)$ , called the *pleating locus* of  $g$ , such that  $g$  is totally geodesic on  $S - \lambda$ , and maps the leaves of  $\lambda$  to geodesics. (One thinks of the surface as being bent along the leaves of  $\lambda$ , although this bending is complicated since the leaf space of  $\lambda$  in any small neighborhood is typically a Cantor set.)

In general, fixing a homotopy class of maps  $[f: S \rightarrow N]$ , we say that a lamination  $\lambda \in \mathcal{GL}(S)$  is *realizable* in  $[f]$  if there is a pleated surface  $g: (S, \rho) \rightarrow N$  with  $g \in [f]$ , such that  $\lambda$  is mapped geodesically by  $g$ .

We note some facts as a proposition (see [32, 7] for proofs).

**PROPOSITION 2.4.** *Fix a  $\pi_1$ -injective homotopy class  $[f: S \rightarrow N]$ , and assume that  $N$  has no cusps.*

1. *Every homotopically non-trivial simple closed curve in  $S$  is realizable in  $[f]$ .*
2. *If  $\lambda$  is realizable in  $[f]$  then the leaves of its geodesic image are uniquely determined.*
3. *If  $\lambda \subset \mu$  and  $\lambda$  is realizable, then  $\mu$  is realizable.*

*Lamination lengths.* If  $\lambda \in \mathcal{ML}(S)$  is realizable in  $[f]$  we can denote by  $l_N(\lambda)$  the length of its image (keeping the homotopy class  $[f]$  implicit), measured as in the case of laminations on surfaces, by pulling back length along the leaves from  $N$ , and integrating the product of this with the transverse measure. Equivalently, we can just set  $l_N(\lambda) = l_\rho(\lambda)$ , where  $\rho$  is the metric induced on any pleated surface mapping  $\lambda$  geodesically. In general, we note that  $l_N(\mu) \leq l_\rho(\mu)$  for an arbitrary pleated surface  $g: (S, \rho) \rightarrow N$  in  $[f]$ , and  $\mu \in \mathcal{ML}(S)$ .

This definition extends the standard notion of length of the geodesic representative of  $f(\gamma)$  for  $\gamma \in \mathcal{S}(S)$ , and is again continuous in  $\lambda$ . In fact it is still continuous at an unrealizable lamination  $\lambda \in \mathcal{ML}(S)$ , if we set  $l_N(\lambda) = 0$  ([32, 34, 4]).

One more piece of notation we will use is the following: if  $f: S \rightarrow N$  is a continuous map (actually, Lipschitz or differentiable in all the cases we'll consider), and  $\gamma \in \mathcal{S}(S)$ , let  $l_f(\gamma)$  denote the minimal  $N$ -length of an image by  $f$  of a representative of  $\gamma$ . (We refrain from letting  $\gamma$  range freely in  $\mathcal{ML}(S)$  to avoid the technicalities that may arise with non-hyperbolic metrics on  $f(S)$ ). If  $f: (S, \rho) \rightarrow N$  is a pleated map then  $l_f = l_\rho$ , and at any rate we always have  $l_f \geq l_N$ .

### 2.4 Geometric tameness

The convex hull of a hyperbolic three-manifold is the smallest convex submanifold  $C(N) \subset N$  for which inclusion is a homotopy equivalence. The theory of *geometrically finite* manifolds, those for which  $C(N)$  has finite volume (or, in our cusp-free case, for which  $C(N)$  is compact) is well-understood (see [3, 20]). To understand geometrically infinite manifolds we must resort to Thurston's notion of *geometric tameness*.

Restrict, for ease of exposition, to the case that  $N$  has no cusps. There is a compact 3-dimensional submanifold  $M \subset N$ , called the *Scott core* (see [29]), whose inclusion is

a homotopy-equivalence. The ends of  $N$  are in one-to-one correspondence with the components of  $N - M$  or, equivalently, the components of  $\partial M$ . We say that an end of  $N$  is *geometrically finite* if it has a neighborhood that misses  $C(N)$ .

We say an end of  $N$  is *simply degenerate* if it has a neighborhood homeomorphic to  $S \times \mathbf{R}$ , where  $S$  is the corresponding component of  $\partial M$ , and if there is a sequence of pleated surfaces homotopic in this neighborhood to the inclusion of  $S$ , and exiting every compact set.  $N$  is called *geometrically tame* if all of its ends are either geometrically finite or simply degenerate. In particular  $N$  is then homeomorphic to the interior of  $M$ .

Suppose that  $\partial M$  is incompressible (the inclusion is a  $\pi_1$ -injection). Group-theoretically this may be stated as the fact that  $\pi_1(N)$  has no non-trivial decomposition as a free product. Bonahon proved in [4] that, in this case,  $N$  is geometrically tame. For further discussion, and progress in other cases, see Canary [6].

We shall always work with  $\pi_1$ -injective maps of surfaces, so after lifting to an appropriate cover we may always assume that  $\pi_1(N) = \pi_1(S)$ . Thus, by Bonahon's theorem we know that  $N$  is geometrically tame, and in particular homeomorphic to  $S \times \mathbf{R}$ .

3. ENERGY AND HARMONIC MAPS

For a differentiable map  $f: M \rightarrow N$  between two Riemannian manifolds the *energy* is defined as

$$\mathcal{E}(f) = \frac{1}{2} \int_M |df|^2 dv(M).$$

A map is *harmonic* if it is a stationary point for this functional in the space of maps. The theory of existence and regularity for such maps is extensive and begins (at this level of generality) with [10].

In our situation it suffices to know that if  $M$  is a compact surface of genus at least 2 with Riemannian metric and  $N$  is a complete hyperbolic manifold, then any homotopy class of maps from  $M$  to  $N$  which is injective on the level of fundamental groups (actually much less is needed) has a unique, energy minimizing, harmonic representative (see [24] for references).

Further (when  $M$  is two dimensional), both energy and harmonicity do not depend strictly on the metric of  $M$  but only on its conformal class. Thus these notions are properly defined on Riemann surfaces and we may say that given an incompressible homotopy class  $[f: S \rightarrow N]$  and a point  $[\sigma] \in \mathcal{T}(S)$  there is an energy minimizing harmonic map  $f_\sigma: S \rightarrow N$  in  $[f]$ , unique up to reparametrizations of the domain (if we fix the representative  $\sigma$  of  $[\sigma]$ ,  $f_\sigma$  is in fact unique).

If we set

$$\mathcal{E}(\sigma) = \mathcal{E}(f_\sigma)$$

we obtain a continuous, and even smooth, function on  $\mathcal{T}(S)$  (see [27]).

We shall need a simple lemma which is really a standard fact about negatively-curved surfaces.

LEMMA 3.1. *If  $f: M \rightarrow N$  is an incompressible harmonic map from a Riemann surface to a hyperbolic three-manifold whose injectivity radii are at least  $\epsilon_0$  then*

$$\text{diam}(f(M)) \leq D$$

where  $D$  depends on  $\chi(M)$  and  $\epsilon_0$ .

*Proof of 3.1.* In [24], a bound

$$\text{Area}(f(M)) \leq 2\pi|\chi(M)|$$

is obtained, using a family of smooth, non-positively curved metrics  $\rho_\varepsilon$  converging as  $\varepsilon \rightarrow 0$  to the pullback metric  $\rho = f^*h$  on  $M$  (this tactic is unnecessary if  $f$  is an immersion, because then  $\rho$  itself is a smooth metric of curvature at most  $-1$  at any point). The metric  $\rho_\varepsilon$  is pointwise bigger than  $\rho$ , and  $\lim_{\varepsilon \rightarrow 0} \text{Area}(\rho_\varepsilon) \leq 2\pi|\chi(M)|$ .

Consider  $\varepsilon$  such that  $\text{Area}(\rho_\varepsilon) \leq 2\pi|\chi(M)| + 1$ . Let  $x, y$  be two points in  $S$  and  $L$  the shortest  $\rho_\varepsilon$ -geodesic segment connecting them. Since  $\rho_\varepsilon > \rho$  pointwise, we know the injectivity radius of  $\rho_\varepsilon$  is at least  $\varepsilon_0$  at any point. This implies that  $L$  has an embedded  $\varepsilon_0/2$ -neighborhood. For otherwise, there is a geodesic arc of length at most  $\varepsilon_0$  connecting two points of  $L$ , and the resulting loop is either shorter than  $2\varepsilon_0$ , in which case it is homotopically trivial and contradicts the non-positive curvature of  $\rho_\varepsilon$ , or is longer, in which case it contradicts the choice of  $L$  (the new arc is a short cut).

Thus (using the non-positive curvature to bound from below the area of the neighborhood),  $\varepsilon_0 l_\rho(L) \leq (2\pi|\chi(M)| + 1)$ , and the lemma follows. □

In [24], we prove

**THEOREM 3.2.** *There is a constant  $C$  depending only on the topological type of  $S$ , such that*

$$\frac{1}{2} \frac{l_f^2(\lambda)}{E_\sigma(\lambda)} \leq \mathcal{E}(f_\sigma) \leq \frac{1}{2} \frac{l_N^2(\mu_\sigma)}{E_\sigma(\mu_\sigma)} + C,$$

where  $\lambda$  is any element of  $\mathcal{L}(S)$  and  $\mu_\sigma$  is a particular element of  $\mathcal{ML}(S)$ , determined by the map.

