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Polar Covariants of Plane Cubics and Quartics

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CONTENTS

INTRODUCTION

According to classical invariant theory a covariant is a rational map from the space of projective hypersurfaces of certain degree n in $\mathbb{P}^r$ to the space of hypersurfaces of some degree m in $\mathbb{P}^r$ that is equivariant with respect to the group of projective transformations. The best known example of a covariant, from which the invariant theory originates, is the discriminant of quadratic forms. It vanishes on the set of singular quadrics. Composing the $(r-2)$-iterated polar map $a \to P_a \ldots P_a(F)$ with the discriminant map we obtain the Hessian covariant $F \to He(F)$. It is the locus of points $a \in \mathbb{P}^r$ such that the Hessian matrix of F of second partials at the point a is not invertible. Here the polarization $P_a(F)$ of a polynomial $F(x_0, \ldots, x_r)$ with respect to a point $a = (a_0, \ldots, a_n)$ is defined by the usual formula $P_a(F) = \sum a_i \partial F/\partial x_i$. This gives some general construction of covariants: take a known invariant of forms of smaller degree and compose it with the

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polarization map. Another example of this construction is the Clebsch covariant S of plane quartic curves. Here we compose the first polarization map $a \to P_a(F)$ with the Aronhold invariant of plane cubics I that vanishes exactly at the locus of anharmonic cubics. The value of this covariant at a quartic F is the locus of points x such that the first polar $P_x(F)$ is projectively isomorphic to the Fermat cubic or its degeneration. A natural question arises: what is the degree of the obtained map from the space of quartics to itself? The answer is 36, as was shown by Gaetano Scorza in 1899 [Sc2]. An expert in the theory of curves immediately recognizes this number as the number of even theta characteristics on a curve of genus 3. In fact, Scorza shows that the covariant quartic $S(F)$ carries naturally such a characteristic ϑ, and the map $F \to (S(F), \vartheta)$ is a birational map from the space of quartics to its cover of degree 36 parametrizing the pairs consisting of a non-singular quartic and an even theta characteristic on it. This leads to a natural birational equivalence between the corresponding moduli spaces $\mathcal{M}_3$ and $\mathcal{M}_3^{ev}$, a rather surprising result, although both spaces are expected to be rational. There is an analog of this for cubic curves: the map that sends a cubic curve to its Hessian is a rational self-cover of degree 3 whose fibre over a non-singular cubic is naturally identified with three non-trivial 2-torsion divisor classes (=even theta characteristics in this case).

The initial plan of this work was to give a modern proof of the result of Scorza. As it turned out, the original argument of Scorza that depends heavily on his earlier work in the theory of apolarity of plane quartic curves [Sc1] does not explain clearly the injectivity of his map. In this paper we give another proof that is independent of the theory of apolarity. As is customary today we pay great attention to analyzing possible degenerate cases by studying throrougly the set of definition of the Scorza map. In particular, we extend the Scorza isomorphism to the larger set of what we call weakly non-degenerate quartics. A non-degenerate curve of even degree is defined by non-vanishing of the catalecticant invariant, a certain analog of the discriminant for forms of higher degree. A weakly non-degenerate curve is allowed to have vanishing catalecticant but to be a general curve with this property in some precise sense. Another beautiful characterization of such curves is given by a theorem of G. Lüroth. The equation of a weakly non-degenerate but degenerate plane quartic can be written as a sum of five (six if non-degenerate) powers of linear forms. A degenerate curve with ths property is called a Clebsch curve in honor of A. Clebsch who was the first to observe that the five, suggested by counting constants, is not enough for a general quartic. The problem of presentation of a form as a sum of powers of linear forms is one of the main applications of the theory of apolarity. As is well-known for binary forms (see [Ku-Ro] for a modern account), it is much less known in the case of forms of larger numbers of variables. We give an account of this theory in the present

paper. The covariant quartic of a Clebsch quartic is a Lüroth quartic that is characterized by the condition that it can be circumscribed around a complete pentagon. The relationship between Lüroth quartics and rank 2 vector bundles on the projective plane was discovered earlier by W. Barth [Ba]. Extending the Scorza result, we prove that the restriction of the Scorza map to the locus of Clebsch quartics is injective.

A possible extension of Scorza's result to curves of higher genus is very promising. Based on an idea of E. Ciani [Ci2], Scorza himself gives a beautiful construction of a certain quartic hypersurface in $\mathbb{P}^{g-1}$ attached to a pair (X, ϑ), where $X \subset \mathbb{P}^{g-1}$ is a canonical curve of genus g and ϑ is an even theta characteristic on it [Sc3]. In case $g = 3$ this gives the inverse of the Scorza map. Scorza's construction depends on three assumptions on the pair (X, ϑ) that he failed to verify for a single curve of genus $g > 3$; they are fulfilled for a general curve of genus 3. Although we succeeded in showing the latter we still cannot verify the former. So, the Scorza construction remains conditional. What are these mysterious quartic hypersurfaces of Scorza?

Notations. The following notation will be most frequently used in the paper:

V: a complex vector space of dimension $r+1$; V^*, its dual space;

$\mathbb{P}^r = \mathbb{P}(V) = |V|$: the projective space of lines in V;

$\check{\mathbb{P}}^r = \mathbb{P}(V^*) = |V^*|$: the dual projective space of hyperplanes in V;

H_α: the hyperplane in $\mathbb{P}^r$ corresponding to a point $\alpha \in \check{\mathbb{P}}^r$;

$\check{H}_a = \{\alpha \in \mathbb{P}^r : a \in H_\alpha\}$: the hyperplane in $\check{\mathbb{P}}^r$ corresponding to a point $a \in \mathbb{P}^r$;

$PGL(V)$: the group of linear projective transformations of $\mathbb{P}(V)$;

$S^n(V)$, $S^n(V^*)$: the symmetric powers of V and V^*;

$|S^n(V^*)|$, $|S^n(V)|$: the projective spaces associated to $S^n(V^*)$, $S^n(V)$;

$N(n) = \dim |S^n(V^*)| = \binom{n+r}{r} - 1$;

$P_a(F)$: the polar hypersurface of F with respect to a point $a \in \mathbb{P}^r$;

$P_\Phi(F)$: the polar hypersurface with respect to a hypersurface Φ in $\check{\mathbb{P}}^r$;

$\mathscr{P}_F$: the linear system of polar hypersurfaces;

$\langle \Phi, F \rangle$: the apolarity pairing between $\Phi \in S^n(V^*)$ and $F \in S^n(V)$;

$V(\Phi; F) = \{\Phi' \in S^{n-k}(V) : \langle \Phi\Phi', F \rangle = 0\}$;

$X_s(F)$: the variety of polar s-polyhedra of a hypersurface F;

$He(F)$: the Hessian of a hypersurface F in $\mathbb{P}^r$;

$St(F)$: the Steinerian of a hypersurface F in $\mathbb{P}^r$;

s_F: the Steiner rational map $He(F) \dashrightarrow St(F)$;

$\mathrm{Cat}(F)$: the catalecticant matrix of a hypersurface of even degree in $\mathbb{P}^r$;

$\mathbf{C}$: the catalecticant invariant $F \to \det(\mathrm{Cat}(F))$;

$\mathbf{I}_4$: the Aronhold invariant of degree 3 of plane cubics;

$\mathbf{S}_4$: the Clebsch quartic covariant of plane quartics;

$S(F) = \mathbf{S}_4(F)$ if $\mathbf{S}_4(F) = \mathbf{S}_4(F)_{\mathrm{red}} \neq \mathbb{P}^2$, the Clebsch covariant quartic of F;

$T(F)$: a symmetric correspondence on $S(F)$ defined by polohessians;

Γ: the Clebsch contravariant of class 6 of plane quartics;

$\Gamma(F)$: the image of $T(F)$ in $\mathbb{P}^2$ defined by the map $(a, b) \to P_a P_b(F)$;

ϑ: a non-effective theta characteristic on a curve of genus g;

T_ϑ: a symmetric correspondence on a curve of genus g defined by ϑ;

$d(\vartheta)$: half of the degree of the mapping $T_\vartheta \to \Gamma(\vartheta)$.

1. Polars of Hypersurfaces

(1.1) Let V be a vector space of dimension $r+1$. Recall that its symmetric power $S^n(V)$ is defined to be the quotient of the tensor power $V^{\otimes n}$ by the subspace spanned by the tensors $t - \sigma(t)$, where σ runs through the group Perm_n of permutations on n letters. The symmetrization map

$$V^{\otimes n} \to V^{\otimes n}, \qquad t \mapsto \sum_{\sigma \in \mathrm{Perm}_n} \sigma(t),$$

factors through $S^n(V)$ and defines the *polarization map.*

$$pl_n^*: S^n(V) \to V^{\otimes n}.$$

Its image consists of symmetric tensors. Replacing V by its dual space V^*, we obtain the map

$$pl_n: S^n(V^*) \to V^{*\otimes n} \cong (V^{\otimes n})^*,$$

whose image equals the subspace $\mathrm{Sym}_n(V)$ of symmetric n-linear forms on the space V. The later space is naturally isomorphic to the space $S^n(V)^*$. Restricting the projection map $V^{*\otimes n} \to S^n(V^*)$ to the subspace $\mathrm{Sym}_n(V)$, we obtain the *restitutution map*

$$r_n: S^n(V)^* \cong \mathrm{Sym}_n(V) \to S^n(V^*).$$

It is easy to verify that the compositions $pl_n \circ r_n$ and $r_n \circ pl_n$ are both equal to $n!$ times the identity map. Thus, if the characteristic of the ground field is zero (as in our case), the polarization map is bijective onto $S^n(V)^*$. We redefine the polarization map by multiplying it by $1/n!$.

(1.2) By choosing a basis $u_0, ..., u_r$ in V, and its dual basis $x_0, ..., x_r$ in V^*, we will identify the spaces $S^n(V)$ (resp. $S^n(V^*)$) with the space of homogeneous polynomials of degree n in $u_0, ..., u_r$ (resp. $x_0, ..., x_r$). The image of the tensor $u_{i_1} \otimes ... \otimes u_{i_n} \in V^{\otimes n}$ in the quotient space $S^n(V)$ is the monomial $u_{i_1} ... u_{i_n}$. The set of zeroes of any non-zero polynomial $F \in S^n(V^*)$ is a *hypersurface of degree n* in the projective space $\mathbb{P}(V) = \mathbb{P}^r$ associated to V, an element $\Phi \in S^n(V)$ defines a hypersurface in the dual space, called an *enveloping hypersurface of class n*. Abusing notation, we will often identify a hypersurface with the corresponding homogeneous polynomial that it defines.

The polarization $pl_n(F)$ of a polynomial $F \in S^n(V^*)$ is the unique symmetric multi-linear function $\tilde{F}(x, y, ..., z)$ on V^n such that for all $x \in V$

$$F(x) = \tilde{F}(x, ..., x).$$

By fixing the first k variables $a, b, ..., c$ in $\tilde{F}$, and making equal the remaining ones, we obtain the kth *mixed polar of F with respect to the points a, b, ..., c*,

$$P_{a, b, ..., c}(F)(x) = \tilde{F}(a, b, ..., c, x, ..., x).$$

Note the symmetry of this notation in the subscripts. We use $P_{a^i}(F)$ to denote i-times repeated polar with respect to a point a. Clearly,

$$P_{a, b, ..., c}(F) = P_a(P_b(\cdots(P_c(F)))).$$

One easily verifies that the *first polar* of F with respect to the point $a = a_0 u_0 + \cdots + a_n u_n \in V$ is equal to

$$P_a(F) = n \sum_{i=0}^{r} a_i \partial F / \partial x_i.$$

Therefore

$$P_{a, b, ..., c}(F) = \frac{(n-k)!}{n!} \sum_{i_1, ..., i_k} a_{i_1} b_{i_2} \cdots c_{i_k} \partial^k F / \partial x_{i_1} \cdots \partial x_{i_k}.$$

(1.3) In coordinate-free terms one can define the mixed polars as follows.

There is a natural inclusion:

$$\mathrm{Sym}_n(V) \subset V^{*\otimes k} \otimes \mathrm{Sym}_{n-k}(V) \subset V^{*\otimes n}.$$

Composing the polarization map pl_n with the map $1 \otimes r_{n,k}$, we obtain a linear map

$$S^n(V^*) \to V^{*\otimes k} \otimes S^{n-k}(V^*), \qquad (*)$$

or, equivalently, a linear map

$$pl_{k,n}: V^{\otimes k} \otimes S^n(V^*) \to S^{n-k}(V^*).$$

We will call it the kth *polarization map*. It is immediately verified that

$$pl_{k,n}(a \otimes b \otimes \cdots \otimes c \otimes F) = P_{a,b,\ldots,c}(F).$$

When $k = n$, we get the linear map

$$S^n(V^*) \to V^{*\otimes n},$$

which is, of course, our polarization map pl_n.

Composing the map $pl_k^* \otimes 1 : S^k(V) \otimes S^n(V^*) \to V^{\otimes k} \otimes S^n(V^*)$ with $pl_{k,n}$, we obtain a linear map

$$spl_{k,n} : S^k(V) \otimes S^n(V^*) \to S^{n-k}(V^*).$$

It is easy to see that the polarization maps are equivariant with respect to the natural representations of the general linear group $GL(V)$ on the source and the target of the maps. In particular, the linear system $\mathscr{P}(F)$ of polars $P_a(F)$ is $PGL(V)$-invariant. For every point $a \in \mathbb{P}^r$

$$P_a(F) \cap F = \{v \in F : a \in T(F)_v\},$$

where $T(F)_v$ is the tangent hyperplane to F if v is a non-singular point of F, or the whole space otherwise. Each singular point of F is a base point of $\mathscr{P}(F)$; the corresponding ideal is the adjoint (or Jacobian) ideal of F at v. The image of F under the rational map defined by $\mathscr{P}(F)$ is the dual variety of F.

(1.4) DEFINITION. Let $\Phi \in S^k(V)$ and $F \in S^n(V^*)$. The homogeneous polynomial

$$P_\Phi(F) := spl_{k,n}(\Phi, F) \in S^{n-k}(V^*)$$

is called the *polar of F with respect to Φ*.

In the case when $\Phi = (\sum a_i u_i) \ldots (\sum c_i u_i) = \check{H}_a \ldots \check{H}_c$ is the product of k linear polynomials the polar of Φ with respect to F is equal to the kth mixed polar $P_{a,\ldots,c}(F)$.

(1.5) DEFINITION. The pairing

$$spl_{n,n} : S^n(V) \otimes S^n(V^*) \to S^0(V^*) \cong \mathbb{C},$$

is called the *apolarity pairing*.

For any $\Phi \in S^n(V)$ and $F \in S^n(V^*)$, we set

$$spl_{n,n}(\Phi, F) = \langle \Phi, F \rangle.$$

Note that the apolarity pairing can be viewed as the map

$$S^n(V^*) \to S^n(V)^*,$$

which is, of course, the polarization map pl_n. In particular, it is non-degenerate.

The apolarity pairing is the keystone of the theory of apolarity between hypersurfaces in $\mathbb{P}^r$ and enveloping hypersurfaces in $\check{\mathbb{P}}^r$. Of course, when $n = 1$, this is just the usual pairing

$$V \otimes V^* \to \mathbb{C}, \qquad (v, f) \to f(v),$$

between the space V and its dual space V^*.

Explicitly, if $\Phi = u^{\mathbf{i}} = u_0^{i_0} \dots u_r^{i_r}$ and $F = x^{\mathbf{j}} = x_0^{j_0} \dots x_r^{j_r}$ are monomials of degree n, then

$$\langle u^{\mathbf{i}}, x^{\mathbf{j}} \rangle = \begin{cases} \binom{n}{\mathbf{i}}^{-1} & \text{if } \mathbf{i} = \mathbf{j} \\ 0 & \text{otherwise.} \end{cases}$$

This extends, by linearity, to an explicit formula for the apolarity pairing.

(1.6) LEMMA. *Let* $t = t_0 u_0 + \dots + t_n u_n \in V$, *and* $\check{H}_t \in S^1(V^*)$ *be the corresponding linear function on* V. *Then for any* $F \in S^n(V^*)$

$$\langle \check{H}_t^n, F \rangle = F(t).$$

More generally, if $a_1, \dots, a_n \in V$, *then*

$$\langle \check{H}_{a_1} \dots \check{H}_{a_n}, F \rangle = \tilde{F}(a_1, \dots, a_n),$$

where $\tilde{F} \in \mathrm{Sym}_n(V)$ *is the total polarization of* F.

Proof. This follows immediately from the definitions.

(1.7) PROPOSITION. *Let* $\Phi \in S^k(V)$, $\Phi' \in S^{n-k}(V)$, *and* $F \in S^n(V^*)$. *Then*

$$\langle \Phi', P_\Phi(F) \rangle = \langle \Phi\Phi', F \rangle.$$

Proof. Since $P_\Phi(F)$ is linear with respect to Φ, it suffices to verify this for

$$\Phi = \check{H}_{a_1} \dots \check{H}_{a_k}, \qquad \Phi' = \check{H}_{b_1} \dots \check{H}_{b_{n-k}}.$$

In view of Lemma (1.6), the right-hand side is equal to $\tilde{F}(a_1, ..., a_k, b_1, ..., b_{n-k})$. In its turn, $P_\Phi(F_n) = P_{a_1, ..., a_k}(F) = \tilde{F}(a_1, ..., a_k, x, ..., x)$, and the left-hand side

$$\langle \check{H}_{b_1} \ldots \check{H}_{b_{n-k}}, \tilde{F}(a_1, ..., a_k, x, ..., x) \rangle = \tilde{F}(a_1, ..., a_k, b_1, ..., b_{n\ k})$$

equals the same.

(1.8) COROLLARY. *Let* $\Phi \in S^k(V)$, $F \in S^n(V^*)$, *and let*

$$V(\Phi; F) = \{\Phi' \in S^{n-k}(V) : \langle \Phi\Phi', F \rangle = 0\}.$$

Then $G \in S^{n-k}(V^*)$ *is equal to the polar* $P_\Phi(F)$ *of* F *with respect to* Φ *if and only if for any* $\Phi' \in V(\Phi; F)$

$$\langle \Phi', G \rangle = 0.$$

Proof. The apolarity pairing is non-degenerate, so by Proposition (1.7) the dual subspace of $V(\Phi; F) \subset S^{n\ k}(V)$ is spanned by $P_\Phi(F)$.

(1.9) COROLLARY. *Let* $\Phi \in S^k(V)$, $F \in S^n(V^*)$. *Then*

$$P_\Phi(F)(a) = 0 \Leftrightarrow \langle \Phi, P_a k(F) \rangle = 0.$$

Proof. Let $\Phi' = \check{H}^{n\ k}_a$. Then $P_{a^k}(F) = P_{\Phi'}(F)$. Applying (1.7) twice, we obtain

$$\langle \Phi, P_{a^k}(F) \rangle = \langle \check{H}^{n\ k}_a \Phi, F \rangle = \langle \check{H}^{n\ k}_a, P_\Phi(F) \rangle.$$

It remains to use Lemma (1.6).

(1.10) DEFINITION. An envelope Φ of class $n-k$ is called a kth *anti-polar* of a hyperplane H with respect to $F \in S^n(V^*)$ if $H^k = P_\Phi(F)$.

(1.11) COROLLARY. $\Phi \in S^k(V)$ *is a* k*th anti-polar of a hyperplane* H *with respect to* $F \in S^n(V^*)$ *if and only if* $H^{n\ k} \in V(\Phi; F)^\perp$ (*see* (1.8)).

(1.12) EXAMPLES. The first anti-polar of a hyperplane H with respect to a quadric F is the hyperplane $\check{H}_a$, where $a \in \mathbb{P}^r$ is dual to H with respect to F (= intersection of all polars $P_x(F)$, here $x \in H$).

The first anti-polar of a hyperplane H with respect to a cubic F is the quadric Φ in $\check{\mathbb{P}}^r$ that is apolar to polar quadrics of F with respect to all points on H. The second anti-polar of a hyperplane H with respect to a cubic F is the hyperplane $\check{H}_b$, where b is the point of intersection of all double polars $P_{a,a}(F)$, $a \in H$.

2. Apolarity

(2.1) Definition. An envelope $\Phi \in S^k(V)$ is called *apolar* to a hypersurface $F \in S^n(V^*)$ if for all $\Phi' \in S^{n-k}(V)$

$$\langle \Phi\Phi', F \rangle = 0.$$

Equivalently (1.7), Φ is apolar to F if and only if

$$P_\Phi(F) = 0.$$

(2.2) Example. Let $F = (\alpha_0 x_0 + \ldots + \alpha_n x_n)^n = H_\alpha^n$. Then $\Phi \in S^n(V)$ is apolar to F if and only if

$$\Phi(\alpha) = 0.$$

This follows from Lemma (1.6) (where we replace V by V^*). For example, H_α is apolar to $\check{H}_a \Leftrightarrow a \in H_\alpha \Leftrightarrow \alpha \in \check{H}_a$.

Similarly $\Phi = \check{H}_a^n = (a_0 u_0 + \ldots + a_n u_n)^n$ is apolar to $F \in S^n(V^*)$ if and only if $F(a) = 0$.

(2.3) Let $F \in S^n(V^*)$, and let

$$AP_k(F) \subset S^k(V)$$

be the linear space of apolars of F of degree k. By definition, it is equal to the kernel of the linear map

$$ap_k(F)\colon S^k(V) \to S^{n-k}(V^*), \qquad \Phi \to P_\Phi(F).$$

Thus we expect that $AP_k(F)$ is of dimension $N(k) - N(n-k)$; hence F always admits a non-zero apolar of class k if $N(k) > N(n-k)$. As is easy to see, the latter is equivalent to the inequality $k > \frac{1}{2}n$. If $n = 2k$, $AP_k(F)$ is the kernel of a linear map between two spaces of the same dimension; hence it is not zero if and only if a certain determinant vanishes. If $k < \frac{1}{2}n$, we expect that $AP_k(F) = \{0\}$.

(2.4) Definition. The matrix $\mathrm{Cat}(F)$ of the linear map $ap_k(F)$ with respect to a basis of monomials $u^{\mathbf{i}}$ of $S^k(V)$ and a basis of monomials $x^{\mathbf{j}}$ of $S^{n-k}(V^*)$ (ordered in some way) is called the kth *catalecticant matrix* of F. If $n = 2k$, the determinant of this matrix is called the *catalecticant* of F.

Note that, up to the sign, the definition of the catalecticant does not depend on the ordering of monomials $u^{\mathbf{i}}$ and $x^{\mathbf{j}}$.

Explicitly, if $\Phi = u^{\mathbf{j}} = u_0^{j_0} \cdots u_r^{j_r}$ then

$$ap_k(F)(u^{\mathbf{j}}) = P_\Phi(F) = \sum_{\mathbf{i}} \binom{n-k}{\mathbf{i}} c_{\mathbf{ij}} x^{\mathbf{i}},$$

hence, by (1.7),

$$\langle u^{\mathbf{k}+\mathbf{j}}, F\rangle = \langle u^{\mathbf{k}}, P_\Phi(F)\rangle = \sum \binom{n-k}{\mathbf{i}} c_{\mathbf{ij}} \langle u^{\mathbf{k}}, u^{\mathbf{i}}\rangle = \begin{cases} c_{\mathbf{ij}} & \text{if } \mathbf{k} = \mathbf{i} \\ 0 & \text{otherwise.} \end{cases}$$

So, we obtain that the (i, j)-entry of the catalecticant matrix $\mathrm{Cat}(F)$ is equal to

$$c_{\mathbf{ij}} = \langle u^{\mathbf{i}+\mathbf{j}}, F\rangle = a_{\mathbf{i}+\mathbf{j}},$$

where $F = \sum \binom{n}{\mathbf{r}} a_{\mathbf{r}} x^{\mathbf{r}}$. Notice the symmetry $c_{\mathbf{ij}} = c_{\mathbf{ji}}$.

(2.5) *Remark.* The function

$$\mathbf{C}\colon S^{2k}(V^*) \to \mathbb{C}, \qquad F \to \det(\mathrm{Cat}(F))$$

is a polynomial function of degree $N(k)+1$ on the space $S^{2k}(V^*)$. It is non-zero [Ia] and is invariant with respect to the natural action of the group $SL(V)$ of matrices with determinant 1. We will call it the *catalecticant invariant* (of level $2k$ and dimension r).

(2.6) EXAMPLE. Let $r=1$, $n=2k$, $F = a_0 x_0^n + \binom{n}{1} a_1 x_0^{n-1} x_1 + \ldots + a_n x^n$. The catalecticant determinant $\det(\mathrm{Cat}(F))$ is equal (up to a non-zero numerical factor) to

$$\begin{vmatrix} a_0 & a_1 & \ldots & a_k \\ a_1 & a_2 & \ldots & a_{k+1} \\ \ldots & \ldots & \ldots & \ldots \\ a_k & a_{k+1} & \ldots & a_n \end{vmatrix}.$$

The determinant of this form is called the *catalecticant determinant* of order $k+1$.

(2.7) EXAMPLE. Let $r=2$, $n=4$,

$$\begin{aligned} F = {} & a_{0000} x_0^4 + 4a_{0001} x_0^3 x_1 + 4a_{0002} x_0^3 x_2 + 6a_{0011} x_0^2 x_1^2 + 6a_{0022} x_0^2 x_2^2 \\ & + 12a_{0012} x_0^2 x_1 x_2 + 4a_{0111} x_0 x_1^3 + 4a_{0222} x_0 x_2^3 \\ & + 12a_{0112} x_0 x_1^2 x_2 + 12a_{0122} x_0 x_1 x_2^2 \\ & + a_{1111} x_1^4 + 4a_{1112} x_1^3 x_2 + 6a_{1122} x_1^2 x_2^2 + 4a_{1222} x_1 x_2^3 + a_{2222} x_2^4. \end{aligned}$$

Then the catalecticant determinant of F is equal (up to a non-zero numerical factor) to

$$\begin{vmatrix} a_{0000} & a_{0011} & a_{0022} & a_{0001} & a_{0002} & a_{0012} \\ a_{0011} & a_{1111} & a_{1122} & a_{0111} & a_{0112} & a_{1112} \\ a_{0022} & a_{1122} & a_{2222} & a_{0122} & a_{0222} & a_{1222} \\ a_{0001} & a_{0111} & a_{0122} & a_{0011} & a_{0012} & a_{0112} \\ a_{0002} & a_{0112} & a_{0222} & a_{0012} & a_{0022} & a_{0122} \\ a_{0012} & a_{1112} & a_{1222} & a_{0112} & a_{0122} & a_{1122} \end{vmatrix}.$$

Here we order the monomials by $(x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2)$.

This determinant is an invariant of degree 6 for ternary quartics, denoted by B in [Sa2, p. 167].

(2.8) DEFINITION. A polynomial $F \in S^{2k}(V^*)$ is called *non-degenerate* if $\det(\mathrm{Cat}(F)) \neq 0$, or, equivalently, if $AP_k(F) = \{0\}$.

(2.9) *Remark.* The catalecticant matrix $\mathrm{Cat}(F)$ is the matrix of the symmetric bilinear form

$$B_F: S^k(V) \times S^k(V) \to \mathbb{C},$$

whose values on $u^k, v^k \in S^k(V)$ with $u, v \in V$ are

$$B_F(u^k, v^k) = \tilde{F}(u, \ldots, u, v, \ldots, v).$$

Let $Q_F \in S^2(S^k(V)^*)$ be the corresponding quadratic form on the spaces $S^k(V)$. It is non-degenerate if and only if F is non-degenerate. For example, if F is a quadratic form, i.e., $k = 1$, then the notion of non-degeneracy coincides with the usual one employed for quadratic forms. So, the catalecticant is a generalization of the notion of the discriminant for forms of even degree greater than 2.

In the projective space $|S^k(V)|$ we can view Q_F as a quadric. Its pre-image under the Veronese map,

$$\mathbb{P}(V) \to |\mathcal{O}_{\mathbb{P}(V)}(k)|^* \cong |S^k(V)|,$$

is equal to the hypersurface F.

(2.10) EXAMPLE. Let $(x_0, x_1, x_2) \to (x_0^2, x_1^2, x_2^2, 2x_0x_1, 2x_0x_2, 2x_1x_2) = (t_0, \ldots, t_5)$ be the Veronese mapping from $\mathbb{P}(V)$ to $|S^2(V)|$. Then our quartic F is equal to the pre-image of the quadric

$$a_{0000}t_0^2 + 2a_{0011}t_0t_1 + 2a_{0022}t_0t_2 + 2a_{0001}t_0t_3 + 2a_{0002}t_0t_4 + 2a_{0012}t_0t_5$$
$$+ a_{1111}t_1^2 + 2a_{1122}t_1t_2 + 2a_{0111}t_1t_3 + 2a_{0112}t_1t_4 + 2a_{1112}t_1t_5$$
$$+ a_{2222}t_2^2 + 2a_{0122}t_2t_3 + 2a_{0222}t_2t_4$$
$$+ 2a_{1222}t_2t_5 + a_{0011}t_3^2 + 2a_{0012}t_3t_4 + 2a_{0122}t_3t_5 + a_{0022}t_4^2$$
$$+ 2a_{0122}t_4t_5 + a_{1122}t_5^2 = 0.$$

The determinant from (2.7) is equal to the discriminant of this quadric.

