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Quantile Constructions for Khinchin's and Pruitt's Theorems and for Doeblin's Universal Laws

SÁNDOR CSÖRGŐ*

Department of Statistics, University of Michigan, Ann Arbor, Michigan

AND

VILMOS TOTIK†

Bolyai Institute, University of Szeged, Szeged, Hungary, and Department of Mathematics, University of South Florida, Tampa, Florida

1. INTRODUCTION

According to the classical result of P. Lévy and A. Ya. Khinchin, a real random variable V, or its distribution function $G(x) = P\{V \le x\}$, $x \in \mathbb{R}$, is infinitely divisible if and only if its characteristic function $\varphi(t) := E(e^{itV}) = \int_{-\infty}^{\infty} e^{itx}\,dG(x)$, $t \in \mathbb{R}$, is of the form

$$\varphi(t) = \exp\left(it\theta - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0}\left\{ e^{itx} - 1 - \frac{itx}{1+x^2}\right\} dL(x) + \int_{0}^{\infty}\left\{ e^{itx} - 1 - \frac{itx}{1+x^2}\right\} dR(x)\right),$$

where $\theta \in \mathbb{R}$ and $\sigma \ge 0$ are uniquely determined constants and $L(\cdot)$ and $R(\cdot)$ are uniquely determined left-continuous and right-continuous non-decreasing functions defined on $(-\infty, 0)$ and $(0, \infty)$, respectively, such that $L(-\infty) = 0 = R(\infty)$ and

$$\int_{-\varepsilon}^{0} x^2\,dL(x) + \int_{0}^{\varepsilon} x^2\,dR(x) < \infty \qquad \text{for all } \varepsilon > 0.$$

Actually, this is Lévy's formula (cf. [6, p. 84]).

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Let Ψ be the set of all non-positive, non-decreasing, right-continuous functions ψ defined on $(0,\infty)$ such that $\int_\varepsilon^\infty \psi^2(s)\,ds < \infty$ for every $\varepsilon > 0$ and introduce

$$\psi_L(s) := \inf\{x < 0 \colon L(x) > s\}, \qquad s > 0,$$

and

$$\psi_R(s) := \inf\{x < 0 \colon -R(-x) > s\} \qquad s > 0,$$

where the infimum of the empty set is taken to be zero. Since $L(\cdot)$ and $\psi_L(\cdot)$ uniquely determine each other and the same is true for $R(\cdot)$ and $\psi_R(\cdot)$, the set $\mathscr{I} := \{(\psi_L, \psi_R, \sigma) \colon \psi_L, \psi_R \in \Psi,\ \sigma \geq 0\}$ of triples describes the class of (infinitely divisible) distributions of all infinitely divisible random variables $V - \theta$ modulo the additive constant $\theta \in \mathbb{R}$. The motivation for introducing the ψ functions is the following. For any $\psi \in \Psi$ and for two independent standard (intensity one) left-continuous Poisson processes $N_1(s)$, $N_2(s)$, $s \geq 0$, introduce the independent random variables

$$W_j(\psi) := \int_1^\infty \left(N_j(s) - s\right) d\psi(s) + \int_0^1 N_j(s)\,d\psi(s) + \Theta(\psi), \qquad j = 1,2,$$

where $\int_a^b := \int_{(a,b]}$ for any $0 \leq a < b < \infty$ and

$$\Theta(\psi) := -\psi(1) + \int_0^1 \frac{\psi(s)}{1+\psi^2(s)}\,ds - \int_1^\infty \frac{\psi^3(s)}{1+\psi^2(s)}\,ds,$$

and consider the random variable

$$V(\psi_L, \psi_R, \sigma) := -W_1(\psi_L) + \sigma Z + W_2(\psi_R),$$

where Z is a standard normal random variable such that $N_1(\cdot)$, Z, and $N_2(\cdot)$ are also independent. (The integrals of the centered Poisson processes on $[1,\infty)$ in the definition of $W_j(\psi)$ exist almost surely as improper Riemann–Stieltjes integrals by the square-integrability condition on $\psi(\cdot)$.) Then by Theorem 3 in [2] (cf. also pp. 291–292 in [3]), $V(\psi_L, \psi_R, \sigma) + \theta \overset{\mathscr{D}}{=} V$, i.e., the random variable $V(\psi_L, \psi_R, \sigma) + \theta$ has the distribution of V above, or, analytically, the characteristic function of $V(\psi_L, \psi_R, \sigma) + \theta$ is $\varphi(\cdot)$ given above.

The above stochastic representation of an arbitrary infinitely divisible random variable V was obtained in the framework of a new unified approach to what is one of the most classical problems of probability theory, the problem of the asymptotic distribution of sums of independent and identically distributed random variables and of lightly trimmed variants of such sums. This "probabilistic approach," presented in [2, 3], is based upon the asymptotic behavior of the uniform empirical distribution function in conjunction with the tail properties of the underlying quantile function.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with a common non-degenerate distribution function $F(x) = P\{X_1 \leq x\}$, $x \in \mathbb{R}$, and quantile function

$$Q(u) = Q_F(u) := \inf\{x\colon F(x) \geq u\}, \qquad 0 < u < 1.$$

(We see that any non-decreasing left-continuous function on the interval $(0, 1)$ is a quantile function of some distribution on the line $\mathbb{R}$.) Suppose that there exist a subsequence $\{n_k\}_{k=1}^{\infty}$ of the sequence $\mathbb{N} = \{1, 2, \ldots\}$ of positive integers, diverging to infinity as $k \to \infty$, a sequence $\{A_k\}_{k=1}^{\infty}$ of positive constants and two, necessarily non-decreasing and right-continuous functions, $\psi_L(\cdot)$ and $\psi_R(\cdot)$ defined on $(0, \infty)$ such that

$$\lim_{k\to\infty} \frac{Q(x/n_k +)}{A_k} = \psi_L(x), \qquad \lim_{k\to\infty} \frac{-Q(1 - y/n_k)}{A_k} = \psi_R(y) \quad (1.1)$$

at each $x > 0$ which is a continuity point of $\psi_L(\cdot)$ and at each $y > 0$ which is a continuity point of $\psi_R(\cdot)$, where for a monotone function $f(\cdot)$ we use the notation $f(t+) = \lim_{s\downarrow t} f(s)$ and $f(t-) = \lim_{s\uparrow t} f(s)$. Assume further that for the same sequences $\{n_k\}_{k=1}^{\infty}$ and $\{A_k\}_{k=1}^{\infty}$ as in (1.1) and for some number $\sigma \geq 0$ we also have

$$\lim_{h\to\infty} \liminf_{k\to\infty} \frac{\sqrt{n_k}\,\sigma(h/n_k)}{A_k} = \sigma = \lim_{h\to\infty} \limsup_{k\to\infty} \frac{\sqrt{n_k}\,\sigma(h/n_k)}{A_k}, \quad (1.2)$$

where for $0 < s < \frac{1}{2}$ and $u \wedge v = \min(u, v)$,

$$\begin{aligned}\sigma^2(s) &:= \int_s^{1-s}\int_s^{1-s} (u \wedge v - uv)\, dQ(u)\, dQ(v) \\ &= s\{Q^s(s) + Q^2(1-s)\} + \int_s^{1-s} Q^2(u)\, du \\ &\quad - \left(s\{Q(s) + Q(1-s)\} + \int_s^{1-s} Q(u)\, du\right)^2. \qquad (1.3)\end{aligned}$$