(We can of course replace  $l_f(\lambda)$  by the possibly smaller  $l_N(\lambda)$ , in which case we may as well take  $\lambda \in \mathcal{ML}(S)$ .)

We call  $\mu_\sigma$  the *maximal-stretch lamination* of  $f_\sigma$ . It is obtained from the horizontal foliation of a certain holomorphic quadratic differential associated to  $f_\sigma$ , namely the  $dz^2$  part of the pullback by  $df$  of the metric of  $N$ . (This foliation may be described by the fact that its leaves are tangent at each point to the direction of greatest stretch of  $df$ . See also [35].)

In addition, we show that, when  $\mathcal{E}(f_\sigma)$  is sufficiently high, the image of  $f_\sigma$  approximates in a fairly strong sense a map that sends  $\mu_\sigma$  to its geodesic representative in  $N$ . In particular  $\mu_\sigma$  is realizable in  $[f]$ . In this paper we will use the following weak version of this fact:

**THEOREM 3.3.** *There exist numbers  $\eta, \mathcal{E}_0$ , depending only on  $\chi(S)$ , such that if  $\mathcal{E}(f_\sigma) \geq \mathcal{E}_0$  then  $f_\sigma(S)$  is no more than  $\eta$  away, in  $N$ , from the geodesic representative of  $\mu_\sigma$  in  $[f]$ .*

(We may in fact choose  $\eta$  to be as small as we like, at the cost of increasing  $\mathcal{E}_0$ .)

If  $f_\sigma$  has low energy this theorem tells us nothing about the location of  $\mu_\sigma$  in  $N$ . However, the following is still true:

**LEMMA 3.4.** *There exists  $D_0 > 0$  such that  $f_\sigma(S)$  is within a distance  $D_0$  in  $N$  of the image of some pleated surface in  $[f]$ .*

*Proof of 3.4.* This is a straightforward consequence of the following theorem from [32]:

**THEOREM 3.5.** (Thurston). *If  $[f: S \rightarrow N]$  is an isomorphism on  $\pi_1$  then every point in the convex hull of  $N$  is within a uniformly bounded distance of the image of a pleated surface in  $[f]$ .*



(Note that Thurston proved this assuming the additional assumption that  $N$  is geometrically tame, but this now follows from Bonahon’s theorem.)

Lifting  $f_\sigma$  to the cover  $\tilde{N}$  of  $N$  corresponding to  $f_*(\pi_1(S))$ , we first argue that the lift  $\tilde{f}_\sigma(S)$  is contained in the convex hull of this cover. This is just the maximum principle argument of [15]—there is a distance-nonincreasing projection of  $\tilde{N}$  to its convex hull, so any energy-minimizing map must be contained there. We then apply Theorem 3.5.

For completeness, let us also give a self-contained proof of Lemma 3.4, which is an observation of Dick Canary. In this proof, the bound  $D_0$  will depend on  $\chi(S)$  and  $\varepsilon_0$ .

It follows from the area bound in Lemma 3.1 that there is some bound  $A(\chi(S))$  such that  $S$  always contains a homotopically non-trivial curve  $\gamma$  with  $l_\sigma(\gamma) \leq A$  (see also Lemma 5.4 in [24]). Lemma 6.4 therefore implies that the geodesic representative  $f(\gamma)^*$  is a bounded distance  $B(A, \varepsilon_0)$  away from  $f_\sigma(\gamma)$ . Thus, setting  $g:(S, \rho) \rightarrow N$  to be a pleated surface homotopic to  $f$  and taking  $\gamma$  to this geodesic (see Theorem 2.4), the proof is complete.

We note also that, using an argument such as that given at the end of Section 8, one can in fact give a proof of Theorem 3.5 in this context, using harmonic maps. □

We conclude this section with an easy estimate for the energy of a map between surfaces, in terms of Teichmüller distance:

LEMMA 3.6. *If  $\sigma$  and  $\rho$  are hyperbolic metrics on  $S$ , such that  $d_T(\sigma, \rho) = \frac{1}{2} \log K$  and  $\text{inj}_\rho(x) \geq \varepsilon_0$  for all  $x \in S$ , then*

$$C_1 K \leq \mathcal{E}(\sigma, \rho) \leq C_2 K$$

where  $\mathcal{E}(\sigma, \rho)$  is the energy of the unique harmonic map from  $(S, \sigma)$  to  $(S, \rho)$  in the homotopy class of the identity. The constants  $C_i$  depend only on  $\varepsilon_0$  and  $\chi(S)$ .

The proof is just a combination of Theorem 3.2 and Lemma 2.1 (together with the fact that  $K \geq 1$ , to remove additive constants).

#### 4. COMPACTNESS THEOREMS

We shall make repeated use of the fact that a variety of spaces of hyperbolic manifolds, and pleated surfaces within them, are compact, in an appropriate geometric topology. We discuss here the concepts and results that we shall need, and refer the reader to [7], [18] and [14a] for a more careful and complete treatment of the topic.

Let us begin with the notion of the geometric topology on spaces of hyperbolic manifolds. Recall, first, the *Chabauty topology* on the set  $\mathcal{G}_n$  of closed subgroups of  $\text{Isom}(\mathbf{H}^n)$ : We say that a sequence  $\{\Gamma_i\}$  converges geometrically to  $\Gamma$  if  $\Gamma$  is the set of all limits of convergent sub-sequences  $\{g_{i_j} \in \Gamma_{i_j}\}$ , and if all elements of  $\Gamma$  are also obtained as limits of sequences  $\{g_i \in \Gamma_i\}$ .  $\mathcal{G}_n$  is compact in this topology. Denote by  $\mathcal{D}_n$  the set of discrete subgroups of  $\text{Isom}(\mathbf{H}^n)$  and by  $\mathcal{D}_n(x_0, \varepsilon)$  the set of discrete subgroups whose elements move a fixed basepoint  $x_0 \in \mathbf{H}^n$  at least  $\varepsilon > 0$ . Then it is also known that the latter is compact.

This gives rise to a topology on  $\mathcal{HF}_n$ , the space of hyperbolic  $n$ -manifolds  $N$  with distinguished baseframes, i.e. with a choice  $y, e_1, \dots, e_n$ , where  $y \in N$  and  $(e_i)$  is an ordered orthonormal basis for  $T_y N$ . The correspondence between  $\mathcal{HF}_n$  and  $\mathcal{D}_n$  is obtained, after fixing a “standard” baseframe  $x_0, (e_i)$  for  $\mathbf{H}^n$ , by associating  $(N, y, (e_i))$  to the unique group  $\Gamma$  such that  $\mathbf{H}^n/\Gamma$  is isometric to  $N$  and the resulting covering map  $\pi: \mathbf{H}^n \rightarrow N$  takes  $x_0, (e_i)$  to  $y, (e_i)$ .

The set of choices of bases  $(e_i)$  for a given  $(N, y)$  produces a compact set (namely  $O(n)$ ) in  $\mathcal{HF}_n$  in this topology, and we shall henceforth automatically identify these choices and

consider just the set of hyperbolic manifolds with basepoints,  $\mathcal{HB}_n$ , with the quotient topology.

An equivalent description of the geometric topology on this set is that  $(N_i, y_i) \rightarrow (N, y)$  if there are sequences  $R_i \rightarrow \infty$ ,  $\delta_i \rightarrow 0$ , and maps  $f_i: (B_{R_i}(y_i), y_i) \rightarrow (N, y)$  which are  $(1 + \delta_i)$ -bilipschitz homeomorphisms onto their images (where  $B_{R_i}(y_i)$  denote  $R_i$ -neighborhoods of the basepoints in  $N_i$ ).

We can now topologize the set of pleated surfaces. Let  $\mathcal{PS}$  denote the set of triples  $(\Gamma_1, \Gamma_2, \tilde{g})$ , where  $\Gamma_1 \in \mathcal{D}_2$ ,  $\Gamma_2 \in \mathcal{D}_3$ , and  $\tilde{g}: \mathbf{H}^2 \rightarrow \mathbf{H}^3$  is a pleated map, equivariant in the sense that there exists a homomorphism  $\varphi: \Gamma_1 \rightarrow \Gamma_2$  such that  $\tilde{g} \circ \gamma = \varphi(\gamma) \circ \tilde{g}$  for all  $\gamma \in \Gamma_1$ . Normalize, also, by requiring that  $\tilde{g}$  maps the standard baseframe in  $\mathbf{H}^2$  to the standard baseframe in  $\mathbf{H}^3$ . Clearly  $\tilde{g}$  descends to a pleated map  $g: S \rightarrow N$ , where  $S = \mathbf{H}^2/\Gamma_1$ ,  $N = \mathbf{H}^3/\Gamma_2$ . To recover the groups from this description, (up to orientation coming from choice of baseframes) we keep track of the images in  $S$  and  $N$  of the standard basepoints. The topology here is defined by convergence in the Chabauty topology of  $\Gamma_1$  and  $\Gamma_2$ , together with uniform convergence on compact sets for the map  $\tilde{g}$ . The basic compactness theorem, due to Thurston [32, 7] is the following.

Fix  $A > 0$  and  $\varepsilon > 0$ . Let  $\mathcal{PS}(A, \varepsilon)$  denote the set of pleated surfaces  $(\Gamma_1, \Gamma_2, \tilde{g})$  where the area of  $\mathbf{H}^2/\Gamma_1$  is at most  $A$ , the injectivity radius of  $\mathbf{H}^2/\Gamma_1$  at the base-point is at least  $\varepsilon$ , and the induced homomorphism  $\varphi: \Gamma_1 \rightarrow \Gamma_2$  is injective.