3. Hessian and Steinerian

(3.1) Let $F \in S^n(V^*)$, and let $\tilde{F} \in \mathrm{Sym}_n(V)$ be its full polarization. With each $v \in V$ one associates a symmetric bilinear form $H(v)$ defined by

$$H(v)(a, b) = \tilde{F}(a, b, v, \ldots, v) = P_{a,b}(F)(v) = \tilde{P}_v^{n-2}(F)(a, b).$$

In other terms, $H(v) \in \mathrm{Sym}_2(V) = S^2(V)^*$ is the composition of the linear maps

$$S^2(V) \to S^{n-2}(V^*) \to \mathbb{C},$$

where the first map is $ap_2(F)$, and the second one is obtained by evaluation of the polynomial at the point v.

Composing the map $v \to H(v)$ with the discriminant map $H(v) \mapsto \mathrm{discr}(H(v))$, we obtain a polynomial function on V of degree $(r+1)(n-2)$. It is called the *Hessian* of F, and is denoted by $He(F)$. By definition,

$$He(F)(v) = 0 \Leftrightarrow H(v) \text{ is degenerate}$$
$$\Leftrightarrow \tilde{F}(a, x, v, \ldots, v) = 0 \text{ for some } a \in V \text{ and any } x \in V.$$

In coordinates, the matrix of $H(v)$ in the basis $u_0, \ldots, u_r$ is equal to

$$\left\| \frac{1}{n(n-1)} \frac{\partial^2 F}{\partial x_i \partial x_j}(v) \right\|,$$

and the Hessian is equal to the functional determinant

$$(n^2-n)^{r+1} \det \left\| \frac{\partial^2 F}{\partial x_i \partial x_j} \right\|.$$

(3.2) The Hessian matrix is the Jacobi matrix for the map

$$\mathrm{grad}(F): \mathbb{P}^r \to \mathbb{P}^r,$$

given by the partials of F. If $v \in F \cap He(F)$, then each $a \in \mathrm{Ker}(H(v))$ belongs to the tangent space $T(F)_v$ at the point v. In fact, $\tilde{F}(a, x, v, ..., v) = 0$ for all x implies, by taking $x = v$, that $P_a(F)(v) = 0$. This shows that every point of intersection of $He(F)$ and F is an *inflection point* of F (i.e., points $x \in F$ for which there exists a line that intersects F with multiplicity $\geqslant 3$). Conversely, every inflection point of F lies on the Hessian.

(3.3) LEMMA. *Let $F \in S^n(V^*)$, $v \in V$, and let $\tilde{F}$ be the full polarization of F. The following properties are equivalent:*

(i) *v is a singular point of F;*

(ii) *$v \in P_a(F)$ for all $a \in V$;*

(iii) *$\tilde{F}(a, v, ..., v) = 0$ for all $a \in V$;*

(iv) *$P_{v^{n-1}}(F) = 0$;*

(v) *v is a singular point of $P_v(F)$.*

Proof. v is a singular point of F if and only if all partials of F vanish at v. The latter is equivalent to (ii) and (iii). Obviously (iii) is equivalent to (iv). Applying (i) $\Leftrightarrow$ (iii) to $P_v(F)(x) = \tilde{F}(v, x, ..., x)$, we obtain that

$$v \in \mathrm{Sing}(P_v(F)) \Leftrightarrow \tilde{F}(v, a, v, ..., v) = \tilde{F}(a, v, v, ..., v) = 0 \qquad \text{for all} \quad a \in V.$$

This proves the equivalence of (iii) and (v).

(3.4) PROPOSITION. *Let $v \in V$. The following properties are equivalent:*

(i) *$He(F)(v) = 0$;*

(ii) *there exists $a \in V$, $a \neq 0$, such that $P_{v^{n-2}a}(F) = 0$;*

(iii) *the polar quadric $P_{v^{n-2}}(F)$ is singular;*

(iv) *there exists $a \in \mathbb{P}$ such that the hypersurface $P_a(F)$ has a singular point at v;*

(v) *there exists $a \in V$, $a \neq 0$, such that $\check{H}_a \check{H}_v^{n-2}$ is polar to F;*

(vi) *$P_{v^{n-2}}(F)$ has a non-zero apolar of class 1 ($= \check{H}_a$ for some $a \in V$).*

Proof. Let $\tilde{F}$ be the full polarization of F. *Then for all $a, x \in V$*

$$H(v)(a, x) = \tilde{F}(a, x, v, ..., v) = P_{v^{n-2}}(F)(a, x) = P_{v^{n-2}a}(F)(x).$$

Clearly (i) holds if and only if there exists $a \in V$ such that $H(v)(a, x) = 0$ for all $x \in V$. The equivalence of (i), (ii), and (iii) follows from this immediately.

By (3.3), $P_a(F)$ has a singular point at v if and only if $\tilde{F}(a, x, v, ..., v)$ for all $x \in V$. This yields (i) $\Leftrightarrow$ (iv). Finally, by (1.7), for any $x \in V$

$$P_{v^{n-2}}(F)(a, x) = 0 \Leftrightarrow \langle \check{H}_a \check{H}_x, P_{v^{n-2}}(F) \rangle = 0 \Leftrightarrow \langle \check{H}_a \check{H}_{v^{n-2}} \check{H}_x, F \rangle = 0.$$

This shows the equivalence of (v), (vi), and (i).

(3.5) Let $F \in S^n(V^*)$, and set

$$X(F) = \{(v, a) \in \mathbb{P}^r \times \mathbb{P}^r : P_{v^{n-2}, a}(F) = 0\}.$$

Its image under the first projection is equal to the Hessian of F. It is either empty, or a hypersurface in $\mathbb{P}^r$, or the whole space. The first case may happen only if $n = 2$; this is the only case when $He(F)$ is a constant polynomial. The image of X under the second projection is called the *Steinerian* of F. We will denote it by $St(F)$. By (3.3):

$$St(F) = \{a \in \mathbb{P}^r : P_a(F) \text{ is singular}\}.$$

Moreover, for any $a \in St(F)$:

$$\mathrm{Sing}(P_a(F)) = \{v \in \mathrm{Hess}(F) : P_{v^{n-2}, a}(F) = 0\}.$$

One can consider the family of polars

$$\mathscr{P} = \{(a, x) \in \mathbb{P}^r \times \mathbb{P}^r : x \in P_a(F)\} \to \mathbb{P}^r, \qquad (a, x) \to a,$$

then interpret the Steinerian as the discriminant locus of this family, that is, the locus of points parametrizing singular fibres. Since the discriminant of any family of hypersurfaces is either empty, or a hypersurface in the base of the family, or the whole base, we obtain that $St(F)$ is either empty (this may happen only if $He(F) = \varnothing$, i.e., if $n = 2$), or a hypersurface in $\mathbb{P}^r$, or the whole space.

By the above, we can view $X(F)$ as the correspondence between the Hessian $He(F)$ and the Steinerian $St(F)$:

$$X(F) = \{(v, a) \in He(F) \times St(F) : v \in \mathrm{Sing}(P_a(F))\}.$$

We wil call it the *Steiner correspondence.*

By Proposition (3.4), for given $v \in He(F)$

$$\begin{aligned} p_1^{-1}(v) &= \{a \in \mathbb{P}^r : v \in \mathrm{Sing}(P_a(F))\} \\ &= \{a \in \mathbb{P}^r : H(v)(a, x) \text{ for all } x \in V\} \cong \mathbb{P}^{c(v)}, \end{aligned}$$

where $c(v) = \mathrm{corank}(H(v)) - 1$.

Similarly, for given $a \in St(F)$

$$p_2^{-1}(a) = \mathrm{Sing}(P_a(F)).$$

For generic F we expect that $c(v) = 0$ for a Zariski open subset of $He(F)$. This allows one to define the *Steiner rational map*:

$$s_F: He \dashrightarrow St(F), \qquad v \mapsto a \text{ such that } v \in \mathrm{Sing}(P_a(F)).$$

If general polar $P_a(F)$ has only one singular point, the Steiner map is birational, and its inverse is given by the formula $a \mapsto \mathrm{Sing}(P_a(F))$.

The next result is a more precise statement about the locus of the set of definition of the Steiner map.

(3.6) LEMMA. *Assume* $He(F) \neq 0$, *and let* v *be a non-singular point of* $He(F)$. *Then*

$$\mathrm{corank}\ H(v) = 1.$$

Proof. Let

$$\mathscr{H}(F) = \{(v, v') \in \mathbb{P}^r \times \mathbb{P}^r : H(v)(v', v') = \tilde{F}(v', v', v, \ldots, v) = 0\}.$$

Then the first projection

$$p_1: \mathscr{H}(F) \to \mathbb{P}^r$$

is a quadric bundle over the open subset U of points $v \in \mathbb{P}^r$ such that $H(v) \neq 0$ (see [Be]). This means that its generic fibre is isomorphic to a non-singular quadric (in an appropriate embedding into a projective bundle over $\mathbb{P}^r$; in our case it is the trivial bundle). The image of the map

$$f: \mathbb{P}^r \to |S^2(V^*)|, \qquad v \to p_1^{-1}(v),$$

intersects the discriminant hypersurface $\mathscr{D}_2$ of singular quadrics transversally at the point $f(v)$. In particular, $f(v)$ is a non-singular point of $\mathscr{D}_2$. It is known (*loc. cit.*, p. 322) that

$$\mathrm{Sing}(\mathscr{D}_2) = \{Q \in |S^2(V^*)| : \mathrm{corank}\ Q > 1\}.$$

This proves the assertion.

(3.7) PROPOSITION. *Assume* $r = 2$. *The following properties are equivalent*:

(i) $St(F) = \mathbb{P}^2$;

(ii) F *has a singular point of multiplicity* $\geqslant 3$, *or* $F = G^2$, *where* $G = 0$ *is non-singular.*

Proof. The implication (ii) $\Rightarrow$ (i) can be checked by direct computation. Let us prove the converse. By Bertini's theorem, the general member of a linear system of hypersurfaces has singular points in the base locus of the system. We apply it to the linear system $\mathscr{P}_F$ of polars of F. Its base points are singular points of F. If $\mathscr{P}_F$ has a fixed irreducible component G, then it is an irreducible multiple component of F. If it is singular, or its multiplicity is greater than 2, of F has another irreducible component, F has a singular point of multiplicity $\geqslant 3$. Otherwise $F = G^2$, where $G = 0$ is non-singular. Assume now that the polar linear system has no fixed components. Then there exists a singular point of F where all polars of F are singular. This implies that the adjoint ideal of the local ring of F at this point is generated by polynomials of multiplicity $\geqslant 2$. This easily implies that the multiplicity of this point > 2.

(3.8) PROPOSITION. *Assume $r = 2$ and F is of degree $\leqslant 4$ with $He(F) = 0$. Then F is the union of concurrent lines.*

Proof. If $F = G^2$, where G is smooth conic, then the singularities of $P_a(F)$ lie in G, so $He(F) \neq 0$. Assume this is not the case. Since $He(F) = 0$ every smooth point of F is an inflection point. As is well known, this happens only if every irreducible component of F with multiplicity 1 is a line (cf. [Kl, p. 174]). This implies that $F = l_1 \dots l_n$, where l_i are linear forms. Suppose F has multiple components. They are concurrent lines unless $F = l_1^2 l_3 l_4$. One easily shows that the Hessian of $x_0^2 x_1 x_3$ is not zero, so the proposition is true for non-reduced F. Now suppose F is reduced. If the assertion is false for F, we can reduce it to the form $x_0 x_1 x_2$ if $n = 3$, or to the form $x_0 x_1 x_2 (a x_0 + b x_1 + c x_2)$ if $n = 4$. In the first case the Hessian is not zero. In the second case the explicit computation of the Hessian shows that its equation has the coefficients $\frac{1}{2}a^3$, $\frac{1}{2}b^3$, and $\frac{1}{2}c^3$ at the monomials $x_0^4 x_1 x_2$, $x_0 x_1^2 x_2$, and $x_0 x_1 x_2^2$, respectively. This is absurd.

(3.9) PROPOSITION. *The following conditions are equivalent:*

(i) *The dimension of the linear system of polars $P_a(F)$, $a \in \mathbb{P}^r$, is less than r;*

(ii) *F is a cone.*

Proof. (i) implies that all partial derivatives of F are linearly dependent. After a change of coordinates, we my assume that $\partial F/\partial x_0 \equiv 0$. This implies that the equation of F is independent of x_0; hence F is a cone. Conversely, if (ii) holds, then we may assume that F is independent of x_0; hence $\partial F/\partial x_0 \equiv 0$.

4. Polar Polyhedra

In this section we discuss some applications of the theory of apolarity to the problem of representing a homogeneous polynomial as a sum of powers of linear forms.

(4.1) Definition. We say that s hyperplanes $H_1, ..., H_s$ in $\mathbb{P}^r$ form a *polar s-polyhedron* of $F \in S^n(V^*)$ if

$$F = l_1^n + ... + l_s^n,$$

where H_i are given by equations $l_i = 0$, and $l_1^n, ..., l_s^n$ are linearly independent in the space $S^n(V^*)$.

(4.1.1) This can be interpreted geometrically as follows. Let

$$v_n : \check{\mathbb{P}}^r \to |S^n(V^*)| \cong \mathbb{P}^{N(n)}$$

be the Veronese map that sends a hyperplane H to the hypersurface H^n, or, in coordinates:

$$v_n(\alpha_0 x_0 + ... + \alpha_r x_r) = (\alpha_0 x_0 + ... + \alpha_r x_r)^n = \sum \binom{n}{\mathbf{i}} \alpha^{\mathbf{i}} x^{\mathbf{i}}.$$

We can view F as a point in the space $|S^n(V^*)|$, and each $H_i^n = \{l_i^n = 0\}$ as a point in the *Veronese variety* $V_{r,n} = v_n(\check{\mathbb{P}}^r)$ in $|S^n(V^*)|$. Then $H_1, ..., H_s$ form a polar s-polyhedron of F if and only if F lies on the secant $(s-1)$-plane of $V_{r,n}$ containing the points $H_1^n, ..., H_s^n$.

Obviously, the "right definition" of a polar polyhedron must include the degenerate case, when the $(s-1)$-secant is tangent to the Veronese variety. We sketch this very briefly. The exact definitions require introducing some technical constructions that are out of scope of the present paper (see [Sch]).

Let $(\mathbb{P}^{N(n)})^{(s)}$ be the sth symmetric product of the space $|S^n(V^*)|$, let U be its open subspace corresponding to linear independent s-tuples, and let

$$f' : U \to G(s-1, \mathbb{P}^{N(n)})$$

be the map to the Grassmannian that sends an s-tuple to the projective $(s-1)$-subspace spanned by it. Let f be the restriction of f' to $(V_{r,n})^{(s)} \cap U$, and let $\overline{(V_{r,n})^{(s)}}$ be the closure of the graph of f in $(V_{r,n})^{(s)} \times G(s-1, \mathbb{P}^{N(n)})$. The first projection defines a birational map

$$p_1 : \overline{(V_{r,n})^{(s)}} \to (V_{r,n})^{(s)} \cong (\check{\mathbb{P}}^r)^{(s)},$$

which is an isomorphism over $(V_{r,n})^{(s)} \cap U$. The second projection

$$p_2 : \overline{(V_{r,n})^{(s)}} \to G(s-1, \mathbb{P}^{N(n)})$$

is equal to f when restricted to $(V_{r,n})^{(s)} \cap U$. The pull-back $p_2^*(\mathbb{P}(\mathscr{S}))$ of the projectivized universal subbundle $\mathbb{P}(\mathscr{S})$ over the Grasmannian defines a projective bundle of rank $s-1$,

$$\pi : \mathrm{Sec}_{s,n} \to \overline{(V_{r,n})^{(s)}};$$

its fibre over a point of $(V_{r,n})^{(s)} \cap U$ is the $(s-1)$-secant spanned by it. Let

$$r_1 : \mathbb{P}(\mathscr{S}) \to \mathbb{P}^{N(n)},$$
$$r_2 : \mathbb{P}(\mathscr{S}) \to G(s-1, \mathbb{P}^{N(n)})$$

be the canonical projections of $\mathbb{P}(\mathscr{S})$. Recall that the fibre of $r_1^{-1}(F)$ is mapped isomorphically by r_2 onto the variety of all $(s-1)$-planes in $\mathbb{P}^{N(n)}$ containing F. They define the two compositions:

$$\varphi_1 : \mathrm{Sec}_{s,n} \to \mathbb{P}(\mathscr{S}) \to \mathbb{P}^{N(n)};$$
$$\varphi_2 : \mathrm{Sec}_{s,n} \to \mathbb{P}(\mathscr{S}) \to G(s-1, \mathbb{P}^{N(n)}).$$

The fibre $\varphi_1^{-1}(F)$ is mapped under π to a closed subvariety $X_s(F)$ of $\overline{(V_{r,n})^{(s)}}$. Its intersection with $U \cap (V_{r,n})^{(s)}$ is equal to the variety of polar s-polyhedrons of F that we denote by $X_s(F)^\circ$. Thus we may call the points in the complement $X_s(F) - X_s(F)^\circ$ *degenerate polar s-polyhedra* of F.

Under p_2 the variety $X_s(F)$ is mapped to the subvariety $Y_s(F)$ of the Grassmann variety. The restriction of this map to $X_s(F)^\circ$ assigns to any polar s-polyhedron the $(s-1)$-secant plane spanned by it.

(4.1.2) EXAMPLE (cf. [ACGH, p. 136]). Assume $r=1$, i.e., we deal with binary forms of degree n. In this case the Veronese variety $V_{1,n}$ is a rational normal curve R_n of degree n in $\mathbb{P}^n$. The symmetric product $(V_{1,n})^{(s)}$ is isomorphic to the projective space $|S^s(V^*)| \cong |\mathscr{O}_{\mathbb{P}_1}(s)| \cong \mathbb{P}^s$. Its points can be viewed as effective divisors $D = \sum n_i x_i$ of degree s on R_n. The open subset $U \cap (V_{1,n})^{(s)}$ is equal to the open Zariski subset parametrizing reduced divisors $\sum x_i$ on R_n. The map $(V_{1,n})^{(s)} \cap U \to G(s-1, \mathbb{P}^n)$ can be extended to the whole $(V_{1,n})^{(s)}$ as follows. If $D = \sum n_i x_i$, where $x_1, \ldots, x_k$ are distinct, then $f(D)$ is the $(s-1)$-plane in $\mathbb{P}^n$ spanned by osculating (n_i-1)-planes of the points x_i. This easily implies that $\overline{(V_{1,n})^{(s)}} = (V_{1,n})^{(s)}$ in this case. By explicit computation of osculating spaces of a rational normal curve, it is easy to see that

$$f(D) = \mathrm{span}\,\{l_1^{n-n_1+1}\varphi_1, \ldots, l_k^{n-n_k+1}\varphi_k\},$$

where x_i corresponds to the powers l_i^n of linear forms, and φ_i are binary forms of degree n_i+1. So, a polar s-polyhedron (resp. degenerate polar s-polyhedron) of F is a reduced divisor (resp. non-reduced) of degree s such that $F \in f(D)$.

(4.2) DEFINITION. A set of points $\Sigma = \{a_1, ..., a_k\}$ in $\mathbb{P}^r$ is said to be in *general position* with respect to hypersurfaces of degree n if the projective dimension of the linear system $|\mathcal{O}_{\mathbb{P}^r}(n) - \Sigma|$ of hypersurfaces passing through Σ is of dimension $N(n)-k$.

(4.2.1) LEMMA. *Let $\Sigma = \{a_1, ..., a_k\}$ be a finite set of points in $\mathbb{P}^r$. The following properties are equivalent:*

(i) *Σ is in general position with respect to hypersurfaces of degree n;*

(ii) *the enveloping hypersurfaces $\check{H}^n_{a_i}$, $i=1, ..., k$, are linearly independent.*

Proof. We know that $a \in F$ if and only if $\langle \check{H}^n_a, F \rangle = 0$. Thus the space of hypersurfaces passing through Σ is dual to the space of enveloping hypersurfaces spanned by $\check{H}^n_{a_i}$, $i=1, ..., k$. This simple remark proves the assertion.

We shall give some useful conditions for powers $l_1^n, ..., l_s^n$ of linear forms to be linearly independent.

(4.2.2) LEMMA. *Suppose no $k \leqslant r+1$ of the linear forms $l_1, ..., l_s \in V^*$ are linearly dependent. Then $l_1^n, ..., l_s^n$ are linearly independent whenever $s \leqslant nr+1$.*

Proof. Suppose

$$F = a_1 l_1^n + ... + a_s l_s^n = 0,$$

where all $a_i \neq 0$. If $s \leqslant r+1$, we should show that this is impossible for all $n \geqslant 1$. Since $l_1, ..., l_s$ are linearly independent, we can find a point at which all l_i, except one l_j, vanish. Plugging this point into the left-hand side, we get that $a_j = 0$. So, the assertion is true for $s \leqslant r+1$. Now we use induction on s. Suppose $s > r+1$. Let x be the point with

$$l_{s-r+1}(x) = ... = l_s(x) = 0.$$

Then

$$P_s(F) = a_1 l_1(x) l_1^{n-1} + ... + a_{s-r} l_{s-r}(x) l_{s-r}^{n-1} = 0.$$

Since $s-r \leqslant (n-1)r+1$, by the induction hypothesis $l_i(x) = 0$ for each $i = 1, ..., s-r$. Thus each set of $r+1$ linear forms l_i has a common zero; hence it is linearly dependent. This contradicts the assumption.

(4.2.3) LEMMA. *Suppose $l_1, ..., l_s$ are s linear forms no two of which are proportional. Then $l_1^n, ..., l_s^n$ are linearly independent whenever $s \leqslant n+1$.*

Proof. The same argument as above that uses induction on s.

(4.2.4) LEMMA. *Assume $r=2$ and $n \geqslant 2$. Let $l_1, ..., l_s$ be linear forms no two of which are proportional, and let $H_1, ..., H_s$ be the corresponding lines. Assume $s \leqslant 2n+2$. Then $l_1^n, ..., l_s^n$ fail to be linearly independent if and only if $s \geqslant n+2$ and either $n+2$ of the lines H_i are concurrent or $s=2n+2$ and all H_i lie on an enveloping conic.*

Proof. By Lemma (4.2.1), $l_1^n, ..., l_s^n$ are linearly independent if an only if $H_1, ..., H_s$ impose independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(n)|$. This easily implies the sufficiency. One immediately verifies the necessity in the case $n=2$ and, by the preceding lemma, in the case $s \leqslant n+1$. Now suppose that no $n+2$ of the lines H_i are concurrent, and neither $s=2n+2$ nor H_i lie on a conic. We prove the lemma by induction on n. Let $n \geqslant 3$ and

$$F = a_1 l_1^n + ... + a_s l_s^n = 0,$$

where all $a_i \neq 0$ (i.e., we take a linear combination with minimal number of summands). Let k be the maximal number of concurrent lines. It does no harm to assume that $l_{s-k+1}(x) = ... = l_s(x) = 0$ for some point x. As in the proof of (4.2.2):

$$P_x(F) = a_1 l_1(x) l_1^{n-1} + ... + a_{s-k} l_{s-k}(x) l_{s-k}^{n-1} = 0.$$

Since $k \geqslant 2$ we have $s-k \leqslant 2(n-1)+2$. Furthermore $k \leqslant n+1$ and $s-k \geqslant 1$ by assumption and by (4.2.3). If $l_1^{n-1}, ..., l_{s-k}^{n-1}$ are linearly independent then $l_i(x)=0$ for all $1 \leqslant i \leqslant s$ which contradicts the assumption that no $n+2$ of the lines H_i are concurrent. So, by induction, $s-k \geqslant n+1$ and either $n+1$ of the lines $H_1, ..., H_{s-k}$ are concurrent or $s-k=2(n-1)+2=2n$ and all $H_1, ..., H_{2n}$ lie on an envelopping conic Φ. Suppose the former case occurs. Then $k=n+1$ and $s=2n+2$ since otherwise $s-k \leqslant (n-1)+1$ and $l_1^{n-1}, ..., l_{s-k}^{n-1}$ would be linearly independent by (4.2.3). Thus $s-k \leqslant n+1$ and $H_1, ..., H_s$ lie on a reducible enveloping conic ($H_1, ..., H_{n+1}$ lie on one of its components and the remaining ones on the other component). If the second case occurs we polarize F with respect to Φ to obtain

$$P_\Phi(F) = a_{s-1} \Phi(l_{s-1}) l_{n-2}^{s-1} + a_s \Phi(l_s) l_{n-2}^{s-2} = 0.$$

By hypothesis, at least one of $\Phi(l_{s-1})$ or $\Phi(l_s)$ is not zero. Therefore either a_{s-1} or a_s is zero. This contradicts our assumption that all $a_i \neq 0$.

We leave it to the reader to state the dual assertions of the previous

lemmas that give some conditions for a set of points to be in general position with respect to hypersurfaces of degree n.

(4.2.5) EXAMPLES. Any set of s distinct points $a_1, ..., a_s$ in $\mathbb{P}^1$ is always in general position. The linear system $|\mathcal{O}_{\mathbb{P}^1}(n) - a_1 - ... - a_s|$ consists of divisors $D + a_1 + ... + a_s$, where $\deg(D) = n - s$, and has dimension $n - s$. A set Σ in $\mathbb{P}^r$ is in general position with respect to hyperplanes if Σ spans a $(k-1)$-plane in $\mathbb{P}^r$. Three points in $\mathbb{P}^2$ are always in general position with respect to curves of any degree $n \geqslant 2$ (4.2.3). Four points in $\mathbb{P}^2$ are in general position with respect to conics if and only if they are not collinear (4.2.4). Eight points in $\mathbb{P}^2$ are in general position with respect to cubics if and only if they are not on a conic (4.2.4). If two cubics intersect transversally at 9 points, then the set of these points is not in general position with respect to cubics.

(4.3) PROPOSITION. *Let* $\Sigma = \{\alpha_1, ..., \alpha_s\} \subset \check{\mathbb{P}}^r$, $F \in S^n(V^*)$. *Then* Σ *is a polar* s*-polyhedron of* F *iff*

(i) $H^n_{\alpha_1}, ..., H^n_{\alpha_s}$ *are linearly independent forms of degree* n;

(ii) $|\mathcal{O}_{\check{\mathbb{P}}^r}(n) - \alpha_1 - ... - \alpha_s| \subset AP_n(F)$;

(iii) $|\mathcal{O}_{\check{\mathbb{P}}^r}(n) - \Sigma'| \not\subset AP_n(F)$ *for any proper subset* Σ' *of* Σ.

Proof. Let us first prove the sufficiency. Assume (i) and (ii) hold. Then

$$\dim |\mathcal{O}_{\check{\mathbb{P}}^r}(n) - \alpha_1 - ... - \alpha_s| = N(n) - s;$$

hence the dimension of the linear system of hypersurfaces of degree n apolar to $|\mathcal{O}_{\check{\mathbb{P}}^r}(n) - \alpha_1 - ... - \alpha_s|$ is equal to s. Since $H^n_{\alpha_i}$ belong to this system (see (2.2)), and are linearly independent,

$$F = \sum_{i=1}^{s} \lambda_i l_i^n$$

for some $\lambda_i \in C$, where $H_i: l_i = 0$. If some $\lambda_i = 0$, then F is apolar to any $\Phi \in |\mathcal{O}_{\check{\mathbb{P}}^r}(n) - \Sigma'|$ for some proper subset Σ' of Σ which contradicts (iii). Replacing l_i by $l_i \sqrt[n]{\lambda_i}$, we obtain that Σ is a polar s-polyhedron of F.

Conversely, let Σ be a polar s-polyhedron of F. By Example (2.2) for any $\Phi \in |\mathcal{O}_{\check{\mathbb{P}}^r}(m) - \alpha_1 - ... - \alpha_s|$, and $\Phi' \in S^{n-m}(V)$ we have $\langle \Phi\Phi', H^m_{\alpha_i} \rangle = 0$. This gives $\langle \Phi\Phi', F \rangle = 0$; hence F is apolar to Φ. This checks (ii). Property (i) holds by definition. If (iii) does not hold for some proper subset Σ' of Σ, then by the first part of the proof Σ' is a polar s'-polyhedron of F. This obviously contradicts (i).

(4.4) LEMMA. *Let* $H_1, ..., H_s$ *be a polar* s*-polyhedron of* F. *Assume*

$\Phi \in |S^m(V)|$ passes through the first $k \leqslant s$ hyperplanes and does not pass through any of the remaining hyperplanes (considered as points of $\check{\mathbb{P}}^r$). Furthermore, suppose that $l_{k+1}^{n-m}, \ldots, l_s^{n-m}$ are linearly independent. Then $\{H_{k+1}, \ldots, H_s\}$ is a polar $(s-k)$-polyhedron of the polar $P_\Phi(F)$ of F with respect to Φ.

Proof. Let

$$F = l_1^n + \ldots + l_s^n.$$

Then

$$P_\Phi(F) = P_\Phi(l_1^n) + \cdots + P_\Phi(l_s^n) = \Phi(l_{k+1}) l_{k+1}^{n-m} + \ldots + \Phi(l_s) l_s^{n-m}).$$

Clearly this proves the assertion.

