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Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control for Discrete-time Systems via Convex Optimization*†

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A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for discrete-time systems is solved by converting it into a convex optimization problem over a finite-dimensional space.

Key Words—Robust control; multiobjective control; convex programming.

Abstract—A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for discrete-time systems is considered, where an upper bound on the $\mathcal{H}_2$ norm of a closed loop transfer matrix is minimized subject to an $\mathcal{H}_\infty$ constraint on another closed loop transfer matrix. Both state-feedback and output-feedback cases are considered. It is shown that these problems are equivalent to finite-dimensional convex programming problems. In the state-feedback case, nearly optimal controllers can be chosen to be static gains. In the output feedback case, nearly optimal controllers can be chosen to have a structure similar to that of the central single objective $\mathcal{H}_\infty$ controller. In particular, the state dimension of nearly optimal output-feedback controllers need not exceed the plant dimension.

1. INTRODUCTION

DESIGN OF CONTROL systems almost invariably involves tradeoffs among competing objectives. It is often the case that the controller is required to meet several different performance and robustness goals, and all of these cannot be met simultaneously. For example, it is intuitively clear that to obtain a greater robust stability margin, it is likely that the performance of the control system needs to be compromised. In classical single loop feedback design, these tradeoffs are performed in terms of the (open) loop transfer function. For instance, stability margins in terms of either the phase/gain margins or the distance of the Nyquist plot to the critical point are traded off against disturbance rejection at low frequencies. Clearly, it is important to develop analytical tools to help the designer understand how the various competing objectives conflict with each other. From this point of view, one should postulate the controller synthesis problem as the problem of studying tradeoffs among competing objectives. For a more detailed discussion of multiobjective controller synthesis as well as additional references, see Boyd and Barratt (1990), Dorato (1991), Khargonekar and Rotea (1991b), and Rotea (1990).

The subject of this paper is a certain constrained optimal controller synthesis problem—the so-called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems can be motivated in many different ways. As a matter of fact, there are many *different* mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems. These problems are one way of analytically formulating the issue of tradeoffs in control system synthesis.

To give a brief description of the various mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems, consider the feedback system shown in Fig. 1. Let $T_{z_iw_i}$, $i = 0, 1$, denote the closed loop transfer matrix from the exogenous input w_i to the controlled output z_i. One mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is to find an internally stabilizing controller $\mathscr{C}$ which minimizes $\|T_{z_0w_0}\|_2$ subject to the constraint $\|T_{z_1w_1}\|_\infty < \gamma$. This problem is equivalent to a problem of optimal nominal performance subject to a robust stability requirement. To be more specific, a controller that solves this mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem will ensure that the closed loop system is robustly stable to all finite-gain stable (possibly nonlinear time-varying) perturbations Δ, interconnected to the system by $w_1 = \Delta z_1$, such that $\|\Delta\|_\infty \le 1/\gamma$. On the other hand, $\|T_{z_0w_0}\|_2$ represents the steady-state variance of the output z_0 when $w_1 = 0$ and w_0 is white noise with unit intensity. Currently no analytic solution to this mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is available. Rotea and Khargonekar (1991a) have

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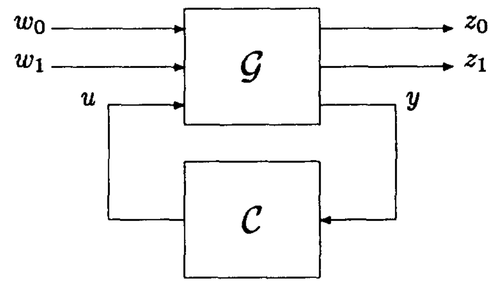

FIG. 1.

obtained some sufficient conditions for the solvability of this mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem in the state-feedback case. While the results in Rotea and Khargonekar (1991a) have been obtained for continuous-time systems, many of them can be extended quite easily to discrete-time systems.

A somewhat different mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem formulation was introduced by Bernstein and Haddad (1989) to combine the LQG and $\mathcal{H}_\infty$ controller design theories. This problem is restricted to the case $w_0 = w_1 =: w$. Instead of minimizing $\|T_{z_0w}\|_2$, they considered the minimization of an "upper bound" for $\|T_{z_0w}\|_2$, subject to the constraint $\|T_{z_1w}\|_\infty < \gamma$. Recently, many papers have appeared that address this mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design problem, see for example, Bambang *et al.* (1990), Doyle *et al.* (1989a), Khargonekar and Rotea (1991a, b), Mustafa and Bernstein (1991), Steinbuch and Bosgra (1991), Yeh *et al.* (1992), Zhou *et al.* (1990) and the references cited therein.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem considered in this paper not only provides a more tractable approach to the problem of minimizing nominal performance subject to a robust stability constraint, but it can also be interpreted as an optimal performance problem. Indeed, as shown by Zhou *et al.* (1990) in the continuous-time case, the "dual" of the auxiliary cost or upper bound of Bernstein and Haddad (1989) is closely related to a system gain from a combination of power and white noise exogenous inputs to the power of the regulated output.

In this paper, we focus on the discrete-time version of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem as formulated by Bernstein and Haddad (1989). While much work has been done on the continuous-time case, the discrete-time case has received much less attention. Indeed, at this time the discrete-time analog of the coupled Riccati equations obtained by Bernstein and Haddad (1989) for the continuous-time case are not available for output-feedback problems. Some results for discrete-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems have been obtained by Bambang *et al.* (1990), Haddad *et al.* (1991), and Mustafa and Bernstein (1991). Mustafa and Bernstein (1991) have considered the static state-feedback problem and derived sufficient conditions for optimality of a state-feedback gain. Bambang *et al.* (1990) and Haddad *et al.* (1991) have considered the static output-feedback problem. It seems that at this time no solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is available in the general dynamic output-feedback case. This is the primary motivation for this paper.

Our approach is as the recent paper by Khargonekar and Rotea (1991a) where a convex optimization approach to the continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem introduced in Bernstein and Haddad (1989) has been developed. A similar approach has been applied earlier by Bernussou *et al.* (1989) to a quadratic stability problem. The starting point is to take a "sub-optimal approach". More specifically, with J denoting the "mixed $\mathcal{H}_2/\mathcal{H}_\infty$" performance measure (a precise definition of J is given in Section 2) let

$$\nu(\mathcal{G}) := \inf\{J : \mathcal{C} \text{ internally stabilizing and } \|T_{z_1w}\|_\infty < \gamma\},$$

denote the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure. Then we consider the following problem:

"Compute $\nu(\mathcal{G})$ and given $\alpha > \nu(\mathcal{G})$, find an internally stabilizing controller $\mathcal{C}$ such that $\|T_{z_1w}\|_\infty < \gamma$, and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure satisfies $J < \alpha$".

The main results of this paper are contained in Sections 4 and 5. The full-information/state-feedback case is considered in Section 4, while the output-feedback case is considered in Section 5. It is shown that if the plant state is available for feedback, one can come arbitrarily close to the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure using constant gain (i.e. nondynamic) state-feedback controllers. In other words, in the state-feedback case, static gain controllers offer the best possible performance.