**THEOREM 4.1.** (Thurston) *The set  $\mathcal{PS}(A, \varepsilon)$  is compact in the geometric topology, for a given  $A > 0$  and  $\varepsilon > 0$ .*

This kind of compactness property is useful, but some information is lost in general under geometric limits, for example the isomorphism type of the groups. Under further restrictions, which are natural for our context, we can keep track of this. (Note that the results we will state are not the most general known).

Let  $S$  be a closed surface of genus at least 2, and consider  $\mathcal{PS}\mathcal{T}(S, \varepsilon)$ , the subset of  $\mathcal{PS}$  for which  $\Gamma_1 \approx \pi_1(S)$  and the injectivity radius of  $\mathbf{H}^2/\Gamma_1$  is at least  $\varepsilon$  at every point. Let  $\{(\Gamma_1^i, \Gamma_2^i, \tilde{g}^i)\}$  be a sequence in  $\mathcal{PS}\mathcal{T}(S, \varepsilon)$ , which converges in  $\mathcal{PS}(2\pi|\chi(S)|, \varepsilon)$  to  $(\Gamma_1, \Gamma_2, \tilde{g})$ . The injectivity radius bound implies that the diameters of  $\mathbf{H}^2/\Gamma_1^i$  are uniformly bounded, so the bilipschitz homeomorphisms given by the convergence of  $\mathbf{H}^2/\Gamma_1^i$  in  $\mathcal{HB}_2$  are eventually defined on all of  $\mathbf{H}^2/\Gamma_1^i$ . Therefore the limit group  $\Gamma_1$  is isomorphic to  $\Gamma_1^i$ , and the same injectivity radius bound holds in the limit. We conclude that  $(\Gamma_1, \Gamma_2, \varepsilon) \in \mathcal{PS}\mathcal{T}(S, \varepsilon)$ , or in other words

**COROLLARY 4.2.**  *$\mathcal{PS}\mathcal{T}(S, \varepsilon)$  is compact in the geometric topology.*

However, we note that under a limit in this space, the isomorphism type of  $\Gamma_2$  may still not be preserved. This is a considerably subtler problem, and it is unknown whether in general an injectivity radius bound on  $\mathbf{H}^3/\Gamma_2$  suffices to control it. However, we can reduce further to a useful special case.

Define  $\mathcal{PS}\mathcal{I}(S, \varepsilon) \subset \mathcal{PS}\mathcal{T}(S, \varepsilon)$  to be the subset for which  $\varphi: \Gamma_1 \rightarrow \Gamma_2$  is required to be a group isomorphism, and the injectivity radius lower bound is applied to all of  $\mathbf{H}^3/\Gamma_2$ , not just  $\mathbf{H}^2/\Gamma_1$ .

Given an element  $(\Gamma_1, \Gamma_2, \tilde{g})$  in  $\mathcal{PS}\mathcal{I}(S, \varepsilon)$ , we may choose an identification  $f: \pi_1(S) \rightarrow \Gamma_1$  such that a fixed set of generators of  $\pi_1(S)$  are mapped to elements of  $\Gamma_1$  with uniformly bounded translation distances—this is guaranteed by the injectivity radius bound on  $\mathbf{H}^2/\Gamma_1$ , and the resulting uniform bound on the diameter of the surface. The same bound then holds for the images of the generators by  $\varphi \circ f$ . If we now consider a sequence

$\Gamma_1^i, \Gamma_2^i, \tilde{g}^i$ ) which converges in  $\mathcal{PST}(S, \varepsilon)$  to  $(\Gamma_1, \Gamma_2, \tilde{g})$ , it is easy to see that the corresponding representations  $R^i = \varphi^i \circ f^i: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$  converge as maps to a discrete, faithful representation  $R$  (this is known as “algebraic convergence”).

It is clear, also, that  $R(\pi_1(S))$  is a subgroup of the geometric limit group  $\Gamma_2$ , so that it only remains to show that  $R$  is onto  $\Gamma_2$ , to conclude that the limit lies in  $\mathcal{PST}(S, \varepsilon)$ . This fact is due to the following theorem of Thurston, a sketch of whose proof can be found in Theorems 9.2 and 9.6.1 of [32] (see also [5]). We state a simplified version here:

**THEOREM 4.3 (Thurston).** *Suppose  $\{\rho_i: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})\}$  is a sequence of discrete faithful representations converging algebraically to a representation  $\rho$ , such that no element of  $\rho_i(\pi_1(S))$  or of  $\rho(\pi_1(S))$  is parabolic.*

*Then  $\rho(\pi_1(S))$  is also the geometric limit of  $\rho_i(\pi_1(S))$ .*

(We note that Thurston states this with the additional assumption that  $\mathbf{H}^3/\rho_i(\pi_1(S))$  are geometrically tame, but this now automatically follows from Bonahon’s theorem).

The condition on parabolics, in our case, follows from the injectivity radius bound on  $\Gamma_2^i$ . (See the remark at the end of this section for a discussion of what can go wrong when this bound is removed.) We conclude, therefore, that the limit lies in  $\mathcal{PST}(S, \varepsilon)$ , and thus:

**COROLLARY 4.4.** (Compactness for surface groups)  *$\mathcal{PST}(S, \varepsilon)$  is compact in the geometric topology.*

Our first application of these compactness results is the following lemma, showing how the internal geometry of a pleated surface gives information about the geometry of the manifold around it.

**LEMMA 4.5.** *Fix  $S$  and  $\varepsilon > 0$ . Given  $b_1, b_2 > 0$  there exists  $A$  such that if  $g: (S, \rho) \rightarrow N$  is a pleated surface,  $g_*$  is an isomorphism on  $\pi_1$ , and  $\text{inj}_N(x) > \varepsilon$  for all  $x \in N$  then the following holds.*

*Let  $\alpha \subset S$  be a simple closed  $\rho$ -geodesic, and let  $\beta$  denote the shortest curve in  $N$  freely homotopic to  $g(\alpha)$  such that  $d_N(\beta, g(S)) \leq b_1$ . Then*

$$l_N(\beta) \leq b_2 \Rightarrow l_\rho(\alpha) \leq A.$$

(Note that, unlike our usual notation, here  $l_N(\beta)$  refers to the length of the actual curve  $\beta$ , rather than its geodesic representative.)

*Proof of 4.5.* If the lemma is false, there is a sequence  $(g_i, \rho_i, N_i, \alpha_i, \beta_i)$  such that  $l_{\rho_i}(\alpha_i) \rightarrow \infty$  but  $d_{N_i}(\beta_i, g_i(S)) \leq b_1$  and  $l_{N_i}(\beta_i) \leq b_2$ . By Corollary 4.4, we can extract a convergent subsequence and a limit in  $\mathcal{PST}(S, \varepsilon)$ . This produces a pleated surface  $g: (S, \rho) \rightarrow N$  where  $\pi_1(N) = g_*(\pi_1(S))$ . Let  $y_i \in g_i(S)$  be basepoints of  $N_i$ ,  $y \in g(S)$  the basepoint for  $N$ , and  $f_i: (B_R(y_i), y_i) \rightarrow (N, y)$  the  $(1 + \delta_i)$ -bilipschitz maps in the definition of the geometric limit.

Note first that  $\beta_i$  must be geodesic except possibly at one point. Otherwise, we could find a shorter homotopic curve that still meets the  $b_1$ -neighborhood of  $g(S)$ . By the length bound on  $\beta_i$  we know that  $\beta_i$  remains in a  $(b_1 + b_2/2)$ -neighborhood of  $g_i(S)$ . Thus, we may extract a further subsequence so that  $f_i(\beta_i) \rightarrow \beta$ , which is also geodesic in  $N$  except possibly at one point. Moreover,  $\beta$  is homotopic to  $f_i(\beta_i)$  for sufficiently high  $i$ , and the homotopy takes place in a small neighborhood of  $\beta$ .

Since  $g_*$  is an isomorphism, there must be a curve  $\alpha \subset S$  such that  $g(\alpha)$  is freely homotopic to  $\beta$  in  $N$  (and thus to  $f_i(\beta_i)$ , for high enough  $i$ ). Pulling this homotopy back by  $f_i$

we obtain a homotopy between  $f_i^{-1}(\alpha)$  and  $\beta_i$ , concluding that  $f_i^{-1}(\alpha)$  is homotopic to  $\alpha_i$ . But  $l_\rho(\alpha_i) \rightarrow \infty$  whereas  $l_\rho(f_i^{-1}(\alpha))$  is bounded, a contradiction.  $\square$

An extension of this is the following, which shows how to compare the geometry of nearby pleated surfaces.

**COROLLARY 4.6.** *Fix  $S$  and  $\varepsilon > 0$ . Given  $a > 0$  there exists  $b > 0$  such that if  $g: (S, \rho) \rightarrow N$  and  $h: (S, \sigma) \rightarrow N$  are homotopic pleated surfaces which are isomorphisms on  $\pi_1$  and  $\text{inj}_N(x) > \varepsilon$  for all  $x \in N$ , then*

$$d_N(g(S), h(S)) \leq a \Rightarrow d_T(\sigma, \rho) \leq b.$$

*Proof of 4.6.* Let  $X = \{\xi_1, \dots, \xi_n\}$  be a collection of simple closed curves in  $S$ . We say that  $X$  is *binding* (see [34]) if their geodesic representatives in a hyperbolic metric cut  $S$  into a union of topological disks (note that this does not depend on the choice of metric). Given  $[\sigma] \in \mathcal{T}(S)$  we define  $l_\sigma(X) = \sum_k l_\sigma(\xi_k)$ .