(4.5) EXAMPLE. Assume $r = 1$, i.e., we deal with binary forms of degree n. Let $H_1, \ldots, H_s$ be a polar s-polyhedron (better, an s-tuple) of F. Then any envelope $\Phi \in |S^n(V)|$ passing through $H_1, \ldots, H_s$ is represented in the form $\check{H}_1 \ldots \check{H}_s \Phi'$. Thus F is apolar to such Φ if and only if $\langle \check{H}_1 \ldots \check{H}_s \Phi', F \rangle = 0$ for any $\Phi' \in |S^{n-s}(V)|$, i.e., $\Phi'' = \check{H}_1 \ldots \check{H}_s$ is apolar to F. Conversely, if $\Phi'' \in |S^s(V)|$ is apolar to F, and Φ'' has only simple roots, then the set of its roots is a polar s-tuple of F. Since $AP_s(F) \neq \{0\}$ for $s > n/2$, we obtain that a general F admits polar s-tuples for any $s > n/2$ (and, of course, $s \leqslant n+1$). If $n = 2k$, F admits a polar k-tuple if and only if the catalecticant of F vanishes. It follows from the construction that the variety $X_s(F)$ of polar s-tuples (including degenerate ones) is isomorphic to the projective space $\mathbb{P}(AP_s(F))$. For general F its dimension is equal to $2s - n - 1$. In particular, we see that a general $F \in |S^{2k+1}(V^*)|$ admits a unique polar $(k+1)$-tuple. It corresponds to the zeroes of the unique apolar of F of class $k+1$ (see [Ku-Ro, El] for details).

In the next two sections we shall study the problem of representation of ternary forms of degree $\leqslant 4$ as a sum of powers of linear forms. We refer to [Ri, Di-St, Di] for some results in the case of ternary forms of higher degree.

5. QUADRICS AND CUBICS

Here we shall study all the previous notions in the case $n \leqslant 3$.

(5.1) The polarization of a quadric $Q \in S^2(V^*)$ is the associated bilinear form

$$b_Q \in \mathrm{Sym}_2(V) = S^2(V)^*,$$

which can be defined by the formula:

$$b_Q(a, b) = \tfrac{1}{2}(Q(a+b) - Q(a) - Q(b)).$$

The polar $P_a(Q)$ is the linear form:

$$b \mapsto b_Q(a, b).$$

In coordinates,

$$Q(x) = \sum h_{ij} x_i x_j,$$

where $\|h_{ij}\|$ is a symmetric scalar matrix equal to $H(v)$ for any $v \in V$, and

$$P_a(Q) = \sum a_i a_{ij} x_j,$$

$$He(Q) = \operatorname{discr}(Q) = |h_{ij}|.$$

The Hessian of Q is the whole space if Q is degenerate, and empty otherwise. The Steinerian of Q is the singular locus of Q (note that we consider the zero linear form as a singular linear form!).

The catalecticant of a quadratic polynomial is equal to its discriminant. The apolar linear envelopes $\check{H} \in \check{\mathbb{P}}^r$ of a degenerate quadric Q correspond to points $a \in \mathbb{P}^r$ contained in the singular locus of Q.

(5.2) From now on in this section we assume that $n = 3$. Let

$$F \in S^3(V^*)$$

be a cubic form. Its first polar $P_a(F)$ is a quadratic form, and its second mixed polar form $P_{a,b}(F)$ is a linear form. The total polarization $\tilde{F} = P_{a,b,c}(F)$ is a trilinear symmetric form

$$\tilde{F} \in \operatorname{Sym}_3(V) = S^3(V)^*,$$

that can be given by the formula:

$$\tilde{F}(a, b, c) = \tfrac{1}{6}[F(a+b+c) - F(a+b) - F(a+c) - F(b+c) + F(a) + F(b) + F(c)].$$

In coordinates, write F in the form

$$F = \sum_{i,j,k} (i, j, k)_3 h_{ijk} x_i x_j x_k,$$

where h_{ijk} is symmetric in the subscripts $i, j, k \in \{0, ..., r\}$, and $(i, j, k)_3$

takes the value 1 if all indices are equal, 3 if only two indices are equal, and 6 if all indices are distinct. Then

$$P_a(F) = \sum h_{ijk} a_i x_j x_k,$$

$$P_{a,b}(F) = \sum h_{ijk} a_i b_j x_k,$$

$$P_{a,b,c}(F) = \sum h_{ijk} a_i b_j c_k.$$

(5.2.1) The Hessian $He(F)$ of a cubic hypersurface is equal to its Steinerian $St(F)$. This immediately follows from Proposition (3.4). The Steiner correspondence

$$X(F) = \{(a, b) \in \mathbb{P}^r \times \mathbb{P}^r : P_{a,b}(F) = 0\}$$

is symmetric. In other words,

$$b \in \mathrm{Sing}(P_a(F)) \Leftrightarrow a \in \mathrm{Sing}(P_b(F)).$$

By Lemma (3.3), the Steiner correspondence $X(F)$ has no united points (i.e., points on the diagonal) if and only if F is a non-singular hypersurface.

(5.3) The set of lines $\langle a, b \rangle$, where $a \in \mathrm{Sing}(P_b(F))$ is an $(r-1)$-dimensional subvariety of the Grassmanian variety $G(2, r+1)$. It is called the *Caylerian variety* of F. One can view this variety as the quotient of the Hessian by the involution s_F. Each such a line can be characterized by the condition that the linear system of polar quadrics $P_a(F)$, $a \in \mathbb{P}^r$, containing this line is of codimension $\geqslant 2$ (instead of codimension 3 for a general line). Indeed, $P_{a,b}(F) = 0$ if and ony if $P_{a,b}(P_x(F)) = 0$ for all $x \in \mathbb{P}^r$; hence $P_x(F)(a) = P_x(F)(b) = 0$ implies that $P_x(F)(c) = 0$ for all $c \in \langle a, b \rangle$. In other words, if a polar quadric $P_x(F)$ passes through the points a and b, it contains the whole line $\langle a, b \rangle$. For every r-dimensional linear system of L of quadrics in $\mathbb{P}^r$ the variety of lines satisfying this property is called the *Reye variety* of L. For example, if $r = 3$, and L is general enough, the Reye variety is isomorphic to an Enriques surface (cf. [Co]).

(5.4) Assume $r = 2$, i.e., F is a cubic curve. Its Hessian is a cubic curve or the whole plane. Applying (3.8) we find that the latter happens if and only if F is a cone, i.e., consists of three concurrent lines (some of them may coincide).

(5.4.1) LEMMA. *Assume $P_a(F) = l^2$ is a double line. Then*

(i) *if $l(a) \neq 0$, F is either a cone, or a cuspidal cubic, or F is projectively isomorphic to the Fermat cubic $x_0^3 + x_1^3 + x_2^3 = 0$;*

(ii) *if $l(a)=0$, F is either a cuspidal cubic, or the union of a smooth conic and its tangent line, or a cone with a double irreducible component.*

Proof. (i) Changing coordinates we may assume that $a=(1,0,0)$ and $l=x_0$. Then

$$P_a(F)=3^{-1}\,\partial F/\partial x_0=x_0^2.$$

This implies that $F-x_0^3=G(x_1,x_2)$ is independent of x_0. By further coordinate change we may assume that one of the following cases occurs:

$$G=x_1^3+x_2^3, \qquad \text{or } x_1^2x_2, \qquad x_1^3, \qquad \text{or } 0$$

(since three points on a projective line do not have moduli). Hence

$$F=x_0^3+x_1^3+x_2^3, \qquad \text{or } x_1^2x_2+x_0^3, \qquad \text{or } x_0^3+x_1^3, \qquad \text{or } x_0^3,$$

i.e., the Fermat cubic, a cuspidal cubic, or a cone.

(ii) In this case, by changing coordinates, we may take $a=(1,0,0)$, $l=x_1$. Therefore

$$P_a(F)=3^{-1}\,\partial F/\partial F x_0=x_1^2,$$

and

$$F=3x_0x_1^2+G(x_1,x_2),$$

where G is a homogeneous form of degree3, or 0. The point $(1,0,0)$ is a double point with the tangent cone equal to x_1^2. Now the statement follows from the known classification of cubic curves.

(5.4.2) PROPOSITION. *Let F be a non-singular cubic curve not isomorphic to the Fermat cubic $x_0^3+x_1^3+x_2^3=0$. Then its Hessian is non-singular, and the Steiner involution has no fixed points.*

Proof. The first fact can be verified by direct computation after reducing the cubic to its Weierstrass form, or better, to its Hesse form

$$F=x_0^3+x_1^3+x_2^3+6\lambda x_0x_1x_2=0$$

(see [We, p. 399]). Here the cubic is non-singular if and only if $8\lambda^3\neq -1$, and is isomorphic to the Fermat cubic if and only if $\lambda^4\neq\lambda$. Computing the Hessian, we easily find

$$He(F)=-\lambda^2(x_0^3+x_1^3+x_2^3)+(1+2\lambda^3)\,x_0x_1x_2=0,$$

which is always non-singular, unless $\lambda^4 = \lambda$. The second assertion follows from Lemma (3.3).

(5.4.3) By explicit computation, it is easy to verify directly the following facts. Assume F is irreducible but singular. Then its Steiner map is defined everywhere if it is a nodal curve, and is not defined at the cusp point, if it is a cuspidal curve. The Hessian is a nodal cubic curve in the first case, and the union of a line l_1 and a double line l_2^2 in the cuspidal case. The line l_1 joins the cusp and the unique non-singular inflection point. The line l_2 is the cuspidal tangent. The polar of the cuspidal cubic with respect to the cusp is l_2^2. The polar of the cubic with respect to the point of intersection of the inflectional tangent with l_2 is equal to l_1^2.

(5.4.4) Assume F is non-singular, and is not isomorphic to the Fermat cubic. Then its Hessian is a non-singular cubic C together with a fixed-point-free Steinerian involution s_F. As is well known, the latter is defined by a 2-torison point $\eta \in \mathrm{Pic}(C)$. By fixing a group law on C, this involution is a translation t_η by a non-trivial point of order 2 corresponding to η. Thus for any point $a \in C$ the polar $P_a(F)$ is the union of two lines intersecting at the point $b = t_\eta(a) \in C$. The line $\langle a, b \rangle$ is a Reye line (5.3); hence it is a component of a singular conic $P_c(F)$ for some $c \in C$. One can show that c is the intersection point of the tangents of C at the points a and b, respectively. The singular point of $P_c(F)$ is the third point of intersection of $\langle a, b \rangle$ and C.

(5.5) Let W be a linear space of dimension $k+1$ and $N \in V^* \otimes S^2(W^*)$. One can view N as a linear map $N: V \to S^2(W^*)$; the image of each point $a \in V$ is a quadric $N(a) \in S^2(W^*)$. Abusing the notation we shall denote the projectivization of N by the same letter and call it an r-dimensional hypernet of quadrics in $\mathbb{P}(W)$. If N is injective this defines an r-dimensional linear subsystem of $|\mathcal{O}_{\mathbb{P}(W)}(2)|$. Let

$$X(N) = \{(a, b) \in \mathbb{P}^r \times \mathbb{P}^k : b \in \mathrm{Sing}(N(a))\}.$$

We call $X(N)$ the Steiner correspondence of N. The hypernet N is given by a symmetric matrix A of size $k+1$ whose entries are linear forms on V. The Hessian subscheme $H(N)$ of $\mathbb{P}^r$ is defined by the equation $\det(A)(v) = 0$. We assume that $H(N) \neq \mathbb{P}^r$, and $H(N)$ is reduced. Then $H(N)$ is equal to the image of $X(N)$ under the first projection $p_1: X(N) \to \mathbb{P}^r$. We define the Steinerian variety $S(N) \subset \mathbb{P}^k$ as the image of $X(N)$ under the second projection $p_2: X(N) \to \mathbb{P}^k$. Note that this terminology is opposite to the one used for polars, but agrees with it if $\deg(F) = 3$. We refer to [B, Ty] for the theory of linear systems of quadrics. If a is a non-singular point of $H(N)$, the quadric $N(a)$ is of corank 1 (see (3.6)). This allows one to define the Steiner map $s_N: H(N)^{ns} \to S(N)$ on the set of non-singular points of

$H(N)$; it sends a point a to the vertex of the quadric $N(a)$. If $a \in H(N)^{ns}$ is a nonsingular point of $H(N)$, the coordinates of the vertex of the quadric $N(a)$ can be given by the complementary minors of any column of A. One can deduce from this that the Steiner map is given by a linear system $L \subset |D|$, where the linear span of $L + L$ in $|2D|$ is cut out by a linear system of adjoints of $H(N)$ of order k corresponding to k-minors of the matrix A [Be, p. 367].

Assume now that $W = V$. We want to characterize linear systems of quadrics $N: V \to S^2(V^*)$ which arise from polars $P_a(F)$ of a cubic hypersurface.

(5.5.1) PROPOSITION. *Let $N \in V^* \otimes S^2(V^*)$ be an r-dimensional linear system of quadrics in $\mathbb{P}(V) = \mathbb{P}^r$ with reduced Hessian hypersurface and let $\tilde{N} \in V^* \otimes \mathrm{Sym}_2(V)$ induce N. Let $s: H(N)^{ns} \to P(V)$ be the Steiner map defined above. Then N is equal to the linear system of polars of a cubic hypersurface F if and only if the equality*

$$\tilde{N}(w, a, s(a)) = 0 \qquad (*)$$

holds for any $a \in H(N)^{ns}$ and any $w \in V$. Moreover F is non-singular if and only if the correspondence $X(N)$ is without united points.

Proof. If $\tilde{N} = \tilde{F}$, where F is a cubic form, then for any $a \in H(N)^{ns}$ we have by the symmetry of $\tilde{F}$ and by the definition of the Steiner map

$$\tilde{F}(w, a, s(a)) = \tilde{F}(a, s(a), w) = \tilde{N}(a, s(a), w) = 0.$$

Now suppose that condition $(*)$ holds. It suffices to prove that

$$\tilde{N}(p, q, r) = N(q, p, r) \qquad (**)$$

for arbitrary r and for every sufficiently general $p, q \in V$. Let $W = \langle p, q \rangle \subset V$ and let $\ell = P(W)$. We can suppose that the line ℓ intersects $H(N)$ transversally in $r+1$ points $p_0, \ldots, p_r$. It is shown in [Be, p. 368], that the points $s(p_0), \ldots, s(p_r)$ are linearly independent. Let us choose coordinates $x_0, \ldots, x_r$ such that $x_i(s(p_j)) = 0$ for $i \neq j$. Then (see *loc. cit.*) by restriction N induces a pencil of quadrics $N_0 \in W^* \otimes S^2(V^*)$ which has diagonal form

$$N_0(u, x) = \sum_{i=0}^{r} \alpha_i(u) x_i^2,$$

where α_i are non-zero linear forms vanishing at p_i. Abusing the notation let us denote by $\{p_i\}$, $\{s(p_i)\}$ some non-zero elements of V that generate the

corresponding lines in $\mathbb{P}(V)$ and such that $\{s(p_i)\}$ forms a dual basis to $\{x_i\}$. Let

$$\tilde{N}_0(u, x, y) = \sum_{i=0}^{r} \alpha_i(u) x_i y_i$$

be the polarization of N_0 with respect to x. Both sides of $(**)$ are linear in r, so it suffices to verify it for any $r = s(p_i)$. Let β_i be the restriction of the coordinate x_i to W. The formula for $\tilde{N}_0$ shows that $\tilde{N}_0(u, v, s(p_i))$ is symmetric with respect to $u, v \in W$ if and only if β_i is proportional to α_i. So we have to prove that $x_i(p_i) = 0$ or, equivalently, that for every i the hyperplane $\langle s(p_0), \ldots, s(p_{i-1}), s(p_{i+1}), \ldots, s(p_r)) \rangle$ contains p_i. We have for every $u \in W$

$$\tilde{N}_0(u, p_i, s(p_i)) = \alpha_i(u) x_i(p_i).$$

The left-hand side equals 0 by hypothesis and $\alpha_i(u) \neq 0$ for general $u \in W$. Thus $x_i(p_i) = 0$. Finally, we use (3.3)(v) to verify that F is non-singular if and only if $X(N)$ is without united points.

(5.6) From now on we assume that $r \geqslant 2$ and that N is general enough so that $H(N)$ is a normal hypersurface. We assume also that $s_N(H(N)^{ns})$ is not contained in a quadric in $\mathbb{P}^k$. This condition is satisfied in the case $r = 2$ [Be, Ty]. By the above remark about adjoints of $H(N)$, we have an injective map:

$$s_N^*: |S^2(W^*)| \to |S^k(V^*)|.$$

We denote by ρ_N one of its linear liftings:

$$\rho_N: S^2(W^*) \to S^k(V^*).$$

(5.6.1) LEMMA. *The tensor*

$$T = (1 \otimes \rho_N)(N) \in V^* \otimes S^k(V^*)$$

defines the linear system of polars of the Hessian H of N. Equivalently, $T \in \mathrm{Sym}_{k+1}(V) \subset V^* \otimes S^k(V^*)$, and $r_{k+1}(T) = H$. Any two linear systems of quadrics satisfying the conditions imposed above, and which have the same Hessian and the same Steiner map, coincide.

Proof. The proof is a straightforward generalization of the case $r = 2$ considered in [Be, Ty]. Let $x \in H^{ns}$ be a non-singular point of H. Then the points of the hyperplane tangent to H at x correspond to the quadrics from N that contain $s_N(x)$ [Be, p. 364]. This immediately implies that

$s_N(x)$ is not a base point, and also that for a general point $p \in \mathbb{P}^r$ such that $s_N(x) \in N(p)$, the line $\langle p, x \rangle$ is tangent to H at x. Thus, by (1.3)

$$s_N^{-1}(N(p)) \cap H^{ns} = P_p(H) \cap H^{ns}.$$

Now, by the argument from above we find that the base locus of the linear system $s_N^*(N) \subset |\mathcal{O}_H(k)|$ is contained in $\mathrm{Sing}(H(N))$; therefore it has no fixed components since $H(N)$ is normal. So, the general member of $s_N^*(N)$ on $H(N)$ is reduced by Bertini's theorem. We have

$$s_N^*(N(p)) = P_p(H) \cdot H$$

and therefore

$$s_N^*(N(p)) = P_p(H),$$

since H is normal of degree $k+1$. The uniqueness property follows from the injectivity of $(1 \otimes \rho_N)$.

(5.6.2) *Remark.* In the case $r = 2$ the linear system $s_N^* |\mathcal{O}_{\mathbb{P}^k}(1)|$ is complete and equals $|L + \vartheta_N|$, where L is the hyperplane section class of $H(N)$ and ϑ_N is an even non-vanishing theta characteristic of $H(N)$ (i.e., $2\vartheta_N = K_H$ and $|\vartheta_N| = \varnothing$). The pair $(H(N), \vartheta_N)$ is called the *Hessian invariant* of N, and it determines N uniquely up to $PGL(W)$-isomorphism. Conversely, given:

(i) a non-singular plane curve $H \subset \mathbb{P}^2$ of degree $k+1$;

(ii) a non-vanishing even theta characteristic ϑ;

(iii) a map $s: H \to \mathbb{P}^k = \mathbb{P}^k = \mathbb{P}(W)$ with $s^* |\mathcal{O}_{\mathbb{P}^k}(1)| = |L + \vartheta|$,

one obtains an isomorphism $\rho = s^*: (W^*) \to S^k(V)$, and defines the net $N = (1 \otimes \rho)^{-1} T$, where T is the polar net of H. Then the Hessian hypersurface and the Steiner map of N coincide with H and s, respectively (see [Be, Ty]).

(5.6.3) *Remark.* In view of (5.6.1) one could ask whether the conditions that the Steiner map transforms $H(N)^{ns}$ to $H(N)$ and is a birational involution are sufficient for a linear system of quadrics $N \in V^* \otimes S^2(V^*)$ with normal Hessian to be the system of polars of a cubic hypersurface. This is not true in general as the following example shows. Consider a cubic hypersurface F has a projective involution σ and, moreover, the Steiner map $s: H(F)^{ns} \to H(F)$ which is a birational involution. Define a linear system of quadrics by

$$N(x, y) = \tilde{F}(\sigma(x), y, y) = 0.$$

One easily shows that $H(N)=H(F)$ and the Steiner map of N is $s_1=\sigma\circ s=s\circ\sigma$. Now, one has $N(w,x,s(x))=0$ for every $w\in V$, so if N were symmetric one would have $N(x,\sigma(s_1(x)),w)=0$ for every $w\in V$ and every $x\in H(N)^{ns}$. Thus $s_1=\sigma\circ s_1$ which is impossible.

(5.7) Let us return to plane cubic curves ($r=2$). Denote by M the open subset of $|S^3(V^*)|$ that consists of non-singular cubic curves, and let $M'\subset M$ be the complement to the subset of cubics that are projectively isomorphic to the Fermat cubic. Let $\tilde{M}\to M$ be the étale covering of degree 3 parametrizing the pairs (C,η), where $C\in M$ and η is a non-zero 2-torsion class in $\mathrm{Jac}(C)$. We have constructed a map

$$h\colon M'\to\tilde{M},$$

that sends a cubic F to the Hessian invariant $(\mathrm{Hess}(F),\eta_F)$ of its net of polar conics.

(5.7.1) THEOREM. *The map $h\colon M'\to\tilde{M}$ is an isomorphism.*

Proof. Let $h(F)=(C,\eta)$. By (5.4.4), the Steiner map $s_F\colon C\to C$ of the net of polars of F equals $t_{\eta'}$ for some 2-torsion point η'. We claim that $\eta'=\eta$. In fact, let $L=x_1+x_2+x_3\in|\mathscr{O}_C(1)|$. Then $s_F^*(L)\in|L+\eta|$ by definition, and we obtain

$$\begin{aligned}s_F^*(L)&=s_F(x_1)+s_F(x_2)+s_F(x_3)\\&=t_{\eta'}(x_1)+t_{\eta'}(x_2)+t_{\eta'}(x_3)\sim L+\eta'.\end{aligned}$$

Therefore $\eta=\eta'$. By Lemma (5.6.1) this shows that $h\colon M'\to\tilde{M}$ is injective. To prove the theorem it suffices to construct the inverse map h^{-1}. Let $(C,\eta)\in\tilde{M}$. The translation map $t_\eta\colon C\to C$, $c\to|c+\eta|$, defines an isomorphism:

$$t_\eta^*\colon|\mathscr{O}_C(2)|\cong|\mathscr{O}_{\mathbb{P}^2}(2)|\to|\mathscr{O}_C(2)|\cong|\mathscr{O}_{\mathbb{P}^2}(2)|.$$

Indeed, if $D=x_1+\ldots+x_6$ is a divisor on C cut out by a unique conic, then

$$t_\eta^*(x_1+\ldots+x_6)=(x_1+\eta)+\ldots+(x_6+\eta)\sim x_1+\ldots+x_6$$

is also cut out by a unique conic. Let ρ_η be a lift of t_η^* to a linear isomorphism:

$$\rho_\eta\colon S^2(V^*)\to S^2(V^*).$$

Let $T\in V^*\otimes S^2(V^*)$ be the first polarization of the equation of C, and let

$$N=(1\otimes\rho_\eta)(T)\in V^*\otimes S^2(V^*).$$

We shall show now that the tensor N defines a net of conics with Hessian C and Steiner map t_η; furthermore it coincides with the net of polars of a unique cubic curve F. We define the inverse map h^{-1} so that it sends (C, η) to F. Following [Be, pp. 374-375], let us consider ρ_η as an element of $S^2(V^*) \otimes S^2(V)$. It defines a quadratic system of conics in $\mathbb{P}(V^*)$. Then as shown in *loc. cit.* the dual system of conics is determined by a product of $N_1 \in V^* \otimes S^2(V^*)$ and a power of the cubic form that vanishes on C. The net of conics N_1 has Hessian equal to C and the Steiner map is t_η. By Lemma (5.6.1) we conclude that $N = N_1$. Now let us show that $\tilde{N} \in V \otimes \mathrm{Sym}_2(V)$ is symmetric. Indeed, by the proof of Proposition (5.5.1) one has to verify that for every line ℓ that intersects transversally C in p_1, p_2, and p_3 the points $t_\eta(p_1)$, $t_\eta(p_2)$, and p_3 are collinear. This holds since

$$t_\eta(p_1) + t_\eta(p_2) + p_3 \sim p_1 + \eta + p_2 + \eta + p_3 \sim p_1 + p_2 + p_3 \in |\mathcal{O}_C(1)|.$$

(5.7.2) COROLLARY. *The variety $\tilde{M}$ of plane non-singular cubics together with an even theta characteristic (= a non-trivial 2-torsion divisor class) is rational.*

(5.7.3) *Remark.* The isomorphism $M' \to \tilde{M}$ is $PGL(V)$-invariant. It descends to a birational isomorphism of the moduli spaces of elliptic curves $\mathcal{M}_1$ and the moduli space $\mathfrak{R}_1$ of pairs (E, η), where E is an elliptic curve and η is a 2-torsion point. Both of these spaces are rational curves (the modular curves H/Γ and $H/\Gamma_0(3)$, respectively), so the existence of such an isomorphism is not surprising. However, our isomorphism is canonical. More explicitly, if we employ the Hesse form

$$x_0^3 + x_1^3 + x_2^3 + 6\lambda x_0 x_1 x_3 = 0,$$

for a representative of an isomorphism class of an elliptic curve F, then, as we noticed already in (5.4.2), $He(F)$ is given by the equation:

$$-\lambda^2(x_0^3 + x_1^3 + x_2^3) + (1 + 2\lambda^3)x_0 x_1 x_2 = 0.$$

Thus the three cubic curves F_i which define the three nets of conics with the Hessian equal to F are

$$F_i: x_0^3 + x_1^3 + x_2^3 + 6\gamma_i x_0 x_1 x_3 = 0, \qquad i = 1, 2, 3,$$

where γ_i is one of the three roots of the equation:

$$2\gamma^3 + 6\gamma^2\lambda + 1 = 0.$$

(5.8) We will need more facts about the polar system of conics of the

non-singular cubic F projectively isomorphic to the Fermat cubic. It is clear that F can be written in the form

$$F: l_0^3 + l_1^3 + l_2^3 = 0, \qquad (*)$$

where $l_i = 0$ are the equations of three non-concurrent lines H_i. The Hessian of F is the union of these three lines. For every point $a \in H_0$ the polar conic $P_a(F)$ is the irreducible conic given by the equation:

$$l_1(a) l_1^2 + l_2(a) l_2^2 = 0.$$

It becomes the double line H_2^2 if $a \in H_0 \cap H_1$. The Steiner map sends each line H_i to the opposite vertex $H_j \cap H_k$. Conversely, let N be a net of conics whose Hessian is a triangle of non-concurrent lines $l_1 l_2 l_3 = 0$, and the conic corresponding to any vertex is equal to the double opposite side. Then N is equal to the polar net of a curve given by the sum of the cubes of some linear forms defining the sides H_i. Indeed, without loss of generality we may assume that the Hessian is given by $x_0 x_1 x_2 = 0$. Then

$$N((1, 0, 0)) = \lambda_0 x_0^2, \qquad N((0, 1, 0)) = \lambda_1 x_1^2, \qquad N((0, 0, 1)) = \lambda_2 x_2^2,$$

where λ_i are some non-zero scalars. This implies that

$$N((a_0, a_1, a_2)) = a_0 \lambda_0 x_0^2 + a_1 \lambda_1 x_1^2 + a_2 \lambda_2 x_2^2$$

which is equal to the polar of the cubic

$$F = \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = (x_0 \sqrt[3]{\lambda_0})^3 + (x_1 \sqrt[3]{\lambda_1})^3 + (x_2 \sqrt[3]{\lambda_2})^3 = 0$$

with respect to the point $a = (a_0, a_1, a_2)$.

(5.9) We see that a non-singular cubic F whose Hessian is equal to a given triangle $l_0 l_1 l_2 = 0$ of non-concurrent lines H_i, $i = 0, 1, 2$, is determined by three non-zero constants $\lambda_0, \lambda_1, \lambda_2$ such that its equation can be given in the form:

$$\lambda_0 l_0^3 + \lambda_1 l_1^3 + \lambda_2 l_2^3 = 0.$$

Let N_i be the pencil of binary quadrics that assigns to every $a \in H_i$ the intersection $H_i \cdot P_a(F)$. Let us see that these three pencils determine the coefficients λ_i, and hence determine uniquely the cubic curve F. For example, the pencil on H_0 is given by

$$Q_0(a) = \lambda_1 l_1(a)\, \bar{l}_1^2 + \lambda_2 l_2(a) \bar{l}_2^2 = 0, \qquad a \in H_0,$$

where $\bar{l}_i$ denotes the restriction of the linear form l_i to H_0. So, it determines uniquely the ratio $\delta_{12} = \lambda_1/\lambda_2$. Similarly the other two pencils determine

uniquely the ratios $\delta_{20}=\lambda_2/\lambda_0$ and $\delta_{01}=\lambda_0/\lambda_1$. Thus the three pencils determine $(\lambda_0, \lambda_1, \lambda_2)$ uniquely up to a scalar factor. As we observed in (5.8), the Steiner map of each N_i interchanges the two vertices p and q of the triangle lying on H_i. The linear subsystem of $|\mathcal{O}_{\mathbb{P}^1}(2)|$ determined by N_i is reconstructed by this property by observing that the latter implies that any non-singular member $\{r_1, r_2\}$ is harmonically conjugate with respect to $\{p, q\}$, i.e., the cross-ratio of (r_1, p, r_2, q) equals -1. Thus, given three non-current lines $H_i: l_i=0$, $i=0, 1, 2$, and two pencils on H_0 and H_1 whose Steiner map is as above, we define a pencil of quadrics on H_2 so that $\delta_{01}\delta_{12}\delta_{20}=1$. Then there is a unique cubic F whose Hessian is $H_0\cup H_1\cup H_2$ and the given pencils on each H_i coincide with the restriction of the pencils $\{P_a(F)\}_{a\in H_i}$ to H_i.