In the full-information feedback case (i.e. both the exogenous input and the system state are available for feedback) there is a significant departure from the continuous-time case. It turns out that in the discrete-time case, one cannot come arbitrarily close to the infimum by taking static state-feedback controllers. The best that one can do is to use static full-information controllers. As a consequence, this result is of little practical interest except that it is critically useful in dealing with the output-feedback case. This situation is similar to that in the single objective standard $\mathcal{H}_\infty$ control problem for discrete-time systems as in Basar and Bernhard (1991), Iglesias and Glover (1991), Limebeer *et al.* (1989), Liu *et al.* (1991) and Stoorvogel (1990).

It is shown that in the state-feedback as well as the full-information case, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal performance $\nu(\mathcal{G})$ and a static gain controller that satisfies $J < \alpha$ (for any $\alpha > \nu(\mathcal{G})$) can be obtained by solving a finite-dimensional convex programming problem over a bounded set of real matrices.

In the output-feedback case, it is shown that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be reduced to a full-information feedback problem for an auxiliary plant, which is obtained from the given plant by solving an ($\mathcal{H}_\infty$ filtering) algebraic Riccati equation. Thus, the output-feedback problem can be reduced to a finite-dimensional convex programming problem over a set of real matrices. It is shown that the output-feedback controllers can always be chosen to have a structure similar to that of the standard $\mathcal{H}_\infty$ central controller. This implies that the order of (nearly) optimal output-feedback controllers need not exceed that of the generalized plant.

While the approach taken here is somewhat similar to the approach of Boyd and Barratt (1990), in that they also reduce such controller synthesis problems to convex optimization problems, there are significant differences between our results and those of Boyd and Barratt (1990). In particular, we reduce the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis problem to a convex optimization problem over a bounded subset of $q \times n$ and $n \times n$ symmetric real matrices, where q and n are, respectively the control input and the state dimensions. We accomplish this reduction of the problem without finite-dimensional approximations of the set of stabilizing controllers or frequency discretizations. Consequently, a solution to our convex programming problem is a global solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem. This is considered to be an important contribution of our work. By comparison, the results of Boyd and Barratt (1990) applied to the present problem would reduce it to a convex optimization problem over the infinite-dimensional space of stable transfer functions.

Next, we briefly introduce notation used in this paper. The symbol $\varnothing$ denotes the empty set. Given a real matrix A, $\|A\|$ denotes its maximum singular value, $\operatorname{tr}(A)$ denotes its trace, and A' its transpose. We will say that a square matrix A is asymptotically stable if all its eigenvalues are inside the open unit disk. For A and B real symmetric matrices, $A > B$ (respectively $A \geq B$) iff the difference $A - B$ is positive-definite (respectively, positive-semi-definite). Linear time-invariant systems described by state space equations and are denoted by the script symbols, whereas the corresponding transfer matrices denoted by italics. For example, $\mathcal{G}$ denotes a system with transfer function G. The Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$ consist of matrix valued functions that are square integrable and essentially bounded, respectively, on the unit circle with analytic extension outside of the unit circle. The norms on these spaces are defined in the usual way.

2. THE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ PERFORMANCE MEASURE

In this section, we will define the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure. This will then be used in setting up the controller synthesis problem in the next section.

Let us begin by considering a finite-dimensional linear time-invariant discrete-time system $\mathcal{T}$ as shown in Fig. 2.

Suppose that $\mathcal{T}$ is internally stable with the discrete-time state-space model:

$$\mathcal{T} := \begin{cases} (\sigma x)(k) := x(k+1) = Fx(k) + Gw(k), \\ z_0(k) = H_0 x(k) + J_0 w(k), \\ z_1(k) = H_1 x(k) + J_1 w(k), \end{cases} \tag{1}$$

where the matrices F, G, H_i and J_i are real and of compatible dimensions, and F has all eigenvalues in the open unit disk. (In the sequel, we will not show the time variable k explicitly in system equations.) Let

$$T_{zw} = \begin{bmatrix} T_{z_0 w} \\ T_{z_1 w} \end{bmatrix},$$

denote the transfer matrix from w to $z = [z_0', z_1']'$.

Let L_c denote the controllability gramian of the pair (F, G), i.e. L_c is the unique solution of the Lyapunov equation

$$F L_c F' + GG' = L_c. \tag{2}$$

Then, as is well known,

$$\|T_{z_0 w}\|_2^2 = \operatorname{tr}(H_0 L_c H_0' + J_0 J_0').$$

Let $\gamma > 0$ be given, and consider the transfer matrix $T_{z_1 w}$. In this paper, we will be interested in the $\mathcal{H}_\infty$ norm bound $\|T_{z_1 w}\|_\infty < \gamma$. The

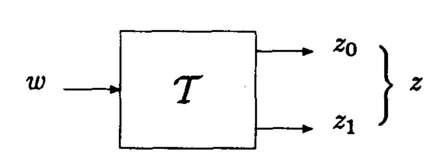

FIG. 2. Diagram for the definition of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure.

following theorem summarizes some useful results on characterizing this norm bound.

Theorem 2.1. Consider the internally stable linear time-invariant discrete-time system given by (1). Then the following statements are equivalent:
(1) $\|T_{z_1w}\|_\infty < \gamma$.
(2) There exists a nonsingular matrix P such that

$$\left\| \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F & G \\ H_1/\gamma & J_1/\gamma \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \right\| < 1. \quad (3)$$

(3) There exists a real symmetric $Y > 0$ such that

$$\begin{bmatrix} F \\ H_1 \end{bmatrix} Y [F' \quad H_1'] + \begin{bmatrix} G \\ J_1 \end{bmatrix} [G' \quad J_1'] < \begin{bmatrix} Y & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \quad (4)$$

Moreover, Y can be chosen to be the same as the one in item 4 below.
(4) There exists a real symmetric $Y > 0$ such that

$$M(Y) := \gamma^2 I - J_1J_1' - H_1YH_1' > 0, \text{ and}$$
$$R(Y) := FYF' - Y + (FYH_1' + GJ_1')M^{-1} \times (H_1YF' + J_1G') + GG' < 0. \quad (5)$$

Moreover, Y can be chosen to be the same as the one in item 3 above.
(5) There exists a real symmetrical $Y \ge 0$ such that

$$M(Y) := \gamma^2 I - J_1J_1' - H_1YH_1' > 0, \text{ and}$$
$$R(Y) := FYF' - Y + (FYH_1' + GJ_1')M^{-1} \times (H_1YF' + J_1G') + GG' = 0, \quad (6)$$

and $F + (FYH_1' + GJ_1')M^{-1}H_1$ is asymptotically stable. (In fact, Y satisfying the above conditions is unique.) Moreover, if $\hat{Y}$ denotes a solution to either (4) or (5), then $Y \le \hat{Y}$.

Proof. The equivalence of items 1 and 5 can be found in Molinari (1975). The equivalence of items 1 and 4 follows from the equivalence of items 1 and 5 and a standard small perturbation argument. The equivalence of items 3 and 4 follows from simple algebraic manipulations and the Schur complement formula. Setting $Y := (P'P)^{-1}$ yields $2 \Rightarrow 3$, and setting $P = Y^{-1/2}$ gives $3 \Rightarrow 2$. Finally, in item 5, the inequality $Y \le \hat{Y}$ follows from Ran and Vreugdenhil (1988). ■

Now suppose $\|T_{z_1w}\|_\infty < \gamma$. Let Y denote the unique real symmetric matrix that satisfies condition 5 in Theorem 2.1. Then, from the definition of the controllability gramian L_c and Theorem 2.1, it follows that

$$0 \le L_c \le Y. \quad (7)$$

Note that this is the best possible upper bound for the controllability gramian that may be defined in terms of the solutions to the various quadratic matrix inequalities in Theorem 2.1.

