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On Some Submodules of the Action of the Symmetric Group on the Free Lie Algebra

HÉLÈNE BARCELO* AND SHEILA SUNDARAM†

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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The free Lie algebra Lie[A] over the complex field, on an alphabet A, is the smallest subspace of the complex linear span of all words in A, which is closed under the bracket operation $[u, v] = uv - vu$. Define Lie_n to be the subspace of the free Lie algebra Lie[1 ... n] spanned by bracketings consisting of words which are permutations of $\{1, ..., n\}$. The symmetric group S_n acts on Lie_n by replacement of letters, giving an $(n-1)!$-dimensional representation isomorphic to the induction $\omega\uparrow_{C_n}^{S_n}$, where C_n is the cyclic group of order n and ω is a primitive nth root of unity. Bracketings in Lie_n may be represented graphically by labelled binary trees with n leaves. Fix a particular unlabelled binary tree T; then the vector subspace spanned by all words corresponding to the $n!$ possible labellings of T is an S_n-module V_T. In this paper we study the representations afforded by certain classes of trees T. We show that the plethysm $V_S[V_T]$ is isomorphic to the submodule corresponding to a tree $S[T]$ which has a natural description in terms of the trees S and T.

INTRODUCTION

The free Lie algebra Lie[1, ..., n] over a field k is defined as follows: take the free algebra over k generated by all words in the letters 1, ..., n (multiplication being concatenation of words); define the *bracket* $[u, v]$ of two words u, v to be $uv - vu$. The subspace Lie[1, ..., n] is now defined as the smallest subspace of the algebra which is closed under the bracket operation, and contains the letters 1, ..., n. The reader is referred to [G, L] for more facts on free Lie algebras. Note that there is a natural left action of the symmetric group S_n on Lie[1, ..., n]: a permutation σ acts on a word w by replacing each occurrence of the letter i in w by $\sigma(i)$.

* Supported by NSF Grant DMS 8903246. Present address: Department of Mathematics, Arizona State University, Tempe, AZ 85287.

† Present address: Department of Mathematics and Computer Science, University of Miami, Coral Gables, FL 33124.

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Define Lie_n to be the subspace of $\mathrm{Lie}[1, ..., n]$ consisting of all linear combinations of words which are permutations of the letters $1, ..., n$. It follows from a 1944 result of Brandt [Br] that, over the complex field, the action of S_n restricted to Lie_n is similar to the $(n-1)!$-dimensional induction $\omega \uparrow_{C_n}^{S_n}$, where C_n is the cyclic group of order n and ω is a primitive nth root of unity. A direct proof of this was given more recently by Klyachko [K].

This paper is an outgrowth of efforts to determine the structure of the representations of S_n afforded by certain subspaces of Lie_n. In what follows we assume that the ground field k has characteristic zero. Combinatorial considerations show that the set of all labelled binary trees with n leaves constitutes a spanning set for Lie_n. We study subspaces which are generated by binary trees of a specific form. The techniques are elementary, involving nothing more than simple manipulations in the group algebra. The reader is assumed to have a sound knowledge of the representation theory of finite groups, and of the symmetric group in particular, as described in [Se, JK]. The authors would like to thank Professor Adriano Garsia for suggesting this question.

1

The following combinatorial scheme of viewing a bracketing in Lie_n, which we illustrate with two examples (Fig. 1), is crucial to our work. (See [G] for a more detailed exposition.)

Given a binary tree T with n leaves ordered from left to right, by attaching a permutation σ to T we mean that we label the leaves of T with $\sigma(1), \sigma(2), ..., \sigma(n)$ from left-most leaf to right-most leaf, in order. We denote this labelled tree by $\sigma(T)$. (In Fig. 1, the permutation 31254 is attached to the tree T). It is clear that Lie_n is the span of the set $\{\sigma(T): \sigma \in S_n\}$ of all binary trees T with n leaves, with all possible permutations of S_n attached.

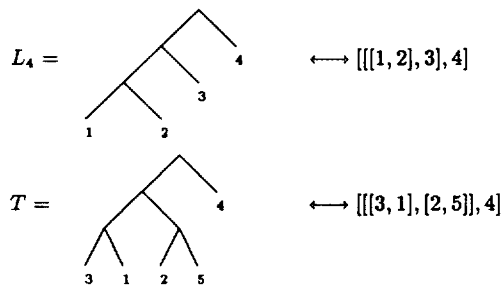


FIG. 1. A bracketing viewed as a labelled binary tree.

Following [G], we define a binary tree to be a *left comb*, if the right child of every internal node is a leaf. We shall denote the left comb in Lie_n by L_n. (See Fig. 2.) We also identify an unlabelled tree T with the tree T which is labelled with the identity permutation. The context of the reference should resolve any ambiguities. The *generator* w_T of a binary tree is the element in the group algebra obtained by writing out the bracketing corresponding to T.

Elementary proofs of the following facts may be found in [G].