The last equation is well known and can be proved by a standard, if somewhat lengthy, integration procedure. Since the finite or infinite limit $\lim_{s\downarrow 0}\sigma(s)$ exists (it is finite if and only if $E(X_1^2) < \infty$), condition (1.2) implies that $\lim_{k\to\infty} A_k = \infty$. Hence $\psi_L(\cdot)$ and $\psi_R(\cdot)$ in (1.1) are necessarily non-positive and it also follows from Lemma 2.5 in [2] that they are square-integrable on any half-line $[\varepsilon, \infty)$, $\varepsilon > 0$. Thus $\psi_L, \psi_R \in \Psi$, if conditions (1.1) and (1.2) hold, and then Theorem 1* in [3] and Theorem 2 in [2] imply that for some constants C_k,

$$\frac{1}{A_k}\left\{\sum_{j=1}^{n_k} X_j - C_k\right\} \xrightarrow{\mathscr{D}} V(\psi_L, \psi_R, \sigma) + \theta \qquad \text{as } k \to \infty, \tag{1.4}$$

where $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution. In fact, for all k large enough, C_k can be taken as

$$C_k = n_k \int_{1/n_k}^{1-1/n_k} Q(u)\, du - \theta A_k.$$

According to classical terminology, this says that under (1.1) and (1.2) the distribution determined by F or Q is in the domain of partial attraction of the type of infinitely divisible distributions pertaining to the triple $(\psi_L, \psi_R, \sigma) \in \mathscr{I}$. In fact, by Theorem 7 in [3] conditions (1.1) and (1.2) are also necessary for (1.4).

Furthermore, Theorem 5 in [2] implies that if for some n_k, $A_k > 0$, and C_k, $k \in \mathbb{N}$, the left side of (1.4) converges in distribution as $k \to \infty$ to some limiting variable W, then W must be equal in distribution to $V(\psi_L, \psi_R, \sigma) + \theta$ for some $(\psi_L, \psi_R, \sigma) \in \mathscr{I}$ and $\theta \in \mathbb{R}$; that is, the limit must be infinitely divisible. This was first proved by Bawly [1] and Khinchin [7]. "The incomparably deeper converse proposition," as Gnedenko and Kolmogorov [6, p. 184] describe it, is provided by

KHINCHIN'S THEOREM. *Every infinitely divisible distribution has a non-empty domain of partial attraction.*

Khinchin's original proof is based on the characteristic function method and is reproduced in [6, pp. 184–186]. Pruitt [8] has given another construction for the proof of Khinchin's theorem that produces an F in the desired arbitrary domain by using the central convergence criterion in [6, p. 116]. One of the aims of the present note is to give a rather explicit inductive construction of a quantile function Q and sequences $\{n_k\}$ and $\{A_k\}$ such that (1.1) and (1.2) are satisfied for any previously given $(\psi_L, \psi_R, \sigma) \in \mathscr{I}$. This is done in the next section, proving Khinchin's theorem.

Khinchin's theorem is in fact a special case of a famous observation of Doeblin [4], that even the intersection of the domains of partial attraction of all infinitely divisible distributions is non-empty.

DOEBLIN'S UNIVERSAL LAWS. *There exist distributions that are in the domain of partial attraction of every infinitely divisible distribution.*

Doeblin's original construction was also given in terms of characteristic functions. In Section 4 we modify the basic construction of the next section in a straightforward fashion to give a universal quantile function Q in still quite an explicit form. In fact, a whole family of universal quantile functions is constructed.

Feller [5] has introduced the class $\mathscr{F}$ of all stochastically compact distributions: $F \in \mathscr{F}$ by definition if there exist $A_n > 0$ and $C_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that every subsequence of $(\sum_{j=1}^{n} X_j - C_n)/A_n$ has a further subsequence which converges in distribution to a non-degenerate (infinitely divisible) limit. One of the three quantile equivalents of his beautiful analytic characterization of $\mathscr{F}$ (cf. Corollary 10 and Section 5 in [2]) is this: $F \in \mathscr{F}$ if and only if for the corresponding $Q = Q_F$ we have

$$\limsup_{s \downarrow 0} \frac{s\{Q^s(s) + Q^2(1-s)\}}{\int_s^{1-s} Q^2(u)\,du} < \infty. \tag{1.5}$$

Pruitt [8] has given an equally beautiful characterization of the class of all possible subsequential limiting laws in (1.4) when F is restricted to the Feller class $\mathscr{F}$. The following is the sufficiency half of the quantile version of Pruitt's theorem.

PRUITT'S THEOREM. *Suppose that for $(\psi_L, \psi_R, \sigma) \in \mathscr{J}$ there is a constant $C > 0$ such that*

$$s\{\psi_L^2(s) + \psi_R^2(s)\} \le C\left(\sigma^2 + \int_s^\infty \{\psi_L^2(t) + \psi_R^2(t)\}\,dt\right), \qquad 0 < s < \infty. \tag{1.6}$$

Then the domain of partial attraction of (ψ_L, ψ_R, σ) contains a stochastically compact distribution.

When put together with the last statement of Corollary 10 in [2] (note also the small correction in the proof pointed out on page 301 in [3]), this result provides an extension of the quantile variant of Pruitt's [8] full theorem for possibly lightly trimmed sums. Pruitt himself proved his original sufficiency statement by checking that his construction for Khinchin's theorem is automatically in $\mathscr{F}$ when his original version of (1.6) is satisfied. In general, we have to modify our simple quantile construction for Khinchin's theorem to obtain a new construction in which (1.6) implies (1.5). This is done in Section 3. The difference between the original and the slightly modified version is quite instructive.

The constructional problems solved in the present note were mentioned on page 328 in [2] and the paper itself has been promised in [3]. So our aim here is to fully round out the quantile theory of, or probabilistic approach to, the asymptotic distribution of sums of independent and identically distributed random variables as presented in those two papers.