The injectivity radius in  $(S, \sigma)$  is at least  $\varepsilon$  at every point, since  $h$  is incompressible. It follows that there is some  $A(\varepsilon)$  such that there exists a binding collection of curves  $X$  with  $l_\sigma(X) \leq A$ . (One can for example take the shortest pair of pants decomposition, and add enough curves to cut the pairs of pants into disks. With more care one could even show that a binding collection of bounded length can be found which consists of only two curves.)

The images  $g(\xi_i)$  and  $h(\xi_i)$  are freely homotopic for  $\xi_i \in X$ , and  $d_N(g(\xi_i), h(\xi_i)) \leq a + 2D$ , where  $D = D(S, \varepsilon)$  is the universal bound on the diameter of a hyperbolic metric on  $S$  with injectivity radii at least  $\varepsilon$ . Further,  $l_N(h(\xi_i)) = l_\sigma(\xi_i) \leq A$  (here again  $l_N(h(\xi_i))$  is the length of the curve and not its geodesic representative), so by direct appeal to the previous lemma we obtain a bound on  $l_\rho(\xi_i)$ , and hence on  $l_\rho(X)$  as well.

The proof is completed by the following lemma.

**LEMMA 4.7.** *Let  $X = \{\xi_1, \dots, \xi_n\}$  be a binding collection of simple closed curves on  $S$ . Given  $A > 0$  there exists  $B > 0$  such that the set*

$$T(X, A) = \{[\sigma] \in \mathcal{T}(S) : l_\sigma(X) \leq A\}$$

*has diameter at most  $B$  in the Teichmüller metric.*

*Proof of 4.7.* This is a well-known fact, and there are many possible proofs. We shall give one that is quick given the tools we have at hand. Define  $i(\alpha, X) = \sum i(\alpha, \xi_k)$ . We shall show

$$C_1 i(\alpha, X) \leq l_\sigma(\alpha) \leq C_2 i(\alpha, X) \tag{4.1}$$

for  $\sigma \in T(X, A)$  and  $\alpha \in \mathcal{S}(S)$ , where  $C_i$  depend on  $A$  and  $\chi(S)$ .

Let  $\alpha^*, \xi_k^*$  denote geodesic representatives in the hyperbolic metric  $\sigma$ . Let  $\{D_j\}$  denote the (disk) components of  $S - \cup \xi_k^*$ , and observe that each  $D_j$  is isometric to a convex set in  $\mathbf{H}^2$ , and

$$\text{diam}_\sigma(D_j) \leq \text{diam}_\sigma(\partial D_j) \leq 2l_\sigma(X) \leq 2A.$$

This bounds the length of any segment of  $\alpha^*$  traversing  $D_j$ , and the number of such traversals is bounded by  $i(\alpha, X)$ . Thus,  $l_\sigma(\alpha) \leq 2Ai(\alpha, X)$ .

On the other hand, each  $\xi_k$  has an embedded regular neighborhood (also known as a collar) of radius at least some constant  $\delta(A)$ , by an application of the thick–thin decomposition (see [16]). It follows immediately that  $l_\sigma(\alpha) \geq \delta(A)i(\alpha, \xi_k)$ . Since there must be some  $\xi_m$  such that  $i(\alpha, \xi_m) \geq i(\alpha, X)/n$ , we have proved the second half of (4.1).

Now consider any two points  $[\sigma], [\rho] \in T(X, A)$ . Because  $X$  is binding,  $i(\alpha, X) > 0$  for any non-trivial  $\alpha$ . Thus, By (4.1), we have

$$\frac{C_1}{C_2} \leq \frac{l_\sigma(\alpha)}{l_\rho(\alpha)} \leq \frac{C_2}{C_1}.$$

By Lemma 2.1 we obtain a corresponding bound on  $E_\sigma(\alpha)/E_\rho(\alpha)$ , and by Theorem 2.2 we get a bound on  $d_T(\sigma, \rho)$ . Note that to use Lemma 2.1 we need to know a uniform lower bound on the injectivity radii of  $\sigma$  and  $\rho$ , but this follows immediately from (4.1).

*Remark.* To properly appreciate the subtlety of Theorem 4.3, let us briefly describe an example, due to Kerckhoff and Thurston [18], in which the lack of a global injectivity radius bound in  $\mathbf{H}^3/\Gamma$  permits the existence of a sequence of groups  $\Gamma_i^2$  whose algebraic limits and geometric limits are different.

Denote by  $QF(\sigma, \tau)$  a quasi-Fuchsian group uniformizing the Riemann surfaces  $(S, \sigma)$  and  $(S, \tau)$  (this group is determined up to conjugation in  $\mathrm{PSL}_2(\mathbf{C})$ ; see e.g. [2]). Let  $\gamma$  be a simple closed curve in  $S$ , and let  $D_\gamma$  denote the action on  $\mathcal{T}(S)$  of a Dehn twist around  $\gamma$ . Fixing  $\sigma$ , let  $\Gamma^n = QF(\sigma, D_\gamma^n(\sigma))$ . This sequence of groups converges algebraically (after an appropriate choice of conjugations) to an isomorphic group  $\Gamma^a$  in which the element corresponding to  $\gamma$  is parabolic. The geometric limit  $\Gamma^g$ , however, contains an additional parabolic element which, together with  $\gamma$ , generates the fundamental group of a torus cusp. In fact  $\mathbf{H}^3/\Gamma^g$  is homeomorphic to  $(S \times \mathbf{R}) \setminus (\gamma \times \{0\})$ .

It is also true that the hyperbolic structures on the boundaries of the convex hull of  $\mathbf{H}^3/\Gamma^n$  remain boundedly near  $\sigma$  and  $D_\gamma^n(\sigma)$ , respectively, by a theorem of Sullivan (see Epstein–Marden [11]). In particular the injectivity radius on the boundaries of the convex hull remains bounded below. Thus, letting  $\Gamma_2^n = \Gamma^n$  and taking  $\mathbf{H}^2/\Gamma_1^n$  isometric to one boundary component of the convex hull of  $\mathbf{H}^3/\Gamma^n$ , we can obtain a sequence in  $\mathcal{PST}(S, \varepsilon)$ , which converges to a configuration in which  $\Gamma_1$  and  $\Gamma_2$  are no longer isomorphic. We leave it to the reader to consider how Corollaries 4.5 and 4.6 fail in this context.

## 5. CONTROLLING PLEATED SURFACES

Before proceeding with the main constructions of the paper, we require some technical tools that relate the placement of pleated surfaces in a 3-manifold to quantities such as the length and intersection numbers of their pleating laminations.

LEMMA 5.1. *Let  $(S, \sigma)$  be a closed Riemann surface and  $\lambda, \mu \in \mathcal{ML}(S)$ . Then*

$$E_\sigma(\lambda)E_\sigma(\mu) \geq i(\lambda, \mu)^2.$$

This is an elementary consequence of the definitions, and we note in passing that Gardiner and Masur [13] have proved a stronger related fact, namely that this inequality is actually an equality if and only if  $\lambda$  and  $\mu$  appear as the horizontal and vertical foliations of a  $\sigma$ -holomorphic quadratic differential; that is, if  $\sigma$  lies on the Teichmüller geodesic determined by  $\lambda$  and  $\mu$ .

*Proof of 5.1.* Consider first the case that  $\lambda$  and  $\mu$  are closed curves, and let  $h$  be the extremal metric for  $\lambda$  (normalized to have area 1). That is,  $h$  is Euclidean except for a finite number of singularities, and admits a foliation whose non-singular leaves are closed Euclidean geodesics homotopic to  $\lambda$ . These leaves form a flat cylinder  $C$ , of height  $E_\sigma(\lambda)^{-1/2}$

and circumference  $E_\sigma(\lambda)^{1/2}$  (see Section 2.1). Any curve homotopic to  $\mu$  must traverse this annulus at least  $i(\lambda, \mu)$  times, and thus its length is at least  $E_\sigma(\lambda)^{-1/2}i(\lambda, \mu)$ . The definition of extremal length now gives

$$E_\sigma(\mu) \geq l_h^2(\mu) \geq \frac{i(\lambda, \mu)^2}{E_\sigma(\lambda)}.$$

This proves the lemma for  $\lambda$  and  $\mu$ , and it follows for any two elements of  $\mathcal{ML}(S)$  by continuity. □

Fixing a hyperbolic manifold  $N$  and a homotopy class  $[f: S \rightarrow N]$ , we may define the following weighted intersection number:

$$I_N(\lambda, \mu) = \frac{i(\lambda, \mu)}{l_N(\lambda)l_N(\mu)}$$

which projects to a continuous function on  $\mathcal{PM}\mathcal{L}(S) \times \mathcal{PM}\mathcal{L}(S)$  (if we allow  $\infty$  as a value, when one of the laminations is unrealizable). Similarly if  $\rho$  is a hyperbolic metric on  $S$  itself we can define  $I_\rho$  (this is really a special case where  $f$  is the identity).