Thus,

$$\|T_{z_0w}\|_2^2 = \operatorname{tr}(H_0L_cH_0' + J_0J_0') \le \operatorname{tr}(H_0YH_0' + J_0J_0').$$

The above inequality motivates the following definition of the *mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure (or cost)* $J(T_{zw})$ for the linear time-invariant system $\mathcal{T}$:

$$J(T_{zw}) := \operatorname{tr}(H_0YH_0' + J_0J_0'). \quad (8)$$

The performance measure defined in (8) is the same as the one considered by Mustafa and Bernstein (1991), Bambang *et al.* (1990), and Haddad *et al.* (1991). (More precisely, this performance measure is one of the costs considered in Mustafa and Bernstein (1991).)

It is easily seen that $J(T_{zw})$ is only a function of the transfer matrix T_{zw}, and does not depend on the choice of realization, as long as such a realization is internally stable. This justifies our notation. Also $\|T_{z_0w}\|_2 \le \sqrt{J(T_{zw})}$, and $\lim_{\gamma\to\infty} \sqrt{J(T_{zw})} = \|T_{z_0w}\|_2$. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure $J(T_{zw})$ is also a function of the parameter γ. In the sequel γ will remain fixed. Therefore, without loss of generality, we set

$$\gamma = 1,$$

for the remainder of this paper. Any other constraint level can be accommodated by simple scaling.

The following result provides an alternative characterization for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure $J(T_{zw})$ that will be useful for establishing some of the results in this paper. The proof of this result is very similar to the proof of Lemma 1.1 in Khargonekar and Rotea (1991a). For the sake of brevity, details are omitted.

Lemma 2.2. Consider the stable system $\mathcal{T}$ defined in (1) and let T_{zw} denote the transfer matrix from w to z. Suppose that $\|T_{z_1w}\|_\infty < 1$. Let $M(.)$, $R(.)$ be given by (5), with $\gamma = 1$. Then

$$J(T_{zw}) = \inf\{\operatorname{tr}(H_0YH_0' + J_0J_0') : Y = Y' > 0 \text{ such that } M(Y) > 0 \text{ and } R(Y) < 0\}.$$

3. THE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL PROBLEM

In this section, we formulate the controller synthesis problem to be solved in this paper. Consider the finite-dimensional linear time-invariant discrete-time feedback system depicted in Fig. 3, where $\mathcal{G}$ is the generalized plant, including weighting functions, and $\mathcal{C}$ is the

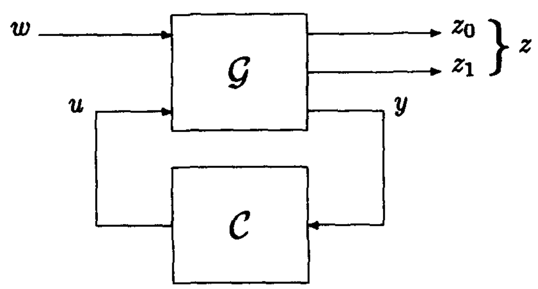

FIG. 3. The synthesis framework.

controller to be designed. The signal w denotes an exogenous input, while z_0 and z_1 denote controlled (i.e. regulated) signals. The signals u and y denote the control input and the measured output, respectively. The transfer matrices of the plant and the controller are denoted by G and C, respectively. Let

$$T_{zw} = \begin{bmatrix} T_{z_0w} \\ T_{z_1w} \end{bmatrix},$$

denote the closed loop transfer matrix, where T_{z_0w} and T_{z_1w} are the closed loop transfer matrices from w to z_0 and w to z_1, respectively.

Definition 3.1. Let $\mathcal{G}$ and $\mathcal{C}$ be the given plant and controller. The controller $\mathcal{C}$ is called admissible (for the plant $\mathcal{G}$) if $\mathcal{C}$ internally stabilizes the plant $\mathcal{G}$. The set of all admissible controllers for the plant $\mathcal{G}$ is denoted by $\mathcal{A}(\mathcal{G})$. Furthermore, we define

$$\mathcal{A}_\infty(\mathcal{G}) := \{\mathcal{C} \in \mathcal{A}(\mathcal{G}) : \|T_{z_1w}\|_\infty < 1\}. \quad (9)$$

In the above notation, "$\mathcal{A}$" stands for "admissible". As is well known, $\mathcal{A}(\mathcal{G}) \neq \emptyset$ if and only if $\mathcal{G}$ is stabilizable from u and detectable from y. Also, the subscript "∞" in $\mathcal{A}_\infty(\mathcal{G})$ stands for the infinity norm constraint on T_{z_1w}.

Consider the feedback system shown in Fig. 3. Given a plant $\mathcal{G}$ and an internally stabilizing controller $\mathcal{C}$, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ cost $J(T_{zw})$ of the closed loop system is a function of the transfer matrix T_{zw} only. We will denote this transfer matrix by $T_{zw}(G, C)$ and define

$$J(G, C) := J(T_{zw}(G, C)),$$

to emphasize on which plant and controller these closed loop quantities depend.

Following Khargonekar and Rotea (1991a), the sub-optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis problem considered in this paper is defined as follows.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. "Calculate the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure

$$\nu(\mathcal{G}) := \inf \{J(G, C) : \mathcal{C} \in \mathcal{A}_\infty(\mathcal{G})\}, \quad (10)$$

and, given any $\alpha > \nu(\mathcal{G})$, find a controller $\mathcal{C} \in \mathcal{A}_\infty(\mathcal{G})$ such that $J(G, C) < \alpha$".

In some cases involving state-feedback, it is natural to also consider memoryless, i.e. static controllers. In such a case, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem is defined in the following way.

Definition 3.2. The set of static admissible controllers satisfying the $\mathcal{H}_\infty$ constraint is denoted by

$$\mathcal{A}_{\infty,m}(\mathcal{G}) := \{\mathcal{C} \in \mathcal{A}_\infty(\mathcal{G}) : C \in R^{q\times p}\}, \quad (11)$$

where $q = \dim(u)$ and $p = \dim(y)$.

The optimal performance over all admissible memoryless controllers is

$$\nu_m(\mathcal{G}) := \inf \{J(G, C) : \mathcal{C} \in \mathcal{A}_{\infty,m}(\mathcal{G})\}. \quad (12)$$

In (11) and (12), the subscript "m" stands for memoryless controllers.

The main results of this paper show that the computation of the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance (10), and the construction of a sub-optimal compensator, can be reduced to the convex optimization problem over a bounded convex subset of a space of real matrices.

4. STATE AND FULL-INFORMATION FEEDBACK PROBLEMS

In this section, we give a solution to the controller synthesis problem formulated previously, for the state/full-information feedback case. Here full-information feedback means that both the state and the exogenous inputs are available to the controller. Even though, such a feedback scheme is not realistic from a practical point of view, the full-information results are instrumental for addressing the more general case of output-feedback.