We shall now study polar polyhedra of quadrics and cubics.

(5.10) Assume $n=2$. A quadratic form F of rank k admits a polar k-polyhedron. This follows from the diagonalization process. Conversely, if F admits a polar k-polyhedron with $k\leqslant r+1$, then its rank is at most k.

(5.11) Let us assume that $n=r=2$. Since the dimension of the space of conics is 5, every conic admits a polar 6-gon. Let F be a conic of rank k. If $k=1$, $F=H^2$, i.e., it admits a 1-polygon. Assume it admits an s-polygon $H_1, \ldots, H_s$, $s>1$. Then $AP_2(F)$ consists of all enveloping conics vanishing at H. Therefore

$$|\mathcal{O}_{\mathbb{P}^2}(2)-H_1-\ldots-H_s|=|\mathcal{O}_{\mathbb{P}^2}(2)-H_1-\ldots-H_s-H|.$$

This shows that the lines $H_1, \ldots, H_s$, H are not in general position with respect to forms of degree 2. Assume $s\leqslant 5$. Applying Lemma (4.2.4), we find that either $s=3$ and H_1, H_2, H_3, and H are concurrent, or $s=5$ and $H_1, \ldots, H_5$, H are on an envelopping conic. These cases do occur as the following examples show:

$$x_0^2=\tfrac{1}{2}(x_0+x_1)^2-x_1^2+\tfrac{1}{2}(x_0-x_1)^2,$$
$$x_0^2=\tfrac{1}{2}(x_0+x_1)^2-\tfrac{1}{2}(x_1+x_2)^2-\tfrac{1}{2}(x_1-x_2)^2+x_2^2+\tfrac{1}{2}(x_0-x_1)^2.$$

A conic F of rank 2 has polar s-polygons for any $s\geqslant 2$. Since all conics of the same rank are projectively equivalent, it suffices to give examples of one such polygon. It is obvious for $s=4$ and $s=3$, by adding x_2^2 (resp. $2x_1^2$) to the first example above. Finally the case $s=5$ is treated by the following example:

$$x_0^2+x_1^2=(x_0+x_2)^2+(x_0+x_1)^2+(x_1+x_2)^2-(x_0+x_1+x_2)^2-x_2^2.$$

(5.11.1) Assume now that F is a non-singular conic. Then F admits ∞^3

polar triangles. This can be seen in different ways. For example, reducing F to the form

$$F = x_0^2 + x_1^2 + x_3^2 = 0,$$

we may apply any orthogonal transformation of variables $x_i \to l_i(x_0, x_1, x_2)$ to obtain

$$F = l_0^2 + l_1^2 + l_3^2.$$

This easily implies that the variety $X_3(F)$ of all polar triangles of F (a subvariety of the symmetric product of $\mathbb{P}^2$) is isomorphic to the coset space $PO(3)/H$, where $PO(3)$ is the projective orthogonal group in 3 variables, and H is its finite subgroup of order 12 consisting of permutations and diagonal matrices. It is easy to see that $PO(3)$ is isomorphic to the 3-dimensional group $PGL(2)$ (via the Veronese map); in its natural action on $|\mathcal{O}_{\mathbb{P}^1}(6)|$ the group H is identified with the stabilizer of the polynomial $x_0x_1(x_0^4 - x_1^4)$. This shows that $X_3(F)$ is isomorphic to the orbit of this polynomial. Note the closure of this orbit is smooth and is isomorphic to a Fano 3-fold of index 2 of degree 5 in $\mathbb{P}^6$ (see [Mu-Um]).

Since all triangles depend on six parameters, we may find a triangle $\{\alpha_1, \alpha_2, \alpha_3\} \subset \check{\mathbb{P}}^2$ such that neither pair of its sides are the sides of some polar triangle of F. We shall also require that any pencil in the net $|\mathcal{O}_{\mathbb{P}^2}(2) - \alpha_1 - \alpha_2 - \alpha_3|$ has four distinct base points. This happens for a general triangle. The linear system $AP_2(F) \cap |\mathcal{O}_{\mathbb{P}^2}(2) - \alpha_1 - \alpha_2 - \alpha_3|$ is a pencil of conics. Let α_4 be its fourth base point. By Proposition (4.3), $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a polar quandrangle of F. Conversely, every polar quandrangle of F is obtained in this way. From this construction we easily derive that the variety $X_4(F)^\circ$ of polar quadrangles is of dimension 6. A similar argument shows that F admits ∞^9 polar pentagons. Finally, F admits ∞^{12} polar hexagons. In fact, take any set of six points $\{a_1, ..., a_6\} \subset \check{\mathbb{P}}^2$ not on a conic such that no proper subset of it is a polar polygon for F. Then the envelopes $H_{a_i}^2$ form a basis in the space $|\mathcal{O}_{\mathbb{P}^2}(2)|$; hence we can write F as a linear combination with non-zero coefficients of the corresponding conics l_i^2.

(5.11.2) *Remark.* There is another way to see the variety $X_3(F)^\circ$ of polar triangles of F. As we have remarked in (4.1.1), it is birationally isomorphic to the variety of trisecant planes of the Veronese surface $V \subset |S^2(V^*)|$ containing F. Projecting from F to a fixed $\mathbb{P}^4 \subset |S^2(V^*)|$, we get that $X_3(F_2)$ is isomorphic to the variety of trisecant lines of a projected Veronese surface. This is a three-dimensional subvariety of the Grassmann variety $G(2, 5)$ of lines in $\mathbb{P}^4$. In the Plücker embedding $G(2, 5) \subset \mathbb{P}^9$ it is of degree 5, and is obtained by cutting out $G(2, 5)$ by three hyperplanes (cf. [Se-Ro, Mu]).

(5.12) One can confirm the computation of the dimensions of the varieties $X_s(F)^\circ$ of polar s-gons of a non-singular conic by counting the dimensions of the fibres of the morphisms $\varphi_1 : \mathrm{Sec}_{s,2} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$ constructed in (5.1.1). We use that, in general,

$$\dim \mathrm{Sec}_{s,n} = rn + s - 1,$$

and that φ_1 is dominant if $r = n = 2$, and $s \geqslant 3$. Similar computations can be made in the case of non-singular quadrics of arbitrary dimension.

(5.13) Let us consider the case of plane cubic curves. If F has a polar 1-gon, it is a triple line, if F has a polar 2-gon, it is the union of three concurrent lines. Assume F has a polar triangle $\{H_1, H_2, H_3\}$ and does not admit a polar 2-gon, i.e., is not the union of three concurrent lines. If $H_1 \cap H_2 \cap H_3 \neq \emptyset$, then, after coordinate change, $H_1 = x_0$, $H_1 = x_1$, and $H_3 = -x_0 - x_1$; hence $F = x_0^3 + x_1^3 - (x_0 + x_1)^3$ is the union of three concurrent lines. Thus H_1, H_2, and H_3 are linearly independent forms, forcing F to be projectively isomorphic to the Fermat cubic.

(5.13.1) Let $FC \subset |S^3(V^*)|$ be the locus of cubic curves projectively isomorphic to the Fermat cubic F. It is the orbit of F with respect to the natural action of $PGL(3)$ on $|S^3(V^*)|$. The isotropy subgroup of the Fermat cubic is a finite group. Thus FC is an open Zariski subset of a hypersurface in $|S^3(V^*)|$. The polynomial of minimal degree vanishing on FC is the Aronhold invariant $\mathbf{I}_4$ of degree 4 of plane cubics given by (see [Sa2, p. 191].

$$\begin{aligned}\mathbf{I}_4 = {} & abcj - (bcde + cafg + abhi) - j(agi + bhe + cdf) \\ & + (afi^2 + ahg^2 + bdh^2 + bie^2 + cgd^2 + cef^2) \\ & - j^4 + 2j^2(fh + id + eg) - 3j(dgh + efi) - (f^2h^2 + i^2d^2 + e^2g^2) \\ & + (ideg + egfh + fhid),\end{aligned}$$

where we write the equation of F in the form:

$$\begin{aligned}F_3 = {} & ax_0^3 + bx_1^3 + cx_2^3 + 3\,dx_0^2x_1 + 3ex_0^2x_2 + 3fx_0x_1^2 \\ & + 3gx_1^2x_2 + 3hx_0x_2^2 + 3ix_1x_2^2 + 6jx_0x_1x_2.\end{aligned}$$

In symbolic form

$$\mathbf{I}_4 = (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta).$$

(5.13.2) PROPOSITION. *Let F be a plane cubic curve. The following properties are equivalent*:

(i) $\mathbf{I}_4(F) = 0$;

(ii) *$F \in FC$, or F is a cuspidal cubic, or a cone, or the union of a conic and a line intersecting the conic non-transversally*;

(iii) *one of the polar conics of F is a double line*;

(iv) *F admits a (possibly degenerate) polar s-polygon with $s \leqslant 3$.*

Proof. By (5.4) we have already (ii) $\Leftrightarrow$ (iii). One verifies directly that the Fermat cubic satisfies (i). This implies that $\mathbf{I}_4(FC) = \{0\}$. Since the closure $\overline{FC}$ of the Fermat locus FC is a hypersurface in $|S^3(V^*)|$, it must be an irreducible component of the quartic hypersurface $\mathbf{I}_4 = 0$; hence it equals the set of zeroes of an invariant polynomial. However, it is known from the theory of invariants that $\mathbf{I}_4$ is an invariant polynomial of minimal positive degree. This shows $\mathbf{I}_4 = 0$ is equal to $\overline{FC}$. One verifies directly that $\mathbf{I}_4$ vanishes on every curve from (ii). On the other hand, the orbits of non-singular curves are closed in the open subset of non-singular curves. This shows that $\overline{FC} - FC$ consists of singular curves. The only singular curves F which do not satisfy (ii) are nodal curves, the union of a conic and a line intersecting the conic transversally, and the union of three non-concurrent lines. One verifies directly, by reducing these curves to a canonical form, that $\mathbf{I}_4(F) \neq 0$. This verifies (i) $\Leftrightarrow$ (ii).

Let $\varphi_1 : \mathrm{Sec}_{3,3} \to \mathbb{P}^9$ be the projection of the 2-secant bundle of the Veronese surface of triple lines to the space of plane cubics. We know that its open Zariski subset maps to the set FC. Since the image of φ_1 is closed, it must be equal to the closure of FC which is the hypersurface $\mathbf{I}_4 = 0$. On the other hand, this image consists of curves admitting a (degenerate) s-gon with $s \leqslant 3$. This shows the equivalence (i) $\Leftrightarrow$ (iv).

(5.13.3) *Remark.* It is easy to verify by direct computation that the singular locus of the hypersurface $\mathbf{I}_4 = 0$ consists of cones.

(5.13.4) *Remark.* Classically a non-singular cubic curve at which the invariant $\mathbf{I}_4$ vanishes was called an *anharmonic cubic*. The reason for this name can be explained as follows. If F_3 is an irreducible curve, it can be reduced to a Weierstrass form:

$$x_2^2 x_0 + x_1^3 + p x_1 x_0^2 + q x_0^3 = 0.$$

Evaluating $\mathbf{I}_4$ on this curve we obtain that $\mathbf{I}_4(F) = 0$ if and only if $p = 0$. If F is non-singular, this means that F is isomorphic to a double cover of $\mathbb{P}^1$ branched over the four points ρ_1, ρ_2, ρ_3, ∞ where $\rho_i^3 = -q$. The cross-ratio

$$R = (\rho_1 - \rho_2)(\rho_3 - \infty)/(\rho_1 - \infty)(\rho_3 - \rho_2) = (\rho_1 - \rho_2)/(\rho_3 - \rho_2) = \rho,$$

where $\rho^3 = -1$, $\rho \neq -1$. Quadruples with this cross-ratio are classically

called anharmonic. They represent one of the two possible $PGL(2)$-orbits of quadruples on $\mathbb{P}^1$ with non-trivial isotropy group. The other one is the orbit of harmonic quadruples which is characterized by the condition that the cross-ratio is equal to -1, $\frac{1}{2}$, or 2. They correspond to elliptic curves with the coefficient $q=0$ in their Weierstrass equation. We will extend the notion of a non-singular anharmonic cubic to all cubics contained in the locus $\mathbf{I}_4=0$.

(5.14) PROPOSITION. *Every cubic curve F admits a (possibly degenerate) polar s-gon with $s\leqslant 4$.*

Proof. The assertion is obvious for cones, since every binary form of degree 3 admits a polar s-gon with $s\leqslant 2$. Assume F is not a cone. It follows from our description of Hessians of cubic curves (5.4.3) that F contains a non-singular point $a\in He(F)\cap F$. If $P_a(F)=H^2$ is a double line, we conclude by (5.13.2). We may therefore assume that the polar conic $Q=P_a(F)$ is of rank 2, and $a\notin \mathrm{Sing}(Q)$ (3.3). We can write Q in the form $l_1^2+l_2^2=0$, where $l_1(a)$ and $l_2(a)$ are not equal to zero. Indeed $P_a(F)=0$, so if $l_1(a)=0$, $l_2(a)=0$ which is impossible since $a\notin \mathrm{Sng}(P_a(F))$. Then

$$P_a(F-l_1(a)\ {}^1l_1^3-l_2(a)\ {}^1l_2^3)=P_a(F)-1l_1^2-l_2^2=0.$$

Changing coordinates in order to have $a=(1,0,0)$, we observe that $F-l_1(a)\ {}^1l_1^3-l_2(a)\ {}^1l_2^3$ depends only on x_1 and x_2; hence we can write:

$$F_3=l_1(a)\ {}^1l_1^3+l_2(a)\ {}^1l_2^3+G_3(x_1,x_2)$$

for some homogeneous binary form of degree 3. Applying (4.5), we can write G_3 as a sum of two cubes of linear forms (or its degeneration). This proves the assertion.

(5.14.1) Let $\{H_1,H_2,H_3,H_4\}$ be a polar quadrangle of a cubic curve F. If three of the lines are linearly dependent, the polar of F with respect to their common intersection point is a double line (the remaining one). By (5.13.2) F admits a polar triangle. Assume the sides H_i are in general linear position. For each point a on a side H_i but not on the other sides, the polar conic $P_a(F)$ admits a polar triangle whose sides are linearly independent. Hence it is a non-singular conic. This shows that the Hessian of F intersects the sides of the complete quadrangle $\{H_1,H_2,H_3,H_4\}$ only at the vertices; in other words the quadrangle is inscribed in $He(F)$. In particular, we see that an anharmonic cubic does not admit a polar quadrangle whose sides are in general position. Indeed its Hessian is the union of three lines or its degeneration, and cannot be circumscribed around a complete quadrangle of lines in general position. However, it admits a polar quadrangle with three linearly dependent sides.

(5.14.2) EXAMPLE. The cubic $x_0x_1x_2=0$ is not anharmonic, so it does not admit polar triangles. However, it admits polar quadrangles; for instance, we may write

$$-24x_0x_1x_2=(x_0-x_1+x_2)^3-(x_0+x_1+x_2)^3$$
$$+(x_0+x_1-x_2)^3+(-x_0+x_1+x_2)^3.$$

(5.14.3) The variety of polar quadrangles of a general cubic curve is birationally isomorphic to the projective plane. This can be seen as follows. Let a_1, a_2, a_3, a_4 be points in $\mathbb{P}^2$ (in general position). Then the linear system of cubic curves through these points consists of curves of the form

$$Q_1H_1+Q_2H_2=0,$$

where Q_1 and Q_2 generate the pencil of conics through $a_1, \ldots, a_4$, and H_1, H_2 are linear forms. Let L be the two-dimensional linear system of apolar conics of F. For any two conics Φ, Φ' in L that intersect transversally, the linear system $\Phi\check{H}+\Phi'\check{H}'$ of apolar cubics is equal to the linear system of enveloping cubics passing through the four points $\Phi_1\cap\Phi'$. This implies that these points form a polar quadrangle of F. Conversely, every polar quadrangle of F is obtained in this way. Thus the variety of polar quadrangles of F is birationally isomorphic to the variety of pencils of conics from L, i.e., the variety of lines in $\mathbb{P}^2$. We refer to [Re-Re] for the geometry of its cover corresponding to ordered quadrangles.

(5.15) We now consider the case of cubic surfaces. If a cubic surface F admits a polar s-polyhedron with $s\leqslant 3$, it is a cone. If F is not a cone and admits a polar 4-polyhedron, it is projectively isomorphic to the Fermat cubic. The $PGL(4)$-orbit of the Fermat cubic is of dimension 15, i.e., of codimension 4 in the space of all cubics. We do not know equations defining this orbit.

(5.15.1) THEOREM (J. Sylvester). *Let* $\dim V=4$. *There exists an open Zariski subset* U *in* $S^3(V^*)$ *such that every* $f\in U$ *can be written in the form*

$$F=l_1^3+l_2^3+l_2^3+l_4^3+l_5^3.$$

Moreover the linear forms l_i *are determined uniquely up to permutation and numerical factors which are cubic roots of unity.*

Proof. See a modern proof in [ShB].

(5.12.2) COROLLARY. *Every cubic surface* F *admits a (possibly degenerate) polar* s*-polyhedron with* $s\leqslant 5$.

Proof. Consider the morphism $\varphi_1 : \mathrm{Sec}_{5,3} \to |\mathcal{O}_{\mathbb{P}^3}(3)|$ from (5.1.1). By Sylvester's theorem this morphism is surjective (in fact, it is a birational morphism).

(5.15.3) Counting constants, we see that for a general cubic the polar pentahedron is in general linear position (i.e., any of its four planes are linearly independent). Then it has 10 vertices that comprise all singular points of the Hessian (it is known that any normal determinantal quartic has at most 10 singular points). The 10 edges $l_{ij} = H_i \cap H_j$ of the pentahedron are contained in the Steinerian (= Hessian). Indeed, for every $a \in l_{ij}$ the quadric $P_a(F)$ is singular (it has a polar 3-polyhedron).

(5.15.4) Let

$$F = l_0^3 + \ldots + l_4^3$$

be a polar pentahedron of a cubic surface F. Let

$$\lambda_0 l_0 + \ldots + \lambda_4 l_4 = 0.$$

be a linear relation between the linear forms l_i. If F is not a cone, it is unique up to proportionality. For every point $a = (a_0, a_1, a_2, a_3) \in \mathbb{P}^3$, the polar quadric $P_a(F_3)$ is given by two equations in $\mathbb{P}^4$:

$$l_0(a)l_0^2 + l_1(a)l_1^2 + l_2(a)l_2^2 + l_3(a)l_3^2 + l_4(a)l_4^2 = 0,$$
$$\lambda_0 l_0 + \lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 + \lambda_4 l_4 = 0$$

Thus $P_a(F)$ is a singular quadric if and only if the matrix

$$\begin{pmatrix} l_0(a)l_0 & l_1(a)l_1 & l_2(a)l_2 & l_3(a)l_3 & l_4(a)l_4 \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix}$$

is of rank 1 at some point $x = (x_0, \ldots, x_3)$. This can be expressed by the equalities:

$$l_a(a) = \lambda_1 / l_0(x), \ldots, l_4(4) = \lambda_4 / l_4(x).$$

Since $\sum \lambda_i l_i(a) = 0$, we obtain the equation of the Hessian surface of F by reducing the equation

$$\sum \lambda_i^2 / l_i = \sum \lambda_i l_i = 0$$

to common denominator:

$$He(F) = \lambda_0^2 l_1 l_2 l_3 l_4 + \lambda_1^2 l_0 l_2 l_3 l_4 + \lambda_2^2 l_0 l_1 l_3 l_4 + \lambda_3^2 l_0 l_1 l_2 l_4 + \lambda_4^2 l_0 l_1 l_2 l_3 = 0$$

[Sa1, p. 176]. One immediately sees from this equation that the complete pentahedron $l_0l_1l_2l_3l_4=0$ is inscribed into the Hessian, its vertices $l_i=l_j=l_k=0$ are the 10 singular points, and its edges $l_i=l_j=0$ are contained in it.

(5.16) Finaly we consider the case of cubic hypersurfaces of dimension $\geqslant 3$. Let $\mathscr{D}$ be the determinantal hypersurface in $|S^3(V^*)| \cong \mathbb{P}^{N(3)}$ parametrizing singular quadrics in $\mathbb{P}^r$. It is well known that the subvariety $\mathscr{D}^i$ of $\mathscr{D}$ parametrizing quadrics of corank i is of codimension $\frac{1}{2}i(i+1)$ [ACGH, p. 102]. Thus, if $r>\frac{1}{2}i(i+1)$, the Hessian of any cubic hypersurface in $\mathbb{P}^r$ contains a point a such that $P_a(F)$ is of corank $\geqslant i$ (unless, by (3.9), F is a cone). Therefore, we can write:

$$P_a(F)=l_0^2+\ldots+l_{r-i}^2.$$

Repeating the argument from (5.14), we may assume that

$$P_a(F-l_0(a)^{-1}l_0^3-\ldots-l_k(a)^{-1}l_k^3)=l_{k+1}^2+\ldots+l_{r-i}^2,$$

for some $k\leqslant r-i$, where $l_{k+1}(a)=\ldots=l_{r-i}(a)=0$. Let us change coordinates in order to have $a=(1, 0, \ldots, 0)$. Then $l_{k+1}, \ldots, l_{r-i}$ are linear forms in $x_1, \ldots, x_r$, and we can write:

$$F=l_0(a)^{-1}l_0^3+\ldots+l_k(a)^{-1}l_k^3+x_0(l_{k+1}^2+\ldots+l_{r-i}^2)+G(x_1, \ldots, x_r).$$

Let $(H_1, \ldots, H_s)$ be a polar polyhedron of the cubic hypersurface G in $\mathbb{P}^{r-1}$. If $k=r-i$, which holds for a generic F, we obtain that F admits a polar $(r-i+s+1)$-polyhedron. If $k<i$, we may use Example (5.14.2) to represent each $x_0(l_i^2+l_j^2)$ as a sum of four cubes and to use the identity

$$6xy^2=(x+y)^3+(x-y)^3-2x^3$$

to represent each $x_0l_i^2$ as a sum of three cubes. This shows that F admits a polar N-polyhedron, where $N=s+k+1+(3/2)(r-i-k)$ if $r-i-k$ is even, and $s+k+1+(3/2)(r-i-k-1)+3$ if $r-i-k$ is odd.

(5.16.1) EXAMPLES. Let $r=4$. It is known that a general cubic hypersurface does not admit polar s-polyhedra for $s\leqslant 7$ (see [Ri, Re]). Using the Sylvester theorem, our inductive argument shows that a general cubic in five variables admits a polar 8-polyhedron.

If $r=5$, we obtain that a general cubic admits a polar 12-polyhedron. We do not know what is the minimal number s such that a cubic hypersurface in $\mathbb{P}^5$ admits a polar s-polyhedron. The constant count shows that $s\geqslant 10$.

6. Plane Quartics

(6.1) Let F be a plane quartic. We denote by $\tilde{F} \in S^4(V)^*$ the corresponding total polarization; thus $F(x) = \tilde{F}(x, x, x, x)$ for any $x \in V$. For every point $a \in \mathbb{P}^2$, the polar $P_a(F) = \tilde{F}(a, x, x, x)$ is a cubic curve. The linear system $\mathscr{P}_F$ of polars of F is a net of cubics (unless F is a cone, (3.9)). Let $\mathscr{D}_3 \subset |S^3(V^*)|$ be the discriminant hypersurface parametrizing singular cubics. It is well known that $\mathscr{D}_3$ is a hypersurface of degree 12 in $\mathbb{P}^9$; its non-singular points parametrize nodal cubics. Its singular locus in codimension 1 consists of two components R_1 and R_2. The singularity of $\mathscr{D}_3$ at the general point of R_1 (resp. R_2) is an ordinary cusp (resp. an ordinary node). The degree of R_1 is equal to 24, and its non-singular points parametrize cuspidal cubics. The degree of R_2 is equal to 21, and its non-singular points parametrize the unions of a conic and a line intersecting it transversally (cf. [Kl2]).

Therefore we see that the Steinerian curve $St(F)$ is either the whole plane, or a curve of degree 12 (counting with multiple components). By (3.7) the first case happens only if F has a triple point, or is a double conic. We expect that for a general F the Hessian $He(F)$ is a non-singular curve of degree 6 and the Steinerian $St(F)$ is an irreducible curve of degree 12 with 24 cusps and 21 nodes. This implies that any polar cubic $P_a(F)$ is either non-singular, or a nodal cubic (a is a non-singular point of $St(F)$), or a reducible cubic with two singular points (a is a node of $St(F)$), or a cuspidal cubic (a is a cusp of $St(F)$). To show that our expectation is right, it suffices to exhibit one quartic F with such properties. We do this in the next example.

(6.1.1) Example. Let

$$F: x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0$$

be the Klein quartic curve with the automorphism group G isomorphic to the group $PSL(2, \mathbb{F}_7)$ of order 168 (cf. [Bur, p. 310 and p. 363]). Its Hessian curve is

$$He(F): 5x_0^2 x_1^2 x_2^2 - x_0 x_1^5 - x_0^5 x_2 - x_1 x_2^5 = 0.$$

It is not difficult to verify that it is non-singular (the easy explicit computations are available upon request). To verify that the Steinerian has the needed properties, we use that the group G acts on both $He(F)$ and $St(F)$. In particular, we use the following elementary fact: the orders of cyclic subgroups of G are 1, 2, 3, 4, and 7. This implies that any orbit of G on a non-singular curve consists of 168, 84, 56, 42, or 24 points. Consider the Steiner map $s_F: He(F) \to St(F)$. Since $He(F)$ is non-singular, and the

Steiner map is surjective, $St(F)$ must be irreducible. Let s': $He(F) \to S'$ be the corresponding map to the normalization S' of $St(F)$. Any singular polar has only isolated singularities, since otherwise $He(F)$ is reducible. The number of isolated singularities of a cubic curve is at most 3. This implies that the degree of the map is at most 3. Since the genus of $He(F)$ equals 10, by Hurwitz's formula, the ramification divisor of s' consists of at most 24 points. The ramification divisor R of s' is G-invariant; hence we obtain that R is either empty or consists of 24 points, in which case S' is of genus 0. On the other hand, it is well known that G acts on $\mathbb{P}^1$ only trivially. This immediately implies that $R = \varnothing$, and s' is an isomorphism. Let $\mathscr{C}$ be the conductor of the normalization map $He(F) \cong S' \to S = St(F)$. We know that $p_a(S) = p_a(S') + \frac{1}{2}\deg(\mathscr{C})$; hence

$$\deg(\mathscr{C}) = (d-1)(d-2) - 20,$$

where d is the degree of S. Again, as above, since $\mathscr{C}$ is a G-invariant divisor, its support $\mathscr{C}_{\text{red}}$ is the union of G-orbits. An elementary computation shows that this implies that either $\mathscr{C} = 0$, in which case S is a non-singular sextic, or $\deg(\mathscr{C}) = 90$, $d = 12$. Consider the former case. This means that every singular point of $St(F)$ has a cuspidal singularity (i.e., it is a double point with the tangent cone equal to a line). Thus all singular polars are cuspidal curves or their degenerations. In particular, each one is contained in the locus of anharmonic cubics. Later on (6.5) we shall see that the locus of points $a \in \mathbb{P}^2$ such that $P_a F$ is an anharmonic cubic is a curve of degree 4. This contradiction allows us to go to the next case, where $St(F)$ is an irreducible curve of degree 12, and the conductor $\mathscr{C}$ is of degree 90. The only way to write $\mathscr{C}$ as a sum of G-invariant divisors is $\mathscr{C}_1 + 2\mathscr{C}_2$, where $\deg(\mathscr{C}_1) = 42$, $\deg(\mathscr{C}_2) = 24$, $\mathscr{C}_i = (\mathscr{C}_i)_{\text{red}}$, $i = 1, 2$. Write $\mathscr{C} = \sum \mathscr{C}_P$, where $\mathscr{C}_P$ is the conductor of a singular point $P \in C$. Let r_P be the number of local branches of S at P and μ_P is the colength of the Jacobian ideal of S at P. It follows from the Jung–Milnor formula $\mu_P = \deg(\mathscr{C}_P) - r_P + 1$ [Mi, Sect. 10] that $\mathscr{C}_P$ is reduced if and only if $\mu_P = 1$, i.e., P is an ordinary quadratic point. This implies that S exactly 21 nodes. The rest of the singular points of S are cusps or their degenerations. In any case each such a point is unibranched (i.e., $r_P = 1$). Thus the local conductor $\mathscr{C}_P$ is supported at one point and is of degree 2. This immediately implies that P is an ordinary cusp. So, $St(F)$ is an irreducible curve of degree 12 with exactly 24 ordinary cusps and 21 ordinary nodes.

(6.2) Let us consider the second polarization map:

$$S^2(V) \to S^2(V^*), \qquad \Phi \mapsto P_\Phi(F).$$

Its kernel is the subspace $AP_2(F)$ of conics apolar to F. Recall from (2.8)

that F is called non-degenerate if $AP_2(F) = \{0\}$. The next condition plays a fundamental role in the study of polar properties of quartic curves.

(6.2.1) DEFINITION. A quartic curve F is called *weakly non-degenerate* if it is non-degenerate or $AP_2(F)$ is spanned by a unique irreducible conic.