PROPOSITION 1.1. 1. *The generator θ_n of the left comb L_n is $(1-\gamma_2)(1-\gamma_3)\cdots(1-\gamma_n)$, where $\gamma_i(1)=i$, $\gamma_i(j)=j-1$ for $j=2, ..., i$, and γ fixes all letters greater than i.*

2. *$\theta_n^2 = n\theta_n$, i.e., $(1/n)\,\theta_n$ is an idempotent in the group algebra kS_n.*

The following observations are immediate: The symmetric group S_n acts on a labelled tree T by permuting the labels. Moreover, for any tree T, the space V_T spanned by all $n!$ labellings of T is invariant under the action of S_n, and is realised as the left ideal of the group algebra kS_n generated by the generator w_T of the tree. In particular, as a left ideal Lie_n is generated by θ_n. Following [G], we call θ_n the *Dynkin* idempotent.

As a trivial application of the preceding remarks, the binary tree with two leaves has word $(12-21)$ and clearly gives the sign representation of S_2.

For the remainder of this paper, "tree" will always refer to a binary tree. Also, we agree to identify two unlabelled trees if one can be obtained from the other by a reflection about a node. It will sometimes be convenient to adopt the following notation: If T_1, T_2, are trees, then $[T_1, T_2]$ denotes the tree whose left subtree is T_1 and whose right subtree is T_2. Hence, for our purposes, $[T_1, T_2] = [T_2, T_1]$. In fact, we will always draw trees in a "left-justified" fashion, in the following sense: The left subtree at any internal node always has at least as many nodes as the right subtree at that node.

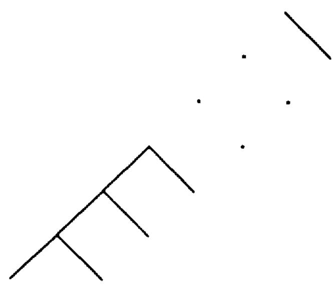


FIG. 2. The left comb L_n.

As a consequence of this convention, we note that given any tree T, it is easy to write down a factorisation of the form of Proposition 1.1(1) for the expression w_T, which we will refer to as the *bracket factorisation* of T. The examples in Fig. 3 should suffice to convince the reader of this fact.

PROPOSITION 1.2. *In the expression w_T of a tree T, the identity permutation occurs exactly once.*

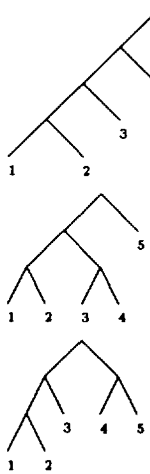
Proof. This is clear when we think about writing out the bracketing corresponding to the tree T (labelled with the identity, left-to-right). ∎

Recall that if α is an idempotent in the group algebra of S_n, then the dimension of the left ideal generated by α is $n!$ times the coefficient of the identity in α. Note that Proposition 1.1(2) and Proposition 1.3, together with the previous remarks, confirm that the dimension of Lie_n is $n!/n = (n-1)!$.

A fundamental result in the theory of free Lie algebras is the construction of an explicit basis, the set of *Lyndon* words (see [L, G]). The general definition need not concern us here; we only point out the following corollary:

THEOREM 1.3. *The $(n-1)!$ permutations which fix 1 index a basis of Lie_n. More precisely, the set of all labelled left combs $\{\sigma(L_n): \sigma \in S_n, \sigma(1) = 1\}$, obtained by attaching σ to the left comb L_n, for all σ such that $\sigma(1) = 1$, forms a basis for Lie_n.*

Proof. It is enough to observe that each $\sigma(L_n)$, when expanded as an



$L_5 =$ $\longleftrightarrow (1-(12))(1-31245))(1-41235)(1-51234)$

$[[L_2, L_2], L_1] =$ $\longleftrightarrow (1-(12))(1-(34))(1-34125)(1-51234)$

$[L_3, L_2] =$ $\longleftrightarrow (1-(12))(1-31245)(1-(45))(1-45123)$

FIG. 3. Examples of bracket factorisation.

element in the group algebra, contains exactly one permutation which fixes '1', namely σ. Hence the left combs indexed by permutations which fix '1' constitute $(n-1)!$ linearly independent elements in the group algebra. But the dimension of Lie_n is $(n-1)!$. These elements must therefore form a basis of Lie_n. ∎

An easy consequence of this combinatorial result is the fact that the restriction of Lie_n to S_{n-1} is isomorphic to the left regular representation kS_{n-1}; this is a well-known property of the representation $\omega\uparrow_{C_n}^{S_n}$, where C_n is the cyclic group of order n and ω is a primitive nth root of unity.

For the purposes of this paper we define a permutation in S_n to be a *Lyndon* word if its first letter is 1.

2

In this section we begin our investigation of the representations V_T, for various trees T.

The first question one might ask is whether any tree besides the left comb L_n generates all of Lie_n. The answer is negative, and is a simple consequence of the following deep result of Kraskiewicz and Weyman ([G, KW]):

THEOREM 2.1. *The multiplicity of the S_n-irreducible indexed by the partition λ of n in* Lie_n *is equal to the number of standard Young tableaux of shape λ with major index congruent to* 1 mod *n*.