We will first ask the question whether the infimum of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure over all dynamic full-information feedback controllers equals the infimum over all static state-feedback controllers. In the continuous-time case, the answer to this question is in the affirmative (Khargonekar and Rotea, 1991a). However, *in the discrete-time*, the answer, in general, turns out to be in the negative. In fact, we will show that the infimum of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure over all dynamic full-information feedback controllers equals the infimum over all static full-information feedback controllers. Thus, the analogy with the continuous-time case breaks down in this sense. As mentioned in the Introduction, a similar situation also occurs in the single objective standard $\mathcal{H}_\infty$ control problem in the discrete-time case.

On the other hand, if only the state of the system is available for feedback, then we will show that the infimum of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure over all dynamic state-feedback controllers equals the infimum over all static state-feedback controllers.

Finally, we will show that the static state/full-information controller synthesis problems may be effectively solved by means of finite-dimensional convex optimization.

Let us begin by considering the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem defined in Section 3 for the following plants (see also Fig. 3):

(1) State-feedback plant: the plant $\mathcal{G}$ is given by the state-space model

$$\mathcal{G}_{sf} := \begin{cases} \sigma x = Ax + B_1 w + B_2 u \\ z_0 = C_0 x + D_{01} w + D_{02} u \\ z_1 = C_1 x + D_{11} w + D_{12} u \\ y = x. \end{cases} \tag{13}$$

(2) Full-information plant: the plant $\mathcal{G}$ is given by the state-space model

$$\mathcal{G}_{fi} := \begin{cases} \sigma x = Ax + B_1 w + B_2 u \\ z_0 = C_0 x + D_{01} w + D_{02} u \\ z_1 = C_1 x + D_{11} w + D_{12} u \\ y = [x' \quad w']'. \end{cases} \tag{14}$$

In the sequel, we let G_{sf} and G_{fi} denote the transfer matrices of (13) and (14), respectively. The subscripts "*sf*" and "*fi*" denote "state-feedback" and "full-information" structure, respectively. The only difference between the state-feedback and full-information plants is in the measurement equation. Note also that no assumptions on problem data, i.e. the matrices introduced in (13)–(14), are imposed.

4.1. *Reduction to memoryless feedback*

Theorem 4.1. Consider the full-information plant $\mathcal{G}_{fi}$ defined in (14). Then

$$\mathcal{A}_\infty(\mathcal{G}_{fi}) \neq \emptyset \Leftrightarrow \mathcal{A}_{\infty,m}(\mathcal{G}_{fi}) \neq \emptyset,$$

where $\mathcal{A}_\infty(\mathcal{G}_{fi})$ and $\mathcal{A}_{\infty,m}(\mathcal{G}_{fi})$ are as in (9) and (11), respectively. In this case,

$$v(\mathcal{G}_{fi}) = v_m(\mathcal{G}_{fi}),$$

where $v(\mathcal{G}_{fi})$ and $v_m(\mathcal{G}_{fi})$ denote the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal costs in (10) and (12), respectively. Furthermore, given any $\alpha > v(\mathcal{G}_{fi})$, there exists a static full-information controller $\mathcal{K} \in \mathcal{A}_{\infty,m}(\mathcal{G}_{fi})$ such that $J(G_{fi}, K) < \alpha$.

Proof. We only need to show that if $\mathcal{A}_\infty(\mathcal{G}_{fi}) \neq \emptyset$, then $\mathcal{A}_{\infty,m}(\mathcal{G}_{fi}) \neq \emptyset$ and $v_m(\mathcal{G}_{fi}) \leq v(\mathcal{G}_{fi})$. Let $\epsilon > 0$ be given. From (10) it follows that there exists $\mathcal{C} \in \mathcal{A}_\infty(\mathcal{G}_{fi})$ such that

$$J(G_{fi}, C) \leq v(\mathcal{G}_{fi}) + \epsilon/2. \tag{15}$$

Let $\mathcal{C}$ be given by

$$\begin{cases} \sigma\xi = A_c\xi + B_{c1}x + B_{c2}w \\ u = C_c\xi + D_{c1}x + D_{c2}w. \end{cases} \tag{16}$$

The closed loop system corresponding to the interconnection of $\mathcal{G}_{fi}$ and $\mathcal{C}$ is given by:

$$\begin{cases} \sigma\eta = F\eta + Gw \\ z_0 = H_0\eta + J_0 w \\ z_1 = H_1\eta + J_1 w, \end{cases}$$

where

$$F := \begin{bmatrix} A + B_2 D_{c1} & B_2 C_c \\ B_{c1} & A_c \end{bmatrix},$$

$$G := \begin{bmatrix} B_1 + B_2 D_{c2} \\ B_{c2} \end{bmatrix},$$

$$H_0 := [C_0 + D_{02}D_{c1} \quad D_{02}C_c],$$

$$J_0 := D_{01} + D_{02}D_{c2},$$

$$H_1 := [C_1 + D_{12}D_{c1} \quad D_{12}C_c],$$

$$J_1 := D_{11} + D_{12}D_{c2}.$$

Since $\mathcal{C} \in \mathcal{A}_\infty(\mathcal{G}_{fi})$, it follows that F is stable and $\|T_{z_1w}(G_{fi}, C)\|_\infty < 1$. Using Theorem 2.1 and Lemma 2.2 we may now conclude that $\exists Y = Y' > 0$ such that:

(1) $M := I - J_1J_1' - H_1YH_1' > 0$

(2) $R(Y) := FYF' - Y + (FYH_1' + GJ_1') \times M^{-1}(H_1YF' + J_1G') + GG' < 0,$

(3) $\mathrm{tr}\,(H_0YH_0' + J_0J_0') \leq J(G_{fi}, C) + \epsilon/2.$ (17)

Now combining (15), and item 3 in (17), we get

$$\mathrm{tr}\,(H_0YH_0' + J_0J_0') \leq v(G_{fi}) + \epsilon. \tag{18}$$

Using the matrix Y introduced in (17), we will construct a memoryless controller $\mathcal{K}$ for $\mathcal{G}_{fi}$. Let $n = \dim(x)$ and $n_c = \dim(\xi)$ and partition Y and $R(Y)$ according to the plant and controller dimensions, i.e.

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2' & Y_3 \end{bmatrix}, \quad R(Y) = \begin{bmatrix} R_1 & R_2 \\ R_2' & R_3 \end{bmatrix},$$

where $\dim(Y_1) = n \times n$, $\dim(Y_2) = n \times n_c$, $\dim(Y_3) = n_c \times n_c$, and similarly for $R(Y)$. Note that $Y_1 > 0$ and $R_1 < 0$. Define the memoryless full-information controller $\mathcal{K}$ by

$$u = K_1x + K_2w,$$

where

$$K_1 := D_{c1} + C_cY_2'Y_1^{-1}, \quad K_2 := D_{c2}. \tag{19}$$

The closed loop system resulting from the interconnection of $\mathcal{G}_{fi}$ and $\mathcal{K}$ is given by

$$\begin{cases} \sigma x = F_m x + G_m w \\ z_0 = H_{0m} x + J_{0m} w \\ z_1 = H_{1m} x + J_{1m} w, \end{cases}$$

where $F_m := A + B_2 K_1$, $G_m := B_1 + B_2 K_2$, $H_{0m} := C_0 + D_{02} K_1$, $J_{0m} := D_{01} + D_{02} K_2$, $H_{1m} := C_1 + D_{12} K_1$, $J_{1m} := D_{11} + D_{12} K_2$, and the gains K_1 and K_2 are given by (19).