(6.3) Let

$$s_2: \mathbb{P}^2 \times \mathbb{P}^2 \to |S^2(V)|, \qquad (a, b) \to \check{H}_a \check{H}_b,$$

be the composition of the Segre map $\mathbb{P}^2 \times \mathbb{P}^2 \to |V \otimes V|$ with the projection $|V \otimes V| \dashrightarrow |S^2(V)|$. The restriction of s_2 to the diagonal is the Veronese map:

$$v_2: \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(2)|^* = |S^2(V)|.$$

Let $(\mathbb{P}^2)^{(2)}$ be the symmetric square of $\mathbb{P}^2$; its singular locus is the image of the diagonal. It is well known that s_2 factors through an isomorphism

$$\bar{s}_2: (\mathbb{P}^2)^{(2)} \xrightarrow{\sim} \check{\mathcal{D}}_2 \subset |S^2(V)|,$$

where $\check{\mathcal{D}}_2$ is the cubic hypersurface parametrizing reducible enveloping conics. It is equal to the secant variety of the Veronese surface $\check{V}_{2,2} = v_2(\mathbb{P}^2) \subset |S^2(V)|$.

(6.3.1) LEMMA. *Assume F is weakly non-degenerate. Then the map*

$$s_2^F: \mathbb{P}^2 \times \mathbb{P}^2 \to |S^2(V^*)|, \qquad (a, b) \mapsto P_{a,b}(F),$$

is everywhere defined. The induced map

$$\bar{s}_2^F(\mathbb{P}^2)^{(2)} \to |S^2(V^*)|$$

is an embedding onto a cubic hypersurface $V_3(F)$ if F is non-degenerate, or is of degree 3 onto the hyperplane $|AP_2(F)^\perp|$ otherwise.

Proof. We know that the map s_2^F is equal to the composition

$$\mathbb{P}^2 \times \mathbb{P}^2 \to |S^2(V)| \dashrightarrow |S^2(V^*)|$$

of the map s_2 and the rational map $|S^2(V)| \dashrightarrow |S^2(V^*)|$ corresponding to the linear map $S^2(V \dashrightarrow S^2(V^*)$, $\Phi \to P_\Phi(F)$. If $AP_2(F) = 0$, the latter map is an isomorphism; hence the asserted property of $\bar{s}_2^F$ follows from the similar property of $\bar{s}_2$. If $AP_2(F) \neq 0$, the map $|S^2(V)| \dashrightarrow |S^2(V^*)|$ is isomorphic to the projection from the point $\{\Phi\}$, where Φ spans $AP_2(F)$. By assumption this point is not in the image of s_2. Thus $\bar{s}_2^F$ is equal to

the composition of the embedding onto a cubic hypersurface and the projection from a point not lying on this hypersurface.

(6.4) LEMMA. *Let* $V_{2,2} \subset |S^2(V^*)|$ *be the Veronese surface of double lines, let* $\mathscr{D}_2$ *be its secant variety of reducible conics, and let* $v_F: \mathbb{P}^2 \to |S^2(V^*)|$ *be the map* $a \to P_{a,a}(F)$. *Then*

$$He(F) = v_F^{-1}(\mathscr{D}_2),$$

$$v_F^{-1}(V_{2,2}) \subset \mathrm{Sing}(He(F)), \qquad \text{if} \quad He(F) \neq \mathbb{P}^2.$$

Proof. The polynomial of degree 3 that vanishes on $\mathscr{D}_2$ is the discriminant of quadratic forms. Hence the polynomial of minimal degree that vanishes on $v_F^{-1}(\mathscr{D}_2)$ is equal to the Hessian of F. This shows that $v_F^{-1}(\mathscr{D}_2)$ and $He(F)$ coincide set-theoretically. It is clear that

$$v_F^{-1}(V_{2,2}) = \{a \in \mathbb{P}^2 : \mathrm{rk}(P_{a,a}(F)) = 1\}.$$

Now the second assertion follows from Lemma (3.6).

(6.5) Let

$$\mathbf{I}_4: S^3(V^*) \to \mathbb{C}$$

be the Aronhold invariant of degree 4 defined on the space of cubics (5.13.1). Composing it with the polarization map

$$V \otimes S^4(V^*) \to S^3(V^*),$$

we obtain the *Clebsch covariant of degree* 4:

$$\mathbf{S}_4 \in S^4(V^* \otimes S^4(V^*)^*).$$

It can be considered as a function on the space of quartics $S^4(V^*)$ of degree 4 in coefficients that takes its values in the space of quartics. By definition

$$\mathbf{S}_4(F)_{\mathrm{red}} = \{a \in \mathbb{P}^2 : P_a(F) \text{ is an anharmonic cubic}\}.$$

In the case when $\mathbf{S}_4(F) = \mathbf{S}_4(F)_{\mathrm{red}} \neq P^2$ we denote by $S(F)$ the quartic curve $S_4(F)$ and will call it the *covariant quartic* of F.

In symbolic notation (see [Sa2, p. 269]),

$$\mathbf{S}_4 = (\alpha, \beta, \gamma)(\alpha, \beta, \delta)(\alpha, \gamma, \delta)(\beta, \gamma, \delta)\alpha_x\beta_x\gamma_x\delta_x.$$

This means that the total polarization of $\mathbf{S}_4$ is an element of $V^{*\otimes 4} \otimes V^{\otimes 4}$,

whose value at a vector $(v_1, v_2, v_3, v_4, \alpha, \beta, \gamma, \delta) \in V^{\otimes 4} \otimes V^{*\otimes 4}$ is equal to the product

$$|\alpha, \beta, \gamma|\,|\alpha, \beta, \delta|\,|\beta, \gamma, \delta|\,|\alpha, \gamma, \delta|\,\alpha(v_1)\beta(v_2)\gamma(v_3)\delta(v_4),$$

where $|\ldots|$ denotes the determinant.

Note that the Clebsch covariant $\mathbf{S}_4$ of quartics is an analog of the Hessian covariant $\mathbf{H}_3$ of cubics. In fact, $F \to He(F)$ is a covariant obtained by composing the discriminant invariant of quadrics with the first polarization map $V \otimes S^3(V^*) \to S^2(V^*)$.

(6.6) Let $\mathrm{Cat}(F)$ be the catalecticant matrix of F in the standard bases of $S^2(V)$ and $S^2(V^*)$ (2.4). Recall that its entries $c_{ij,\,sk}$ are defined by

$$P_\Phi(F) = \sum c_{ij,\,sk} a_{sk} x_i x_j,$$

where $\Phi = \sum a_{sk} u_s u_k$ (see (2.4)). They satisfy the symmetry conditions:

$$c_{ij,\,sk} = c_{ji,\,sk} = c_{ij,\,ks} = c_{sk,\,ij}.$$

Let $H = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$ be a line, and let $\Phi = \sum a_{ij} u_i u_j$ be its second anti-polar, i.e., $P_\Phi(F) = H^2$. Assume $\det(\mathrm{Cat}(F)) \neq 0$, i.e., F is non-degenerate. Then the equation of Φ is

$$\Phi = \sum (C_{ij;\,sk} \alpha_s \alpha_k) u_i u_j;$$

here $(C_{ij;\,sk}) = \mathrm{Cat}(C)^*$ is the cofactor (or adjugate) matrix of $\mathrm{Cat}(F)$. The condition that Φ is reducible is

$$\det \begin{pmatrix} \sum C_{00;\,sk}\alpha_s\alpha_k & \sum C_{01;\,sk}\alpha_s\alpha_k & \sum C_{02;\,sk}\alpha_s\alpha_k \\ \sum C_{10;\,sk}\alpha_s\alpha_k & \sum C_{11;\,sk}\alpha_s\alpha_k & \sum C_{12;\,sk}\alpha_s\alpha_k \\ \sum C_{20;\,sk}\alpha_s\alpha_k & \sum C_{21;\,sk}\alpha_s\alpha_k & \sum C_{22;\,sk}\alpha_s\alpha_k \end{pmatrix} = 0.$$

This is an equation of degree 6 in the coordinates $(\alpha_0, \alpha_1, \alpha_2)$ of H. The left-hand side is a polynomial in the coefficients of F (of degree 15) and a polynomial of degree 6 in α_0, α_1, α_2. It defines a $SL(V)$-equivariant polynomial map of degree 15,

$$\Gamma\colon S^4(V^*) \to S^6(V),$$

i.e., a contravariant of degree 15 and class 6 on the space of quartics. If $\mathrm{crk}(\mathrm{Cat}(F)) > 1$, $\Gamma(F) = 0$. If $\mathrm{crk}(\mathrm{Cat}(F)) = 1$, then $\mathrm{rk}(\mathrm{Cat}(C)^*) = 1$, and we may write

$$C_{ij;\,sk} = \lambda_{ij}\lambda_{sk},$$

for some scalars λ_{ij} satisfying $\lambda_{ij}=\lambda_{ji}$. Then

$$\Gamma(F)=\det\begin{pmatrix}\lambda_{00} & \lambda_{01} & \lambda_{02}\\ \lambda_{10} & \lambda_{11} & \lambda_{12}\\ \lambda_{20} & \lambda_{21} & \lambda_{22}\end{pmatrix}\left(\sum \lambda_{sk}\alpha_s\alpha_k\right)^3.$$

The conic

$$\Phi_0=\sum \lambda_{sk}\alpha_s\alpha_k$$

is the apolar conic of F, i.e., a solution of the equation $\mathrm{Cat}(F)\cdot\Phi=0$. This implies that $\Gamma(F)\neq 0$ if F is weakly non-degenerate but degenerate. Note that Γ is not identically zero (cf. (6.7)). So, we find that the set

$$\Gamma(F)=\{H\in\check{\mathbb{P}}^2 : P_{a,b}(F)=H^2 \text{ for some } a, b\in\mathbb{P}^2\}$$

is equal to $\Gamma(F)_{\mathrm{red}}$ if F is weakly non-degenerate.

(6.6.1) *Remark.* Assume F is not weakly non-degenerate. Then $\Gamma(F)=0$ but $\Gamma(F)\neq\mathbb{P}^2$. In fact, let $\Phi\in AP_2(F)$ and $P_{a,b}(F)=H^2$ for some line H. Then, by (1.7),

$$\langle\Phi, H^2\rangle=\langle\Phi, P_{a,b}(F)\rangle=\langle\Phi H_a H_b, F\rangle=\langle H_a H_b, P_\Phi(F)\rangle=0;$$

hence, by Lemma (1.6), H belongs to Φ. This shows that $\Gamma(F)$ is contained in the base locus of the linear system $|AP_2(F)|$. Thus $\Gamma(F)\neq\check{\mathbb{P}}^2$ for any degenerate F.

(6.6.2) PROPOSITION. *Let F be a non-degenerate quartic. Then $\Gamma(F)\neq 0$.*

Proof. It suffices to show that $\Gamma(F)\neq\mathbb{P}^2$. It is clear that

$$\Gamma(F)=v_2^{-1}(s_2^F(\mathbb{P}^2\times\mathbb{P}^2))\cap V_{2,2}),$$

where $s_2^F:\mathbb{P}^2\times\mathbb{P}^2\to|S^2(V^*)|$ is the morphism defined for any weakly non-degenerate curve F in Lemma (6.3.1), and $v_2:\mathbb{P}^2\to V_{2,2}\subset|S^2(V^*)|$ is the Veronese map. If F is non-degenerate, s_2^F is a double cover onto a cubic hypersurface $W=s_2^F(\mathbb{P}^2\times\mathbb{P}^2)$ that maps isomorphically the diagonal Δ to the singular locus $\mathrm{Sing}(W)$ of W. Assume $\Gamma(F)=\mathbb{P}^2$. Then $V_{2,2}\subset W$, and the restriction of s_2^F over $V_{2,2}$ defines a double cover $\pi: T(F)\to V_{2,2}$ branched along $V_{2,2}\cap\mathrm{Sing}(W)$, where

$$T(F)=(s_2^F)^{-1}(V_{2,2})=\{(a,b)\in\mathbb{P}^2\times\mathbb{P}^2 : \mathrm{rk}(P_{a,b}(F))=1\}.$$

By (3.6) the map $v_2^{-1}: V_{2,2} \to \mathbb{P}^2$ defines an embedding from $V_{2,2} \cap \mathrm{Sing}(W)$ to the set of singular points of the Hessian $He(F)$ unless $He(F) = \mathbb{P}^2$. In the latter case, by (3.8), F is a cone which is obviously a degenerate curve. Assume $T(F)$ is reducible. In this case $T(F)$ consists of two components $T(F)_1$ and $T(F)_2$ each isomorphic to $\mathbb{P}^2$. We know that the degree of the projection $T(F) \to \mathbb{P}^2$, $(a, b) \to a$, is at most 3 (cf. (6.8)). Hence the restriction of each projection to the component $T(F)_i$ is a finite cover $\mathbb{P}^2$ of degree $\leqslant 2$. As is well known, and is easy to prove, the degree of such a map must be equal to 1 So, the degree of $T(F) = T(F)_1 \cup T(F)_2 \to \mathbb{P}^2$ is equal to 2; therefore for any $a \in \mathbb{P}^2$ the polar $P_a(F)$ is a singular anharmonic cubic. This implies that $St(F) = \mathbb{P}^2$. Let us prove that this is impossible. Indeed, according to (3.7) either $F = G^2$ where G is a non-singular conic, of F has a point of multiplicity $\geqslant 3$. In the former case, for general $a \in \mathbb{P}^2$ the polar cubic $P_a(F)$ is the union of a conic and a line that intersects it transversally, so $P_a(F)$ is not anharmonic cubic. In the latter case every $P_a(F)$ is either a cuspidal cubic or the union of a smooth conic and its tangent line. Indeed, otherwise for some $a \in \mathbb{P}^2$ the polar $P_a(F)$ would either be zero or a cone. In the first case F is a cone so it is degenerate. In the second case $P_{b,a}(F) = 0$ for b in the vertex of $P_a(F)$, so F would be degenerate again. Therefore we obtain that every polar $P_a(F)$ has a cuspidal point at some fixed point A which is a base point of the linear system of polars. Its cuspidal tangent line does not move, since otherwise we find a polar with an ordinary quadratic point at A. This shows that in an appropriate coordinate system

$$P_a(F) = l(a)x_0x_1^2 + G_a(x_1, x_2),$$

where $G_a(x, x_2)$ is homogeneous of degree 3 and $l(a)$ is a linear function of a. Now for all zeroes a of l the polar $P_a(F)$ is a cone which contradicts the non-degeneracy of F as we have shown above.

Assume now that $T(F)$ is irreducible. This immediately yields that the branch locus of $\pi: T(F) \to V_{2,2} \cong \mathbb{P}^2$ is a curve of even degree. Since this curve is isomorphic to a multiple component of the Hessian $He(F)$, its degree must be equal to 2. From this we deduce that $T(F)$ is isomorphic to a quadric. Using the previous argument we obtain that the degree of the projection $T(F) \to \mathbb{P}^2$ must be equal to 3. This degree is equal to the self-intersection of a divisor on a minimal resolution of the quadric, which is isomorphic to the minimal ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$. It is easy to verify that the self-intersectio of any divisor on any of these surfaces is even. This contradiction proves the assertion.

(6.6.3) COROLLARY. *Let F be a quartic curve. The following assertions are true*:

(i) $\Gamma(F)=0$ *if and only if* F *is not weakly non-degenerate*;

(ii) $\Gamma(F)\neq\mathbb{P}^2$;

(iii) $T(F)=\{(a,b)\in\mathbb{P}^2\times\mathbb{P}^2 : \mathrm{rk}(P_{a,b}(F))\leqslant 1\}$ *is a curve if* F *is weakly non-degenerate*;

(iv) $\mathbf{S}_4(F)\neq 0$ *if* F *is weakly non-degenerate quartic.*

Proof. (i) It follows from (6.6) that $\Gamma(F)=0$ if F is not weakly non-degenerate and $\Gamma(F)\neq 0$ if F is degenerate but weakly non-degenerate. By (6.6.2) $\Gamma(F)\neq 0$ if F is non-degenerate.

(ii) We know from (6.6) that $\Gamma(F)=\Gamma(\mathbf{F})_{\mathrm{red}}$ if F is weakly non-degenerate. On the other hand, by Remark (6.6.1) $\Gamma(F)\neq\mathbb{P}^2$ for any degenerate quartic.

(iii) Since F is weakly non-degenerate the map

$$\pi\colon T(F)\to\check{\mathbb{P}}^2,$$

which assigns to a pair (a,b) the line H such that $P_{a,b}(F)=H^2$, is well-defined. Its image is the set $\Gamma(F)$ and its degree is 2 if F is non-degenerate and is 6 if F is degenerate. By (ii) $\Gamma(F)\neq\check{\mathbb{P}}^2$ which implies (iii).

(iv) Consider the first projection:

$$p\colon T(F)\to\mathbb{P}^2,\qquad (a,b)\to a.$$

By Proposition (5.13.2), its image equals the set $\{\mathbf{S}_4(F)=0\}$. By (iii) this set is not the whole plane.

(6.6.4) *Remark.* There is another contravariant of class 6 of ternary quartics of degree 3 in the coefficients (see [Sa2, p. 271]). Its value at F is equal to the locus of lines which cut out F in a harmonic quadruple. It was asserted in [Cl] that Γ is equal to the product of this contravariant and the square of the catalecticant invariant $\mathbf{C}^2$. As was remarked in [Ci1] this is wrong (because $\Gamma(F)\neq 0$ for weakly non-degenerate curves with zero catalecticant).

(6.7) EXAMPLE. It is easy to compute the curve $\Gamma(F)$, where F is the Klein curve from Example 6.1.1. By direct computation we find that its equation is

$$\Gamma(F)=5\alpha_0^2\alpha_1^2\alpha_2^2-\alpha_0\alpha_1^5-\alpha_0^5\alpha_2-\alpha_1\alpha_2^5=0,$$

so it coincides with the equation of the Hessian, up to a change of coordinates x_i to the dual coordinates α_i. This is not surprizing, and can be easily deduced from the theory of invariants of the group $PSL(2,\mathbb{F}_7)$ [Bur, p. 363].

(6.8) Let F be a weakly non-degenerate quartic with irreducible Clebsch covariant quartic $S(F)$. By (6.6.3) the curve

$$T(F) = \{(a, b) \in \mathbb{P}^2 \times \mathbb{P}^2 : \mathrm{rk}(P_{a,b}(F)) = 1\}$$
$$= \{(a, b) \in S(F) \times S(F) : \mathrm{rk}(P_{a,b}(F)) = 1\}$$

defines a symmetric correspondence on $S(F)$. If $S(F)$ is not a component of $St(F)$, then the first projection,

$$p: T(F) \to S(F), \qquad (a, b) \to a,$$

is a map of degree 3. In fact, one verifies directly that $p^{-1}(a)$ consists of three points if and only if $P_a(F)$ is projectively isomorphic to the Fermat cubic, $p^{-1}(a)$ consists of two points if and only if $P_a(F)$ is a cuspidal curve, and $p^{-1}(a)$ consists of one point if $P_a(F)$ is the union of a conic and its tangent line. In the last case the point b must be the singular point of $P_a(F)$. Note that $P_a(F)$ is never a cone, since otherwise $P_{a,b}(F) = 0$ for some b; hence F is not weakly non-degenerate. Finally note that $T(F)$ has no united points (i.e., points on the diagonal) if $He(F)$ is non-singular. So, we obtain:

(6.8.1) PROPOSITION. *Let F be a weakly non-degenerate quartic with non-singular Hessian and irreducible $S(F)$ which is not contained in $St(F)$. Then*

$$T(F) = \{(a, b) \in S(F) \times S(F) : \mathrm{rk}(P_{a,b}(F)) = 1\}$$

is a symmetric correspondence of degree (3, 3) *without united points.*

(6.8.2) *Remark.* For general F the correspondence $T(F)$ is a non-singular curve. Indeed this is true for the Klein curve from Example (6.1.1). As we saw in Example (6.7) the curve $\Gamma(F)$ is isomorphic to the Hessian $He(F)$ and hence is non-singular. Since $T(F)$ has no united points, the double cover $T(F) \to \Gamma(F)$ is unramified; hence $T(F)$ is non-singular.

(6.9) Assume $S(F)$ is non-singular; for every point $a \in S(F)$ we denote by $T(a)$ the divisor corresponding to the point a in the correspondence $T(F)$. Assume first that $T(a)$ is reduced, i.e., $P_a(F)$ is a non-singular anharmonic cubic. Then $T(a) = p_1 + p_2 + p_3$, where the three points p_1, p_2, and p_3 are the vertices of the Hessian triangle of lines of F. We will call the triangle formed by the lines $\langle p_1, p_2 \rangle$, $\langle p_1, p_2 \rangle$, and $\langle p_1, p_3 \rangle$, the *polohessian triangle* of the point $a \in S$. If no confusion arises, we denote it by $T(a)$. It is inscribed in $S(F)$, and its vertices are the points p_1, p_2, and p_3 (see Fig. 1). We know that

$$T(a) = He(P_a(F));$$

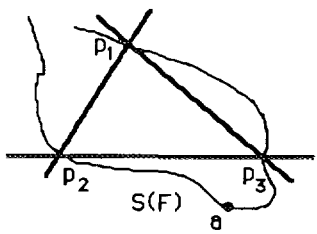

FIGURE 1

in fact, $P_{a,p_1}(F)=2\langle p_2, p_3\rangle$, $P_{a,p_2}(F)=2\langle p_1, p_3\rangle$, and $P_{a,p_3}(F)=2\langle p_1, p_2\rangle$.

(6.9.1) It follows from the definition that the curve $\Gamma(F)$ parametrizes the sides of polohessian triangles of F. According to (6.3.1), for every line $H\in\Gamma(F)$ there exists an unordered pair $\{a, b\}$ of points on S such that $P_{a,b}(F)=H^2$. If $He(F)$ is non-singular then $a\neq b$. This pair is unique if F is non-degenerate. There are three such pairs if F is weakly non-degenerate. We have (see the proof of (7.6))

$$T(a)=b+p+q,$$
$$T(b)=a+p'+q',$$

where the common side $H=\langle p, q\rangle=\langle p', q'\rangle$ of these triangles is opposite to the vertex b in $T(a)$ and to the vertex a in $T(b)$. We have

$$T(a)-b+T(b)-a=p+q+p'+q'\sim K_{S(F)}.$$

Figure 2 describes the situation when all four points p, q, p', and q' are distinct.

(6.9.2) Now let $T(a)=2p+q$, i.e., $P_a(F)$ is a cuspidal irreducible cubic. We may also assume that $P_a(F)=x_1^2x_0+x_2^3=0$. Then easy computations show that $p=(1, 0, 0)$ is the cusp, $q=(0, 0, 1)$ is the intersection point of the cuspidal and the inflectional tangents, $He(P_a(F))=x_2x_1^2$, the union of

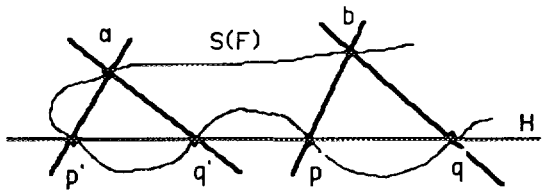

FIGURE 2

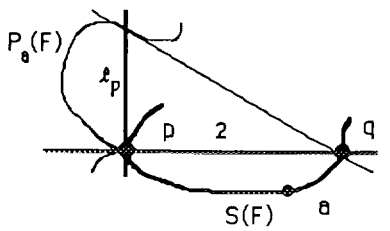

FIGURE 3

the cuspidal tangent and the line ℓ joining the cusp with the inflection point. By (6.9.1), $T(q) = a + x + y$, where $2p + x + y \in |K_{S(F)}|$. The line which cuts out this divisor is the tangent line ℓ_p to S at p. Also it equals the second polar $P_{a,q}(F) = P_{q,a}(F)$ of F; hence it coincides with ℓ. We will call $\mathrm{Hess}(P_a(F)) = \ell_p + 2\langle p, q\rangle$ the *degenerate polohessian triangle* of the point $a \in S$. Note that the point p lies on the Hessian of F, since it is a singular point of the polar $P_a(F)$ (see Fig. 3).

(6.9.3) Finally, $T(a)$ could be equal to $3p$. In this case, $P_a(F)$ is the union of an irreducible conic and its tangent line ℓ. The Hessian of this cubic is the tangent line taken with multiplicity 3. The point p is the tangency point. As in the previous case, considering $T(p)$, we obtain that ℓ equals the tangent line ℓ_p to S at the point p. We will call $\mathrm{Hess}(P_a(F)) = \ell_p^3$ the *degenerate polohessian triangle* of the point $a \in S$ (see Fig. 4). Again, we have $p \in He(F) \cap S$.

(6.9.4) Note that the number of degenerate triangles is at most 24. This follows easily from the Hurwitz formula applied to the cover $p: T(F) \to S(F)$. By Remark (6.8.2) the curve $T(F)$ is non-singular for generic F. One can prove that the non-singularity of $S(F)$ and $He(F)$ implies the non-singularity of $T(F)$. Using Remark (6.8.2) we obtain that for general F the genus of the curve $T(F)$ equals 19. The triangle $T(a)$ is degenerate if and only if the polar cubic $P_a(F)$ is singular, and the point a belongs to the branch locus of p. The previous analysis shows that for each such triangle $T(a)$ the unique singular point of $P_a(F)$ belongs to

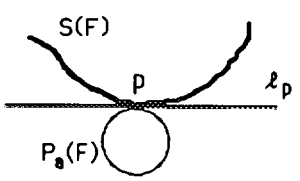

FIGURE 4

$He(F) \cap S(F)$. Note that the set $B = \{a \in S(F) : P_a(F) \text{ is singular}\}$ is equal to $S(F) \cap St(F)$. Each point of B is a cusp of $St(F)$. Thus generically we expect that the set B consists exactly of 24 cusps of $St(F)$, and $He(F)$ intersects $S(F)$ transversally at 24 cusps of cuspidal polar cubics.

(6.9.5) We leave it to the reader to draw pictures similar to Fig. 2 in the cases where the polohessian triangles degenerate. One can show that the point $(a, b) \in T(F)$ is a singular point of the correspondence $T(F)$ if and only if $T(a) = 2b + x$, $T(b) = 2a + y$, and $\langle a, b\rangle = \langle b, x\rangle = \langle a, y\rangle$. In this case the common side $H = \langle a, b\rangle$ is a multiple side in each degenerate triangle.

(6.10) Let F be a plane quartic curve. It admits a polar 1-gon if and only if it is a multiple line. It admits a polar 2-gon if and only if it is the union of four concurrent lines. If it admits a polar triangle, then we can write it in the form:

$$l_1^4 + l_2^4 + l_3^4 = 0.$$

If the linear forms l_i are linearly independent, F is isomorphic to the Fermat quartic. If they are not, F is the union of four concurrent lines.

(6.11) DEFINITION. A quartic admitting a polar quadrangle is called a *Capolary quartic*.

(6.11.1) A general Capolary quartic is isomorphic to a quartic of the form:

$$x_0^4 + x_1^4 + x_2^4 + (ax_0 + bx_1 + cx_2)^4 = 0.$$

In particular, Capolary quartics depend on 3 moduli, so the locus of Capolary quartics is of codimension 3 in the space of all quartics. The polar of a Capolary quartic F with respect to any vertex of its polar quadrangle is a cubic equal to the union of three concurrent lines. For every point x on a side of its polar quadrangle, the polar cubic admits a polar triangle; hence x belongs to the covariant quartic $S(F)$. This shows that $S(F)$ is equal to the union of four lines.

(6.11.2) Any Capolary quartic is not weakly non-degenerate. In fact, if

$$F = l_1^4 + l_2^4 + l_3^4 + l_4^4,$$

any enveloping conic passing through the points $l_i = 0$ is apolar to F. Hence $\mathrm{crk}(\mathrm{Cat}(F))$ is equal to 2 if the lines $l_1 = 0, \ldots, l_4 = 0$ are not concurrent, and equals 3 otherwise.

We refer to [Ci3] for many interesting properties of Capolary quartics.

(6.12) Suppose F admits a polar pentagon. If four of the sides are concurrent then $\mathbf{S}_4(F)=0$. Indeed, for every $a\in\mathbb{P}^2$ the polar cubic $P_a(F)$ has a polar pentagon with four concurrent sides. Let c be the point of concurrency. Then $P_c(P_a(F))=H^2$ for some line H, so by (5.4.1) $P_a(F)$ is an anharmonic cubic. If no four of the sides are concurrent, then $\mathbf{S}_4(F)\neq 0$. In fact, one easily shows that four of the sides are in general linear position. Thus for every point a on the remaining side the polar cubic $P_a(F)$ is not anharmonic (5.14.1). Now, suppose that no four of the sides are concurrent but there are three sides which are concurrent. Then the polar of F with respect to the concurrency point a admits a polar 2-gon; hence $P_a(F)$ is a cone. By (7.5) the covariant quartic $S(F)$ is singular. Also, F is not weakly non-degenerate. In fact, the reducible enveloping conic passing through the sides of the polar pentagon is a polar to F.

(6.12.1) DEFINITION. A quartic admitting a polar pentagon whose sides form a complete pentagon is called a *Clebsch quartic*.

Recall that a *complete pentagon* is formed by 5 lines intersecting pairwisely in 10 distinct points (the *vertices* of the pentagon). In other words a complete pentagon is the set of 5 lines in general linear position (no 3 are concurrent).

(6.12.2) THEOREM. *Let F be a plane quartic curve. The following conditions are equivalent*:

(i) *F is a Clebsch quartic*;

(ii) *F is weakly non-degenerate but degenerate*;

(iii) *$AP_2(F)$ is spanned by an irreducible conic*;

(iv) $\mathbf{C}(F)=0$, $\mathbf{\Gamma}(F)\neq 0$.