The reader unfamiliar with the theory of standard Young tableaux should consult [JK]. The major index of a tableau is defined to be the sum of all the *descents* in the tableau, i.e., the sum of all entries i such that $i+1$ appears below i in the tableau (we write our tableaux so that the largest part of the partition is the length of the highest row, and the smallest part is the length of the lowest row).

We are now ready to prove:

PROPOSITION 2.2. *If T is any tree not equivalent to the left comb L_n (up to reflections), then the space V_T is a proper subspace of* Lie_n.

Proof. The first observation is that any tree that is not equivalent to the left comb via reflections must have a node a which has a left child l_a and a right child r_a, each of which has a left and a right child. Equivalently, there are at least two internal nodes both of which have right and left children which are leaves. Let T be such a tree. It is not hard to see that the word of T, obtained by reading it off the tree as in Example 1.3, is of the form $w_T = (1 - 12)\,\alpha(1 - (i, i+1))\,\beta$, for some $i > 2$, where α, β consist of factors of the form $(1 - \sigma)$, and α contains only permutations that fix the letters $i, i+1, ..., n$. (Note that, because of our convention to write trees so that the left child of any node is always "heavier" than the right child, $(1 - 12)$ always appears as an initial factor of the word of any tree.)

Using Theorem 2.1, it is easy to see that the irreducible corresponding to the partition $(n-1, 1)$ appears in Lie_n with multiplicity 1. We will argue that this component never appears in the space V_T, for trees T as above. It suffices to show that the central idempotent corresponding to the shape $(n-1, 1)$ annihilates the expression w_T. In fact, we claim that for every standard Young tableau t of shape $(n-1, 1)$, the product $e_t w_T$ vanishes, where e_t is the Young symmetriser

$$e_t = \sum_{\gamma \in C_t} (\operatorname{sgn} \gamma)\,\gamma \sum_{\sigma \in R_t} \sigma,$$

where C_t (respectively R_t) denotes the column (respectively row) stabiliser of t, that is, the set of permutations on k letters which leave the elements in each column (respectively row) of t unchanged.

There are two cases:

(i) The letter '2' occurs in the first row of the tableau t, in which case one sees that $(1 + (12))$ is a right factor of e_t (the cyclic group generated by the transposition (12) being a subgroup of the row stabiliser). But then, within the product $e_t w_T$, we have the factors

$$(1 + 12)\,w_T = (1 + 12)(1 - 12)\,\alpha(1 - (i, i+1))\,\beta = 0.$$

(ii) Otherwise the '2' occurs in the unique box in the second row, which forces both i, $i+1$ to be in the first row. Now $(1 + (i, i+1))$ can be written as a right factor of e_t; thus in the product $e_t w_T$ we now have the factors

$$(1 + (i, i+1))\,w_T = (1 - (12))\,\alpha(1 + (i, i+1))(1 - (i, i+1))\,\beta = 0.$$

This proves our claim. ∎

EXAMPLE 2.3. (i) The only tree with 4 leaves not equivalent to the left comb L_4 is $S = [L_2, L_2]$. Hence S generates a proper subspace of Lie_4. By Theorem 2.1, one easily computes the decomposition $\mathrm{Lie}_4 = (3, 1) \oplus (2, 1, 1)$. It follows immediately from the proof of Proposition 2.2 that $V_{[L_2, L_2]} = (2, 1, 1)$.

(ii) The two trees T with 5 leaves which are not equivalent to the left comb L_5 are drawn in Fig. 3.

Another easy but crucial observation is the following:

PROPOSITION 2.4. *Suppose T_1, T_2 are trees on n_1 and n_2 leaves respectively, and let T be the tree whose left subtree is T_1, and whose right subtree is T_2. Then the $S_{n_1+n_2}$-module V_T is a homomorphic image of the induced module $(V_{T_1} \otimes V_{T_2}) \uparrow_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}}$.*

Proof. Let w_{T_i} denote the word of the tree T_i, for $i = 1, 2$, as usual. Then the word of T is easily seen to be $w_T = w_{T_1} \sigma w_{T_2} \sigma^{-1}(1 - \sigma)$, where, written as a word, σ is the permutation $n_1 + 1\ n_1 + 2 \ldots n_1 + n_2\ 12 \ldots n_1$. The induced module $(V_{T_1} \otimes V_{T_2}) \uparrow_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}}$ is realised as the left ideal $kS_{n_1+n_2} w_{T_1} \sigma w_{T_2} \sigma^{-1}$, and there is an obvious module epimorphism $kS_{n_1+n_2} w_{T_1} \sigma w_{T_2} \sigma^{-1} \to kS_{n_1+n_2} w_{T_1} \sigma w_{T_2} \sigma^{-1}(1 - \sigma) = kS_{n_1+n_2} w_T$. ∎

This important observation allows us to give an upper bound for the dimension of the space V_T, and also to estimate the multiplicities of the irreducible components in V_T. In the following small examples we were able to determine the decomposition into irreducibles completely, using Young symmetriser computations for the final equality.