Define $Q := Y_3 - Y_2' Y_1^{-1} Y_2$. Since, $Y > 0$ it follows that $Q > 0$ by taking Schur complement. Simple algebraic manipulations show that the "1–1 block" of $R(Y)$ satisfies

$$\begin{aligned} R_1 = F_m Y_1 F_m' - Y_1 + G_m G_m' + B_2 C_c Q C_c' B_2' \\ + (F_m Y_1 H_{1m}' + G_m J_{1m}' + B_2 C_c Q C_c' D_{12}') M^{-1} \\ \times (H_{1m} Y_1 F_m' + J_{1m} B_{1m}' + D_{12} C_c Q C_c' B_2') < 0, \end{aligned} \quad (20)$$

where

$$M = I - J_{1m} J_{1m}' - H_{1m} Y_1 H_{1m}' - D_{12} C_c Q C_c' D_{12}' > 0. \quad (21)$$

Since $Y_1 > 0$, (20), (21) and the implication $4 \Rightarrow 3$ of Theorem 2.1 imply that Y_1 satisfies

$$\begin{bmatrix} F_m \\ H_{1m} \end{bmatrix} Y_1 [F_m' \quad H_{1m}'] + \begin{bmatrix} G_m \\ J_{1m} \end{bmatrix} [G_m' \quad J_{1m}'] + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} C_c Q C_c' [B_2' \quad D_{12}'] < \begin{bmatrix} Y_1 & 0 \\ 0 & I \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} F_m \\ H_{1m} \end{bmatrix} Y_1 [F_m' \quad H_{1m}'] + \begin{bmatrix} G_m \\ J_{1m} \end{bmatrix} [G_m' \quad J_{1m}'] < \begin{bmatrix} Y_1 & 0 \\ 0 & I \end{bmatrix}, \quad (22)$$

since $Q > 0$.

Now using (22) and $Y_1 > 0$, a simple Lyapunov argument shows that F_m is a stable matrix. Further, from implication $3 \Rightarrow 1$ in Theorem 2.1, it follows that $\|T_{z_1 w}(G_{fi}, K)\|_\infty < 1$. Consequently, the memoryless controller defined in (19) satisfies $\mathcal{K} \in \mathcal{A}_{\infty,m}(\mathcal{G}_{fi})$.

It is easy to verify that

$$H_{0m} Y_1 H_{0m}' + J_{0m} J_{0m}' = H_0 Y H_0' + J_0 J_0' - D_{02} C_c Q C_c' D_{02}'.$$

Since $Q > 0$, we may now conclude that

$$\mathrm{tr}\,(H_{0m} Y_1 H_{0m}' + J_{0m} J_{0m}') \le \mathrm{tr}\,(H_0 Y H_0' + J_0 J_0'),$$

which, together with (22), implies

$$\mathrm{tr}\,(H_{0m} Y_1 H_{0m}' + J_{0m} J_{0m}') \le \nu(\mathcal{G}_{fi}) + \epsilon.$$

Using the last inequality, $Y_1 > 0$, the implication $3 \Rightarrow 4$ in Theorem 2.1, Lemma 2.2, and (18) we obtain

$$\nu_m(\mathcal{G}_{fi}) \le J(G_{fi}, K) \le \mathrm{tr}\,(H_{0m} Y_1 H_{0m}' + J_{0m} J_{0m}') \le \nu(\mathcal{G}_{fi}) + \epsilon.$$

Since ϵ is arbitrary, we conclude that

$$\nu_m(\mathcal{G}_{fi}) \le \nu(\mathcal{G}_{fi}).$$

The last part of the theorem now follows from definitions. ■

Theorem 4.1 applies in the case where both the state and the exogenous inputs are available for feedback. This is rarely, if every, true in practice. The principal motivation for this result comes from the output-feedback case. As will be seen in the next section, Theorem 4.1 is a key result in the derivation of the output feedback solution.

However, in applications one often comes across problems where the state vector is available for feedback. From this point of view the following results on the state-feedback case is much more useful.

Theorem 4.2. Consider the state-feedback plant $\mathcal{G}_{fi}$ defined in (13). Then

$$\mathcal{A}_\infty(\mathcal{G}_{sf}) \ne \emptyset \Leftrightarrow \mathcal{A}_{\infty,m}(\mathcal{G}_{sf}) \ne \emptyset,$$

where $\mathcal{A}_\infty(\mathcal{G}_{sf})$ and $\mathcal{A}_{\infty,m}(\mathcal{G}_{sf})$ are as in (9) and (11), respectively. In this case

$$\nu(\mathcal{G}_{sf}) = \nu_m(\mathcal{G}_{sf}),$$

where $\nu(\mathcal{G}_{sf})$ and $\nu_m(\mathcal{G}_{sf})$ denote the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal costs defined in (10) and (12), respectively. Furthermore, given any $\alpha > \nu(\mathcal{G}_{sf})$, there exists $\mathcal{K} \in \mathcal{A}_{\infty,m}(\mathcal{G}_{sf})$, such that $J(G_{sf}, K) < \alpha$.

A proof of this result can be constructed from the proof of Theorem 4.1 by setting B_{c2} and D_{c2} equal to zero in the definition of the controller $\mathcal{C}$ given in (16). Details are omitted for the sake of brevity.

4.2. *A convex optimization approach to static feedback problem*

In this subsection we will show that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem with state/full-information feedback can be reduced to a convex optimization problem over a convex bounded set of real matrices. That is, given $\alpha > \nu_m(\mathcal{G}_{fi})$, real matrices K_1 and K_2 such that $J(G_{fi}, [K_1 \ K_2]) < \alpha$ can be found by solving a finite-dimensional convex programming problem. We will consider the full-information case. *The state-feedback case follows by taking $K_2 = 0$ in the analysis below.*

With reference to the full-information plant defined in (14), let $n = \dim(x)$, $q = \dim(u)$, $p = \dim(w)$. Let Σ denote the set of all real $n \times n$ symmetric matrices, and define

$$\Omega := \{(W, Y, K_2) \in R^{q\times n} \times \Sigma \times R^{q\times p} : Y > 0\}. \quad (23)$$

Note that Ω is a strictly convex open subset of $R^{q\times n} \times \Sigma \times R^{q\times p}$. Given $(W, Y, K_2) \in \Omega$ define

$$f(W, Y, K_2) := \operatorname{tr}((C_0 Y + D_{02} W) Y^{-1} (C_0 Y + D_{02} W)' + (D_{01} + D_{02} K_2)(D_{01} + D_{02} K_2)'). \quad (24)$$

Given any $(W, Y, K_2) \in \Omega$, define

$$L(W, Y, K_2) := \begin{bmatrix} AY + B_2 W \\ C_1 Y + D_{12} W \end{bmatrix} Y^{-1} \begin{bmatrix} AY + B_2 W \\ C_1 Y + D_{12} W \end{bmatrix}' + \begin{bmatrix} B_1 + B_2 K_2 \\ D_{11} + D_{12} K_2 \end{bmatrix} \begin{bmatrix} B_1 + B_2 K_2 \\ D_{11} + D_{12} K_2 \end{bmatrix}' - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}. \quad (25)$$

Now consider the set of real matrices:

$$\Phi(\mathcal{G}_{fi}) := \{(W, Y, K_2) \in \Omega : L(W, Y, K_2) < 0\}, \quad (26)$$

and the constrained optimization problem

$$\psi(\mathcal{G}_{fi}) := \inf\{f(W, Y, K_2) : (W, Y, K_2) \in \Phi(\mathcal{G}_{fi})\}. \quad (27)$$

Note that $\psi(\mathcal{G}_{fi}) \geq 0$ since $f \geq 0$ on Ω, as can be seen from (24). We now state the main result of this subsection.