Proof. (ii)$\Leftrightarrow$(iii) by definition of a weakly non-degenerate quartic curve. (iii)$\Leftrightarrow$(iv) was observed in (6.6) (recall that $\mathbf{C}$ is the catalecticant invariant). If F is a Clebsch quartic, its polar pentagon is formed by five points in the dual plane in general linear position. The unique enveloping conic Φ_0 through these points is necessarily non-singular and apolar to F. Suppose Φ is an enveloping conic which is apolar to $F=l_1^4+\ldots+l_5^4$. Then

$$P_\Phi(F)=\Phi(l_2)l_1^2+\ldots+\Phi(l_5)l_5^2=0.$$

Since $l_1,\ldots,l_5$ are in general position (as points in the dual plane), we may apply Lemma (4.2.2) to obtain that $\Phi(l_1)=\ldots=\Phi(l_5)=0$. Hence Φ coincides with Φ_0. This proves (i)$\Rightarrow$(iii).

Let us prove the converse. Let Φ be the unique non-singular apolar conic of F. By (6.6), $\Phi=\Gamma(F)$ parametrizes the sides of the polohessian triangles

$T(a) = He(P_a(F))$ of F. For a general $\alpha \in \Phi$ there exist three unordered pairs (a, b) such that

$$P_{a,b}(F_4) = H_\alpha^2.$$

Moreover, $a \in T(b)$, $b \in T(a)$. Since the line $\check{H}_a$ intersects Φ at two points, each point a is a vertex of two sides of polohessian triangles (instead of six for a general quartic!). Let $\alpha \in \Phi$, $\alpha \notin \check{F}$, i.e., H_α is not tangent to F. Then there are six non-degenerate polohessians T_i such that H_α is a side of each of them. They are $T(a_i)$, $T(b_i)$, where $P_{a_i,b_i}(F) = H^2$, $i = 1, 2, 3$. Let $p_1, \ldots, p_4$ be the four points of intersection of H_α with F. Every T_i has three sides; one of them is H_α, two others pass through some p_j and p_k. Since each p_i lies on only two sides, we obtain that the six polohessians T_i are composed of five lines forming a complete pentagon inscribed into S. Thus S has ∞^1 inscribed complete pentagons.

Let P be one such pentagon. For each of its five sides H_i there is a pair of points (a_i, b_i) such that

$$P_{a_i,b_i}(F) = H_i^2, \qquad i = 1, \ldots, 5.$$

Since each pair (a_j, b_j), $j \neq i$, contains a point on H_i,

$$P_{a_j,b_j,a_i,b_i}(F) = 0, \qquad j \neq i.$$

Moreover, H_i does not contain a_i, b_i; therefore

$$P_{a_i,b_i,a_i,b_i}(F) \neq 0.$$

Now, since H_i^2 are linearly independent by (4.2.2), the assertion will follow from the following:

(6.12.3) LEMMA. *Let F satisfy* (iii) *in* (6.12.2). *Let (a_i, b_i) be five pairs of points such that*:

(i) $Q_i = P_{a_i,b_i}(F)$ *are linearly independent*;

(ii) $P_{a_j,b_j,a_i,b_i}(F) = 0$ *for* $i \neq j$;

(iii) $P_{a_i,b_i,a_i,b_i}(F) \neq 0$.

Then there exist some constants $\lambda_i \neq 0$, $i = 1, \ldots, 5$, *such that*:

$$F = \lambda_1 Q_1^2 + \ldots + \lambda_5 Q_5^2.$$

Proof. Since the space of conics apolar to Φ is 5-dimensional, for every pair of points (a, b) we can write

$$P_{a,b}(F) = t_1 P_{a_1,b_1}(F) + \ldots + t_5 P_{a_5,b_5}(F) \qquad (1)$$

for some constants t_i, $i=1, ..., 5$. Polarizing both sides with respect to (a_i, b_i), and using (ii), we get

$$P_{a_i, b_i, a, b}(F) = t_i P_{a_i, b_i, a_i, b_i}(F), \qquad i=1, ..., 5. \tag{2}$$

Setting $a=b=x$, a generic point of $\mathbb{P}^2$, we find:

$$F = P_{x, x}(F).$$

Hence if we plug $a=b=x$ into (1) and use (2) we obtain

$$F = \lambda_1 Q_1^2 + ... + \lambda_5 Q_5^2,$$

where

$$\lambda_i = (P_{a_i, b_i, a_i, b_i}(F))^{1}.$$

(6.13) Let us now compute the covariant quartic $S(F)$ of a Clebsch quartic F. Let

$$F = l_1^4 + l_2^4 + l_3^4 + l_4^4 + l_5^4 = 0$$

be its polar complete pentagon. Then for any $a \in \mathbb{P}^2$ we have

$$P_a(F) = \sum l_i(a) l_i^3.$$

By definition, up to a numerical factor,

$$\mathbf{S}_4(F) = \mathbf{I}_4(P_a(F)) = \mathbf{I}_4\left(\sum l_i(a) l_i^3\right).$$

The symbolic expression for the invariant $\mathbf{I}_4$ of cubics is

$$\mathbf{I}_4 = (\alpha, \beta, \gamma)(\beta, \gamma, \delta)(\alpha, \gamma, \delta)(\alpha, \beta, \delta).$$

This means the following. If we polarize $Q = P_a(F)$,

$$\tilde{Q} = \sum l_i(a) l_i \otimes l_i \otimes l_i,$$

and write down $\tilde{Q} \otimes \tilde{Q} \otimes \tilde{Q} \otimes \tilde{Q} \in V^{*\otimes 12}$ as a linear combination of the tensors

$$(\alpha \otimes \alpha \otimes \alpha) \otimes (\beta \otimes \beta \otimes \beta) \otimes (\gamma \otimes \gamma \otimes \gamma) \otimes (\delta \otimes \delta \otimes \delta),$$

then the value of $\tilde{\mathbf{I}}_4$ at such a tensor is given by the product of determinants

$$|\alpha, \beta, \gamma|\, |\alpha, \beta, \delta|\, |\beta, \gamma, \delta|\, |\alpha, \gamma, \delta|,$$

where we identify the linear forms from V^* with the column of its coefficients. In our case, we obtain

$$\tilde{\mathbf{I}}_4(\tilde{Q}\otimes\tilde{Q}\otimes\tilde{Q}\otimes\tilde{Q}) = \tilde{\mathbf{I}}_4\left(\sum_{j=1}^{5}\ \bigotimes_{j\neq i} l_i(a) l_i \otimes l_i \otimes l_i\right) = \sum_{j=1}^{5} k_l \prod_{j\neq i} l_i(a),$$

where

$$k_r = \prod_{\substack{i<j<k \\ r\notin\{i,j,k\}}} \det(l_i, l_j, l_k) \neq 0, \qquad r = 1, \ldots, 5.$$

Replacing a by x, we derive the equation of $S(F)$:

$$S(F) = \sum_{j=1}^{5} k_j \prod_{j\neq i} l_i(x) = 0.$$

It is clear that $S(F)$ is circumscribed around the complete pentagon $l_1, \ldots, l_5$. The equation of $S(F)$ can be, informally, put in the form

$$k_1 l_1^{\ -1} + k_2 l_2^{\ -1} + k_3 l_3^{\ -1} + k_4 l_4^{\ -1} + k_5 l_5^{\ -1} = 0.$$

(cf. [Sa2, p. 269; Lü, p. 46]).

(6.13.1) DEFINITION. A *Lüroth quartic* is a quartic which can be circumscribed around a complete pentagon.

We shall now show that every Lüroth quartic is realized as the quartic covariant of a Clebsch quartic.

(6.13.2) LEMMA. *Let $l_1, \ldots, l_5 \in V^*$ be in general position. Then the five quartic forms $l_2 l_3 l_4 l_5, l_1 l_3 l_4 l_5, \ldots, l_1 l_2 l_3 l_4$ are linearly indepedent.*

Proof. It is easy to verify that any four of these forms are linearly independent. Indeed, let the first four be linearly dependent. Then

$$\alpha l_2 l_3 l_4 + \beta l_1 l_3 l_4 + \gamma l_1 l_2 l_4 + \delta l_1 l_2 l_3 = 0.$$

It does no harm to assume that $l_1 = x_0$, $l_2 = x_1$, $l_3 = x_2$, $l_4 = ax_0 + bx_1 + cx_2$, $a, b, c \neq 0$. Then, collecting the coefficients at each monomial, we find

$$\beta a = \beta b = \gamma b = \gamma c = \delta a = \delta c = 0.$$

This obviously implies the assertion.

Now we may assume that

$$a_1 l_2 l_3 l_4 l_5 + a_2 l_1 l_3 l_4 l_5 + a_3 l_1 l_2 l_4 l_5 + a_4 l_1 l_2 l_3 l_5 + a_5 l_1 l_2 l_3 l_4 = 0,$$

where all a_i are different from zero. Then, as we saw in (6.13), the covariant quartic of $\alpha_1 l_1^4 + \ldots + \alpha_5 l_5^4$ is equal to

$$k_1\alpha_2\alpha_3\alpha_4\alpha_5 l_2 l_3 l_4 l_5 + \ldots + k_5\alpha_1\alpha_2\alpha_3\alpha_4 l_1 l_2 l_3 l_4, \qquad (*)$$

where k_i are defined in (6.13). It is clear that we can choose $(\alpha_1, \ldots, \alpha_5)$ such that

$$\alpha_2 \ldots \alpha_5 k_1 = a_1, \ldots, \alpha_1 \ldots \alpha_4 k_5 = a_5.$$

Thus the assertion will follow from the fact that $\mathbf{S}_4(F) \neq 0$ for any Clebsch quartic (use (6.12.2) and (6.6.3), or argue as follows. Let $F = l_1^4 + \ldots + l_5^4$ be Clebsch quartic. Then

$$P_a(F) = l_1(a) l_1^3 + \ldots + l_4(a) l_4^3$$

for any a with $l_5(a) = 0$, $l_i(a) \neq 0$, $i \neq 5$. Thus $P_a(F)$ admits a polar quadrangle; hence it is not anharmonic cubic (5.14.1), i.e. $\mathbf{S}_4(F) = \mathbf{I}_4(P_a(F)) \neq 0$).

(6.13.3) COROLLARY. *Let $\mathscr{H} = \{H_1, \ldots, H_5\}$ be a complete pentagon. Then $\mathbf{S}_4$ transforms injectively the set of Clebsch quartics with polar pentagon $\mathscr{H}$ into the set of Lüroth quartics with inscribed pentagon $\mathscr{H}$.*

Proof. Let H_i be given by a linear form l_i. Then every Clebsch quartic F with polar pentagon $\{H_1, \ldots, H_5\}$ can be written as $F = \alpha_1 l_1^4 + \ldots + \alpha_5 l_5^4$. Using (*) from (6.13.2), the previous lemma, and the fact that the map $(\alpha_1, \ldots, \alpha_5) \to (k_1\alpha_2\alpha_3\alpha_4\alpha_5, \ldots, k_5\alpha_1\alpha_2\alpha_3\alpha_4)$ defines an automorphism of the open subset of $\mathbb{C}^5$ defined by $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \neq 0$, we prove the corollary.

(6.13.4) THEOREM (J.Lüroth). *Let F be a Clebsch quartic. Then its covariant quartic $S(F) = \mathbf{S}_4(F)$ is a Lüroth quartic. Every Lüroth quartic is obtained in this way.*

Proof. The polar of F with respect to any vertex of its polar pentagon P is a cubic admitting a polar triangle. Therefore the covariant quartic $S(F)$ passes through the ten vertices of P. Hence P is inscribed in $S(F)$.

Let C be a Lüroth quartic and let $l_1, \ldots l_5 = 0$ be its inscribed complete pentagon. Then the five quartics $l_1 \ldots \hat{l}_i \ldots l_5 = 0$, $i = 1, \ldots, 5$, are linearly independent (6.13.2) and pass through the ten vertices of the pentagon. The dimension of the linear system of quartics with these ten base points is equal to 4. Indeed, by (4.2.4) it is larger than 4 if either six base points are collinear, or all of them lie on a conic. In each case they are not realized as the set of vertices of a complete pentagon. So, the equation of C can be written in the form

$$C = \lambda_1 k_1 l_1^{-1} + \ldots + \lambda_5 k_5 l_5^{-1} = 0,$$

where k_i are defined as in (6.13). Then we consider the quartic

$$F = \lambda_1^{-1} l_1^4 + \ldots + \lambda_5^{-1} l_5^4,$$

and verify that $S(F) = C$.

(6.13.5) PROPOSITION. *A general Lüroth quartic is non-singular.*

Proof. It is known that the Hessian of a general cubic surface is a normal surface. As we saw in (5.15.4) it is circumscribed around a pentahedron of planes. Thus a general plane section of the Hessian is a non-singular Lüroth quartic.

(6.14) Let us now study polar hexagons of a quartic curve F. Let

$$\varphi_1 : \mathrm{Sec}_{6,4} \to |\mathcal{C}_{\mathbb{P}^2}(4)|$$

be the projection of the 6-secant bundle of the Veronese surface $V_{2,4} \subset \mathbb{P}^{14}$ (4.1.1). This map is surjective. Indeed, its image is an irreducible closed subset of $|\mathcal{C}_{\mathbb{P}^2}(4)|$ containing the locus of Clebsch quartics. Thus it is equal either to the closure of the locus of Clebsch quartics (= the catalecticant hypersurface) or to the whole space of quartics. It is easy to exclude the former possibility. Indeed take any six linear forms $l_1, \ldots, l_6$ such that the corresponding lines do not lie on an enveloping conic. Then the quartic

$$F = l_1^4 + \ldots + l_6^4 = 0$$

admits a polar hexagon and is non-degenerate. Indeed, if $P_\Phi(F) = 0$ for some conic Φ, we get

$$P_\Phi(F) = \Phi(l_1) l_1^2 + \ldots + \Phi(l_6) l_6^2 = 0.$$

Since l_i^2 are linearly independent (4.2.4), we find that $\Phi(l_i) = 0$, $i = 1, \ldots, 6$. This contradicts our choice of the linear forms l_i showing that the image of $\mathrm{Sec}_{6,4}$ contains a point not on the catalecticant hypersurface. A non-degenerate quartic does not admit polar (possibly degenerate) pentagons, since otherwise it belongs to the closure of the locus of Clebsch quartics. Thus we obtain that every non-degenerate quartic admits a polar hexagon (possibly degenerate). Finally we observe that the dimension of the general fibre of the map φ is equal to 3; hence a general quartic admits ∞^3 polar hexagons.

(6.14.1) Assume that F is non-degenerate and admits a polar hexagon $H_1, \ldots, H_6$, and let $\alpha_1, \ldots, \alpha_6$ be the corresponding points in $\check{\mathbb{P}}^2$. Let Φ be an enveloping conic passing through $\alpha_1, \ldots, \alpha_5$. Then $P_\Phi(F)$ has H_6 as its polar 1-gon; hence it is equal to the double line H_6^2. In other words, each

enveloping conic Φ_i passing through $\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_6$ is the second anti-polar of the line H_i. Note that, since F is non-degenerate, $\alpha_1, ..., \alpha_6$ do not lie on a conic; hence $\Phi_i \neq \Phi_j$, $i \neq j$, and $\Phi_i \cap \Phi_j = \{\alpha_1, ..., \alpha_6\} - \{\alpha_i, \alpha_j\}$, $\alpha_i \notin \Phi_i$.

This suggests the following construction of a polar hexagon of F:

(6.14.2) THEOREM. *Let F be a non-degenerate quartic. Let $H_1 = H_{\alpha_1}$ be a line in $\mathbb{P}^2$ and let Φ_1 be its second anti-polar enveloping conic with respect to F. Assume that α_1 does not belong to Φ_1. Let $H_2 = H_{\alpha_2}$ be such that $\alpha_2 \in \Phi_1$ and let Φ_2 be its second anti-polar with respect to F. Assume Φ_1 intersects transversally Φ_2. Let $\alpha_3, ..., \alpha_6$ be the four intersection points of Φ_1 and Φ_2. Then $H_{\alpha_1}, ..., H_{\alpha_6}$ is a polar hexagon of F.*

Proof. Let L be the linear system of enveloping quartics passing through the six constructed points $\alpha_1, ..., \alpha_6$. By Lemma (4.2.4) it is of dimension 8, unless $\alpha_1, ..., \alpha_6$ lie on the same line (then it is of dimension 9). In our case the latter is impossible. It contains two four-dimensional linear systems formed by quartics of the form $\Phi_1 K$ (resp. $\Phi_2 K'$), where $\alpha_1 \in K$, $\alpha_2 \in K'$. By assumption, Φ_1 and Φ_2 have no common irreducible components. This implies that the two four-dimensional linear systems intersect along the unique quartic $\Phi_1 \Phi_2$, and hence span L. Since

$$\langle \Phi_1 K, F \rangle = \langle K, P_{\Phi_1}(F) \rangle = \langle K, H_{\alpha_1}^2 \rangle = K(\alpha_1) = 0,$$
$$\langle \Phi_2 K', F \rangle = \langle K', P_{\Phi_2}(F) \rangle = \langle K', H_{\alpha_2}^2 \rangle = K'(\alpha_2) = 0,$$

we obtain that $\langle F, \Psi \rangle = 0$ for all $\Psi \in L$. Thus we can use Proposition (4.3) to conclude that $H_{\alpha_1}, ..., H_{\alpha_6}$ is a polar hexagon of F.

(6.14.3) Let us see what can go wrong in this construction. First of all, we begin by taking a line H satisfying the property that its second anti-polar Φ does not contain it. This is easy to fulfil. Since F is non-degenerate, we find the equation of Φ by applying the inverse of the catalecticant matrix $\mathrm{Cat}(F)$ to H^2 (see (6.6)),

$$\Phi_1 = \sum (C_{ij;sk} \alpha_s \alpha_k) u_i u_j,$$

where $H = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$, and $(G_{ij;sk}) = \mathrm{Cat}(F)^*$ is the cofactor matrix of $\mathrm{Cat}(F)$. The condition that Φ_1 vanishes at H is expressed by the equation

$$\sum C_{ij;sk} \alpha_s \alpha_k \alpha_i \alpha_j = 0, \qquad (*)$$

so this is an equation of degree 4 in the coordinates of H. The left-hand side of $(*)$ is a contravariant Ω of quartics of class 4 and degree 5 in the coefficients of F. This is a "new" covariant, not mentioned in [Sa2].

Contrary to a statement from [Cl], it differs from the product of the quartic contravariant σ and the cubic invariant A of Salmon (see [Ci1]). Obviously

$$\mathbf{\Omega}(F)=0 \Leftrightarrow \operatorname{crk}(\operatorname{Cat}(F))>1.$$

In particular, for a non-degenerate quartic F, the locus $\Omega(F)$ of "bad" lines H is the curve $\mathbf{\Omega}(F)=0$. Note that the correspondence $F \to \Omega(F)=\{\mathbf{\Omega}(F)=0\}$ is a new kind of duality for quartics. It corresponds to the usual duality between quadrics in $|S^4(V^*)|$ and quadrics in $|S^4(V)|$ (see (2.9)). Under this duality quartics with $\operatorname{crk}(\operatorname{Cat}(F))=1$ correspond to double conics

$$\Omega(F)=\left(\sum \lambda_{ij}\alpha_i\alpha_j\right)\left(\sum \lambda_{sk}\alpha_s\alpha_k\right)=\left(\sum \lambda_{ij}\alpha_i\alpha_j\right)^2=0,$$

where $C_{ij;\,sk}=\lambda_{ij}\lambda_{sk}$ as in (6.6). Here

$$\Gamma(F)=\sum \lambda_{ij}\alpha_i\alpha_j=0$$

is the apolar conic of F. So, we obtain that, for quartics F with $\dim(AP_2(F))=1$, $\Omega(F)$ is the square of an irreducible conic if F is a Clebsch quartic, and is the square of a reducible conic otherwise.

Thus we should start our construction of a polar hexagon by taking any line not in the curve $\Omega(F)$. The next difficulty occurs when for every $\alpha_2 \in \Phi_1$ the second apolar conic Φ_2 is not transversal to Φ_1. We do not know whether this is avoidable, even in the case when Φ_1 is irreducible (for this we have to take H_1 not in the set $\Gamma(F)$). The case when Φ_2 is tangent to Φ_1 leads to degenerate hexagons.

(6.15) DEFINITION. A line H is called a *good line* for a non-degenerate quartic F if F admits a polar hexagon with one side equal to H, and the second anti-polar of F with respect to H is a non-singular conic not containing H (i.e., $\Omega(H)\neq 0$).

It easily follows from (6.14.1) and (6.14.2) that, if F admits at least one polar hexagon, the set of its good lines is a non-empty open Zariski subset of $\mathbb{P}^2$.

(6.15.1) PROPOSITION. *Let H be a good line with respect to a non-degenerate quartic F. Then the variety of all pentagons $H_1, \ldots, H_5$ such that $H, H_1, \ldots, H_5$ is a polar hexagon of F is one-dimensional. It consists of all polar pentagons of a unique Clebsch quartic F'.*

Proof. Let $H_1 = H, H_2, ..., H_6$ be a polar hexagon of F. Suppose $H_1, H_2', ..., H_6'$ is another polar hexagon of F. Then we can write F in two ways,

$$F = l_1^4 + l_2^4 + ... + l_6^4 = \lambda l_1^4 + m_1^4 + ... + m_6^4 = 0,$$

where $H_i = \{l_i = 0\}$, $i = 1, ..., 6$, $H_i' = \{m_i = 0\}$, $i = 2, ..., 6$, and λ is a non-zero constant. Hence

$$(1 - \lambda) l_1^4 = \sum_{i=2}^{6} (l_i^4 - m_i^4). \qquad (*)$$

Let Φ (resp. Φ') be the enveloping conic through the points $H_2, ..., H_6$ (resp. $H_2', ..., H_6'$). Then $P_\Phi(F) = \Phi(H_1) l_1^2$, $P_{\Phi'}(F) = \lambda \Phi'(l_1) l_1^2$. Since F is non-degenerate, we get $\Phi = \Phi'$. Hence Φ contains the 10 lines $H_2, ..., H_6$, $H_2', ..., H_6'$. If $\lambda \neq 1$, $(*)$ implies also that $\Phi(l_1) = 0$ hence $P_\Phi(F) = 0$. This is impossible. Therefore $\lambda = 1$,

$$l_2^4 + ... + l_6^4 = m_2^4 + ... + m_6^4.$$

By definition of a good line, Φ is a non-singular conic. Hence the lines $\{H_2, ..., H_6\}$ and $\{H_2', ..., H_6'\}$ are complete pentagons which are polar pentagons of the same Clebsch quartic F'. It follows from Theorem (6.14.2) that the set of pentagons which together with a fixed line form a polar sextic of F is one-dimensional. This proves our assertion.

(6.15.2) COROLLARY. *Let F be a quartic admitting a polar hexagon, and let H be a good line with respect to F. Then there exists a unique Clebsch quartic F' which is* 4*-tangent to F at the points of intersection of F and H.*

(6.16) DEFINITION. A polar hexagon of a non-degenerate plane quartic is called *concurrent* if it contains three concurrent sides.

Note that no four lines of a polar hexagon of a non-degenerate quartic are concurrent.

(6.16.1) PROPOSITION. *Let $H_1, ..., H_6$ be a concurrent polar hexagon of F and let $a = H_1 \cap H_2 \cap H_3$ be a concurrent point. Then a belongs to the covariant quartic $S(F)$ of F, and the triangle formed by H_4, H_5, and H_6 is a polar triangle of the polar of F with respect to a. If Φ is the second anti-polar of H_1 with respect to F, then the line $\check{H}_a$ intersects Φ at two points corresponding to H_2 and H_3.*

Proof. The triangle formed by H_4, H_5, an H_6 is a polar triangle of $P_a(F)$, so $a \in S(F)$. We know that H_i, $i \neq 1$, lie on the second anti-polar Φ of H_1, The three lines H_1, H_2, and H_3 belong to the pencil of lines through

the point a, i.e., belong to the line $\check{H}_a$ in the dual plane. Hence H_2 and H_3 are the intersection points of $\mathscr{H}_a$ with Φ.

(6.16.2) COROLLARY. *A polar hexagon is concurrent if and only if one of its sides is a side of a polohessian of F.*

Proof. The necessity was shown in the proof of the previous proposition. Let $H_1, ..., H_6$ be any polar hexagon of F which contains a side of a polohessian triangle of F, say H_1. Then the remaining lines lie on the second anti-polar Φ of H_1. Since $P_{a,b}(F) = H_1^2$ for some $a, b \in S(F)$, we get $\Phi = \check{H}_a \check{H}_b$. This shows that three of these lines belong to either $\check{H}_a$ or $\check{H}_b$.

We refer to [Mu] for the geometry of the variety $X_6(F)$ of polar hexagons of a general quartic.

7. THE SCORZA MAP

Here we define a birational map from the space of plane quartics to its finite cover parametrizing the pairs (C, ϑ), where C is a non-singular quartic and ϑ is an even theta characteristic on C.

(7.1) Recall that an even (non-vanishing) theta characteristic on a non-singular non-hyperelliptic curve X of genus g is a non-effective divisor class ϑ such that 2ϑ belongs to the canonical class of X. By Riemann–Roch, for every point $x \in X$, the linear system $|x + \vartheta|$ consists of a unique effective divisor $T_\vartheta(x)$ of degree g. This defines a correspondence $T_\vartheta \in \mathrm{Div}(X \times X)$. Define the map

$$\alpha: X \times X \to \mathrm{Jac}(X),$$

by sending a pair (x, y) to the divisor class $\mathrm{cl}(y - x)$. Let $W_{g-1} \subset \mathrm{Pic}^{g-1}(X)$ be the hypersurface of effective divisor classes of degree $g - 1$, and let $\Theta = W_{g-1} - \vartheta \subset \mathrm{Jac}(X)$ be its translate. Define a correspondence $T'_\vartheta \in \mathrm{Div}(X \times X)$ by setting

$$T'_\vartheta = \alpha^*(\Theta).$$

The restriction of T'_ϑ to the fibre $\{x\} \times X$, $x \in X$, is equal to $\zeta = \alpha_x^*(\Theta)$, where $\alpha_x(y) = \mathrm{cl}(y - x)$. By Riemann's theorem,

$$\zeta \sim x + \vartheta,$$

so $\zeta = |x + \vartheta| = T_\vartheta(x)$, since $h^0(x + \vartheta) = 1$. Thus $T'_\vartheta = T_\vartheta$. If σ is the involution of $X \times X$ which permutes the factors, then

$$\sigma^* T_\vartheta = \alpha^*((-\mathrm{id}_{\mathrm{Jac}(X)})^* \Theta) = \alpha^*(\Theta) = T_\vartheta,$$

so T_ϑ is a symmetric correspondence. It has no united points, since the existence of a point $(x, x) \in \mathrm{Supp}(T_\vartheta)$ implies that ϑ is effective. Let p_1, p_2 be the projelctions $X \times X \to X$. Using the seesaw theorem one obtains,

$$T_\vartheta \sim \Delta + p_1^*(\vartheta) + p_2^*(\vartheta). \qquad (*)$$

Finally note that T_ϑ is a connected curve. In fact, $\alpha(X \times X) \cdot \Theta$ is connected because it is an ample divisor. Since $0 \notin \Theta$, and α is an embedding outside the diagonal of $X \times X$, we obtain that T_ϑ is connected.

(7.1.1) LEMMA. *Let* $x \in X$ *and* $y \in \mathrm{Supp}(T_\vartheta(x))$. *Let* $T_\vartheta(x) = y + \xi$, $T_\vartheta(y) = \eta + x$. *Then* $\xi + \eta \in |K_X|$.

Proof. We have

$$\vartheta = \xi + y - x,$$
$$\vartheta = \eta + x - y.$$

Adding up, we obtain the assertion.

(7.1.2). Let us identify X with its canonical model in $\mathbb{P}^{g-1}$. We shall refer to the divisor $T_\vartheta(x) = y_1 + \ldots + y_g$ as a *ϑ-polyhedron* attached to x with *vertices* $y_1, \ldots, y_g$. The hyperplane $\langle T_\vartheta(x) - y_i \rangle$ will be called the *face* of $T_\vartheta(x)$ opposite to the vertex y_i. Note that this definition is meaningful even when $T_\vartheta(x)$ is not reduced; here, for any effective divisor D on X, we denote by $\langle D \rangle$ the intersecion of all hyperplanes in $\mathbb{P}^{g-1}$ which cut out the divisors $D + D' \in |K_X|$ on the canonical model of X. We can interpret the previous lemma by saying that for any $(x, y) \in \mathrm{Supp}(T_\vartheta)$, the opposite faces in the polyhedra $T_\vartheta(x)$ and $T_\vartheta(y)$ coincide.

(7.1.3) LEMMA. *A point* $(x, y) \in T_\vartheta$ is singular if and only if $|T_\vartheta(x) - 2y| \neq \emptyset$, $x \in \langle T_\vartheta(x) - y \rangle$, *or equivalently* $|T_\vartheta(y) - 2x| \neq \emptyset$, $y \in \langle T_\vartheta(y) - x \rangle$.