2. Construction for Khinchin's Theorem

Consider any triple $(\psi_L, \psi_R, \sigma) \in \mathscr{S}$. We have to construct a non-decreasing and left-continuous function, i.e., a quantile function Q on $(0,1)$, a subsequence $\{n_k\}_{k=2}^{\infty} \subset \mathbb{N}$ and constants $A_k > 0$, $k = 2, 3, \ldots$, to satisfy (1.1) and (1.2). The Q to be constructed will be of the form

$$Q(u) := \begin{cases} Q_L(u) + Q_\sigma(u), & 0 < u \le \frac{1}{2}, \\ Q_R(u) + Q_\sigma(u), & \frac{1}{2} < u < 1, \end{cases} \tag{2.1}$$

where Q_L and Q_R do not depend on $\sigma \ge 0$ and

$$Q_\sigma(u) := \begin{cases} -\dfrac{\sigma}{\sqrt{2}} \dfrac{1}{\sqrt{u}}, & 0 < u \le \frac{1}{2}, \\ \dfrac{\sigma}{\sqrt{2}} \dfrac{1}{\sqrt{1-u}}, & \frac{1}{2} < u < 1. \end{cases} \tag{2.2}$$

As an initial step in the construction of Q_L and Q_R, set $a_2 := \frac{1}{8}$, $b_2 := \frac{1}{2}$ and $n_2 := 4$, so that $n_2 a_2 = \frac{1}{2}$ and $n_2 b_2 = 2$, put $A_2 := 2\sqrt{\log 4}$, and define

$$Q_L(u) := A_2\{\psi_L(n_2 u -) - \tfrac{1}{2}\} \qquad \text{if } u \in \left[a_2, \tfrac{1}{2}\right]$$

and (2.3)

$$Q_R(u) := A_2\{-\psi_R(n_2 - n_2 u) + \tfrac{1}{2}\} \qquad \text{if } u \in \left(\tfrac{1}{2}, 1 - a_2\right].$$

Then Q is well defined, i.e., it is non-decreasing and left-continuous on the interval $[a_2, 1 - a_2] = [\frac{1}{8}, \frac{7}{8}]$.

Suppose now that for some $k \ge 3$, the function Q is already defined on the interval $[a_{k-1}, 1 - a_{k-1}]$, where $0 < a_{k-1} < \frac{1}{2}$, and that $n_2 < \cdots < n_{k-1}$ and $A_2 < \cdots < A_{k-1}$ have also been defined. Our task will be to choose $n_k > n_{k-1}$ large enough to make the following definitions proper:

Let $A_k > A_{k-1}$ and $0 < a_k < b_k < a_{k-1}$ be defined by

$$A_k := \sqrt{n_k \log n_k}, \qquad n_k a_k = 1/k, \qquad n_k b_k = k, \tag{2.4}$$

and set

$$Q_L(u) := \begin{cases} A_k\left\{\psi_L(n_k u -) - \dfrac{1}{k}\right\}, & u \in [a_k, b_k], \\ A_{k-1}\left\{\psi_L(n_{k-1}a_{k-1} -) - \dfrac{1}{k-1}\right\}, & u \in (b_k, a_{k-1}], \end{cases}$$

and (2.5)

$$Q_R(u) := \begin{cases} A_{k-1}\left\{-\psi_R(n_{k-1}a_{k-1}) + \dfrac{1}{k-1}\right\}, & u \in [1 - a_{k-1}, 1 - b_k], \\ A_k\left\{-\psi_R(n_k - n_k u) + \dfrac{1}{k}\right\}, & u \in (1 - b_k, 1 - a_k]. \end{cases}$$

Of course, if $n_k > n_{k-1}$ then $A_k > A_{k-1}$ and $0 < a_k < b_k$ in any case. If we choose n_k large enough so that $n_k/n_{k-1} > (k-1)k$, then $b_k < a_{k-1}$. Furthermore, by the definition of A_{k-1} and A_k, we can choose n_k large enough to make

$$\frac{A_k}{A_{k-1}} \geq \frac{-\psi_L\left(\dfrac{1}{k-1} -\right) + \dfrac{1}{k-1}}{-\psi_L(k -) + \dfrac{1}{k}},$$

$$\frac{A_k}{A_{k-1}} \geq \frac{-\psi_R\left(\dfrac{1}{k-1}\right) + \dfrac{1}{k-1}}{-\psi_R(k) + \dfrac{1}{k}}.$$

This implies by (2.4) that $Q_L(u) \leq Q_L(b_k +)$ for all the points $u \in [a_k, b_k]$ and $Q_R(1 - b_k) \leq Q_R(u)$ for all $u \in (1 - b_k, 1 - a_k]$. Hence, with such a choice of n_k and by (2.1), Q is a non-decreasing and left-continuous function defined on $[a_k, 1 - a_k]$. Letting $k \to \infty$, our Q will now be inductively defined on the whole interval $(0, 1)$ as a proper quantile function. Note that Q is continuous at a_k and $1 - a_k$ for all $k = 2, 3, \ldots$.

Furthermore, for every $x \in (n_k a_k, n_k b_k) = 1/k, k)$, $k = 2, 3, \dots,$ we have

$$\frac{Q\left(\frac{x}{n_k}+\right)}{A_k} = \psi_L(x) - \frac{1}{k} - \frac{\sigma}{\sqrt{2x}} \frac{1}{\sqrt{\log n_k}}$$

and (2.6)

$$\frac{-Q\left(1-\frac{x}{n_k}\right)}{A_k} = \psi_R(x) - \frac{1}{k} - \frac{\sigma}{\sqrt{2x}} \frac{1}{\sqrt{\log n_k}}.$$

Hence (1.1) follows trivially at all continuity points $x > 0$ and $y > 0$ of ψ_L and ψ_R, respectively.

We will now impose further growth conditions on the sequence $\{n_k\}$ to also satisfy (1.2). These conditions will be asymptotic in nature, arising from the behavior of $\sqrt{n_k}\sigma(h/n_k)/A_k = \sigma(h/n_k)/\sqrt{\log n_k}$. For a sequence of positive numbers w_k, the symbol $o_h(w_k)$ will denote a sequence of functions of $h > 0$, indexed by $k = 2, 3, \dots,$ such that $\lim_{n\to\infty} \lim_{k\to\infty} o_h(w_k)/w_k = 0$. The asymptotic behavior of $\sigma^2(h/n_k)$ will be determined by investigating the terms on the right side of (1.3) after substituting $s = h/n_k$.