The next lemma is our most crucial technical tool, and will be used repeatedly in the proof of Theorem A. It shows that measured laminations that are realized far apart in a three-manifold must have some definite amount of intersection as curves in  $S$ .

**LEMMA 5.2.** (Distance implies intersection) *Let  $g: (S, \rho) \rightarrow N$  be a pleated surface which induces an isomorphism on  $\pi_1$ , and suppose that  $\text{inj}_N(x) > \varepsilon_0$  at every point  $x$  of  $N$ . Suppose that  $g$  maps  $\lambda \in \mathcal{ML}(S)$  geodesically.*

*There exist constants  $D_1, c_1 > 0$  depending only on  $\chi(S)$  and  $\varepsilon_0$  such that for any  $\mu \in \mathcal{ML}(S)$ ,*

$$\begin{aligned} d_N(\lambda, \mu) \geq D_1 &\Rightarrow I_\rho(\lambda, \mu) \geq c_1 \\ &\Rightarrow I_N(\lambda, \mu) \geq c_1. \end{aligned}$$

Here  $d_N$  denotes the distance in  $N$  between the realizations of  $\lambda$  and  $\mu$  in  $[g]$  (or  $\infty$  if  $\mu$  is unrealizable).

*Proof of 5.2.* Note first that the last implication follows immediately from the fact that  $l_\rho \geq l_N$ .

Let  $D(S, \varepsilon_0)$  again be the universal bound on the diameter of a hyperbolic surface with injectivity radii bounded below by  $\varepsilon_0$ .

Suppose the theorem is false for  $D_1 = D + 1$ . Then there is a sequence of examples  $g_i: (S, \rho_i) \rightarrow N_i$  mapping  $\lambda_i$  geodesically, and  $\mu_i$  such that  $I_{\rho_i}(\lambda_i, \mu_i) \rightarrow 0$  while  $d_{N_i}(\lambda_i, \mu_i) \geq D_1$ .

Applying Corollary 4.4 (Compactness for surface groups), we may restrict to a subsequence that converges in  $\mathcal{PS}\mathcal{F}(S, \varepsilon_0)$ , obtaining a limit pleated surface  $g: (S, \rho) \rightarrow N$ , with  $(1 + \delta_i)$ -bilipschitz maps  $h_i: (S, \rho_i) \rightarrow (S, \rho)$ ,  $f_i: \mathcal{N}_i \rightarrow N$  (where  $\mathcal{N}_i$  are large neighborhoods of  $g_i(S)$  in  $N_i$ ), such that the lifts to the universal covers of  $f_i \circ g_i$  approximate those of  $g \circ h_i$  on large compact sets.

Using the compactness of  $\mathcal{PM}\mathcal{L}(S)$ , we may restrict to a further subsequence so that  $[h_i(\lambda_i)]$  and  $[h_i(\mu_i)]$  converge to  $[\lambda]$  and  $[\mu]$ , respectively, in  $\mathcal{PM}\mathcal{L}(S)$ . Then  $g$  maps  $\lambda$  geodesically (see [7], for example), and  $d_N(\lambda, \mu) \geq D_1$ .

By continuity of the intersection number and length on a surface (see §2.1),  $I_\rho(\lambda, \mu) = 0$ . Therefore  $\mu$  and  $\lambda$  are disjoint, or have common components, so that  $\lambda \cup \mu$  is a lamination. Since  $\lambda$  is realizable,  $\lambda \cup \mu$  is also realizable, by part 3 of Proposition 2.4.

But now we have a contradiction: if  $h:(S, \rho') \rightarrow N$  is a pleated surface homotopic to  $f$  which maps  $\lambda \cup \mu$  geodesically, then  $\text{diam}(h(S)) \leq D$ , whereas  $d_N(\lambda, \mu) > D$ .  $\square$

Now consider the opposite case, when two laminations are realized nearby. In this case the length of each lamination in  $N$  is approximated well by its length in either pleated surface:

LEMMA 5.3. *Suppose  $g:(S, \rho) \rightarrow N$  is a pleated surface inducing an isomorphism on  $\pi_1$ , and  $\text{inj}_N(x) > \varepsilon_0$  for all  $x \in N$ . Given  $D > 0$  there exists  $c$ , depending on  $D, \chi(S)$  and  $\varepsilon_0$ , such that*

$$d_N(\lambda, \mu) \leq D \Rightarrow l_\rho(\mu) \leq cl_N(\mu).$$

*Proof of 5.3.* This is a consequence of Corollary 4.6, which gives a constant  $b$  such that  $d_T(\rho, \sigma) \leq b$ . This bounds the ratio  $l_\rho(\gamma)/l_\sigma(\gamma)$  by a constant  $c(b, \varepsilon_0)$  for all  $\gamma \in \mathcal{ML}(S)$  (by Theorems 2.2 and 2.1). But for  $\mu$  we have  $l_\sigma(\mu) = l_N(\mu)$ , since  $\mu$  is mapped geodesically by  $h:(S, \sigma) \rightarrow N$ . This produces the desired inequality.  $\square$

6. A QUASI-SELF-MAP OF  $\mathcal{T}(S)$

Fix now a complete hyperbolic 3-manifold  $N$  with a bound on injectivity radii  $\text{inj}_N(x) \geq \varepsilon_0 > 0$  for all  $x \in N$ . Fix a closed surface  $S$  and a homotopy class of maps  $[f:S \rightarrow N]$ , injective on the level of fundamental groups.

Secretly, what we want is a map  $\Pi : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$  which assigns to each  $\sigma$  the conformal structure of the pullback metric  $f_\sigma^*h$ , where  $h$  is the hyperbolic metric on  $N$ . The translation distance  $d_T(\sigma, \Pi(\sigma))$  of this map should be directly related to  $\mathcal{E}(\sigma)$ . However, this involves delicate technical issues (for example if  $f_\sigma$  fails to be an immersion) that for our purposes do more harm than good. It turns out, in fact, that a certain amount of indefiniteness in the definition of  $\Pi$  is natural to our results and techniques.

We will define a map

$$\Pi : \mathcal{T}(S) \rightarrow \mathcal{B}_B(\mathcal{T}(S)),$$

where  $\mathcal{B}_B(\mathcal{T}(S))$  is the set of subsets of  $\mathcal{T}(S)$  whose diameter is bounded by a certain  $B > 0$ . The intention is that the pullback metric  $f_\sigma^*h$  determines a point in  $\mathcal{T}(S)$  that lies near  $\Pi(\sigma)$ . However, we will not need to prove this.

Let  $P(\sigma, D)$  denote the set of pleated surfaces  $(g, \rho)$  such that  $g \sim f$  and  $d_N(g(S), f_\sigma(S)) \leq D$ . By Lemma 3.4, there exists  $D_0(\chi(S), \varepsilon_0)$  such that  $P(\sigma, D_0)$  is non-empty, for all  $\sigma$ . Define

$$\Pi(\sigma) = \{[\rho] : \text{there is a pleated surface } (g, \rho) \in P(\sigma, D_0)\}.$$

By Lemma 3.1 these pleated surfaces all lie in a subset of  $N$  of uniformly bounded diameter, and thus by Corollary 4.6,  $\Pi(\sigma)$  has uniformly bounded diameter in  $\mathcal{T}(S)$ . Let us state this as:

LEMMA 6.1. *There is a constant  $B_0 > 0$  depending on  $\chi(S)$  and  $\varepsilon_0$  such that*

$$\text{diam}(\Pi(\sigma)) \leq B_0$$

for any  $[\sigma] \in \mathcal{T}(S)$ .

The first interesting property of  $\Pi$  is the following:

PROPOSITION 6.2. For any  $[\sigma] \in \mathcal{T}(S)$ ,

$$d(\sigma, \Pi(\sigma)) - c_2 \leq \frac{1}{2} \log \mathcal{E}(\sigma) \leq d(\sigma, \Pi(\sigma)) + c_2$$

where  $c_2$  depends only on  $\chi(S)$  and  $\varepsilon_0$ .

*Proof of 6.2.* Let  $[\rho]$  be any point in  $\Pi(\sigma)$ , and  $(g, \rho)$  a pleated surface in  $P(\sigma, D_0)$ . If  $h: (S, \sigma) \rightarrow (S, \rho)$  is the harmonic map isotopic to the identity and  $\mathcal{E}(h)$  is its energy, then by Lemma 3.6,

$$d(\sigma, \rho) - c \leq \frac{1}{2} \log \mathcal{E}(h) \leq d(\sigma, \rho) + c$$

for some  $c(\chi(S), \varepsilon_0)$ .

We now wish to prove an estimate of the form

$$\frac{1}{c'} \mathcal{E}(h) - d \leq \mathcal{E}(\sigma) \leq \mathcal{E}(h). \tag{6.1}$$

This suffices to prove the lemma, because  $\Pi(\sigma)$  has bounded diameter, and because there is a uniform positive lower bound on  $\mathcal{E}(\sigma)$  (which bounds how negative the logarithm can be). This lower bound arises from the lower bound on injectivity radii in  $N$ , together with the first inequality in Theorem 3.2 applied to a short curve in  $(S, \sigma)$ .

One direction of (6.1) is easy. Since the pleated map  $g$  is, in particular, 1-Lipschitz with respect to the metric  $\rho$ , the map  $g \circ h$  gives an upper bound  $\mathcal{E}(\sigma) \leq \mathcal{E}(h)$ . (The technicality of  $g \circ h$  not being differentiable is treated, for example, in [28], where it is sufficient to have  $L^2$  distributional derivatives. Alternatively one could approximate  $g$  by a smooth map with derivatives of norm nearly 1).