Proof. The point (x, y) is singular on T_ϑ if and only Θ does not intersect $\alpha(X \times X)$ transversally at $\alpha(x, y)$. The projectivized tangent space of Θ at $\alpha(x, y)$ is $\langle T_\vartheta(x) - y \rangle$, the projectivized tangent space of $\alpha(X \times X)$ at (x, y) is the line $\langle x + y \rangle$. Thus (x, y) is singular if and only if $\langle x + y \rangle \subset \langle T_\vartheta(x) - y \rangle$. Since the divisor $T_\vartheta(x)$ spans $\mathbb{P}^{g-1}$, we have

$$y \in \mathrm{Supp}(T_\vartheta(x) - y), \qquad x \in \langle T_\vartheta(x) - y \rangle.$$

By symmetry, we must have $x \in \mathrm{Supp}(T_\vartheta(y) - x)$, $y \in \langle T_\vartheta y) - x \rangle$.

(7.1.4) As we have already mentioned, for every $(x, y) \in T_\vartheta$, $h^0(T_\vartheta(x) - y) = 1$; hence, by Riemann's theorem, $\alpha(x, y)$ belongs to the open subset Θ^{ns} of non-singular points of Θ. Define the map

$$\pi: T_\vartheta \to |K_X|,$$

as the composition of the embedding $\alpha|_{T_\vartheta}: T_\vartheta \to \Theta$ and the Gauss map:

$$\gamma: \Theta^{ns} \to |K_X|.$$

Recall that the latter is defined by sending a point $z \in \Theta^{ns}$ to the unique $D \in |K_X|$ containing the effective divisor linearly equivalent to $z + \vartheta$. By definition, the map π sends a pair $(x, y) \in T_\vartheta$ to $\langle T_\vartheta(x) - y \rangle = \langle T_\vartheta(y) - x \rangle$. Denote by $\Gamma(\vartheta)$ the image of π.

It is known that the Gauss map is a finite map of degree $\binom{2g}{g}\frac{2}{1}$ [ACGH, p. 247]. Its branch locus is an open subset of the dual hypersurface of the canonicl model of X. Since T_ϑ is symmetric, the degree of the map

$$\pi: T_\vartheta \to \Gamma(\vartheta)$$

equals $2d(\vartheta) \leqslant \binom{2g}{g}\frac{2}{1}$. In other words, the Gauss map factors through the map

$$\bar{\gamma}: \bar{\Theta} = \Theta/(-\mathrm{id}_{\mathrm{Jac}(X)}) \to |K_X|,$$

of degree $\frac{1}{2}\binom{2g}{g}\frac{2}{1}$, and $d(\vartheta)$ is equal to the degree of the restriction of $\bar{\gamma}$ to the image $\bar{\Gamma}(\vartheta)$ of T_ϑ in $\bar{\Theta}$. The number $d(\vartheta)$ is an interesting invariant of the pair (X, ϑ). By (7.1.1) and (7.1.2), $2d(\vartheta)$ is equal to the number of g-polyhedra of X that have a common face. For example, if $g = 3$, $d(\vartheta) \in \{1, 2, 3\}$. Later on we shall see that for a general quartic curve X one has $d(\vartheta) = 1$ for every ϑ; however, $d(\vartheta)$ may take the value 3 on special pairs (X, ϑ). We do not know an example of a pair (X, ϑ) with $d(\vartheta) = 2$. We refer to [Do-Or] for more geometry related to the surface $\bar{\Theta}$ (the *dianode surface*).

(7.1.5) PROPOSITION. *Assume T_ϑ is a reduced curve. Then its arithmetic genus $p_a(T_\vartheta)$ is given by*

$$p_a(T_\vartheta) = 3g(g-1) + 1.$$

Proof. By adjunction,

$$2p_a(T_\vartheta) - 2 = (T_\vartheta + K_{X \times X}) \cdot T_\vartheta = T_\vartheta \cdot T_\vartheta + 4g(g-1).$$

Now we use that

$$T_\vartheta \cdot T_\vartheta = T_\vartheta \cdot \alpha^*(\Theta) = (\alpha(T_\vartheta) \cdot \Theta)_{\mathrm{Jac}(X)} = (\alpha(X \times X) \cdot \Theta^2)_{\mathrm{Jac}(X)}.$$

It is known that

$$\chi(X \times X) \approx \frac{2}{(g-2)!}\Theta^{g-2}$$

and $\Theta^g = g!$ ([ACGH]; note a mistake in the formula on p. 223). This gives

$$T_\vartheta \cdot T_\vartheta = 2g(g-1),$$

and

$$2p_a(T_\vartheta) - 2 = 6g(g-1).$$

(7.1.6) PROPOSITION. *Assume $d(\vartheta) = 1$ and T_ϑ is a reduced curve. Let $\pi = \pi' \circ h$ be the composition, where $\pi'\colon T_\vartheta \to T_\vartheta/(\tau) = \Gamma(\vartheta)'$ is a factor map by the natural involution of T_ϑ, and $h\colon \Gamma(\vartheta)' \to \Gamma(\vartheta)$ is a birational map. Let $\omega_{\Gamma(\vartheta)'}$ be the canonical sheaf of $\Gamma(\vartheta)'$. Then*

$$\omega_{\Gamma(\vartheta)'} \cong h^*(\mathcal{O}_{\Gamma(\vartheta)}(3)) \otimes \mathcal{O}_{\Gamma(\vartheta)'}(\varepsilon)$$

for some 2-torsion divisor class ε on $\Gamma(\vartheta)'$.

Proof. Since $T_\vartheta \cap \Delta = \varnothing$, because ϑ is not effective, the factor map $\pi'\colon T_\vartheta \to \Gamma(\vartheta)'$ is an unramified double cover. This gives

$$\pi'^*(\omega_{\Gamma(\vartheta)'}) \cong \omega_{T_\vartheta}.$$

Let $a \in X$ and $\check{H}_a$ be the hyperplane in $\check{\mathbb{P}}^{g-1}$ whose points are the hyperplanes in $\mathbb{P}^{g-1}$ which contain the point a. Then

$$\check{H}_a \cap \Gamma(\vartheta) = \{H \in \Gamma(\vartheta) : H \text{ is a face of } T_\vartheta(b),\ b \in T_\vartheta(a),\ a \in H\}.$$

If $T_\vartheta(a) = b_1 + \ldots + b_g$, then $a \in T_\vartheta(b_i)$, and

$$H \in \check{H}_a \cap \Gamma(\vartheta) \Leftrightarrow H = \langle T_\vartheta(b_i) - q \rangle = \pi(b_i, q)$$

for some b_i, $q \in T_\vartheta(b_i)$, $q \neq a$. Let

$$p, p'\colon T_\vartheta \to X, \qquad (a, b) \mapsto a, \qquad (a, b) \mapsto b$$

be the two maps induced by the projections $X \times X \to X$. By definition of the map π, we have

$$\begin{aligned}\pi^*(\check{H}_a \cap \Gamma(\vartheta)) &= (p^*(T_\vartheta(a)) - p'^*(a)) + (p'^*(T_\vartheta(a)) - p^*(a)) \\ &= p^*(T_\vartheta(a) - a) + p'^*(T_\vartheta(a) - a) = p^*(\vartheta) + p'^*(\vartheta).\end{aligned}$$

Restricting (*) from (7.1) to T_ϑ, and taking into account that Δ and T_ϑ are disjoint, we see that $p^*(\vartheta)+p'^*(\vartheta)$ is cut out on T_ϑ by the theta divisor Θ. So, since Δ and T_ϑ are disjoint, we get

$$\pi^*(\mathcal{O}_{\Gamma(\vartheta)}(1)) \cong \mathcal{O}_{T_\vartheta}(\Theta).$$

By the proof of the formula for the genus of T_ϑ (7.1.5),

$$\begin{aligned}\pi'^*(\omega_{\Gamma(\vartheta)'}) &\cong \omega_{T_\vartheta} \cong \mathcal{O}_{T_\vartheta}(\Theta)\otimes(\mathcal{O}_{T_\vartheta}\otimes\mathcal{O}_{X\times X}(K_{X\times X}))\\ &\cong \mathcal{O}_{T_\vartheta}(\Theta)\otimes\mathcal{O}_{T_\vartheta}(p^*(2\vartheta)+p'^*(2\vartheta)) \cong \mathcal{O}_{T_\vartheta}(3\Theta)\\ &\cong \pi^*(\mathcal{O}_{\Gamma(\vartheta)}(3)) \cong \pi'^*(h^*(\mathcal{O}_{\Gamma(\vartheta)}(3))).\end{aligned}$$

This proves the assertion.

(7.1.7) COROLLARY. *Assume $d(\vartheta)=1$ and T_ϑ is a reduced curve. Then $\Gamma(\vartheta)$ is of degree $g(g-1)$ and $\Gamma(\vartheta)'$ is of arithmetic genus $\frac{3}{2}g(g-1)+1$. In particular, if $g=3$, $\Gamma(\vartheta)'\cong\Gamma(\vartheta)$ is a plane sextic.*

(7.1.8) PROPOSITION. *Let $a, b\in X$, and let $T_\vartheta(a)$ and $T_\vartheta(b)$ be the corresponding ϑ-polyhedra. Then the dimension of the linear system of quadrics through the divisor $T_\vartheta(a)+T_\vartheta(b)$ is of dimension $\frac{1}{2}g(g+1)-2g$.*

Proof. We have

$$T_\vartheta(a)\sim a+\vartheta, \qquad T_\vartheta(b)\sim b+\vartheta.$$

Adding up, we find

$$|2K_X - T_\vartheta(a) - T_\vartheta(b)| = |K_X - a - b|.$$

From this we infer that the linear system of quadrics through $T_\vartheta(a)$ and $T_\vartheta(b)$ cuts out on X the linear system $|K_X-a-b|$ of dimension $g-3$. Adding up the dimension of the linear system of quadrics in $\mathbb{P}^{g-1}$ containing X ($\frac{1}{2}g(g+1)-3g+3$), we obtain the needed number.

(7.1.9) *Remark.* If $T_\vartheta(a)$ and $T_\vartheta(b)$ are reduced and have no common points, this proposition can be interpreted by saying that the set $T_\vartheta(a)\cup T_\vartheta(b)$ imposes one less condition on quadrics; alternatively, it is a self-associated unordered set of points in $\mathbb{P}^{g-1}$ (see [Do-Or]).

(7.2) Suppose the covariant quartic $S(F)$ is non-singular. Then every even theta characteristic ϑ on $S(F)$ defines a symmetric correspondence of degree (3, 3) without united points on $S(F)$. Another correspondence of this kind is $T(F)$ defined in (6.8). We shall later show that $T(F)$ is equal to T_ϑ for some ϑ.

(7.2.1) LEMMA. *Let X be a non-singular curve of genus g, and let*

$$T \in \mathrm{Div}(X \times X)$$

be a symmetric effective correspondence without united points, of some valence v and degree (g, g). Suppose that $h^0(X, T(x)) = 1$ for general point $x \in X$. Then there exists a unique even theta characteristic ϑ on X such that $T = T_\vartheta$.

Proof. Recall that a correspondence T of degree (a, b) is of valence v for some integer v if the divisor class $T(x) + vx$ is independent of x. Equivalently, this means that T induces the endomorphism $-v\,\mathrm{id}_{\mathrm{Jac}(X)}$ of the Jacobian variety $\mathrm{Jac}(X)$ of X, or T is algebraically equivalent on $X \times X$ to the divisor

$$E + F - v\Delta,$$

where E and F are divisors whose support consists of fibres of the two projections $X \times X \to X$, and Δ is the diagonal. By the Cayley–Brill formula [Gr-Ha, p. 287], the number of united points of T is equal to $a + b + 2vg$, which implies in our case that $v = -1$. Let us denote the divisor class $T(x) - x$ by ϑ, and prove that it is an even theta characteristic. By assumption $h^0(T(x)) = 1$ for general x; since $x \notin \mathrm{Supp}(T(x))$, we obtain that $T(x) - x$ is not efective for general x. This gives that $h^0(\vartheta) = 0$. Let

$$\Theta' = (-\mathrm{id}_{\mathrm{Jac}(X)})^*(\Theta),$$

where $\Theta = W_{g-1} - \vartheta$ is as in (7.1). By the inversion part of Riemann's theorem we conclude as in (7.1) that $T = \alpha^*(\Theta')$ and $\sigma^* T = \alpha^*(\Theta)$. By hypothesis, the correspondence T is symmetric, so

$$\alpha^*(\Theta') = \alpha^*(\Theta). \qquad (*)$$

Let $\Theta' = \Theta + e$ for some $e \in \mathrm{Jac}(X)$. Restricting $(*)$ to a fibre $\{x\} \times X$ and again using Riemann's theorem, we obtain $e = 0$. So $\Theta' = \Theta$; therefore ϑ is a theta characteristic.

(7.3) Let F be a Clebsch quartic with non-singular covariant quartic $C = S(F)$. By the proof of (6.12.2) the family $X_5(F)$ of polar pentagons of F is equal to the family of pentagons inscribed in C. Its sides are the sides of the polohessian triangles of F, and hence are parametrized by the apolar conic Φ of F. Furthermore any general $H \in \Phi$ belongs to a unique polar pentagon. Therefore $X_5(F)$ is an irreducible rational curve. This implies that every sufficiently general $x \in C$ is a vertex of a polar pentagon of F. Such a pentagon is unique since the three sides not containing x are the sides of the polohessian triangle $He(P_x(F))$ and the other sides are

obtained from the polohessian triangle of the vertices of $He(P_x(F))$. Assigning to each $x \in C$ the unique pentagon that has x as its vertex, we obtain a map $\alpha: C \to \mathbb{P}^1$ of degree 10 which can be reconstructed from the correspondence $T \subset C \times C$ and vice versa.

(7.3.1) LEMMA. *There is a unique even theta characteristic ϑ on C such that $T = T_\vartheta$ (see (7.1)).*

Proof. Let $x \in C$ be sufficiently general and $y \in T(x) = y + p_1 + p_2$. Then the line $\langle p_1, p_2 \rangle$ is a side of the pentagon $\alpha(x)$ and does not contain x and y. Therefore it is a side of the triangle $T(y)$ and by the description of the polohessian triangles given above we have

$$T(x) - y + T(y) - x \in |K_C|. \qquad (*)$$

Let us consider the map $f: C \to \mathrm{Pic}^{(2)}(C)$ defined by $f(x) = \mathrm{cl}(T(x) - x)$. If $T(x) = y_1 + y_2 + y_3$ we conclude by $(*)$ that $f(y_i) = f(y_j)$, $i, j = 1, 2, 3$. Applying the same argument for $T(y_i)$ and so forth we conclude that $f(x) = f(x')$ for any two vertices of a complete pentagon from $X_5(F)$. Thus $f = \beta \circ \alpha$ for some map $\beta: \mathbb{P}^1 \to \mathrm{Pic}^{(2)}(C)$ showing that $f(C)$ is a point $\vartheta \in \mathrm{Pic}^{(2)}(C)$. By $(*)$ we conclude that $2\vartheta = K_C$. For general $x \in C$ the points of $T(x) = |\vartheta + x|$ are not collinear so that $h^0(\vartheta) = 0$.

Let M^C be the variety of Clebsch quartics F with non-singular covariant quartic $S(F)$. By (6.13.5) and (6.13.4), M^C is not an empty open Zariski subset in the variety of all Clebsch quartics (which is an open subset of the catalecticant hypersurface $\mathbf{C} = 0$). The restriction of the covariant $\mathbf{S}_4$ to M^C defines a morphism,

$$S: M^C \to M^L,$$

where M^L is the variety of non-singular Lüroth quartics. According to the previous lemma we can lift this morphism to a morphism

$$Sc^C: M^C \to \tilde{M}^L,$$

where $\tilde{M}^L$ is the variety of pairs (C, ϑ), where C is a nonsingular Lüroth quartic, and ϑ is an even theta characteristic defined by the polohessian triangles of some Clebsch quartic in $S^{-1}(C)$.

(7.3.2) THEOREM. *The map*

$$Sc^C: M^C \to \tilde{M}^L$$

is an isomorphism.

Proof. It suffices to show the injectivity of the map Sc^C. Suppose $Sc^C(F) = Sc^C(F') = (C, \vartheta)$. The correspondence T_ϑ determines uniquely the map $\alpha: C \to \mathbb{P}^1$ of degree 10 with the property that T_ϑ transforms any fibre to itself. Thus $X_5(F) = X_5(F')$. It remains to apply Corollary (6.13.3).

(7.3.3) COROLLARY. *The map*

$$Sc^C: M^C \to M^L, \qquad F \mapsto S(F)$$

is finite. In particular, the closure of M^L *in* $|\mathcal{O}_{\mathbb{P}^2}(4)|$ *is a hypersurface.*

(7.3.4) *Remark.* The degree of the hypersurface M^L of Lüroth quartics is equal to 54 (see [Mo, LP, TT]).

(7.4) Let C be a non-singular Lüroth quartic; an even theta characteristic ϑ on C will be called *pentagonal* if (C, ϑ) is in the image of M^C under the map Sc^C. Note that the invariant $d(\vartheta)$ defined in (7.1.4) is equal to 3 for any pentagonal characteristic. We do not know whether every even theta characteristic on C with $d(\vartheta) = 3$ is pentagonal. Each pentagonal characteristic defines a one-dimensional family of complete pentagons inscribed in C. The sides of these pentagons are parametrized by an open subset of $\Gamma(F)$. Each complete pentagon inscribed in C belongs to some family of pentagons defined by some pentagonal characteristic (see (6.13.4)). The number of pentagonal theta characteristics on a general Lüroth quartic is equal to the degree of the map $S^C: M^C \to M^L$. It is known to be equal to 1 [LP]. It is known that for some curves the preimage of some Lüroth quartic consists of more than one point (e.g., for desmic quartics; see [Bat]). According to [Ba] the moduli space $M(2, 0, 4)$ of stable rank 2 vector bundles on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 4$ is birationally isomorphic to the variety of pairs (C, ϑ), where C is a Lüroth quartic, and ϑ is a pentagonal even theta characteristic. By Theorem (7.3.2), it is birationally isomorphic to M_C. The rationality of $M(2, 0, 4)$ follows easily from the rationality of the catalecticant hypersurface.

(7.4.1) THEOREM. *Let C be a non-singular Lüroth quartic and let ϑ be its pentagonal theta characteristic. Then there exists a cubic surface K in $\mathbb{P}^3$ such that the web of polar quadrics of K contains a net whose Hessian invariant is equal to (C, ϑ).*

Proof. Let $C = S(F)$, where $F = l_1^4 + \ldots + l_5^4$. Then

$$C = k_1 l_1^{-1} + \ldots + k_5 l_5^{-1},$$

where $k_r = \Pi_r \notin \{i, j, k\} \det(l_i, l_j, l_k)$.

Let K be the cubic surface in $\mathbb{P}^3 = \mathbb{P}(W)$ with equation

$$K = L_1^3 + \ldots + L_5^3 = 0,$$

where L_i are linear forms in four variables $y_0, \ldots, y_4$ satisfying

$$\sum \lambda_i L_i = 0, \qquad \lambda_i^2 = k_i, \qquad i = 1, \ldots, 5.$$

We know from (5.15.4) that its Hessian is given by the equation

$$He(K) = \sum \lambda_i^2 L_i^{-1} = 0.$$

Let

$$\sum_{i=1}^{5} a_i l_i = 0$$

be one of the non-trivial linear relations between the forms l_i in which all coefficients are non-zero. The condition that each three forms l_i are linearly independent allows us to do it. Choose λ_i satisfying

$$\lambda_i^3 = k_i a_i, \qquad i = 1, \ldots, 5,$$

and let

$$i: V \to W$$

be the linear map whose transpose map $W^* \to V^*$ sends each L_i to $(\lambda_i^2/k_i) l_i$. Then the Hessian of the net $\mathcal{N} = \{P_{i(a)}(K)\}_{a \in V}$ is equal to C.

Let $s: C \to \mathbb{P}(W)$ be the Steiner map, which assigns to each point $a \in C$ the singular point $s(a)$ of the quadric $P_{i(a)}(K)$. We know from (5.5.2) that for any plane H in $\mathbb{P}(W)$ its inverse image $s^*(H)$ is a divisor on C linearly equivalent to $K_C + \vartheta'$, where ϑ' is the theta characteristic that, together with C, defines the Hesse invariant of the net $\mathcal{N}$. We shall show that $\vartheta' = \vartheta$ by choosing a special H and computing the divisor $s^*(H)$. In view of (5.15.4), for any $a \in C$ the coordinates $(y_0, \ldots, y_4)$ of the point $s(a)$ satisfy the relation:

$$L_i(y) L_i(i(a)) = \lambda_i, \qquad i = 1, \ldots, 5.$$

Pick the plane H with the equation:

$$\lambda_1 L_1(y) + \lambda_2 L_2(y) + \lambda_3 L_3(y) = (-\lambda_4 L_4(y) - l_5 L_5(y)) = 0.$$

Then

$$\frac{\lambda_1^2}{L_1(i(a))} + \frac{\lambda_2^2}{L_2(i(a))} + \frac{\lambda_3^2}{L_3(i(a))} = -\frac{\lambda_4^2}{L_4(i(a))} - \frac{\lambda_5^2}{L_5(i(a))} = 0$$

and the divisor $2s^*(H)$ is cut out in $\mathbb{P}(V)$ by the conic

$$k_1 l_2 l_3 + k_2 l_1 l_3 + k_3 l_1 l_2 = 0,$$

and the line

$$k_4 l_5 + k_5 l_4 = 0.$$

Writing the equation of the Lüroth quartic in the form

$$C: l_1 l_2 l_3 (k_4 l_5 + k_5 l_4) + l_4 l_5 (k_1 l_2 l_3 + k_2 l_1 l_3 + k_3 l_1 l_2) = 0,$$

we observe that the line is tangent to C at the point $p = l_4 \cap l_5$ and cuts out the divisor

$$D_1 = 2p + q_1 + q_2,$$

for some points q_1 and q_2. The conic cuts out the divisor

$$D_1 = 2(p_1 + p_2 + p_3) + q_1 + q_2,$$

where $p_1 = l_2 \cap l_3$, $p_2 = l_1 \cap l_3$, $p_3 = l_1 \cap l_2$ are the vertices of the triangle $T_3(p)$ (see (7.3.1). Therefore, $s^*(H)$ cuts out the divisor

$$p_1 + p_2 + p_3 + p + q_1 + q_2 \sim T_3 + K_C - p = p + \vartheta + K_C - p \sim \vartheta + K_C.$$

(7.4.2) *Remark.* The assertion that a Lüroth quartic is equal to the Hessian of a net of polars of a cubic surface was proven first by W. Frahm [Fr].

(7.5) LEMMA. *Assume $S(F)$ is non singular. Then*

(i) *F is weakly non-degenerate;*

(ii) *the correspondence $T(F) \subset S(F) \times S(F)$ is of degree (3, 3).*

Proof. (i) Assume F is not weakly non-degenerate. Then $P_{a,b}(F) = 0$ for some a, $b \in \mathbb{P}^2$. Indeed if $\dim AP_2(F) = 1$, this follows from the definition. If $AP_2(F) > 1$, then the linear system of apolar conics $|AP_2(F)|$ contains a reducible conic. The polar cubic $P_a(F)$ is a cone. It is easy to verify, by using the explicit formula for $\mathbf{I}_4$ in (5.13.1), that the corresponding point in the locus $\mathbf{I}_4{}^{-1}(0)$ of anharmonic cubics is a singular point. Since the polar linear system $\mathscr{P}_F \cong \mathbb{P}^2$ is not contained in the $\mathbf{I}_4{}^{-1}(0)$ (otherwise $\mathbf{S}_4(F) = 0$), we obtain that a is a singular point of $S(F)$.

(ii) For a general F and a general point a from $S(F)$ the polar cubic $P_a(F)$ is a non-singular Fermat cubic since this holds for Clebsch quartics. So (ii) holds for general quartics. Assume (ii) is false for some F with non-

singular $S(F)$. Then for any point $a \in S(F)$ the polar cubic $P_a(F)$ is singular. There exists a one-parameter family (F_t, a_t) with $a_t \in S(F_t)$ and $P_{a_t}(F_t)$ non-singular for $t \neq 0$ such that $F_0 = F$ and $a_0 = a$. The lines of $He(P_a(F))$ are limits of the sides of the triangles $He(P_{a_t}(F_t))$ as $t \to 0$, so there exists $b \in S(F)$ such that $P_{b,a}(F) = H^2$ and the line H is tangent to $S(F)$. Since $H \in \Gamma(F)$, and $\Gamma(F)$ is a curve of degree $\leqslant 6$ (6.6.2), this shows that the dual curve of $S(F)$ is a component of $\Gamma(F)$. But this is absurd since the degree of the dual curve of a non-singular quartic is equal to 12.

(7.6) THEOREM. *Assume $S(F)$ is non-singular. Then the correspondence*

$$T(F) = \{(a, b) \in S(F) \times S(F) : \operatorname{rk} P_{a,b}(F) = 1\}$$

is equal to the correspondence T_ϑ defined by a unique even theta characteristic on S.

Proof. Let us first consider the case where F is a general quartic and let $C = S(F)$. We claim that for any $x \in C$ and $y \in \operatorname{Supp}(T(x))$

$$T(x) - y + T(y) - x \in |K_C|. \qquad (*)$$

It suffices to prove this for general $x \in C$. For such x the polar $P_x(F)$ is a non-singular Fermat cubic, the sides of the polohessian triangle $He(P_x(F))$ intersect transversally C, and for any $y \in T(x)$ the divisors $T(x) - y$ and $T(y) - x$ have no common points. All these properties hold for general F since they hold for the Clebsch quartics. Now for x, y as above $P_{x,y}(F) = H^2$, where H is the opposite side of y in $T(x)$ and of x in $T(y)$. Therefore $H \cdot C = x_1 + x_2 + y_1 + y_2$, where $T(x) = y + y_1 + y_2$ and $T(y) = x + x_1 + x_2$. This proves $(*)$. Let us consider the map $f: C \to \operatorname{Pic}^{(2)}(C)$ given by $f(x) = T(x) - x$. Using $(*)$ we see that $f(y) = f(y_1) = f(y_2)$ for the vertices of $T(x)$. Thus either $f(C)$ is a point or $f(C)$ is a curve of genus $\leqslant 2$. The case of genus 0 is impossible since rational curves do not lie on an abelian variety. The easy argument counting constants shows that the coverings of curves of genus 1 or 2 form a subvariety of codimension $\geqslant 2$ of the space of quartics. Now we can appeal to (7.3.3) to get that the closure of $S(|S^4(V^*)|)$ contains the hypersurface of Lüroth quartics. This shows that for general F the image of f is a point ϑ. By $(*)$ we find that $2\vartheta = K_C$ and clearly $h^0(\vartheta) = 0$. Let $U \subset |S^4(V^*)|$ be the open Zariski subset of curves F with non-singular $S(F)$. By (7.5) for each $F \in U$ the correspondence $T(F)$ is symmetric and of degree (3, 3). Now for general $a \in S$ we have $T(F)(a) = p_1 + p_2 + p_3$, where p_i are distinct. We can find a one-parameter family of triples (F_t, S_t, a_t) such that $S_t = S(F_t)$, $(F_0, S_0, a_0) = (F, S(F), a)$, and for $t \neq 0$ the curve F_t has the property that

$T(F_t) = T_{\vartheta_t}$ for some even theta characteristic ϑ_t on S_t. Let $T(F_t)(a_t) = p_1(t) + p_2(t) + p_3(t)$. Then

$$\vartheta_t = \mathrm{cl}(p_1(t) + p_2(t) + p_3(t) - a_t)$$

is an even theta characteristic on S_t for $t \neq 0$. A smooth specialization of an even theta characteristic is an even theta characteristic [Mum]. So,

$$\vartheta \sim p_1 + p_2 + p_3 - a.$$

Since the number of theta characteristics is finite, ϑ does not depend on a. This shows that $T(F) = T_\vartheta$.

(7.7) Let M° be the open Zariski subset of $|S^4(V^*)|$ which consists of plane quartics F with non-singular covariant quartic $S(F)$. By (7.5) every quartic $F \in M^\circ$ is weakly non-degenerate. The Clebsch covariant $\mathbf{S}_4$ is well-defined on M°, and the curve $S(F) = \mathbf{S}_4(F)$ carries a unique even theta characteristic ϑ such that the correspondence $T(F)$ on $S(F)$ defined by the polohessian triangles coincides with the correspondence T_ϑ. This defines a map, the *Scorza map*,

$$Sc: M^\circ \to M^{cr}, \qquad F \to (S, \vartheta),$$

where M^{cr} is the variety parametrizing the pairs (S, ϑ), where S is a non-singular quartic, and ϑ is an even theta characteristic on it.

(7.7.1) LEMMA. *The variety M^{cr} is an irreducible variety, and its natural projection to the variety M of non-singular plane quartic curves is an unramified covering of degree* 36.

Proof. The latter assertion is well known; the number of even theta characteristics on any curve of genus g is $2^{g-1}(2^g + 1)$. As we explained in (5.7) the assignment of the Hesse invariant (X, ϑ) to a net of quadrics in $\mathbb{P}^3 = \mathbb{P}(W)$ defines a birational isomorphism between M^{cr} and the orbit space of an open Zariski subset of $\mathbb{P}(V^* \otimes S^2(W^*))$ by $PGL(W)$. The latter variety is irreducible.