First note that since $\psi_L, \psi_R \in \Psi$, we have

$$|\psi_L(h-)| \le \left(\frac{2}{h}\int_{h/2}^{h} \psi_L^2(s)\, ds\right)^{1/2} = o\left(\frac{1}{\sqrt{h}}\right),$$

and similarly, $|\psi_R(h)| = o(1/\sqrt{h})$ as $h \to \infty$. Since for every fixed $h > 0$ and all k large enough, $a_k n_k = 1/k \le h < b_k n_k = k$, so

$$Q\left(\frac{h}{n_k}\right) = A_k\left\{\psi_L(h-) - \frac{1}{k}\right\} - \frac{\sigma}{\sqrt{2}}\sqrt{\frac{n_k}{h}}$$

and

$$Q\left(1-\frac{h}{n_k}\right) = A_k\left\{-\psi_R(h) + \frac{1}{k}\right\} + \frac{\sigma}{\sqrt{2}}\sqrt{\frac{n_k}{h}};$$

by (2.4) we have

$$\frac{h}{n_k}\left\{Q^2\left(\frac{h}{n_k}\right)+Q^2\left(1-\frac{h}{n_k}\right)\right\}$$
$$=\mathcal{O}\left(\frac{h}{n_k}o\left(\frac{1}{h}\right)n_k\log n_k+\frac{h}{n_k}o\left(\frac{1}{\sqrt{h}}\right)\sqrt{n_k\log n_k}\right).$$

Hence

$$\frac{h}{n_k}\left\{Q^2\left(\frac{h}{n_k}\right)+Q^2\left(1-\frac{h}{n_k}\right)\right\}=o_h(\log n_k) \tag{2.7}$$

is automatic. Similarly,

$$\frac{h}{n_k}\left\{Q\left(\frac{h}{n_k}\right)+Q\left(1-\frac{h}{n_k}\right)\right\}=\mathcal{O}\left(\frac{h}{n_k}\sqrt{n_k\log n_k}\,o\left(\frac{1}{\sqrt{h}}\right)+\frac{h}{n_k}\frac{\sqrt{n_k}}{\sqrt{h}}\right);$$

hence

$$\frac{h}{n_k}\left\{Q\left(\frac{h}{n_k}\right)+Q\left(1-\frac{h}{n_k}\right)\right\}=o_h\left(\sqrt{\log n_k}\right) \tag{2.8}$$

is also automatic.

On the other hand, for all fixed $h>0$ and k large enough, using (2.4), (2.5), (2.1), and (2.2),

$$\int_{h/n_k}^{1-h/n_k}|Q(u)|\,du=\int_{h/n_k}^{b_k}|Q(u)|\,du+\int_{b_k}^{1-b_k}|Q(u)|\,du$$
$$+\int_{1-b_k}^{1-h/n_k}|Q(u)|\,du$$
$$=\mathcal{O}(A_kb_k\{|\psi_L(h-)|+|\psi_R(h)|\})$$
$$+\mathcal{O}\left(A_{k-1}\left\{\left|\psi_L\left(\frac{1}{k-1}-\right)\right|+\left|\psi_R\left(\frac{1}{k-1}\right)\right|\right\}\right)$$
$$=\mathcal{O}\left(\frac{k\sqrt{\log n_k}}{\sqrt{n_k}}\{|\psi_L(h-)|+|\psi_R(h)|\}\right)$$
$$+\mathcal{O}\left(A_{k-1}\left\{\left|\psi_L\left(\frac{1}{k-1}-\right)\right|+\left|\psi_R\left(\frac{1}{k-1}\right)\right|\right\}\right),$$

since the contribution of $Q_\sigma(\cdot)$ is only

$$\int_{h/n_k}^{1-h/n_k} |Q_\sigma(u)|\, du = \sqrt{2}\,\sigma \int_{h/n_k}^{1/2} \frac{1}{\sqrt{u}}\, du \le \sqrt{2}\,\sigma \int_0^{1/2} \frac{1}{\sqrt{u}}\, du = 2\sigma.$$

Thus

$$\int_{h/n_k}^{1-h/n_k} |Q(u)|\, du = o_h\left(\sqrt{\log n_k}\right), \tag{2.9}$$

provided that the sequence $\{n_k\}$ grows fast enough, depending on how fast $\psi_L(s-)$ and $\psi_R(s)$ go to $-\infty$, if at all, as $s \downarrow 0$. Putting this together with (2.8), we see that

$$\left(\frac{h}{n_k}\left\{Q\left(\frac{h}{n_k}\right) + Q\left(1 - \frac{h}{n_k}\right)\right\} + \int_{h/n_k}^{1-h/n_k} Q(u)\, du\right)^2 = o_h(\log n_k) \tag{2.10}$$

if $\{n_k\}$ is chosen to grow fast enough.

Finally, to analyze the behavior of $\int_{h/n_k}^{1-h/n_k} Q^2(u)\, du$, we first look at

$$\int_{h/n_k}^{1/2} Q^2(u)\, du = \int_{h/n_k}^{b_k} Q^2(u)\, du + \int_{b_k}^{b_k^{1/k}} Q^2(u)\, du + \int_{b_k^{1/k}}^{1/2} Q^2(u)\, du,$$

where for any fixed $h > 0$, k is taken large enough to make $h/n_k < \frac{1}{2}$. By (2.4) and (2.5) we have

$$\begin{aligned}\int_{h/n_k}^{b_k} Q_L^2(u)\, du &= (\log n_k) \int_h^k \left(\psi_L(s) - \frac{1}{k}\right)^2 ds \\ &= (\log n_k)\, \mathcal{O}\left(\int_h^k \psi_L^2(s)\, ds + \frac{1}{k}\right) = o_h(\log n_k)\end{aligned}$$

automatically since $\psi_L \in \Psi$. Also, from (2.2) and (2.4),

$$\int_{h/n_k}^{b_k} Q_\sigma^2(u)\, du = \frac{\sigma^2}{2} \log \frac{b_k n_k}{h} = \frac{\sigma^2}{2} \log \frac{k}{h} = o_h(\log n_k)$$

if $\{n_k\}$ grows fast enough. By (2.1) the last two relations imply that

$$\int_{h/n_k}^{b_k} Q^2(u)\, du = o_h(\log n_k)$$

if $\{n_k\}$ grows fast enough, so the first term will not count.

Next, using (2.4) and (2.5) again,

$$\int_{b_k}^{1/2} Q_L^2(u)\,du = \mathscr{O}\left(A_{k-1}^2 \psi_L^2\left(\frac{1}{k-1}-\right)\right) = o(\log n_k)$$

by the choice of $\{n_k\}$ for (2.9), and thus we also have

$$\int_{b_k^{1/k}}^{1/2} Q_L^2(u)\,du = o(\log n_k),$$

and by (2.2) and (2.4) again,

$$\int_{b_k^{1/k}}^{1/2} Q_\sigma^2(u)\,du = \frac{\sigma^2}{2}\log\frac{1}{2b_k^{1/k}} < \frac{\sigma^2}{2k}\log\frac{n_k}{k} = o(\log n_k).$$

Hence

$$\int_{b_k^{1/k}}^{1/2} Q^2(u)\,du = o(\log n_k)$$

if $\{n_k\}$ is fast enough, so the third term will not count either.