EXAMPLE 2.5. For convenience, given a partition λ of S_n, we write simply $(\lambda_1, \lambda_2, \ldots)$ for the S_n-irreducible indexed by λ. Consider the two trees which generate proper subspaces of Lie_5. (See Fig. 3). By Theorem 2.1, we know that Lie_5 decomposes into irreducibles as $(4, 1) \oplus (3, 2) \oplus (3, 1, 1) \oplus (2, 2, 1) \oplus (2, 1, 1, 1)$.

(i) The tree $T = [[L_2, L_2], L_1]$. By Proposition 2.4, V_T is a homomorphic image of $V_S \uparrow^{S_5}$, where S is the tree in Example 2.3(i). The Littlewood–Richardson rule (see [JK]) shows that $V_S \uparrow^{S_5}$ decomposes into $(3, 1, 1) \oplus (2, 2, 1) \oplus (2, 1, 1, 1)$. A Young symmetriser computation now confirms that each of these irreducibles appears at least once in V_T, and hence we have a space of dimension 15, with

$$V_{[[L_2, L_2], L_1]} = (3, 1, 1) \oplus (2, 2, 1) \oplus (2, 1, 1, 1) \simeq V_{[L_2, L_2]} \uparrow^{S_5}.$$

(ii) The tree $T = [L_3, L_2]$. Again, V_T is a homomorphic image of $(\mathrm{Lie}_3 \otimes \mathrm{Lie}_2) \uparrow$, which decomposes as $(3, 2) \oplus (3, 1, 1) \oplus (2, 2, 1) \oplus (2, 1, 1, 1)$ (a space of dimension 20). Another calculation with Young symmetrisers shows that in fact $V_T = (3, 2) \oplus (3, 1, 1) \oplus (2, 2, 1) \oplus (2, 1, 1, 1) \simeq (\mathrm{Lie}_3 \otimes \mathrm{Lie}_2) \uparrow$.

We conclude this section with a brief discussion of the enumerative implications of Theorem 2.1. We shall need the following notation. We write $\lambda \vdash n$ for a partition $\lambda = (\lambda_1 \geqslant \lambda_3 \geqslant \ldots)$. Also write f^λ for the number of standard Young tableaux (SYT) P of shape λ. Finally let maj(P) denote the major index of the standard tableau P, as defined earlier.

Equating dimensions in the statement of Theorem 2.1 gives the following identity:

$$(n-1)! = \sum_{\lambda \vdash n} f^{\lambda} \# \{P\colon P \text{ is a SYT of shape } \lambda \text{ and } \mathrm{maj}(P) \equiv 1 \bmod n\}. \qquad (1)$$

Recall that the Robinson–Schensted correspondence establishes a similar identity

$$n! = \sum_{\lambda \vdash n} (f^{\lambda})^2,$$

which in turn may be derived algebraically from the decomposition of the left regular representation of S_n into irreducibles. It is natural to ask whether one can give a combinatorial proof of (1) by refining this correspondence. We do this in the next theorem. We assume the reader is familiar with the Robinson–Schensted insertion algorithm (see [JK]).

THEOREM 2.6. *There is a bijection between the set of Lyndon words in* S_n *and the set of pairs* (P, Q) *of standard Young tableaux of the same shape* λ, *for all partitions* $\lambda \vdash n$, *such that* $\mathrm{maj}(P) \equiv 1 \bmod n$.

Proof. The reader familiar with the mechanics of the Robinson–Schensted algorithm will recall the following property. Suppose the image of a permutation w is the pair of tableaux (P, Q). We write $w \leftrightarrow (P, Q)$. Then

$$w^{-1}(i) > w^{-1}(i+1) \Leftrightarrow i+1 \text{ appears in a strictly lower row than } i, \text{ in the tableau } P.$$

Equivalently, i is a descent of the permutation w^{-1} iff i is a descent of the tableau P. Defining the major index maj(τ) of a permutation τ to be the sum of its descents, this implies that

$$w \leftrightarrow (P, Q) \Rightarrow \mathrm{maj}(w^{-1}) = \mathrm{maj}(P).$$

In particular, for any i, the sets

$$\{(P, Q)\colon P, Q \text{ are SYT of the same shape } \lambda, \text{ for any } \lambda \vdash n, \text{ and } \mathrm{maj}(P) = i\}$$

and

$$\{w \in S_n\colon \mathrm{maj}(w^{-1}) = i\}$$

are in bijection via the Robinson–Schensted correspondence. (i)

Since the number of Lyndon words in S_n is $(n-1)!$, it suffices to construct a bijection

$$f: \{\sigma \in S_n : \sigma(1) = 1\} \to \{w \in S_n : \text{maj}(w^{-1}) \equiv 1 \bmod n\}.$$

For $\sigma \in S_n$, let $[\sigma]$ denote the class of circular rearrangements of σ, that is,

$$[\sigma] = \{\sigma, \sigma\gamma, \sigma\gamma^2, ..., \sigma\gamma^{n-1}\},$$

where $\gamma = \gamma_n^{-1}$ in the notation of Proposition 1.1(i), i.e., $\gamma = 2\ 3 \ldots n-1\ n\ 1$, when written as a word. Also, let $[\sigma]^{-1} = \{\sigma, \gamma\sigma, \gamma^2\sigma, ..., \gamma^{n-1}\sigma\}$ denote the inverses of the class $[\sigma]$.