(7.8) THEOREM (G. Scorza). *The Scorza map is an injective birational isomorphism.*

Proof. Since M° and M^{cr} have the same dimension, it suffices to prove that the Scorza map is injective. Let us first prove the injectivity of the Scorza map restricted to the locus $(M^\circ)_{nd}$ of non-degenerate quartics. Let $F \in (M^\circ)_{nd}$ and $S = S(F)$. By (7.5) there exists an open non-empty subset S' of S such that for every $a \in S'$ the polohessian $T(a)$ is non-degenerate with no sides tangent to S. If $Sc(F) = Sc(F') = (S, \vartheta)$ for some $F' \neq F$, then for

any $a \in S'$, $P_a(F)$ and $P_a(F')$ are non-singular anharmonic cubics whose Hessian is equal to the triangle of lines whose vertices are the three points in $T(a) = T_\vartheta(a) \sim \vartheta + a$. Suppose we show, using (5.9), that this implies that $P_a(F) = P_a(F')$ for all $a \in S'$. Since S' spans $\mathbb{P}^2$, we obtain that $P_a(F) = P_a(F')$ for all $a \in \mathbb{P}^2$. This obviously implies that $F' = F$.

Take a general point $a \in S'$, let $T(a) = p_1 + p_2 + b$, and let $H = \langle p_1 + p_2 \rangle$ be the side of $T(a)$ opposite to the vertex b. Since a is general, the triangle $T(b)$ is non-degenerate, and hence equal to $a + q_1 + q_2$, where $q_1, q_2 \in H$, and $\{p_1, p_2\} \cap \{q_1, q_2\} = \varnothing$. Let $T(q_1) = b + r_1 + r_2$ (again we may assume that $T(q_1)$ is non-degenerate, and $\{r_1, r_2\} \cap \{a, q_2\} = \varnothing$). Then its side $\langle r_1 + r_2 \rangle$ is equal to the side $\langle a + q_2 \rangle$ of $T(b)$. The four lines

$$l_1 = \langle b, r_1 \rangle, \qquad l_2 = \langle b, r_2 \rangle, \qquad l_3 = \langle b, p_1 \rangle, \qquad l_4 = \langle b, p_2 \rangle,$$

are all different (see Fig. 5). Indeed, $l_1 \neq l_2$, $l_3 \neq l_4$ because all triangles are non-degenerate. If $l_1 = l_3$, then the triangles $T(a)$, $T(q_1)$, and $T(r_2)$ have a common side equal to l_1. This is impossible by (6.9.1). Similarly we verify that $l_1 \neq l_4$, $l_2 \neq l_3$, $l_2 \neq l_4$. Let

$$P_{q_1, a}(F) = P_{a, q_1} = m_1 + m_2.$$

We know that

$$P_{p_1, a}(F) = l_4^2, \qquad P_{p_2, a}(F) = l_3^2.$$

So, the lines l_3, m_1, l_4, m_2 cut out four points on H with cross-ratio equal to -1 (5.9). Similarly, the lines l_1, m_1, l_2, m_2 cut out four points on $\langle r_1, r_2 \rangle$ with cross-ratio -1. Consider the pencil of divisors $\alpha + \beta$ of degree 2 on H such that the cross-ratio $(l_1 \cap H, \alpha, l_2 \cap H, \beta) = -1$, and another pencil characterized by the condition that the cross-ratio $(l_3 \cap H, \alpha, l_4 \cap H, \beta) = -1$. These pencils are different since $\{l_1, l_2\} \neq \{l_3, l_4\}$. Their intersection is equal to the divisor $m_1 \cap H + m_2 \cap H$. Now, for the pencil of

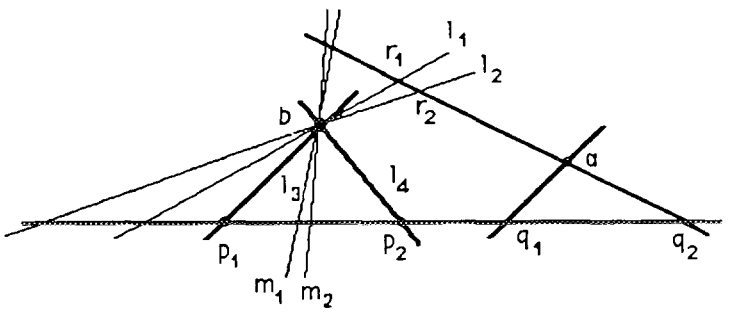

FIGURE 5

quadrics on H cut out by the polars $P_{x,a}(F)$ with $x \in H$ we know that to p_1 corresponds $2p_2$, to p_2 corresponds $2p_1$, and to q_1 corresponds $\alpha + \beta = m_1 \cap H + m_2 \cap H$ that is determined uniquely by the lines l_1, l_2, l_3, l_4 and H as above. So, the pencil is completely determined by the set of polohessian triangles which is the data given by (S, ϑ) via T_ϑ. Using (5.9) this shows that $P_a(F)$ can be reconstructed from (S, ϑ). Now we have $M^{\circ} = (M^{\circ})_{nd} \cup M^{c}$, and we have shown in (7.3.2) that the Scorza map Sc is injective when restricted to M^{c}. If $F \in (M^{\circ})_{nd}$ and $F' \in M^{c}$, it is impossible that $Sc(F) = Sc(F') = (X, \vartheta)$ since in the former case $d(\vartheta) = 1$ and in the latter case $d(\vartheta) = 3$. This proves the theorem.

(7.9) One repeats the argument from the proof of the Scorza theorem to construct explicitly F. In fact, we reconstruct from (S, ϑ) a family of anharmonic cubics $\{C(a)\}_{a \in S'}$, where $S' = \{a \in S: T(a) \text{ is reduced}\}$. The polar triangle of each such a cubic is the triangle $T(a)$. Since S' spans $\mathbb{P}^2$, we will reconstruct the linear sysem of polars

$$U \in V \otimes S^3(V^*),$$

which arises from a symmetric tensor

$$F \in S^4(V^*)$$

representing our quartic curve F.

We return to another explicit construction for the inverse of the Scorza map in Section 9.

(7.10) For every pair (S, ϑ) in the image of the Scorza map, the value of the invariant $d(\vartheta)$ defined in (7.1.4) is equal to 1 or 3. The latter case happens only if F is degenerate. In his proof of Theorem (7.8) (see Sect. 9) Scorza implicitly assumed that for a general curve X there exists an even theta characteristic ϑ with $d(\vartheta) = 1$. This is not obvious and, in fact, follows from Theorem (7.6). A priori, a pair (S, ϑ) with $d(\vartheta) = 2$ may exist. What is an intrinsic characterization of special ϑ for which $d(\vartheta) = 3$?

(7.11) COROLLARY. *Let $\mathcal{M}_3$ be the moduli space of curves of genus 3, and let $\mathcal{M}_3^{ev}$ be the moduli space of curves of genus 3 with an even theta characteristic. Then $\mathcal{M}_3$ and $\mathcal{M}_3^{ev}$ are birationally isomorphic.*

Proof. The Scorza map is $PGL(V)$-equiinvariant. Passing to the quotient, we obtain the assertion.

(7.12) *Remark.* The previous fact is implicitly contained in [Mu]. The question of rationality of either space is still open (cf. [Do]).

(7.13) COROLLARY. *The degree of the Clebsch covariant* S_4: $|S^4(V^*)| \dashrightarrow |S^4(V^*)$ *is equal to* 36.

8. AN EXAMPLE

In this section, following [Ci5] we shall find all quartics F such that $S(F)$ is a fixed Klein curve with 168 automorphisms.

(8.1) We will need the following explicit formula for the values of the covariant S_4 on quartics from the following family:

$$F = ax_0^4 + bx_1^4 + cx_2^4 + 6fx_1^2x_2^2 + 6gx_0^2x_1^2 + 6hx_0^2x_1^2.$$

We have (see [Sa2, p. 270])

$$S_4(F) = a'x_0^4 + b'x_1^4 + c'x_2^4 + 6f'x_1^2x_2^2 + 6g'x_0^2x_1^2 + 6h'x_0^2x_1^2,$$

where

$$\begin{aligned} a' &= 6g^2h^2, \qquad b' = 6h^2f^2, \qquad c' = 6f^2g^2, \\ f' &= bcgh - f(bg^2 + ch^2) - ghf^2 \\ g' &= acfh - g(ch^2 + af^2) - fhg^2, \\ h' &= abfg - h(af^2 + bg^2) - fgh^2. \end{aligned}$$

(8.2) Let C be the Klein quartic given by the equation as in (6.1.1):

$$x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = 0.$$

We claim that its covariant quartic $S(C)$ coincides with C. In fact, if F is given by an equation of the form

$$F = Ax_0^3x_1 + Bx_1^3x_2 + Cx_2^3x_0,$$

then

$$P_a(F) = a_0(3Ax_0^2 + Cx_2^3) + a_1(3Bx_1^2x_2 + Ax_0^3) + a_2(3Cx_2^2 + Bx_1^3),$$

and, applying the formula for $\mathbf{I}_4$, we find

$$\mathbf{I}_4(P_a(F)) = ABC(Aa_0^3a_1 + Ba_1^3a_2 + Ca_2^3a_0).$$

Thus, replacing (a_0, a_1, a_2) by (x_0, x_1, x_2), we find an equation of the covariant quartic $S(F)$

$$S(F) = ABC(Ax_0^3x_1 + Bx_1^3x_2 + Cx_2^3x_0).$$

In particular, if $A = B = C$, we obtain $S(F) = F$. Note that if C' is the value at C of some covariant or contravariant of degree 4, then C' is projectively isomorphic to C. Indeed, it is well known that every plane quartic with 168 automorphisms is projectively isomorphic to the Klein quartic [Bur, p. 364]. Much more non-trivial is the fact that $S(F) \cong F$ if and only if F is a double conic or belongs to the $PGL(3)$-orbit of the Klein quartic [Ci4].

(8.3) Following [Ci5] we shall find quartics F such that $S(F) = C$. According to Theorem (7.6) we expect to find 36 such curves F. They correspond to even theta characteristics on C. We have found already one such curve, the curve C itself. This defines an even theta characteristics ϑ_0 which is invariant with respect to the whole group of automorphisms G of C.

Consider the family of plane quartics from (8.1) with $a = b = c$ and $f = g = h$:

$$F_\lambda = x_0^4 + x_1^4 + x_2^4 + 6\lambda(x_0^2x_1^2 + x_0^2x_1^2 + x_1^2x_2^2). \qquad (*)$$

Every curve from this family is invariant with respect to the octahedral group O_{24} of order 24 that consists of transformations $(x, y, z) \to (\pm x_0, \pm x_1, \pm x_2)$ and permutations of the coordinates. It is easy to see that every plane quartic invariant with respect to this group of transformations belongs to the above pencil. It is well known that O_{24} has only one isomorphism class of 2-dimensional linear projective representations. Since the group $G = \mathrm{Aut}(C)$ contains a subgroup isomorphic to O_{24}, the Klein curve C is projectively isomorphic to a curve F_λ from pencil (*). Since $S(F_\lambda) = F_\lambda$, using the formulae of (8.1) we easily find:

$$6\lambda^3 + \lambda^2 + 2\lambda - 1 = 0.$$

One of the solutions is $\lambda = 1/3$; it gives a double conic. Two other solutions are

$$\lambda = \frac{-1 \pm i\sqrt{7}}{4}.$$

Then F_λ is invariant with respect to the following transformation of order 7:

$$(x_0, x_1, x_2) \to (-\lambda(x_1 - x_0) + 2x_2, -\lambda(x_0 - x_1) + 2x_2, \lambda^2(x_0 + x_1)).$$

So each of the two curves is projectively isomorphic to the Klein curve C. We fix one of them, say corresponding to $\lambda = (-1 + i\sqrt{7})/4$. Now we shall

try to find F among the curves of the family from (8.1). If $S(F)=F_\lambda$ we must have:

$$a'=b'=c', \qquad f'=g'=h', \qquad f'/a'=\lambda.$$

This implies

$$f^2=g^2=h^2. \tag{**}$$

If $f=g=h$, then $a=b=c$, and we get that $F=F_\mu$, where

$$6\mu^2\lambda+\mu^2+2\mu-1=0.$$

This gives two solutions for μ. One is $\mu=\lambda$ that has been already accounted for. The second is

$$\mu=\frac{5-i\sqrt{7}}{16}.$$

The corresponding curve F_μ is invariant with respect to the octahedral group so that its G-orbit consists of seven curves each having C as its covariant quartic. There are two conjugacy classes of octahedral subgroups in G (see [Bur]). So we must have seven more curves F with $S(F)=C$. Since any two octahedral groups are conjugate in $PGL(3)$, there exists a system of projective coordinates (y_0, y_1, y_2) such that F and C can be written in the form

$$y_0^4+y_1^4+y_2^4+\mu(y_0^2y_1^2+y_0^2y_2^2+y_1^2y_2^2)=0.$$

As above C corresponds to the value $\mu=(-1\pm i\sqrt{7})/4$ and F corresponds to the value $\mu'=(5\mp i\sqrt{7})/16$. It is easy to see that $\mu=(-1\pm i\sqrt{7})/16$ and $\mu'=(5+i\sqrt{7})/16$ (otherwise the transformation $x\to y$ leaves the original Klein quartic invariant). Furthermore the second group of seven projectively equivalent quartics is not projectively equivalent to the first one.

Now we shall consider the cases when $f=g=-h$, and so forth. Then

$$-ac+h(a+c)+h^2=ab-h(a+b)-h^2=-bc+h(b+c)+h^2=6\lambda h^2;$$

hence

$$(a-b)(h-c)=0.$$

The case $h=c$ leads to $2h^2=6\lambda h^2$ which is absurd since $\lambda\neq 1/3$. So $a=b$, and we obtain

$$(a/h)^2-2(a/h)-1-6\lambda=0,$$

$$-(a/h)(c/h)+(a/h)+(c/h)+1-6\lambda=0.$$

This allows us to find two solutions:

$$a/h = (5 + i\sqrt{7})/2, \qquad c/a = -(1 + 3i\sqrt{7})/4;$$
$$a/h = (-1 - i\sqrt{7})/2, \qquad c/a = -2.$$

One can verify that the first solution leads to an equation which is invariant with respect to an octahedral group. This gives one of the 14 curves that have been found earlier. The second solution gives the curve

$$x_0^4 + x_1^4 - 2x_2^4 + 6\beta(x_0^2x_2^2 + x_1^2x_2^2 - x_0^2x_1^2) = 0,$$

where $\beta = (-1 + i\sqrt{7})/4$. It is invariant with respect to a subgroup of G of order 8. All subgroups of order 8 are conjugate in G, and this gives us 21 projectively equivalent curves F with $S(F) = C$. Now all the even theta characteristics are accounted for. We have

$$36 = 1 + 7 + 7 + 21,$$

and this partition of 36 describes the decomposition of the set of even theta characteristics into G-orbits.

(8.4) One immediately verifies that each curve F with $S(F) = C$ is not a Clebsch curve. In other words, C is not a Lüroth quartic. In fact, a Klein curve does not belong to the closure of the locus of Lüroth quartics. This immediately implies that the map given by the covariant $\mathbf{S}_4$ is dominant. This fact can be used for another proof of Theorem (7.6).

(8.5) *Remark.* The existence of an invariant even theta characteristic on a Klein curve can be verified in various ways. We refer to [Bu] for one of them. Another way is to use the representation theory of $G = SL(2, \mathbb{F}_7)$ to construct a G-invariant net of quadrics in $\mathbb{P}^3$. Its Hessian (C, ϑ) is a Klein curve with a G-invariant even theta characteristic.

9. Canonical Curves and Quartic Hypersurfaces

(9.1) Let (X, ϑ) be a pair consisting of a non-singular canonical curve $X \subset \mathbb{P}^{g-1} = \mathbb{P}(V)$, $V = H^0(X, \mathcal{O}_X(K_X))^*$, and a non-effective theta characteristic ϑ on X. Assume the following properties are satisfied:

(A1) $d(\vartheta) = 1$.

(A2) Let $T_\vartheta \in \mathrm{Div}(X \times X)$ be the correspondence on X defined in Section (7.1), and let

$$\Gamma(\vartheta) = \{H \in \check{\mathbb{P}}^{g-1} : H \text{ is a face of some } \vartheta\text{-polyhedron of } X\}$$

be the image of the map $\pi: T_\vartheta \to \check{\mathbb{P}}^{g-1}$, $(a, b) \mapsto \langle T_\vartheta(a) - p_b \rangle$. Then $\Gamma(\vartheta)$ is not contained in a quadric.

(A3) T_ϑ is reduced.

In the case $g = 3$ this is satisfied for any pair (X, ϑ) with $d(\vartheta) = 1$, in particular for any pair which is the value of the Scorza map at a non-degenerate quartic F (see (6.3.1)). Indeed, it suffices to show that $\deg \Gamma(\vartheta) \leqslant 6$ because in this case the dual cuve $\check{X}$ is of degree 12 and hence cannot be contained in $\Gamma(\vartheta)$, and so for general $a \in X$ none of the sides of $T(a)$ are tangent to X. By (7.1.3) we obtain (A3). Since $d(\vartheta) = 1$ we obtain using Corollary (7.1.7) that $\deg \Gamma(\vartheta) = 6$. Then (A2) is fulfilled as well. Now let us prove that $\deg \Gamma(\vartheta) \leqslant 6$. Consider the universal family $\mathscr{X} \to M^{cr}$ of plane quartics with even theta characteristic. Using a relative version of (7.1) one can define a relative correspondence $\mathscr{T} \in \mathrm{Pic}(\mathscr{X} \times_{M^{cr}} \mathscr{X})$ by the pull-back of the relative theta divisor. One defines as in (7.1.4) a relative Gauss map. Let $U \subset M^{cr}$ be a Zariski open subset that consists of pair $(X, \vartheta) = Sc(F')$ for non-degenerate quartics F' (7.6). For such (X, ϑ) the curve $\Gamma(\vartheta)'$ is of degree 6 (see (6.3.1)). By specialization for every $(X, \vartheta) \in M^{cr}$ the curve $\Gamma(\vartheta)'$ is the support of a divisor of degree 6; therefore $\deg \Gamma(\vartheta)' \leqslant 6$. This proves the assertion.

Next we shall discuss a remarkable contruction of G. Scorza that assigns to each pair (X, ϑ) satisfying (A1)–(A3) a certain quartic hypersurface in $\mathbb{P}^{g-1}$. In case $g = 3$, this construction gives the inverse of the Scorza map. Naturally, his construction assumes that the set of such pairs (X, ϑ) is not empty. This is true for $g = 3$, but we have not been able to verify it for $g > 3$.

This makes all the results in this section (except when $g = 3$), as well as in [Sc3], conditional. From now on we shall assume that (X, ϑ) is a pair satisfying (A1)–(A3).

(9.2) PROPOSITION. *Let H be a hyperplane such that the divisor $H \cap X = q_1 + \ldots + q_{2g-2} \in |K_X|$ is reduced. Then the $g(2g-2)$ faces H_i of the ϑ-polyhedra $T_\vartheta(q_i)$, $i = 1, \ldots, 2g-2$, are cut out by an enveloping quadric Q in $\check{\mathbb{P}}^{g-1}$.*

Proof. Let H be a common face of two non-degenerate polyhedra $T_\vartheta(a)$ and $T_\vartheta(b)$. By Lemma (7.1.1) H cuts out X in the union of two divisors $p_1 + \ldots + p_{g-1}$ and $r_1 + \ldots + r_{g-1}$, where $T_\vartheta(a) = b + p_1 + \ldots + p_{g-1}$, $T_\vartheta(b) = a + r_1 + \ldots + r_{g-1}$. Then the hyperplanes passing through the point a (resp. b) contain the remaining faces of $T_\vartheta(b)$ (resp. $T_\vartheta(a)$). One easily checks that each face of $T_\vartheta(r_i)$ and $T_\vartheta(p_i)$ contains one of the points a, b. Since $\deg(\Gamma(\vartheta)) = g(g-1)$ (7.1.7), the assertion is true; the needed quadric Q is the union of two hyperplanes $\check{H}_a$ and $\check{H}_b$.

Let $p: T_\vartheta \to X \subset \mathbb{P}^{g-1}$ be the canonical projection and let $\pi: T_\vartheta \to \Gamma(\vartheta) \subset \check{\mathbb{P}}^{g-1} = |K_X|^*$ be as in (9.1). We can interpret the previous observation as follows. Let $H \in \Gamma(\vartheta)$, $D_H \in |K_X|$ be the corresponding canonical divisor, and let $p^*(D_H)$ be its pre-image under p. Then $\pi_* p^*(D_H)$ is cut out by some quadric $Q \in |\mathcal{O}_{\mathbb{P}^{g-1}}(2)|$. Let U be the Zariski open subset of $\check{\mathbb{P}}^{g-1}$ which consists of hyperplanes H such that $p^*(D_H)$ does not contain singular points of T_ϑ. By Lemma (7.1.3) each hyperplane H not containing a branch point of the map p belongs to U. Let $\mathrm{Div}^{2g(g-1)}(\Gamma(\vartheta))'$ be the variety of divisors on $\Gamma(\vartheta)$ with support outside of the singular locus of $\Gamma(\vartheta)$. The variety $W \subset \mathrm{Div}^{2g(g-1)}(\Gamma(\vartheta))'$ of divisors $\bar{D}_H = \pi_* p^*(D_H)$, $H \in U$, is unirational, and hence spans a linear system $L = g^r_{2g(g-1)}$ of divisors on $\Gamma(\vartheta)$. We want to show that L equals the restriction of the linear system $|\mathcal{O}_{\mathbb{P}^{g-1}}(2)|$ to the curve $\Gamma(\vartheta)$. Let $f: \check{\mathbb{P}}^{g-1} \to L$ be the map $H \to \bar{D}_H$, and let γ be a general line in $\check{\mathbb{P}}^{g-1}$, i.e., a pencil of hyperplanes in $\mathbb{P}^{g-1}$. Let $H_0 \in \Gamma(\vartheta)$ be a common face of two polyhedra $T_\vartheta(a)$ and $T_\vartheta(b)$. Then H_0 is contained in exactly two divisors $\bar{D}_H$ of the curve $f(\gamma)$; they correspond to the two hyperplanes passing through a and b. This shows that the pre-image (under f) of the hyperplane in L of divisors containing H_0 intersects γ in two points. This implies that for any hyperplane $L' \subset L$ its pre-image under f is a quadric. This gives that the dimension r of L is at most $\frac{1}{2}g(g+1)-1$. Now observe that $L \cap |\mathcal{O}_{\Gamma(\vartheta)}(2)|$ contains the divisors $\bar{D}_H$, where H is a face of a general ϑ-polyhedron $T_\vartheta(a)$. They are cut out by the quadrics $Q = \check{H}_a \check{H}_b$, $(a, b) \in T_\vartheta$. These quadrics must span the space of quadrics in $\check{\mathbb{P}}^{g-1}$. If not, there exists a quadric in $\mathbb{P}^{g-1}$ apolar to all quadrics $\check{H}_a \check{H}_b$. This would imply that for a fixed $a \in X$ the points $b \in T_\vartheta(a)$ lie a hyperplane. However $\langle T_\vartheta(a) \rangle = \mathbb{P}^{g-1}$, since otherwise the theta characteristic ϑ is effective. So, thanks to assumption (A2), $\dim L \cap |\mathcal{O}_{\Gamma(\vartheta)}(2)| \geqslant \dim |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| = \frac{1}{2}g(g+1)-1$. This forces L to be equal to the restriction of the linear system $|\mathcal{O}_{\mathbb{P}^{g-1}}(2)|$ to $\Gamma(\vartheta)$ which proves the assertion.

(9.2.1)) *Remark.* We do not know whether the restriction homomorphism $|\mathcal{O}_{\mathbb{P}^{g-1}}(2)| \to |\mathcal{O}_{\Gamma(\vartheta)}(2)|$ is an isomorphism. This is true if $g = 3$ since $\Gamma(\vartheta)$ is a plane sextic.

(9.3) Let L be the restriction of $|\mathcal{O}_{\mathbb{P}^{g-1}}(2)|$ to $\Gamma(\vartheta)$ and let

$$f: \check{\mathbb{P}}^{g-1} \to L$$

be the map $H \to \bar{D}_H$ constructed in Proposition (9.1). As we saw in its proof this map is given by a linear system of quadrics in $\mathbb{P}^{g-1}$, and its image spans the target space. Therefore, f defines a projective isomorphism

$$|\mathcal{O}_{\mathbb{P}^{g-1}}(2)|^* = |S^2(V^*)| \to |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| \cong |S^2(V)|$$

which can be lifted to a linear isomorphism:

$$\bar{f}: S^2(V^*) \to S^2(V).$$

We shall show that the inverse of this map is equal to the polarization map

$$S^2(V) \to S^2(V^*), \qquad \Phi \to P_\Phi(F),$$

for a unique quartic hypersurface F.

(9.3.1) THEOREM. *There exists a unique quartic hypersurface F in $\mathbb{P}^{g-1}$ such that the polarization map $\Phi \mapsto P_\Phi(F)$ defines an isomorphism $S^2(V) \to S^2(V^*)$ whose inverse coincides with the map*

$$A: |\mathcal{C}_{\mathbb{P}^{g-1}}(2)| \to |\mathcal{C}_{\mathbb{P}^{g-1}}(2)|$$

defined on a general quadric H^2 of rank 1 by the formula

$$A(H^2) = H_1 + \ldots + H_{2g(g-1)},$$

where $H_1, \ldots, H_{2g(g-1)}$ are the faces of the ϑ-polyhedra $T_\vartheta(x_i)$, $x_i \in H \cap X$.

Proof. It suffices to show that the tensor $U(x, y, z, w) \in S^2(V^*) \otimes S^2(V^*)$ defined by the linear isomorphism $\bar{f}^{-1}: S^2(V) \to S^2(V^*)$ is symmetric. Then it arises from the polarization of a unique quartic polynomial F.

Fix a reduced ϑ-polyhedron $T_\vartheta(a) = p_1 + \ldots + p_g$, and let $H_i = \langle T_\vartheta(a) - p_i \rangle$ be its faces. Then, as we saw in the proof of Proposition (9.1), by definition,

$$A(H_i^2) = \check{H}_a \check{H}_{p_i}.$$

Choose coordinates such that H_i are the coordinate hyperplanes $x_i = 0$. Then for any $y = (\lambda_0, \ldots, \lambda_{g-1})$

$$A\left(\sum \lambda_i x_i^2\right) = \left(\sum \lambda_i u_i\right)\left(\sum a_i u_i\right),$$

where $a = (a_0, \ldots, a_{g-1})$. Therefore we have

$$U(a, y, x, x) = \sum \lambda_i x_i^2 = P_y\left(\sum x_i^3\right).$$

Thus $U(a, y, x, x)$ can be considered as a polarization of a unique cubic $K(a) = U(a, x, x, x)$. Therefore, the tensor $U(x, y, z, w)$ is symmetric in y, z, w. Since it is symmetric in x, y and in z, w, we find that it is symmetric.

(9.4) DEFINITION. The unique quartic F attached to a pair (X, ϑ) satisfying the assumptions (A1) (A3) is called the *Scorza quartic* of the pair (X, ϑ).

(9.4.1) COROLLARY. *The image of the Scorza map contains all pairs (X, ϑ) where X is a non-singular quartic and ϑ is an even theta characteristic with $d(\vartheta) = 1$. For such a pair the Scorza quartic F is non-degenerate and $Sc(F) = (X, \vartheta)$. It is the unique quartic with these properties.*

Proof. We have shown above that the pair (X, ϑ) satisfies (A1)–(A3). The Scorza quartic F is non-degenerate since it is obtained from the isomorphism $S^2(V) \to S^2(V^*)$. The map $f: \check{\mathbb{P}}^2 \to |\mathcal{O}_{\mathbb{P}^2}(2)|$ assigns to a line H its second anti-polar conic Φ with respect to F. When H is a face of a ϑ-polyhedron of X, $\Phi = \check{H}_a \check{H}_b$ for some $(a, b) \in T_\vartheta$, implying that $P_{a,b}(F) = H^2$ where $H \in \Gamma(\mathbb{P})$. This shows that $S(F) \supset X$. Now according to (6.6.2) $S(F)$ is a curve of degree 4, so $S(F) = X$. By the construction of the Scorza map we have that $Sc(F) = (X, \vartheta)$. If F' is another non-degenerate quartic with $Sc(F') = (X, \vartheta)$ then $F = F'$ by (7.8).

(9.5) What is the special property characterizing Scorza quartics F? As in (9.4.1) we verify that for any $(a, b) \in T_\vartheta$

$$P_{a,b}(F) = H^2,$$

where $H \in \Gamma(\vartheta)$. In the other words, the map $\pi: T_\vartheta \to \Gamma(\vartheta)$ coincides with the mixed polarization map $(a, b) \to P_{a,b}(F)$. Let N be the locus of quartics with the property that the set

$$\{(a, b) \in \mathbb{P}^{g-1} \times \mathbb{P}^{g-1} : \mathrm{rk}(P_{a,b}(F)) = 1\}$$

is a curve. The locus of quadrics of rank 1 is of codimension $\frac{1}{2}g(g+1) - g$ in the space of all quadrics. Considering the polarization map

$$\mathbb{P}^{g-1} \times \mathbb{P}^{g-1} \times |\mathcal{O}_{\mathbb{P}^{g-1}}(4)| \to |\mathcal{O}_{\mathbb{P}^{g-1}}(2)|,$$

we easily find that the expected codimension of N in $|\mathcal{O}_{\mathbb{P}^{g-1}}(4)|$ equals $\frac{1}{2}g(g+1) - 3g + 3$. Since all Scorza quartics belong to N, a general quartic is not a Scorza quartic for $g > 3$. If $g = 4$, we expect that the locus of Scorza

quartics is of codimension 1. What is this hypersurface parametrizing the Scorza quartic surfaces?

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