As to the middle term, by (2.2) and (2.4),

$$\int_{b_k}^{b_k^{1/k}} Q_\sigma^2(u)\,du = \frac{\sigma^2}{2}\log\frac{b_k^{1/k}}{b_k} = \frac{\sigma^2}{2}\left(1-\frac{1}{k}\right)\log\frac{n_k}{k}$$
$$= \frac{\sigma^2}{2}\log n_k + o(\log n_k).$$

On the other hand,

$$\inf_{b_k \le u \le b_k^{1/k}} |Q_\sigma(u)| = \frac{\sigma}{\sqrt{2}}\frac{1}{\sqrt{b_k^{1/k}}} = \frac{\sigma}{\sqrt{2}}\left(\frac{n_k}{k}\right)^{1/(2k)}$$

while

$$\sup_{b_k \le u \le b_k^{1/k}} |Q_L(u)| = \mathscr{O}\left(A_{k-1}\left|\psi_L\left(\frac{1}{k-1}-\right)\right|\right).$$

Hence we can choose $\{n_k\}$ to grow fast enough to ensure that

$$\frac{1}{k}|Q_\sigma(u)| > |Q_L(u)| \qquad \text{for all } u \in [b_k, b_k^{1/k}].$$

With such a choice,

$$\int_{b_k}^{b_k^{1/2}} Q^2(u)\,du = (1+o(1))\int_{b_k}^{b_k^{1/k}} Q_\sigma^2(u)\,du = \frac{\sigma^2}{2}\log n_k + o(\log n_k).$$

Collecting the three estimates,

$$\int_{h/n_k}^{1/2} Q^2(u)\,du = \frac{\sigma^2}{2}\log n_k + o_h(\log n_k)$$

if $\{n_k\}$ is chosen to grow fast enough. Since the construction of Q on the intervals $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$ is completely symmetric, except that we use ψ_L and ψ_R on these intervals, respectively, we see that the sequence $\{n_k\}$ can be chosen so fast, depending on the behavior of ψ_R, that we also have

$$\int_{1/2}^{1-h/n_k} Q^2(u)\,du = \frac{\sigma^2}{2}\log n_k + o_h(\log n_k).$$

Hence, if $\{n_k\}$ satisfies both criteria, then

$$\int_{h/n_k}^{1-h/n_k} Q^2(u)\,du = \sigma^2 \log n_k + o_h(\log n_k). \tag{2.11}$$

The last estimate along with (2.7) and (2.10), when substituted into (1.3), now gives

$$\sigma^2\left(\frac{h}{n_k}\right) = \sigma^2 \log n_k + o_k(\log n_k) \tag{2.12}$$

if $\{n_k\}$ grows fast enough. Therefore, by the definition of A_k in (2.4), the second requirement in (1.2) also follows.

3. Construction for Pruitt's Theorem

Let $(\psi_L, \psi_R, \sigma) \in \mathscr{I}$ be given such that $(\psi_L, \psi_R, \sigma) \neq (0,0,0)$ and (1.6) holds for some $C > 0$. We need to construct a quantile function Q on $(0,1)$, a subsequence $\{n_k\} \subset \mathbb{N}$ and constants $A_k > 0$, $k \in \mathbb{N}$, such that along with (1.1) and (1.2) we also have (1.5).

It will be clear from the first case below that our construction for Khinchin's theorem in Section 2 actually works for Pruitt's theorem; i.e., for the constructed Q we have (1.5) along with (1.1) and (1.2) if (1.6) holds, whenever $\sigma > 0$, that is, when the infinitely divisible limit in Pruitt's class has a non-degenerate normal component. However, when $\sigma = 0$ the validity of (1.5) is generally violated even under (1.6) in this construction because (setting $h = 1$, say) the ratio of the two $o(\log n_k)$ terms in (2.7) and (2.11) may go to infinity. We now introduce a simple general modification of the basic construction that will make the $o_h(\log n_k)$ term in (2.11) "large enough."

Let $\gamma(u)$, $0 < u \le \frac{1}{2}$, be an arbitrary continuous, non-negative function such that $\lim_{u \downarrow 0} \gamma(u) = 0$ and that the function $\gamma(u)/\sqrt{u}$ is non-increasing on $(0, \frac{1}{2}]$. The modification consists in redefining $Q_\sigma(\cdot)$ in (2.2) as

$$Q_\sigma(u) := \begin{cases} -\dfrac{\sigma}{\sqrt{2}} \dfrac{1}{\sqrt{u}} - \dfrac{\gamma(u)}{\sqrt{u}}, & 0 < u \le \frac{1}{2}, \\ \dfrac{\sigma}{\sqrt{2}} \dfrac{1}{\sqrt{1-u}} + \dfrac{\gamma(1-u)}{\sqrt{1-u}}, & \frac{1}{2} < u < 1. \end{cases} \tag{3.1}$$

With this new Q_σ we define Q in (2.1) as before, where Q_L and Q_R remain defined by (2.3)–(2.5) exactly as before. The inductive choice of $\{n_k\}$ goes through word for word just as well, with trivial adjustments concerning the new term in Q_σ, so we have (1.1) and (1.2) for any $\gamma(\cdot)$ as described above.

To prove (1.5) it clearly suffices to show that there exist a constant $C_0 > 0$ and an $s_0 \in (0, \frac{1}{2})$ such that

$$\max\left(sQ_\sigma^2(s), sQ_\sigma^2(1-s)\right) \le C_0\left(\sigma^2 + \int_s^{1-s} Q^2(u)\,du\right), \qquad 0 < s < s_0,$$

and

$$s\left\{Q_L^2(s) + Q_R^2(1-s)\right\} \le C_0\left(\sigma^2 + \int_s^{1-s} Q^2(u)\,du\right), \qquad 0 < s < s_0. \tag{3.2}$$

Since

$$sQ_\sigma^2(s) = sQ_\sigma^2(1-s) = \left(\frac{\sigma}{\sqrt{2}} + \gamma(s)\right)^2 \le C_\sigma, \qquad 0 < s < \tfrac{1}{2},$$

for some constant $C_\sigma > 0$, the first of these two requirements is trivially satisfied for all $\sigma \ge 0$. Hence we have to work only for (3.2). In order to establish (3.2) for our construction it is enough to consider the case when $s \in (a_k, b_k]$ for some $k = 2, 3, \ldots,$ because if $s \in (b_{k+1}, a_k]$ for some $k = 2, 3, \ldots,$ then $sQ_L^2(s) \le a_k Q_L^2(a_k)$ and $sQ_R^2(1-s) \le a_k Q_R^2(1-a_k)$, following from (2.5). However, we now separate the two cases when $\sigma > 0$ and $\sigma = 0$.

Case $\sigma > 0$. In this case we claim that (3.2) and hence (1.5) hold for any admissible function $\gamma(\cdot)$ in (3.1); in particular they hold when $\gamma(\cdot)$ is the zero function as in the original construction.