To construct the mapping f, we proceed as follows. Given a Lyndon word $\sigma \in S_n$, find a w in $[\sigma]$ such that w^{-1} has major index congruent to 1 modulo n. We claim that such a w exists and is unique, thereby enabling us to define the image $f(\sigma) = w$.

Indeed a result of Klyachko [G, Lemma 4.1] states that for any $\alpha \in S_n$, we have

$$\text{maj}(\gamma\alpha) \equiv \text{maj}(\alpha) - 1 \bmod n.$$

and hence, if $\text{maj}(\sigma^{-1}) = m \bmod n$, the set of major indices (modulo n) of the elements of $[\sigma]^{-1}$ is $\{m, m-1, ..., m-n+1\} \bmod n = \{0, 1, 2, ..., n-1\}$.

Hence for any i, $0 \leqslant i \leqslant n-1$, there is a unique element in $[\sigma]^{-1}$ with major index congruent to i modulo n. (ii)

In particular, letting w^{-1} be the unique element with $\text{maj}(w^{-1}) \equiv 1 \bmod n$, we set $f(\sigma) = w$. To see that f is one-to-one, note that if σ and α are two distinct Lyndon words, clearly $[\sigma] \cup [\alpha] = \varnothing$ which in turn implies that $[\sigma]^{-1} \cup [\alpha]^{-1} = \varnothing$. To show that f is onto: if $w \in S_n$ is such that w^{-1} has major index congruent to 1 modulo n, there is a unique Lyndon word σ which is a circular rearrangement of w. It is clear that $f(\sigma) = w$. ∎

Remark. Note that from observations (i) and (ii) of the preceding proof, we have in fact established the following identity, for each $i = 0, 1, ..., n-1$:

$$(n-1)! = \sum_{\lambda \vdash n} f^{\lambda} \# \{P : P \text{ is a SYT of shape } \lambda \text{ and } \text{maj}(P) \equiv i \bmod n\}. \quad (2)$$

Again, the algebraic origin of this formula is a theorem of Kraskiewicz and Weyman [KW], which gives the decomposition into irreducibles of the representations $V_i = \omega^i \uparrow_{C_n}^{S_n}$, for $i = 0, 1, ..., n-1$. Here C_n denotes a cyclic subgroup of S_n of order n and ω is a primitive nth root of unity. Their result states that for each partition λ of n, the multiplicity of the λ-irreducible in V_i is the number of standard Young tableaux of shape λ with major index congruent to i modulo n.

3

We now turn our attention to trees obtained by "composition," i.e., given two trees S, T, construct the tree $S[T]$ obtained by attaching to each leaf of S a copy of T. If the subspaces generated by trees were permutation representations, it would be natural to expect the space $V_{S[T]}$ to be a wreath product of the spaces V_S and V_T. It turns out that this is in fact true for the tree subspaces of Lie_n. We first establish some notation. If H is a subgroup of S_m, and G is any group, the wreath product of H with G (with H acting on m copies of G), will be denoted by $H[G]$. (The notation in [JK] is $GwrH$.) Recall that the H-module V can be lifted to a representation of $H[G]$, and that the G-module W canonically yields the representation $(\bigotimes^m W)^\sim$ of $H[G]$. If V, W are representations of S_m, S_n, respectively, we shall denote by $V[W]$ the induction to S_{mn} of the $S_m[S_n]$-module $(\bigotimes^m W)^\sim$, that is,

$$V[W] = \left(\left(\bigotimes^m W\right)^\sim \otimes V\right)\uparrow_{S_m[S_n]}^{S_{mn}}.$$

THEOREM 3.1. *If S, T are binary trees on m, n leaves, respectively, then the representation $V_{S[T]}$ of S_{mn} afforded by the tree $S[T]$ is*

$$V_{S[T]} = V_S[V_T] = \left(\left(\bigotimes^m V_T\right)^\sim \otimes V_S\right)\uparrow_{S_m[S_n]}^{S_{mn}}.$$

Proof. We shall again view all spaces as left ideals in the group algebra. Let w_U denote the generator of the space spanned by a tree of the form U. Thus w_S is an element of kS_m, w_T is an element of kS_n. Denote by I_k the interval $\{n(k-1)+1, ..., nk\}$, for $k = 1, ..., m$. Let σ_k denote the permutation in S_{mn} which interchanges the intervals I_1 and I_k, i.e.,

$$\sigma_k(i) = n(k-1)+i, \qquad i = 1, ..., k,$$

$$\sigma_k(n(k-1)+i) = i, \qquad i = 1, ..., k,$$

and

$$\sigma_k(j) = j \qquad \text{otherwise.}$$

Also, σ_1 is the identity in S_{mn}. Clearly the σ_k are all involutions, for $k > 1$.