To prove the bound in the other direction, consider first the case where  $\mathcal{E}(\sigma) \geq \mathcal{E}_1$ , where  $\mathcal{E}_1$  is a constant to be named later.

Since  $(g, \rho)$  was an arbitrary member of  $P(\sigma, D_0)$ , if we assume that  $\mathcal{E}_1$  is at least as large as the constant  $\mathcal{E}_0$  in Theorem 3.3, we may choose  $g$  to be a pleated surface which maps  $\mu$  geodesically, where  $\mu = \mu_\sigma$  is the maximal-stretch lamination for  $f_\sigma$ .

Let  $h$  be as before, and let  $\mu'$  denote the maximal-stretch lamination for  $h$  (the theorems in [24] hold for two dimensional targets as a special case—see also [23]).

We first show that  $\mu'$  and  $\mu$  have small intersection number relative to their lengths in  $(S, \rho)$ . Lemma 5.1 gives

$$E_\sigma(\mu)E_\sigma(\mu') \geq i(\mu, \mu')^2.$$

Dividing by the squares of the  $\rho$ -lengths,

$$\frac{E_\sigma(\mu)}{l_\rho^2(\mu)} \frac{E_\sigma(\mu')}{l_\rho^2(\mu')} \geq I_\rho(\mu, \mu')^2.$$

Noting that  $l_\rho(\mu) = l_N(\mu)$  and applying Theorem 3.2 to both  $\mu$  and  $\mu'$ , we obtain

$$I_\rho(\mu, \mu')^2 \leq \frac{1}{4(\mathcal{E}(\sigma) - C)(\mathcal{E}(h) - C)}$$

for sufficiently high  $\mathcal{E}(\sigma)$ . Note, by the easy direction of (6.1), that  $\mathcal{E}(h)$  is then high as well. Thus we may choose  $\mathcal{E}_1$  such that  $I_\rho(\mu, \mu')$  is no larger than  $c_1$ , the constant in Lemma 5.2 (Distance implies intersection). It then follows that

$$d_N(\mu, \mu') \leq D_1$$

where the distance is between the geodesic realizations of the two laminations. Lemma 5.3



then tells us that

$$l_\rho(\mu') \leq cl_N(\mu')$$

where  $c$  depends on  $\chi(S)$ ,  $\varepsilon_0$  and  $D_1$ . Another invocation of Theorem 3.2 tells us that

$$\mathcal{E}(h) \leq \frac{1}{2} \frac{l_\rho^2(\mu')}{E_\sigma(\mu')} + C \leq \frac{1}{2} c^2 \frac{l_N^2(\mu')}{E_\sigma(\mu')} + C \leq c^2 \mathcal{E}(\sigma) + C.$$

There remains the case where  $\mathcal{E}(\sigma) \leq \mathcal{E}_1$ . Let  $(g, \rho)$  again denote an arbitrary member of  $P(\sigma, D_0)$ . In this case it suffices to bound  $\mathcal{E}(h)$  by any fixed  $\mathcal{E}_2$ . This follows from the compactness results of Section 4, by an argument similar to that of Corollary 4.6, as follows.

In view of Lemma 3.6 relating energy of surface maps to Teichmüller distance, to bound  $\mathcal{E}(h)$  it suffices to bound  $d_T(\sigma, \rho)$ . Consider a binding collection of curves  $\{\xi_i\}$  in  $S$ , as in the proof of Lemma 4.6, such that  $l_\sigma(\xi_i) \leq a = a(\varepsilon_0)$ . It follows that  $E_\sigma(\xi_i) \leq a'(a, \varepsilon_0)$  by Lemma 2.1, and therefore the left side of Theorem 3.2 implies that

$$l_\sigma^2(\xi_i) \leq 2a' \mathcal{E}_1.$$

We now apply Lemma 4.5 to argue that there exists a uniform  $B$  such that  $l_\rho(\xi_i) \leq B$ . It follows, by Lemma 4.7, that  $d_T(\rho, \sigma)$  is uniformly bounded, and we are done.  $\square$

The next lemma shows that the harmonic map associated to a metric induced by a pleated surface is of low energy, and located near the pleated surface.

LEMMA 6.3. *There is a constant  $D_2$  depending on  $\chi(S)$  and  $\varepsilon_0$  such that, if  $g : (S, \rho) \rightarrow N$  is a pleated surface homotopic to  $f$ , then*

$$\mathcal{E}(\rho) \leq 2\pi|\chi(S)|$$

and

$$d_N(g(S), f_\rho(S)) \leq D_2.$$

*Proof of 6.3.* The first inequality is immediate, as in the proof of Proposition 6.2, from the existence of the 1-Lipschitz map  $g$ .

It therefore follows from Proposition 6.2 that, for some  $\tau \in \Pi(\rho)$ ,  $d_T(\rho, \tau) \leq \frac{1}{2} \log 2\pi|\chi(S)| + c_2$ . Let  $h : (S, \tau) \rightarrow N$  be a pleated surface in  $[f]$ , such that  $d_N(h(S), f_\rho(S)) \leq D_0$ . It remains to bound  $d_N(g(S), h(S))$  using the bound on  $d_T(\rho, \tau)$ . This is essentially a converse to Lemma 4.6, and can be shown using the following simple lemma:

LEMMA 6.4. *Given  $A$  there exists  $B(A, \varepsilon_0)$  such that, if  $\alpha$  is a homotopically non-trivial curve in  $N$  with length  $l_N(\alpha) \leq A$ , then  $d_N(\alpha, \alpha^*) \leq B$ , where  $\alpha^*$  is the geodesic representative of  $\alpha$ .*

*Proof of 6.4.* Suppose  $d(\alpha, \alpha^*) > R$ . Lift  $\alpha^*$  to a geodesic  $\tilde{\alpha}^*$  in  $\mathbf{H}^3$ , and consider the corresponding lift  $\tilde{\alpha}$  of  $\alpha$ , which lies outside an  $R$ -neighborhood of  $\tilde{\alpha}^*$ . Since the closest-point projection to a geodesic in  $\mathbf{H}^3$  is contracting by a factor of at least  $\cosh R$  outside this neighborhood, we may conclude that  $l_N(\alpha) \geq \cosh R l_N(\alpha^*)$ . By the injectivity radius assumption on  $N$ , and the hypothesis on  $l_N(\alpha)$ , we obtain  $A \geq 2\varepsilon_0 \cosh R$ , which bounds  $R$  as needed.  $\square$

Now let  $\gamma$  be the non-trivial homotopy class in  $S$  with minimal  $\sigma$ -length. Then  $l_\rho(\gamma)$  and  $l_\tau(\gamma)$  are bounded by some fixed  $L(\chi(S), \varepsilon_0, d_T(\rho, \tau))$ . Let  $\gamma_*$  be the common geodesic representative in  $N$  of  $g(\gamma)$  and  $h(\gamma)$ . Applying Lemma 6.4 to both  $g(\gamma)$  and  $h(\gamma)$ , we obtain a bound on  $d_N(g(S), h(S))$ .  $\square$

Our goal is to show that  $\Pi$  is, in a certain approximate sense, like a closest-point projection to a convex set in  $\mathcal{F}(S)$ . Usually a projection fixes the points in its image, but in our quasi-situation we can only hope for an approximate version of this. For any positive number  $a$ , define

$$\mathcal{L}_a = \{[\sigma] \in \mathcal{F}(S) : d(\sigma, \Pi(\sigma)) \leq a\}.$$

Then the points in  $\mathcal{L}_a$  are “nearly fixed”, and the next proposition shows that  $\Pi$  has the approximate properties of a closest-point projection to  $\mathcal{L}_a$ , for appropriate  $a$ .

**PROPOSITION 6.5. (Quasi-projection)** *There exist constants  $a_0, a_1, a_2$  depending on  $\chi(S)$  and  $\varepsilon_0$ , such that*

1.  $\Pi(\sigma) \in \mathcal{L}_{a_0}$  for any  $[\sigma] \in \mathcal{F}(S)$ .
2. For any  $[\sigma], [\tau] \in \mathcal{F}(S)$ ,

$$d(\sigma, \Pi(\sigma)) \leq d(\sigma, \tau) + d(\tau, \Pi(\tau)) + a_1.$$

3. For  $[\sigma] \in \mathcal{F}(S)$ ,

$$d(\sigma, \Pi(\sigma)) - a_2 \leq d(\sigma, \mathcal{L}_{a_0}) \leq d(\sigma, \Pi(\sigma))$$

*Proof of 6.5.* Part (1) follows from Lemma 6.3, applied to any  $\rho \in \Pi(\sigma)$ , and from Proposition 6.2. In particular if  $a_0 = \frac{1}{2} \log 2\pi |\chi(S)| + c_2$  then  $d(\rho, \Pi(\rho)) \leq a_0$  for any  $\rho \in \Pi(\sigma)$ .

To prove part (2), consider  $[\rho] \in \Pi(\tau)$ . Then  $\rho$  represents a metric on a pleated surface in  $N$ , so the energy to map  $(S, \sigma)$  to  $(S, \rho)$  gives an upper bound for  $\mathcal{E}(\sigma)$ , as in the proof of Proposition 6.2. Applying that proposition we have

$$d(\sigma, \Pi(\sigma)) \leq d(\sigma, \rho) + c_2.$$

The rest is just the triangle inequality together with the diameter bound on  $\Pi(\tau)$ .