In order to see this, suppose that for some $k = 2, 3, \dots,$

$$s \in (a_k, b_k] = \left(\frac{1}{kn_k}, \frac{1}{n_k}\right) \cup \left[\frac{1}{n_k}, \frac{k}{n_k}\right].$$

If $s \in [1/n_k, k/n_k]$, then $s = x/n_k$ for some $x \in [1, k]$ and hence by (2.4) and (2.5), with equality for $s < b_k = k/n_k$ and a possible inequality for $Q_R^2(1 - b_k)$ when $s = b_k$ in the first line,

$$\begin{aligned} s\{Q_L^2(s) + Q_R^2(1-s)\} &\le \frac{x}{n_k}(n_k \log n_k)\left\{\left(\psi_L(x-) - \frac{1}{k}\right)^2 \right. \\ &\qquad \left. + \left(-\psi_R(x) + \frac{1}{k}\right)^2\right\} \\ &\le 2x\left\{\psi_L^2(x-) + \psi_R^2(x) + \frac{2}{k^2}\right\}\log n_k. \quad (3.3) \end{aligned}$$

Since $x\psi_L^2(x-) \to 0$ and $x\psi_R^2(x) \to 0$ as $x \to \infty$, we obtain

$$\begin{aligned} s\{Q_L^2(s) + Q_R^2(1-s)\} &\le 2\left\{x\psi_L^2(x-) + x\psi_R^2(x) + \frac{2}{k}\right\}\log n_k \\ &\le K \log n_k \end{aligned}$$

for some constant $K > 0$. But in the derivation of (2.11) we have actually chosen $\{n_k\}$ to grow fast enough to ensure the asymptotic equality

$$\int_{b_k}^{1-b_k} Q^2(u)\,du \sim \sigma^2 \log n_k \qquad \text{as } k \to \infty,$$

that is, that (2.11) is also valid with the formal choice $h = k$. Hence there exists an integer $k_0 \ge 2$ such that

$$\int_s^{1-s} Q^2(u)\,du \ge \frac{\sigma^2}{2}\log n_k \qquad \text{if } s \in (a_k, b_k], \qquad k \ge k_0. \quad (3.4)$$

Putting together the two bounds, we see that (3.2) holds for some $C_0 > 0$

and $s_0 \in (0, \frac{1}{2})$. In this part, on the second interval $[1/n_k, k/n_k]$, we did not even have to use condition (1.6).

If, on the other hand, $s \in (a_k, 1/n_k)$ and hence $s = x/n_k$ for some $x \in (1/k, 1)$, then again by (2.4) and (2.5) we have (3.3) with an equality in the first line. Therefore, if $s \in (a_k, 1/n_k)$ is a continuity point of $Q_L(\cdot)$, then

$$s\{Q_L^2(s) + Q_R^2(1-s)\} \le s\{x\psi_L^2(x) + x\psi_R^2(x) + \tfrac{1}{2}\}\log n_k,$$

while by (3.4), (2.4), (2.5) and by changing variables twice,

$$\begin{aligned}
\int_s^{1-s} Q^2(u)\,du &\ge \int_{b_k}^{1-b_k} Q^2(u)\,du + \int_s^{b_k} Q_L^2(u)\,du + \int_{1-b_k}^{1-s} Q_R^2(u)\,du \\
&\ge \left\{\frac{\sigma^2}{2} + \int_x^k \left(\psi_L(u) - \frac{1}{k}\right)^2 du \right. \\
&\qquad \left. + \int_{n_k-k}^{n_k-x} \left(-\psi_R(n_k - v) + \frac{1}{k}\right)^2 dv\right\}\log n_k \\
&= \left\{\frac{\sigma^2}{2} + \int_x^k \left(\psi_L(y) - \frac{1}{k}\right)^2 dy \right. \\
&\qquad \left. + \int_x^k \left(\psi_R(y) - \frac{1}{k}\right)^2 dy\right\}\log n_k \\
&\ge \left\{\frac{\sigma^2}{2} + \int_x^k \left[\psi_L^2(y) + \psi_R^2(y)\right] dy\right\}\log n_k \\
&\ge \left\{\frac{\sigma^2}{4} + \int_x^\infty \left[\psi_L^2(y) + \psi_R^2(y)\right] dy\right\}\log n_k \\
&\ge \frac{1}{4}\left\{\sigma^2 + \int_x^\infty \left[\psi_L^2(y) + \psi_R^2(y)\right] dy\right\}\log n_k, \qquad (3.5)
\end{aligned}$$

provided that $k \ge k_1$ for some $k_1 \ge k_0$ large enough, where we used the square-integrability property of ψ_L and ψ_R. Using now the upper and lower bounds and condition (1.6) for the upper bound, we see that (3.2) is true again for some $C_0 > 0$ and $0 < s_0 < \frac{1}{2}$ for every continuity point $s \in (0, s_0)$ of Q_L. But then the left-continuity of Q_L implies the same for all $s \in (0, s_0)$.

Case $\sigma = 0$. In this case we will chose the function $\gamma(\cdot)$ in $Q_0(\cdot)$ given in (3.1) is a suitable way. In fact, we let $\gamma(\cdot)$ to be any continuous non-negative function on $(0, \frac{1}{2}]$ such that $\gamma(u)/\sqrt{u}$ is non-increasing on $(0, \frac{1}{2}]$ and $\lim_{u \downarrow 0} \gamma(u) = 0$, for which

$$\gamma(u) = \gamma_k > 0 \qquad \text{for } u \in \left[b_k, b_k^{1/k}\right], \qquad k = 2, 3, \ldots,$$

where the sequence $\{\gamma_k\}$ converges to zero as $k \to \infty$ slowly enough to satisfy the inequalities

$$\gamma_k^2 \geq \max\left(4\int_{\sqrt{k}/2}^{\infty}\left[\psi_L^2(u) + \psi_R^2(u)\right] du + \frac{4}{k}, \frac{8}{Ck^{3/2}}\right), \qquad k = 2, 3, \ldots, \tag{3.6}$$

where $C > 0$ is the constant from condition (1.6) with $\sigma = 0$. Functions $\gamma(\cdot)$ satisfying all these properties clearly exist.

Presently, for all $k = 2, 3, \ldots,$

$$\begin{aligned}\int_{b_k}^{1-b_k} Q^2(u)\, du &\geq \int_{b_k}^{b_k^{1/k}} Q_0^2(u)\, du + \int_{1-b_k^{1/k}}^{1-b_k} Q_0^2(u)\, du \\ &= 2\int_{b_k}^{b_k^{1/k}} \frac{\gamma_k^2}{u}\, du \\ &= 2\gamma_k^2\left(1 - \frac{1}{k}\right)\log\frac{n_k}{k} \\ &\geq \gamma_k^2 \log\frac{n_k}{k}.\end{aligned}$$

Since $\{n_k\}$ grows faster than $\{k\}$, there exists an integer $k_0 \geq 2$ such that

$$\int_s^{1-s} Q^2(u)\, du \geq \frac{\gamma_k^2}{2}\log n_k \qquad \text{if } s \in (a_k, b_k], \qquad k \geq k_0, \tag{3.7}$$

a lower bound that will play the role of (3.4).