Theorem 4.3. Consider the system $\mathcal{G}_{fi}$ defined in (14) with transfer matrix G_{fi}. Let $\mathcal{A}_{\infty,m}(\mathcal{G}_{fi})$ be the set of static controllers defined in (11). Then,

$$\mathcal{A}_{\infty,m}(\mathcal{G}_{fi}) \neq \emptyset \Leftrightarrow \Phi(\mathcal{G}_{fi}) \neq \emptyset,$$

where $\Phi(\mathcal{G}_{fi})$ is given by (26). In this case

$$\nu_m(\mathcal{G}_{fi}) = \psi(\mathcal{G}_{fi}),$$

where $\nu_m(\mathcal{G}_{fi})$ and $\psi(\mathcal{G}_{fi})$ are as in (12) and (27), respectively. Furthermore, given any $\alpha > \nu_m(\mathcal{G}_{fi})$, there exists a triple $(W, Y, K_2) \in \Phi(\mathcal{G}_{fi})$ such that the static full-information controller

$$K := [WY^{-1} \quad K_2],$$

satisfies

$$\mathcal{K} \in \mathcal{A}_{\infty,m}(\mathcal{G}_{fi}) \quad \text{and} \quad J(G_{fi}, K) < \alpha.$$

This result is a direct and straightforward generalization of Theorem 4.2 of Khargonekar and Rotea (1991a). Proof is omitted.

In the remainder of this section we will show that the optimization problem defined in (27) is convex.

Lemma 4.4. Let Ω denote the set defined in (23). The mapping $f : \Omega \to R^+$ defined in (24) is a real-analytic convex function on Ω.

Proof. Using (24) f may be rewritten as

$$f(W, Y, K_2) = \operatorname{tr}(C_0 Y C_0') + 2\operatorname{tr}(C_0' D_{02} W) + \operatorname{tr}(D_{02} W Y^{-1} W' D_{02}') + \operatorname{tr}(D_{01} + D_{02} K_2)(D_{01}' + K_2' D_{02}').$$

In Khargonekar and Rotea (1991a) it was shown that the first three terms in this expression are convex in Ω. Clearly, the term $\operatorname{tr}(D_{01} + D_{02} K_2)(D_{01}' + K_2' D_{02}')$ is convex in K_2. The convexity of $f(.)$ now follows. The fact that $f(.)$ is real analytic follows from its definition. ■

Lemma 4.5. Let $L : \Omega \to \Sigma$ denote the matrix-valued mapping defined in (25). Then L is a convex mapping. Consequently, the constraint set $\Phi(\mathcal{G}_{fi})$ defined in (26) is convex.

Proof. Let $(W, Y, K_2) \in \Omega$. Using (25), it is easy to see that $L(W, Y, K_2)$ can be rewritten as

$$L(W, Y, K_2) = \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} Y \\ W \end{bmatrix} Y^{-1} [Y' \quad W'] \times \begin{bmatrix} A' & C_1' \\ B_2' & D_{12}' \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ D_{11} & D_{12} \end{bmatrix} \times \begin{bmatrix} I \\ K_2 \end{bmatrix} [I \quad K_2'] \begin{bmatrix} B_1' & D_{11}' \\ B_2' & D_{12}' \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}$$
$$= \tilde{F} \tilde{W} Y^{-1} \tilde{W}' \tilde{F}' + \tilde{G} \tilde{K} \tilde{K}' \tilde{G}' - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix},$$

where

$$\tilde{F} := \begin{bmatrix} A' & C_1' \\ B_2' & D_{12}' \end{bmatrix}, \quad \tilde{G} := \begin{bmatrix} B_1 & B_2 \\ D_{11} & D_{12} \end{bmatrix},$$
$$\tilde{W} := [Y' \quad W']', \quad \tilde{K} := [I \quad K_2']'.$$

By Proposition E.7.f in Marshall and Olkin (1979) the mappings $(\tilde{W}, Y) \to \tilde{F}\tilde{W}Y^{-1}\tilde{W}'\tilde{F}'$, and $\tilde{K} \to \tilde{G}\tilde{K}\tilde{K}'\tilde{G}'$, are convex on their domains (here $Y = Y' > 0$). Since the maps $(W, Y) \to [Y' \quad W']'$ and $K_2 \to [I \quad K_2']'$ are affine linear, the convexity of L follows. Finally, the convexity of $\Phi(\mathcal{G}_{fi})$ follows from the convexity of L. ■

Lemma 4.6. Consider the set $\Phi(\mathcal{G}_{fi})$ defined in (26). Assume that D_{12} has full column rank. Suppose that for all z inside the open unit disc, the system matrix

$$\begin{bmatrix} zI - A & B_2 \\ -C_1 & D_{12} \end{bmatrix},$$

has full column rank. Then the set $\Phi(\mathcal{G}_{fi})$ is bounded.

Proof. We need to show that there exist positive constants $m_1, m_2, m_3 < \infty$ such that

$$(W, Y, K_2) \in \Phi(\mathcal{G}_{fi}) \Rightarrow \|W\| \le m_1,$$
$$\|Y\| \le m_2, \quad \|K_2\| \le m_3.$$

Let $(W, Y, K_2) \in \Phi(\mathcal{G}_{fi})$. From definitions of $\Phi(\mathcal{G}_{fi})$ and $L(W, Y, K_2)$, it follows that $D_{12}K_2K_2'D_{12}' < I$, which implies that $\|D_{12}K_2\| < 1$. Since D_{12} is full column rank there exists a constant $m_3 < \infty$, such that

$$\|K_2\| \le m_3 < \infty.$$

Now define the matrices

$$T := \begin{bmatrix} I & -B_2XD_{12}' \\ 0 & I - D_{12}XD_{12}' \end{bmatrix},$$
$$\tilde{C}_1 := (I - D_{12}XD_{12}')C_1,$$
$$\tilde{A} := A - B_2XD_{12}'C_1,$$

where $X := (D_{12}'D_{12})^{-1}$. Premultiply L by T and postmultiply it by T'. After simple algebraic manipulations we obtain

$$\begin{bmatrix} \tilde{A} \\ \tilde{C}_1 \end{bmatrix} Y[\tilde{A}' \quad \tilde{C}_1'] - \begin{bmatrix} Y & 0 \\ 0 & I - D_{12}XD_{12}' \end{bmatrix} \le 0. \quad (28)$$