Observe that the map $\phi: S_m \to S_{mn}$ which sends the transposition $(1, i)$ to σ_i for all i, is an injective group homomorphism, i.e., the subgroup generated by the σ_i is isomorphic to S_m. Extend ϕ linearly to the group algebra level, and consider the image $\phi(w_S)$ of the generator w_S of V_S. Note that $(1-\sigma_2)(1-\sigma_2\sigma_3)\cdots(1-\sigma_2\sigma_3\cdots\sigma_m)$ is just the image of the Dynkin idempotent θ_m under ϕ (by definition, $\theta_m = \prod_{i=2}^{m}(1-\gamma_i)$, where $\gamma_i = (1,2)(1,3)\ldots(1,i)$).

If the bracket factorisation of S is $w_S = (1-\tau_2)\ldots(1-\tau_m)$, for some permutations τ_i, then a moment's reflection shows that the generator of the ideal $V_{S[T]}$ is

$$\begin{aligned} w_{S[T]} = w_T(\sigma_2 w_T \sigma_2)&(1-\phi(\tau_2)) \\ &(\sigma_3 w_T \sigma_3)(1-\phi(\tau_2\tau_3)) \\ &(\sigma_4 w_T \sigma_4)(1-\phi(\tau_2\tau_3\tau_4)) \\ &\ldots \\ &(\sigma_m w_T \sigma_m)(1-\phi(\tau_2\tau_3\ldots\tau_m)). \end{aligned}$$

Clearly

1. If $i \neq k > 1$, then σ_i commutes with $\sigma_k w_T \sigma_k$, since the latter element contains permutations which act only on I_k (and fix all other letters).
2. Each $\phi(\tau_i)$ belongs to the subgroup generated by $\sigma_1, ..., \sigma_i$.

Hence we may rewrite the generator of $V_{S[T]}$ as

$$\begin{aligned} w_{S[T]} &= \prod_{i=1}^{m} (\sigma_i w_T \sigma_i)(1-\phi(\tau_2))(1-\phi(\tau_2\tau_3)) \\ &\quad \times (1-\phi(\tau_2\tau_3\tau_4))\ldots(1-\phi(\tau_2\tau_3\ldots\tau_m)) \\ &= \prod_{i=1}^{m} (\sigma_i w_T \sigma_i)\, \phi(w_S). \end{aligned}$$

Recall that, as a subgroup of S_{mn}, $S_m[S_n]$ can be identified with the normaliser $N_{S_{mn}}(\times^m S_n)$ of $\times^n S_n$ in S_{mn}. Hence to prove the theorem it suffices to show that

(i) $w_{S[T]}$ is an element of $N_{S_{mn}}(\times^m S_n)$, and

(ii) in the group algebra of the normaliser, the ideal generated by $w_{S[T]}$ affords a representation which is isomorphic to the inner tensor product $(\bigotimes^m V_T)^{\sim} \otimes V_S$.

An easy computation shows that σ_k is in $N_{S_{mn}}(\times^m S_n)$ for all $k = 1, ..., m$: it suffices to check that for any m elements x_i in S_n, the element

$\sigma_k \prod_{i=1}^{m}(\sigma_i x_i \sigma_i)\sigma_k$ sends the interval I_1 to itself, and the interval I_k to itself. Hence (i) follows, from the remarks preceding the computation of $w_{S[T]}$.

It is also easy to see that the identification of $S_m[S_n]$ with $N_{S_{mn}}(\times^m S_n)$ of $\times^n S_n$ in S_{mn} is achieved by the isomorphism

$$(x_1, ..., x_m; (1, k)) \to \left(\prod_{i=1}^{m} \sigma_i x_i \sigma_i\right) \sigma_k,$$

where, following the notation in [JK], $(x_1, ..., x_m; \alpha)$ denotes a typical element of $S_m[S_n]$, for $x_i \in S_n$, $\alpha \in S_m$. Recall that (by definition), $S_m[S_n]$ acts on $(\bigotimes^m V_T)^\sim \otimes V_S$ by

$$(x_1, ..., x_m; \alpha) \cdot \left(\left(\bigotimes^{m} v_i\right) \otimes v\right) = \left(\bigotimes_{i=1}^{m} x_i \cdot v_{\alpha^{-1}(i)}\right) \otimes \alpha \cdot v.$$

Now consider the effect of multiplying $w_{S[T]}$ on the left by an element $(\prod_{i=1}^{m} \sigma_i x_i \sigma_i)\sigma_k$ of the normaliser. The key point here is that σ_k commutes with $\prod_{i=1}^{m}(\sigma_i w_T \sigma_i)$, for all $k = 1, ..., m$. (In view of 1 above, it suffices to check that σ_k commutes with $w_T \sigma_k w_T \sigma_k$; but this is clear since $w_T \sigma_k w_T \sigma_k = \sigma_k w_T \sigma_k w_T$.) Hence we have

$$\begin{aligned}\left(\prod_{i=1}^{m} \sigma_i x_i \sigma_i\right) \sigma_k w_S[T] &= \left(\prod_{i=1}^{m} \sigma_i x_i \sigma_i\right) \prod_{i=1}^{m} (\sigma_i w_T \sigma_i \sigma_k)\, \phi(w_S) \\ &= \left(\prod_{i=1}^{m} \sigma_i x_i w_T \sigma_i\right) \sigma_k \phi(w_S).\end{aligned}$$