Now we break $(a_k, b_k]$, $k = 2, 3, \ldots,$ in the following way:

$$(a_k, b_k] = \left(\frac{1}{kn_k}, \frac{k}{n_k}\right] = \left(\frac{1}{kn_k}, \frac{\sqrt{k}}{n_k}\right) \cup \left[\frac{\sqrt{k}}{n_k}, \frac{k}{n_k}\right].$$

If $s \in [\sqrt{k}/n_k, k/n_k]$, so that $s = x/n_k$ for some $x \in [\sqrt{k}, k]$, then for such an x we have (3.3) and hence

$$\begin{aligned} s\{Q_L^2(s) + Q_R^2(1-s)\} &\le 4\left\{\frac{x}{2}[\psi_L^2(x-) + \psi_R^2(x)] + \frac{1}{k}\right\}\log n_k \\ &\le 4\left\{\int_{x/2}^{x}[\psi_L^2(u) + \psi_R^2(u)]\,du + \frac{1}{k}\right\}\log n_k \\ &\le 4\left\{\int_{\sqrt{k}/2}^{\infty}[\psi_L^2(u) + \psi_R^2(u)]\,du + \frac{1}{k}\right\}\log n_k \\ &\le \gamma_k^2 \log n_k, \end{aligned}$$

where the last inequality is by (3.6). This and (3.7) together give (3.2) for some $s_0 \in (0, \frac{1}{2})$, $\sigma = 0$, and $C_0 = 2$. Again, we did not need (1.6) here.

If, on the other hand, $s \in (a_k, \sqrt{k}/n_k)$, then $s = x/n_k$ for some $x \in (1/k, \sqrt{k})$ and we again have (3.3) for such an x. Therefore, if s is a continuity point of $Q_L(\cdot)$ then, using (3.6) and (1.6),

$$\begin{aligned} s\{Q_L^2(s) + Q_R^2(1-s)\} &\le 2\left\{x\psi_L^2(x) + x\psi_R^2(x) + \frac{2}{k^{3/2}}\right\}\log n_k \\ &\le 2C\left\{\int_x^{\infty}[\psi_L^2(y) + \psi_R^2(y)]\,dy + \frac{\gamma_k^2}{4}\right\}\log n_k, \end{aligned}$$

while, using (3.7), the argument in (3.5) now yields

$$\begin{aligned} \int_s^{1-s} Q^2(u)\,du &\ge \left\{\frac{\gamma_k^2}{2} + \int_x^k[\psi_L^2(y) + \psi_R^2(y)]\,dy\right\}\log n_k \\ &\ge \left\{\frac{\gamma_k^2}{4} + \int_x^{\infty}[\psi_L^2(y) + \psi_R^2(y)]\,dy\right\}\log n_k \end{aligned}$$

whenever $k \ge k_0$, where the last inequality is by (3.6) again. Hence (3.2) follows again for $C_0 = 2C$ and some $s_0 \in (0, \frac{1}{2})$, for every continuity point $s \in (0, s_0)$ of Q_L and hence for every $s \in (0, s_0)$.

4. Construction of Doeblin's Universal Laws

We will construct a quantile function Q, a subsequence $\{n_k\}_{k=2}^{\infty} \subset \mathbb{N}$ and a sequence $A_k = A_{n_k} > 0$, $k = 2, 3, \ldots$, such that for each $(\psi_L, \psi_R, \sigma) \in \mathscr{I}$ there will exist a subsequence $\{n_{k_j}\}_{j=1}^{\infty} \subset \{n_k\}_{k=2}^{\infty}$ so that (1.1) and (1.2) hold along this subsequence as $j \to \infty$. Hence the law defined by this quantile function Q will be in the domain of partial attraction of every infinitely divisible law. This construction is also a direct modification of the basic one in Section 2. Having one such universal Q, we will point out at the end of the paper that there are in fact many.

First we note that there exists a sequence $\Psi_0 = \{\psi_2, \psi_3, \dots\} \subset \Psi$ which is dense in Ψ with respect to weak convergence: for every $\psi \in \Psi$ there is a subsequence $\{l_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $\psi_{l_j} \Rightarrow \psi$ as $j \to \infty$; i.e., $\psi_{l_j}(x) \to \psi(x)$ as $j \to \infty$ at every continuity point x of ψ. For example, as can be seen by any standard proof of Helley's selection theorem, Ψ_0 can be chosen as the countable set of all step-functions $\psi \in \Psi$ which have only a finite number of jumps (and hence by the square-integrability condition are zero for all large enough argument), each taking place at a rational point and which have only non-positive rational values. Now let

$$\mathscr{I}_0 := \bigcup_{j=2}^{\infty} \bigcup_{k=2}^{\infty} \bigcup_{l=2}^{\infty} \{(\psi_j, \psi_k, \sigma_l)\},$$

where $\{\sigma_2, \sigma_3, \dots\}$ is the sequence of all non-negative rational numbers in an arbitrary listing, and let

$$\{(\psi_{m,L}, \psi_{m,R}, \sigma_m)\}_{m=2}^{\infty} = \mathscr{I}_0$$

be an enumeration of $\mathscr{I}_0$. Thus the sequence $\mathscr{I}_0$ is dense in the "space" $\mathscr{I}$: for every $(\psi_L, \psi_R, \sigma) \in \mathscr{I}$ there is a subsequence $\{m_j\}_{j=1}^{\infty} \subset \{2, 3, \dots\}$ such that

$$\psi_{m_j, L} \Rightarrow \psi_L, \qquad \psi_{m_j, R} \Rightarrow \psi_R, \qquad \sigma_{m_j} \to \sigma \qquad \text{as } j \to \infty. \tag{4.1}$$

Finally, let π: $\{2, 3, \dots\} \mapsto \{2, 3, \dots\}$ be any mapping such that $\operatorname{card}(\pi^{-1}(k)) = \infty$ for every $k \in \{2, 3, \dots\}$; the inverse image of every $k \in \mathbb{N} \setminus \{1\}$ is of infinite cardinality; i.e., k is the image of an infinite number of integers in $\mathbb{N} \setminus \{1\}$.