To prove the boundedness of Y, we are going to use Theorem 2.1 of Section 2 and Theorem 3.1 in Ran and Vreugdenhil (1988). To be able to use these theorems, we first need to show that $I - \tilde{C}_1Y\tilde{C}_1' > 0$. It follows from (28) that $\tilde{C}_1Y\tilde{C}_1' + D_{12}XD_{12}' - I \le 0$, and so $I - \tilde{C}_1Y\tilde{C}_1' \ge 0$. Now suppose there exists a vector $y \ne 0 \in R^p$ such that

$$(I - \tilde{C}_1Y\tilde{C}_1')y = 0. \quad (29)$$

Multiplying the inequality $\tilde{C}_1Y\tilde{C}_1' + D_{12}XD_{12}' - I \le 0$ by y on the right and y' on the left, we get $y'D_{12}XD_{12}'y = 0$. Since X is invertible, it follows that

$$D_{12}'y = 0 \quad \text{and} \quad \tilde{C}_1'y = C_1'y. \quad (30)$$

Also by taking the $(2, 2)$ sub-block of the inequality $L < 0$, we get

$$I - (C_1Y + D_{12}W)Y^{-1}(C_1Y + D_{12}W)' - D_{12}K_2K_2'D_{12}' > 0. \quad (31)$$

Premultiplying (31) by y' and postmultiplying it by y yields

$$y'(I - C_1YC_1')y > 0.$$

Since $C_1'y = \tilde{C}_1'y$, it now follows that

$$0 = y'(I - \tilde{C}_1Y\tilde{C}_1')y = y'(I - C_1YC_1')y > 0.$$

This contradicts (29), and therefore

$$\tilde{X} := I - \tilde{C}_1Y\tilde{C}_1' > 0. \quad (32)$$

Using (32) and taking Schur complement of the $(2, 2)$ block in (28), it follows that

$$\tilde{R} := \tilde{A}Y\tilde{A}' - Y - \tilde{A}Y\tilde{C}_1'\tilde{X}^{-1}\tilde{C}_1Y\tilde{A}' \le 0. \quad (33)$$

Now the rank assumption on the system matrix ensures that the pair $(\tilde{C}_1, \tilde{A})$ has no unobservable modes inside the unit disc. Using a simple extension of Theorem 3.1 in Ran and Vreugdenhil (1988) and the observability of the stable modes of $(\tilde{C}_1, \tilde{A})$, it follows that there exists a real symmetric matrix Y_- (depending only on $\tilde{A}$, B and $\tilde{C}_1$) such that $-Y \ge Y_-$, or $Y \le -Y_-$. Let $m_1 = \|Y_-\|$, then since $Y > 0$, $\|Y\| \le m_1 < \infty$.

Finally, from equation (31) it follows that

$$(C_1Y + D_{12}W)Y^{-1}(C_1Y + D_{12}W)' < I$$
$$\Rightarrow \|(C_1Y + D_{12}W)Y^{-1/2}\| < 1$$
$$\Rightarrow Y^{-1/2}(C_1Y + D_{12}W)(C_1Y + D_{12}W)'Y^{-1/2} < I$$
$$\Rightarrow \|C_1Y + D_{12}W\| < \sqrt{\|Y\|}.$$

Since $\|Y\| \le m_1$ and D_{12} has full column rank, we conclude that there exists $m_2 < \infty$ such that

$$\|W\| \le m_2 < \infty. \quad \blacksquare$$

5. OUTPUT-FEEDBACK CASE

In this section we will solve the synthesis problem defined in Section 3 for the output-feedback case. We will show that the problem can be reduced to solving one algebraic Riccati equation, and a convex optimization problem similar to the one in Section 4. In the following subsection we introduce a technical result that will be needed in order to prove the main theorem of this section.

5.1. *Preliminaries*

The next result is an extension of Redheffer's lemma (see, for example, Iglesias and Glover (1991) and Stoorvogel (1990)) to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure for discrete-time systems.

Consider the feedback interconnection of Fig.

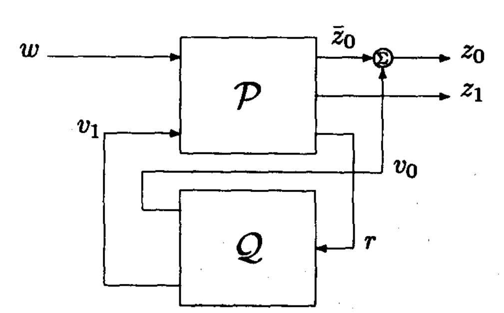

FIG. 4. System interconnection in Lemma 5.1.

4, where

$$\mathcal{P} := \begin{cases} \sigma\eta = A\eta + B_1 w + B_2 v_1 \\ \bar{z}_0 = C_0\eta \\ z_0 = \bar{z}_0 + v_0 \\ z_1 = C_1\eta + D_{11}w + D_{12}v_1 \\ r = C_2\eta + D_{21}w + D_{22}v_1, \end{cases} \quad (34)$$

and

$$\mathcal{Q} := \begin{cases} \sigma x = \hat{A}x + \hat{B}r \\ v_0 = \hat{C}_0 x + \hat{D}_0 r \\ v_1 = \hat{C}_1 x + \hat{D}_1 r. \end{cases} \quad (35)$$

The matrices in the state-space equations (34) and (35) are real and of compatible dimensions. Let P and Q denote the corresponding transfer matrices, and partition them as

$$P = \begin{matrix} \bar{z}_0 \\ z_1 \\ r \end{matrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}, \quad Q = \begin{matrix} v_0 \\ v_1 \end{matrix} \begin{bmatrix} Q_0 \\ Q_1 \end{bmatrix}. \quad (36)$$

Lemma 5.1. Consider the feedback system shown in Fig. 4, where $\mathcal{P}$ and $\mathcal{Q}$ are given by (34) and (35), respectively. Let T_{zw} denote the closed loop transfer matrix from w to $z = (z_0, z_1)$. Suppose that $\mathcal{P}$ is internally stable, and let L_c denote the controllability gramian of the pair $(A, [B_1 \;\; B_2])$, i.e.

$$AL_cA' + B_1B_1' + B_2B_2' = L_c. \quad (37)$$

Suppose also that D_{12} is square and nonsingular, that $A - B_2D_{12}^{-1}C_1$ is a stable matrix, and

$$[B_1 \;\; B_2]\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}' + AL_c[C_1' \;\; C_2'] = 0, \quad (38)$$

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}' + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} L_c[C_1' \;\; C_2'] = I. \quad (39)$$

Then the following statements are equivalent:

(i) The feedback system in Fig. 4 is well-posed, internally stable, and $\|T_{z_1w}\|_\infty < 1$.

(ii) $\mathcal{Q}$ is internally stable and $\|Q_1\|_\infty < 1$.

If the above conditions hold, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ costs $J(T_{zw})$ (with respect to z_0) and $J(Q)$ (with respect to v_0), are defined and they satisfy

$$J(T_{zw}) = \operatorname{tr}(C_0L_cC_0') + 2\operatorname{tr}(\hat{D}_0C_2L_cC_0') + J(\mathcal{Q}). \quad (40)$$

A continuous-time version of the above result is in Khargonekar and Rotea (1991a). The discrete-time case differs from the continuous-time case in one important aspect. In the continuous-time case the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ cost $J(T_{zw})$ of the feedback interconnection of $\mathcal{P}$ and $\mathcal{Q}$, is the sum of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ cost of $\mathcal{Q}$ and the $\mathcal{H}_2$ norm of the "1–1" block of the inner plant $\mathcal{P}$. This decomposition fails in the discrete-time case; there is an additional cross term (which depends on $\mathcal{Q}$) in (40).

Proof. The proof of the equivalence of the statements (i) and (ii) can be found in Doyle *et al.* (1989b); Iglesias and Glover (1991) and Stoorvogel (1990).