The right side of part (3) follows directly from part (1). To obtain the left side, apply part (2) to any  $[\tau] \in \mathcal{L}_{a_0}$ . □

### 7. SLOW HARMONIC MAPS

We are now ready to demonstrate the central property of the map  $\Pi$ —it is distance-decreasing, on the average, far away from  $\mathcal{L}_{a_0}$ .

Denote by  $\mathcal{N}_r(X)$  an  $r$ -neighborhood in the Teichmüller metric of a subset or point  $X$  in  $\mathcal{F}(S)$ . We shall also abuse notation slightly by writing  $\Pi(X)$  to mean  $\bigcup_{x \in X} \Pi(x)$ , for  $X \subset \mathcal{F}(S)$ .

**THEOREM 7.1. (High energy maps move slowly)** *There are constants,  $B_1, B_2$  depending on  $\chi(S)$  and  $\varepsilon_0$  such that:*

*For  $[\sigma] \in \mathcal{F}(S)$  and  $r > 0$ , if*

$$d(\mathcal{N}_r(\sigma), \mathcal{L}_{a_0}) > B_1$$

*then*

$$\text{diam}(\Pi(\mathcal{N}_r(\sigma))) \leq B_2.$$

(Recall that  $d(X, Y)$  is the *shortest* distance between points in  $X$  and  $Y$ .)

An interpretation of this is that “in the large”, the average speed of the harmonic map  $f_\sigma$  (and equivalently of  $\Pi$ ) goes to zero when  $\mathcal{E}(\sigma)$  goes to infinity—in fact, the average speed at  $\sigma$  is inversely proportional to  $d(\sigma, \mathcal{L}_{a_0}) \sim d(\sigma, \Pi(\sigma)) \sim \log \mathcal{E}(\sigma)$ .

*Proof of 7.1.* Let  $\sigma_1 = \sigma$ , and take any  $\sigma_2 \in \mathcal{N}_r(\sigma)$ . As the proof progresses we shall indicate how to choose the constants  $B_i$ .

Let  $\mu_i$  be the maximal-stretch lamination of  $f_{\sigma_i}$  (for  $i = 1, 2$ ). If  $C, \mathcal{E}_0$  are the constants in Theorem 3.2 and Theorem 3.3, let  $\mathcal{E}_1 = \max(\mathcal{E}_0, 2C)$ . Then, when  $\mathcal{E}(\sigma_i) \geq \mathcal{E}_1$ , Theorem 3.2 gives

$$\frac{1}{2} \frac{l_N^2(\lambda)}{E_{\sigma_i}(\lambda)} \leq \mathcal{E}(\sigma_i) \leq \frac{l_N^2(\mu_i)}{E_{\sigma_i}(\mu_i)} \quad (7.1)$$

for any  $\lambda \in \mathcal{ML}(S)$ .

Let us then require that  $B_1 \geq \frac{1}{2} \log \mathcal{E}_1 + c_2$ , where  $c_2$  is the constant in Proposition 6.2. Applying Proposition 6.2 and Proposition 6.5 (Quasi-projection) (part 3), we may conclude

$$\mathcal{E}(\sigma_i) \geq \mathcal{E}_1.$$

To show that  $\Pi(\sigma_1)$  and  $\Pi(\sigma_2)$  are close, we shall first show that

$$d_N(f_{\sigma_1}(S), f_{\sigma_2}(S)) \leq D_1 + 2(\eta + D), \quad (7.2)$$

where  $D_1$  is the constant in Lemma 5.2 (Distance implies intersection).

If  $\eta$  is the constant of Theorem 3.3, and  $D$  is the universal bound on the diameter of a pleated surface with injectivity radii at least  $\varepsilon_0$ , we have

$$d_N(\mu_1, \mu_2) \geq d_N(f_{\sigma_1}(S), f_{\sigma_2}(S)) - 2(\eta + D)$$

where the left-hand side is distance between the geodesic representatives in  $N$ .

Now suppose, by contradiction, that (7.2) fails to hold. Then  $d_N(\mu_1, \mu_2) \geq D_1$  and it follows by Lemma 5.2 that

$$I_N(\mu_1, \mu_2) \geq c_1.$$

By Lemma 5.1,

$$E_{\sigma_1}(\mu_1)E_{\sigma_1}(\mu_2) \geq c_1^2 l_N^2(\mu_1)l_N^2(\mu_2).$$

Now, note that  $d(\sigma_1, \sigma_2) \leq r$  implies, via Theorem 2.2, that

$$\frac{E_{\sigma_1}(\mu_2)}{E_{\sigma_2}(\mu_2)} \leq e^{2r}, \quad (7.3)$$

so

$$\frac{l_N^2(\mu_1)}{E_{\sigma_1}(\mu_1)} \frac{l_N^2(\mu_2)}{E_{\sigma_2}(\mu_2)} \leq \frac{e^{2r}}{c_1^2}$$

or, via (7.1)

$$\mathcal{E}(\sigma_1)\mathcal{E}(\sigma_2) \leq \frac{e^{2r}}{c_1^2}. \quad (7.4)$$

Using (7.1) again, we have

$$\mathcal{E}(\sigma_2) \geq \frac{1}{2} \frac{l_N^2(\mu_1)}{E_{\sigma_2}(\mu_1)} \geq \frac{1}{2} \frac{l_N^2(\mu_1)}{e^{2r} E_{\sigma_1}(\mu_1)} \geq \frac{1}{2e^{2r}} \mathcal{E}(\sigma_1)$$

This implies

$$\mathcal{E}(\sigma_1) \leq \frac{\sqrt{2}}{c_1} e^{2r}$$

or, combining with Proposition 6.5 (part 3) and Proposition 6.2,

$$\begin{aligned} d(\sigma_1, \mathcal{L}_{a_0}) &\leq \frac{1}{2} \log \mathcal{E}(\sigma_1) + c_2 \\ &\leq r + \frac{1}{2} \log \frac{\sqrt{2}}{c_1} + c_2. \end{aligned}$$

Thus, let us choose  $B_1 = \max(\frac{1}{2} \log \frac{\sqrt{2}}{c_1}, \frac{1}{2} \log \mathcal{E}_1) + c_2$ . Then, since the hypothesis of the theorem and the triangle inequality imply

$$d(\sigma_1, \mathcal{L}_{a_0}) \geq r + B_1,$$

we obtain a contradiction at this point.

Therefore, we must conclude that (7.2) holds so that the images of  $f_{\sigma_1}$  and  $f_{\sigma_2}$  are uniformly nearby in  $N$ . Now applying Corollary 4.6, it follows that there is a fixed  $B$  such that

$$\text{diam}(\Pi(\sigma_1) \cup \Pi(\sigma_2)) \leq B.$$

Setting  $B_2 = 2B$ , we have a bound on  $\text{diam}(\Pi(\mathcal{N}_r(\sigma_1)))$ . □

### 8. QUASI-PROJECTIONS AND THREE-MANIFOLDS

In this section we will prove our main theorem. The main step is to complete our understanding of the map  $\Pi$  by showing that the set  $\mathcal{L}_{a_0}$  where  $\mathcal{E}$  is bounded is “quasi-convex”.

As before, assume a fixed homotopy class  $[f: S \rightarrow N]$  into a hyperbolic 3-manifold  $N$  with injectivity radius bounded from below by  $\varepsilon_0$ .

**THEOREM 8.1.** (Quasi-convexity) *For each  $r > 0$  there exists  $r' > 0$  such that if  $[\sigma], [\tau] \in \mathcal{N}_r(\mathcal{L}_{a_0})$  then the Teichmüller geodesic  $\overline{\sigma\tau}$  connecting them lies in  $\mathcal{N}_{r'}(\mathcal{L}_{a_0})$ .*

*Note.* By Proposition 6.5,

$$\mathcal{L}_{a_0+r} \subseteq \mathcal{N}_r(\mathcal{L}_{a_0}) \subseteq \mathcal{L}_{a_0+a_2+r},$$

so we may conclude equivalently that  $[\sigma], [\tau] \in \mathcal{L}_{a_0+r}$  implies  $\overline{\sigma\tau} \subseteq \mathcal{L}_{a_0+a_2+r}$ .

*Proof of 8.1.* The proof is a standard argument in synthetic geometry, in which Theorem 7.1 plays the central role. The main idea is that a path venturing far away from  $\mathcal{L}_{a_0}$  is inefficient since its image under  $\Pi$  is much shorter. Let us quantify this with the following corollary of Theorem 7.1:

**LEMMA 8.2.** *Let  $L \subset \mathcal{F}(S)$  be a geodesic segment lying outside  $\mathcal{N}_b(\mathcal{L}_{a_0})$ , where  $b \geq B_1 + kB_2, k > 1$ . Then, if  $x$  and  $y$  are the endpoints of  $L$ ,*

$$\text{diam}(\Pi(x) \cup \Pi(y)) \leq \frac{1}{2k} d(x, y) + B_2.$$

*Proof of 8.2.* Partition  $L$  by points  $x = x_1, \dots, x_n, x_{n+1} = y$  such that  $d(x_i, x_{i+1}) = kB_2$  for  $i < n$  and  $d(x_n, y) \leq kB_2$ . Thus  $d(x, y) = (n - 1 + \delta)kB_2$ , for  $0 < \delta \leq 1$ .