We are now ready to define Q, $\{n_k\}_{k=2}^{\infty}$, and $\{A_k = A_{n_k}\}_{k=2}^{\infty}$. Let n_2, a_2, b_2, and A_2 be as in (2.3) and set

$$Q_L(u) := A_2\{\psi_{\pi(2), L}(n_2 u -) - \tfrac{1}{2}\}, \qquad u \in [a_2, \tfrac{1}{2}],$$
$$Q_R(u) := A_2\{-\psi_{\pi(2), R}(n_2 - n_2 u) + \tfrac{1}{2}\}, \qquad u \in (\tfrac{1}{2}, 1 - a_2],$$

and

$$Q(u) := \begin{cases} Q_L(u) + Q_{\sigma_{\pi(2)}}(u), & a_2 < u \le \tfrac{1}{2}, \\ Q_R(u) + Q_{\sigma_{\pi(2)}}(u), & \tfrac{1}{2} < u \le 1 - a_2, \end{cases}$$

where $Q_\sigma(\cdot)$, for any $\sigma \geq 0$, is given in (2.2). As in Section 2, for $k \geq 3$ define inductively a_k, b_k, and A_k by (2.4), set

$$Q_L(u) := \begin{cases} A_k\left\{\psi_{\pi(k),L}(n_k u -) - \dfrac{1}{k}\right\}, & u \in [a_k, b_k], \\ A_{k-1}\left\{\psi_{\pi(k),L}(n_{k-1}a_{k-1} -) - \dfrac{1}{k-1}\right\}, & u \in (b_k, a_{k-1}], \end{cases}$$

and

$$Q_R(u) := \begin{cases} A_{k-1}\left\{-\psi_{\pi(k),R}(n_{k-1}a_{k-1}) + \dfrac{1}{k-1}\right\}, \\ \qquad\qquad u \in [1 - a_{k-1}, 1 - b_k], \\ A_k\left\{-\psi_{\pi(k),R}(n_k - n_k u) + \dfrac{1}{k}\right\}, \\ \qquad\qquad u \in (1 - b_k, 1 - a_k], \end{cases}$$

and finally put

$$Q(u) := \begin{cases} Q_L(u) + Q_{\sigma_{\pi(k)}}(u), & a_k < u \leq a_{k-1}, \\ Q_R(u) + Q_{\sigma_{\pi(k)}}(u), & 1 - a_{k-1} < u \leq 1 - a_k. \end{cases}$$

The function Q is left continuous on $(a_k, 1 - a_k]$, and the values $Q(a_{k-1}+)$ and $Q(1 - a_{k-1})$ depend only on the finite sequence $\{(\psi_{\pi(j),L}, \psi_{\pi(j),R}, \sigma_{\pi(j)})\}_{j=2}^{k-1}$. Therefore, n_k can be chosen large enough, depending on the whole triple $(\psi_{\pi(k),L}, \psi_{\pi(k),R}, \sigma_{\pi(k)})$, to make $Q(\cdot)$ nondecreasing on the whole $(a_k, 1 - a_k]$. It is no longer continuous at a_{k-1} and $1 - a_{k-1}$, in general, and in this step we also ensure, by choosing n_k even larger if necessary, that $b_k^{1/k} \leq a_{k-1}$ instead of the earlier requirement that $b_k < a_{k-1}$. The procedure inductively defines a proper quantile function on the interval $(0, 1)$.

In this construction (2.6) becomes

$$\frac{Q\left(\dfrac{x}{n_k}+\right)}{A_k} = \psi_{\pi(k),L}(x) - \frac{1}{k} - \frac{\sigma_{\pi(k)}}{\sqrt{\log n_k}}\frac{1}{\sqrt{2x}}$$

and (4.2)

$$\frac{-Q\left(1 - \dfrac{x}{n_k}\right)}{A_k} = \pi_{\pi(k),R}(x) - \frac{1}{k} - \frac{\sigma_{\pi(k)}}{\sqrt{\log n_k}}\frac{1}{\sqrt{2x}}$$

for every $x \in (n_k a_k, n_k b_k) = (1/k, k)$, $k = 2, 3, \dots$. Following through the rest of Section 2, we see that the sequence $\{n_k\}$ can be chosen to grow fast enough, depending on the behavior of the whole infinite sequence $\{(\psi_{\pi(k), L}, \psi_{\pi(k), R}, \sigma_{\pi(k)})\}_{k=2}^{\infty}$, to ensure instead of (2.12) that

$$\frac{\sqrt{n_k}\,\sigma(h/n_k)}{A_k} = \sigma_{\pi(k)} + o_h(1) \tag{4.3}$$

as $k \to \infty$ and then $h \to \infty$.

We now claim that the Q constructed is universal. Let (ψ_L, ψ_R, σ) be an arbitrary triple in $\mathscr{S}$. Then there exists a subsequence $\{m_j\}_{j=1}^{\infty} \subset \{2, 3, \dots\}$ such that (4.1) holds. Let $k_1 \in \pi^{-1}(m_1)$ be fixed arbitrarily. Then choose $k_2 \in \pi^{-1}(m_2)$ such that $k_2 > k_1$, and in general, if $k_1 < \cdots < k_{j-1}$ are already chosen, pick $k_j \in \pi^{-1}(m_j)$ so that $k_j > k_{j-1}$. This is always possible by the property that $\operatorname{card}(\pi^{-1}(m)) = \infty$ for every $m \in \{2, 3, \dots\}$. Since $\pi(k_j) = m_j$ for all $j \in \mathbb{N}$, (4.2) and (4.3) give that

$$\frac{1}{A_{k_j}} Q\left(\frac{x}{n_{k_j}} +\right) = \psi_{m_j, L}(x) - \frac{1}{k_j} - \frac{\sigma_{m_j}}{\sqrt{\log n_{kj}}} \frac{1}{\sqrt{2x}}$$

and

$$\frac{1}{A_{kj}} Q\left(1 - \frac{x}{n_{kj}}\right) = \psi_{m_j, R}(x) - \frac{1}{k_j} - \frac{\sigma_{m_j}}{\sqrt{\log n_{kj}}} \frac{1}{\sqrt{2x}}$$

for every $x \in (1/k_j, k_j)$, $j = 1, 2, \dots$, and

$$\frac{\sqrt{n_{k_j}}}{A_{k_j}} \sigma\left(\frac{h}{n_{k_j}}\right) = \sigma_{m_j} + o_h(1)$$

as $j \to \infty$ and then $h \to \infty$, since $k_j \to \infty$ as $j \to \infty$. For the same reason, using (4.1), these imply

$$\frac{1}{A_{k_j}} Q\left(\frac{\cdot}{n_{k_j}} +\right) \Rightarrow \psi_L(\cdot), \qquad \frac{1}{A_{k_j}} Q\left(1 - \frac{\cdot}{n_{k_j}}\right) \Rightarrow \psi_R(\cdot)$$

as $j \to \infty$, and

$$\lim_{h\to\infty} \liminf_{j\to\infty} \frac{\sqrt{n_{k_j}}}{A_{k_j}} \sigma\left(\frac{h}{n_{k_j}}\right) = \sigma = \lim_{h\to\infty} \limsup_{j\to\infty} \frac{\sqrt{n_{k_j}}}{A_{k_j}} \sigma\left(\frac{h}{n_{k_j}}\right),$$

proving the claim.

We note in conclusion that a whole family of universal quantile functions can be obtained if in the construction above we use at each step $Q_{\sigma_{\pi(k)}}$, $k = 2, 3, \ldots$, as given in (3.1), rather than (2.2), with a function $\gamma(\cdot)$ satisfying the properties described there. Then for any such $\gamma(\cdot)$ the resulting Q is universal.

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