Denote the subgroup generated by the $\{\sigma_k\}$ by H; then the left ideal $kH\phi(w_S)$ is clearly isomorphic to V_S; while $kS_n w_T$ is of course isomorphic to V_T. Hence (ii) follows. ▮

EXAMPLE 3.2. Write χ_V for the character of a representation V. The above theorem says that the character $\chi_{V_{S[T]}}$ is given by the plethysm $\chi_{V_S}[\chi_{V_T}]$. Using well-known techniques for computing plethysms, we obtain, for instance, the following decompositions. Recall that L_n denotes the left comb with n leaves. If λ is any partition of n, we write χ_λ for the character of the S_n-irreducible indexed by λ. Also, by Theorem 2.1, we have the following decompositions into irreducibles: $\chi_{\mathrm{Lie}_2} = \chi_{(1,1)}$, $\chi_{\mathrm{Lie}_3} = \chi_{(2,1)}$. For convenience, we write χ_S for the character of the representation associated to a binary tree S.

(i) Take $S=T=L_2$. Then the tree $S[T]=[L_2][L_2]=[L_2, L_2]$ generates a 3-dimensional representation of S_4 whose character may be computed as the plethysm

$$\chi_{\mathrm{Lie}_2}[\chi_{\mathrm{Lie}_2}]=\chi_{(1,1)}[\chi_{(1,1)}]=\chi_{(2,1,1)},$$

in agreement with the calculation in Example 2.3.

(ii) Let $S=L_3$, and $T=L_2$. Then the tree $L_3[L_2]$ generates a 30-dimensional representation of S_6 whose character is given by

$$\chi_{L_3[L_2]}=\chi_{(3,2,1)}+\chi_{(2,2,1,1)}+\chi_{(2,1,1,1,1)}.$$

(iii) Now let $S=L_2$, $T=L_3$. The tree $L_2[L_3]$ generates a 40-dimensional representation of S_6 whose character is given by

$$\chi_{L_2[L_3]}=\chi_{(4,1,1)}+\chi_{(3,3)}+\chi_{(3,2,1)}+\chi_{(2,2,1,1)}.$$

As a consequence of one observation in the proof of Theorem 3.1, we obtain:

PROPOSITION 3.3. *If S, T are trees such that the generators w_S, w_T are, up to a scalar, idempotents, i.e., there exist integers p, q such that $w_S^2=pw_S$, $w_T^2=qw_T$, then the generator $w_{S[T]}$ is also an idempotent, up to a scalar.*

Proof. (Notation as in the preceding proof.) The key point is that all the σ_k, and hence $\phi(w_S)$, commute with $\prod_{i=1}^m(\sigma_i w_T \sigma_i)$. One immediately has that $w_{S[T]}^2=p^n q w_{S[T]}$. ∎

Note that, combined with Proposition 1.1, this enables us to verify that the dimensions check with Theorem 3.1.

From the construction of the wreath product module, one easily obtains a basis for the space $V_{S[T]}$, S, T as above, as follows:

COROLLARY 3.4. *Let $\mathbf{B}=(B_1, B_2, ..., B_m)$ be any partition of the integers from $\{1 \ldots mn\}$ into m blocks each of size n, where we list the blocks so that they are in increasing order of the smallest element. Write $B_i=\{b_{i1}<b_{i2}<\ldots<b_{in}\}$. (Note that the number of such partitions is $(mn)!/m!\,(n!)^m$.) Suppose the d_T permutations $\{\rho_i, i=1 \ldots d_T\}$ index a basis of V_T, i.e., each labelling of T with ρ_i is a basis element of V_T, and suppose the d_S permutations $\{\alpha_i, i=1 \ldots d_S\}$ likewise index a basis for V_S. For any k from $1 \ldots m$, let $B_k^{\rho_i}$ denote the word $b_{k,\rho_i(1)}b_{k,\rho_i(2)}\cdots b_{k,\rho_i(n)}$. Then the set of trees $S[T]$ labelled with the permutations*

$$B^{\rho_i}_{\alpha_j(1)}B^{\rho_i}_{\alpha_j(2)}\cdots B^{\rho_i}_{\alpha_j(m)}$$

obtained by concatenation (in the order indicated) of the words $B^{\rho_i}_{\alpha_j(k)}$, for all the $(mn)!/m!\,(n!)^m$ partitions **B** *as above, and all $i=1, ..., d_T$, $j=1, ..., d_S$, forms a basis for $V_{S[T]}$.*

Recall that for any tree T and any permutation $\rho \in S_n$, we denote by $\rho(T)$ the tree obtained by labelling T from left to right with the permutation ρ.