Now,  $d(x_i, \mathcal{L}_{a_0}) \geq b$ , so  $d(\mathcal{N}_{kB_2}(x_i), \mathcal{L}_{a_0}) \geq b - kB_2 \geq B_1$ . Since  $x_{i-1}, x_{i+1} \in \mathcal{N}_{kB_2}(x_i)$  for  $1 < i \leq n$ , we may apply Theorem 7.1 to conclude that

$$\text{diam}(\Pi(x_{i+1}) \cup \Pi(x_{i-1})) \leq B_2.$$

Applying the triangle inequality we easily obtain

$$\text{diam}(\Pi(x) \cup \Pi(y)) \leq \frac{n+1}{2} B_2.$$

Since  $n = d(x, y)/kB_2 + 1 - \delta$ , we get

$$\text{diam}(\Pi(x) \cup \Pi(y)) \leq \frac{1}{2k} d(x, y) + \left(1 - \frac{\delta}{2}\right) B_2$$

from which the statement follows. □

We assume now, with no loss of generality, that  $r \geq B_1 + kB_2$ . Let  $[\sigma], [\tau]$  be as in the statement of the theorem, and suppose that  $d(\rho, \mathcal{L}_{a_0}) > r$  for some  $[\rho] \in \overline{\sigma\tau}$ . Then, since  $d(\cdot, \mathcal{L}_{a_0})$  is a continuous function,  $\rho$  must be contained in a segment  $\overline{xy} \subseteq \overline{\sigma\tau}$  such that  $d(x, \mathcal{L}_{a_0}) = d(y, \mathcal{L}_{a_0}) = r$  and  $d(\cdot, \mathcal{L}_{a_0}) > r$  on  $\overline{xy}$ .

The triangle inequality, Lemma 8.2 and Lemma 6.5 (part 3) give:

$$\begin{aligned} d(x, y) &\leq d(x, \Pi(x)) + \text{diam}(\Pi(x) \cup \Pi(y)) + d(\Pi(y), y) \\ &\leq \frac{1}{2k} d(x, y) + B_2 + 2(r + a_2). \end{aligned}$$

Setting  $k = 1$ , say, we have

$$d(x, y) \leq 2B_2 + 4(r + a_2).$$

Now, since  $\rho$  lies somewhere in the geodesic segment  $\overline{xy}$ , we have

$$\begin{aligned} d(\rho, \mathcal{L}_{a_0}) &\leq \frac{1}{2} d(x, y) + r \\ &\leq B_2 + 3r + 2a_2. \end{aligned}$$

Since  $\rho$  was any point in  $\overline{\sigma\tau}$ , we can set  $r'$  equal to this bound, completing the proof. □

We conclude with a restatement, and proof, of the main theorem. Define, after fixing  $[f: S \rightarrow N]$ ,

$$\mathcal{P} = \{[\rho] \in \mathcal{T}(S) : \text{there is a pleated surface } g: (S, \rho) \rightarrow N \text{ in } [f]\}$$

and

$$\mathcal{H}_K = \{[\sigma] \in \mathcal{T}(S) : \mathcal{E}(\sigma) \leq K\}.$$

**THEOREM A.** *Let  $[f: S \rightarrow N]$  be a  $\pi_1$ -injective homotopy class of maps of a closed surface  $S$  of genus  $g > 2$  to a hyperbolic 3-manifold  $N$  with  $\text{inj}_N(x) \geq \varepsilon_0 > 0$  for all  $x \in N$ .*

*Then there is a Teichmüller geodesic  $L$  in  $\mathcal{T}(S)$  and constants  $A, B$  depending only on  $\chi(S)$  and  $\varepsilon_0$ , such that*

1.  $L \subset \mathcal{H}_A$ ,
2.  $\mathcal{P} \subset \mathcal{N}_B(L)$ .

*Furthermore, let  $\hat{N}$  be the cover of  $N$  for which  $\pi_1 \hat{N} = f_*(\pi_1 S)$ . Then the following cases arise:*

- (a) *If  $\hat{N}$  is geometrically finite,  $L$  is a finite segment.*
- (b) *If  $\hat{N}$  has one simply degenerate end, then  $L$  is a ray.*
- (c) *If  $\hat{N}$  has two simply degenerate ends,  $L$  is an infinite geodesic.*

*Proof of Theorem A.* Note that Proposition 6.2 implies

$$\mathcal{H}_{\exp 2(a-c_2)} \subset \mathcal{L}_a \subset \mathcal{H}_{\exp 2(a+c_2)},$$

and that Proposition 6.5 implies

$$\mathcal{P} \subset \mathcal{L}_{a_0},$$

so these sets are all essentially equivalent. In particular, to obtain (1) we will find  $L$  such that  $L \subset \mathcal{L}_a$  for some  $a$ .

We may as well assume, taking a lift of  $f$  if necessary, that  $\hat{N} = N$ . Let  $e$  be an end of  $N$ . If  $e$  is simply degenerate, then (see Section 2.4), there is a sequence  $\{(g_i, \rho_i)\}$  of pleated surfaces in  $[f]$  such that  $g_i(S)$  is eventually contained in any neighborhood of  $e$ . If  $e$  is geometrically finite, let  $(g_i, \rho_i)$  denote the pleated surface determined by the boundary of the convex hull that corresponds to  $e$  (so there is no dependence on  $i$ ). Do the same for the other end  $e'$ , obtaining a sequence  $\{(g'_i, \rho'_i)\}$ .

As in Proposition 6.5,  $\rho_i, \rho'_i \in \mathcal{L}_{a_0}$  for all  $i$ . Thus the Teichmüller geodesic segments  $L_i = \overline{\rho_i \rho'_i}$  are contained in  $\mathcal{L}_{a_3}$  for a fixed  $a_3$ , by Theorem 8.1 (Quasi-convexity). If  $N$  is in case (a) the  $L_i$  are all equal and we take  $L = L_i$  for any  $i$ . If not, we have to find a convergent subsequence. This is again easy in case (b): the  $L_i$  share a common endpoint, so a subsequence must converge to a ray  $L$ , by Lemma 2.3. In case (c) we must show that all the  $L_i$  intersect a fixed compact set, and again apply Lemma 2.3.

Let  $(g, \rho)$  denote any fixed pleated surface in  $[f]$ , and let  $y \in g(S)$  be any point on its image. We will show that, for sufficiently large  $i$  there must be some  $\sigma_i \in L_i$  such that  $y \in f_{\sigma_i}(S)$ .

Fix an identification of  $N$  with  $S \times \mathbf{R}$  such that  $y \in S \times \{0\}$ . We claim that, for sufficiently large  $i$ ,  $f_{\rho_i}(S)$  is contained in  $S \times [1, \infty)$ , and  $f_{\rho'_i}(S)$  is contained in  $S \times (-\infty, -1]$ . This is because  $g_i(S)$  is eventually in  $S \times [K, \infty)$  for any  $K > 0$ , and by Lemma 6.3  $f_{\rho_i}(S)$  is a uniformly bounded distance from  $g_i(S)$  (similarly for  $\rho'_i$ ).

Let  $[\sigma_i(t)]$ ,  $t \in [0, 1]$  parametrize  $L_i$ , and define  $F: S \times [0, 1] \rightarrow S \times \mathbf{R}$  by  $F(x, t) = f_{\sigma_i(t)}(x)$ . Let  $\pi: S \times \mathbf{R} \rightarrow S \times [-1, 1]$  be the retraction defined by  $\pi(x, s) = (x, s)$  for  $|s| \leq 1$  and  $\pi(x, s) = (x, \text{sgn}(s))$  for  $|s| \geq 1$ . Then  $\pi \circ F: S \times [0, 1] \rightarrow S \times [-1, 1]$  is a proper homotopy equivalence, and its restriction to the boundary is a homotopy equivalence on each component, since the boundaries of  $S \times [0, 1]$  are incompressible. It follows that  $\pi \circ F$  has degree 1 on the boundary, and therefore degree 1 on the interior. In particular  $y$  lies in its image, and must therefore be in the image of  $F$ .

In other words, for some  $[\sigma_i] \in L_i$  we have  $f_{\sigma_i}(S) \cap g(S) \neq \emptyset$ .

By definition,  $[\rho] \in \Pi(\sigma_i)$ , and since  $d(\sigma_i, \Pi(\sigma_i)) \leq a_3 + a_2$  (Proposition 6.5) we conclude that  $L_i \cap \mathcal{N}_{a_1+B_0}(\rho)$  is non-empty. By Lemma 2.3 a subsequence of  $\{L_i\}$  must therefore converge to an infinite geodesic  $L$ , which is contained in  $\mathcal{L}_{a_1}$ , proving conclusion (1).

Fixing this convergent subsequence of  $L_i$  we can repeat the same argument, replacing  $\rho$  by any other  $\tau \in \mathcal{P}$ , and concluding that  $\mathcal{N}_{a_1+B_0}(\tau) \cap L_i \neq \emptyset$  for sufficiently large  $i$ . Thus,

$$\mathcal{P} \subset \mathcal{N}_{a_1+B_0}(L),$$

giving conclusion (2). Note that this works in cases (a) and (b) as well, where the boundary conditions for the map  $F$  are automatically satisfied on the geometrically finite ends.  $\square$

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