THEOREM 3.5. *Consider the tree T whose left subtree (at the top node) is the plethysm $L_m[L_n]$, and whose right subtree is a single node. If $n>1$, the subspace V_T affords a representation of S_{mn+1} isomorphic to $V_{L_m[L_n]}\uparrow^{S_{mn+1}}_{S_{mn}}$.*

Proof. Note first that the condition $n>1$ is necessary, since it is not the case that Lie_{m+1} is obtained by inducing Lie_m. By Proposition 2.3, V_T is a homomorphic image of $V_{L_m[L_n]}\uparrow^{S_{mn+1}}_{S_{mn}}$, so we need only show that there are $(mn+1)\dim(V_{L_m[L_n]})$ linearly independent trees of the form T.

For convenience we denote the tree $L_m[L_n]$ by U. Denote by $\{\alpha_j(U)=T_j: j=1, ..., (mn)!/mn^m\}$ the basis elements of $V_{L_m[L_n]}$ constructed as in Corollary 3.3; observe that since Lie_p has a basis obtained by attaching Lyndon words to L_p, the permutations α_j all fix '1'.

Clearly the set $\mathscr{B}=\{(mn+1, i)\,\alpha_j(U): i=1, ..., (mn+1),$ all $j\}$ is a spanning set for V_T. Moreover, since the α_j are all distinct Lyndon words, it follows that the subset $\tilde{\mathscr{B}}$ of $\mathscr{B}$ obtained by taking the mn permutations $(mn+1, i)\,\alpha_j$ for $i\neq 1$, is linearly independent. It is also easy to see (by extracting Lyndon words in each bracketing) that the set $\mathscr{B}\backslash\tilde{\mathscr{B}}$ is linearly independent, because each tree in the set contributes a Lyndon word containing a unique subset of the letters $\{1, ..., mn+1\}$. Hence it suffices to show that no linear combination of trees in the set $\mathscr{B}\backslash\tilde{\mathscr{B}}$ can be written as a linear combination of trees in the set $\tilde{\mathscr{B}}$.

Suppose such a dependence relation exists, with the trees in $\tilde{\mathscr{B}}$ on the right-hand side. Let α_j be such that $(mn+1, 1)\,\alpha_j(T)$ appears on the left-hand side. The Lyndon words contributed by this tree are those of the form $(1, mn+1)\,\alpha_j w_U \gamma_{mn+1}$. It is clear from the independence of the set $\mathscr{B}\backslash\tilde{\mathscr{B}}$ that these Lyndon words cannot be cancelled out by trees on the left-hand side alone. In particular, the Lyndon words $(1, mn+1)\,\alpha_j\gamma_{mn+1}$ survive. Hence on the right-hand side we must have trees labelled with permutations of the form $(1, mn+1)\,\alpha_j\gamma_{mn+1}$. (Note that these trees can be written as a linear combination of trees in $\tilde{\mathscr{B}}$.) But each such tree in turn contributes a word of the form $mn+1\ 1\ \alpha_2 \ldots \alpha_{mn}$. It is easy to see that such words can never arise on the left-hand side, so that our hypothetical dependence relation leads to a contradiction. ∎

We conclude by applying the algebraic and combinatorial techniques used in this paper to the computation of a subspace of Lie_6.

EXAMPLE 3.6. Consider the space V_T, where T is the tree T in Lie_6 given by $T=[[[L_2, L_2], L_1], L_1]$. Set $U=[[L_2, L_2], L_1]$. In a previous calculation (Example 2.5) we showed that V_U is a 15-dimensional space. From Proposition 2.4, it is clear that if $\{\rho_i(U): \rho \in S_5, i=1 \ldots 15\}$ is a basis of V_U, then a spanning set for V_T is obtained by taking $\{((j, 6)\rho_i)(T): \rho \in S_5, i=1 \ldots 5, j=1, ..., 6\}$. A hand calculation shows in fact that any tree T labelled with a permutation α such that $\alpha(6)=1$ (i.e., '1' on the right-most leaf) is a linear combination of trees with any letter other than '1' on the right-most leaf, and further that the subset of 75 trees $\{((j, 6)\rho_i)(T): \rho_i \in S_5, i=1 \ldots 5, j=2, ..., 6\}$ is linearly independent, and therefore a basis for V_T.

We can explicitly calculate the decomposition of V_T as follows. First, from Proposition 2.3, we know that V_T is a quotient of $V_U \uparrow^{S_6}$; combining the result of a previous computation (Example 2.5) with the Littlewood-Richardson rule, the latter representation decomposes as $(4, 1, 1) \oplus 2(3, 2, 1) \oplus 2(3, 1, 1, 1) \oplus (2, 2, 2) \oplus 2(2, 2, 1, 1) \oplus (2, 1, 1, 1, 1)$. On the other hand (by Theorem 2.1), in Lie_6, the multiplicity of the irreducible $(3, 1, 1, 1)$ is one, and that of $(2, 2, 2)$ is zero. Throwing out the $(2, 2, 2)$-irreducible, as well as one copy of the $(3, 1, 1, 1)$-irreducible, we are left with a space of dimension exactly 75, the dimension of V_T. Hence

$$V_T=(4, 1, 1) \oplus 2(3, 2, 1) \oplus (3, 1, 1, 1) \oplus 2(2, 2, 1, 1) \oplus (2, 1, 1, 1, 1).$$

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