Suppose now that either one of these statements is true. For simplicity, the rest of the proof will be done under the assumption that P_{22} is strictly proper, i.e. $D_{22} = 0$. Let $\psi = [x' \;\; \eta']'$ denote the state of the composite system. It is easy to check that the system resulting from the interconnection of $\mathcal{P}$ and $\mathcal{Q}$ is given by

$$\begin{cases} \sigma\psi = F\psi + Gw \\ z_0 = H_0\psi + J_0w \\ z_1 = H_1\psi + J_1w, \end{cases} \quad (41)$$

where

$$F := \begin{bmatrix} \hat{A} & \hat{B}C_2 \\ B_2\hat{C}_1 & A + B_2\hat{D}_1C_2 \end{bmatrix},$$

$$G := \begin{bmatrix} \hat{B}D_{21} \\ B_1 + B_2\hat{D}_1D_{21} \end{bmatrix},$$

$$H_0 := [\hat{C}_0, \; C_0 + \hat{D}_0C_2],$$

$$H_1 := [D_{12}\hat{C}_1 \;\; C_1 + D_{12}\hat{D}_1C_2],$$

$$J_0 := \hat{D}_0D_{21}, \quad J_1 := D_{11} + D_{12}\hat{D}_1D_{21}.$$

Note that because of internal stability, all eigenvalues of F are inside the open unit disc. Moreover, $\|T_{z_1w}\|_\infty < 1$.

In order to establish formula (40), we need to determine the stabilizing solution of the ARE corresponding to the condition $\|T_{z_1w}\|_\infty < 1$. That is, the real symmetric matrix Y such that

$$M(Y) := I - J_1J_1' - H_1YH_1' > 0,$$

$$R(Y) := FYF' - Y + (FYH_1' + GJ_1')M^{-1} \times (H_1YF' + J_1G') + GG' = 0, \quad (42)$$

and $F + (FYH_1' + GJ_1')M^{-1}H_1$ is asymptotically stable. Since $\hat{A}$ is asymptotically stable $\|Q_1\|_\infty < 1$, there exists a real symmetric matrix $\hat{Y}$ such that

$$\hat{M}(\hat{Y}) := I - \hat{D}_1\hat{D}_1' - \hat{C}_1\hat{Y}\hat{C}_1' > 0,$$

$$\hat{R}(\hat{Y}) := \hat{A}\hat{Y}\hat{A}' - \hat{Y} + (\hat{A}\hat{Y}\hat{C}_1' + \hat{B}\hat{D}_1')\hat{M}^{-1} \times (\hat{C}_1\hat{Y}\hat{A}' + \hat{D}_1\hat{B}') + \hat{B}\hat{B}' = 0, \quad (43)$$

and $\hat{A} + (\hat{A}\hat{Y}\hat{C}_1' + \hat{B}\hat{D}_1')\hat{M}^{-1}\hat{C}_1$ stable.

Let L_c be given by (37) and set

$$Y := \begin{bmatrix} \hat{Y} & 0 \\ 0 & L_c \end{bmatrix}. \quad (44)$$

After some algebra and using equations (43) and (37)–(39), we obtain $M(Y) = D_{12}\hat{M}(\hat{Y})D_{12}' > 0$

and $R(Y)=0$. Moreover,

$$F+(FYH_1'+GJ_1')M^{-1}H_1 = \begin{bmatrix} \hat{A}+(\hat{A}\hat{Y}\hat{C}_1'+\hat{B}\hat{D}_1')\hat{M}^{-1}\hat{C}_1 & * \\ 0 & A-B_2D_{12}^{-1}C_1 \end{bmatrix},$$

where the "* block" is not relevant. Since $\hat{A}+(\hat{A}\hat{Y}\hat{C}_1'+\hat{B}\hat{D}_1')\hat{M}^{-1}\hat{C}_1$ and $A-B_2D_{12}^{-1}C_1$ are both asymptotically stable, we conclude that the real symmetric matrix Y defined in (44) is the unique stabilizing solution of the ARE (42).

Next we compute the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ cost $J(T_{zw})$. From equations (39), (41) and (44), it follows that

$$\begin{aligned} J(T_{zw}) &= \operatorname{tr}(H_0YH_0'+J_0J_0') = \operatorname{tr}(C_0L_cC_0') \\ &\quad + \operatorname{tr}(\hat{C}_0\hat{Y}\hat{C}_0'+\hat{D}_0\hat{D}_0') \\ &\quad + \operatorname{tr}(\hat{D}_0C_2L_cC_0'+C_0L_cC_2'\hat{D}_0') \\ &= \operatorname{tr}(C_0L_cC_0')+J(Q) \\ &\quad + 2\operatorname{tr}(\hat{D}_0C_2L_cC_0'). \quad \blacksquare \end{aligned} \tag{45}$$

5.2. *Main result*

We now consider the output-feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis problem. Suppose the plant $\mathcal{G}$ in Fig. 3 is given by the state-space model:

$$\mathcal{G} := \begin{cases} \sigma x = Ax+B_1w+B_2u \\ z_0 = C_0x+D_0u \\ z_1 = C_1x+D_1u \\ y = C_2x+D_2w, \end{cases} \tag{46}$$

where all the matrices in (46) are constant real matrices of compatible dimensions. We will also make the following assumptions:

(A1) The triple (C_2, A, B_2) is stabilizable and detectable.

(A2) For each complex number z, such that $|z|=1$, the matrix

$$\begin{bmatrix} zI-A & -B_1 \\ C_2 & D_2 \end{bmatrix},$$

has full row rank.

Note that we have also assumed that there are no feedthrough terms from w to z_i, or u to y. Even though it is possible to include these terms, we have chosen not to do so, to keep the presentation as simple as possible. The results given below may be combined with those in Stoorvogel (1990) to obtain formulae for the most general case.

Suppose there exists admissible controller $\mathcal{C}$ such that the closed loop system is internally stable and $\|T_{z_1w}\|_\infty<1$, i.e. $\mathcal{C}\in\mathcal{A}_\infty(\mathcal{G})$. Then it follows (Stoorvogel, 1990) that there exists a (unique) real symmetric matrix $Q\geq 0$ such that

$$V := C_2QC_2'+D_2D_2'>0, \tag{47}$$

$$R := I-C_1QC_1'+C_1QC_2'V^{-1}C_2QC_1'>0, \tag{48}$$

and Q satisfies the following discrete-time algebraic Riccati equation:

$$AQA'-Q+B_1B_1'-[AQC_2'+B_1D_2' \quad AQC_1'] \times G(Q)^{-1}\begin{bmatrix} C_2QA'+D_2B_1' \\ C_1QA' \end{bmatrix}=0, \tag{49}$$

where

$$G(Q) := \begin{bmatrix} D_2D_2' & 0 \\ 0 & -I \end{bmatrix}+\begin{bmatrix} C_2 \\ C_1 \end{bmatrix}Q[C_2' \quad C_1'].$$

Moreover, the matrix

$$A-[AQC_2'+B_1D_2' \quad AQC_1']G(Q)^{-1}\begin{bmatrix} C_2 \\ C_1 \end{bmatrix}, \tag{50}$$

is asymptotically stable.

Given a real symmetric matrix Q, define the auxiliary full-information system:

$$\mathcal{G}_{fi}(Q) := \begin{cases} \sigma x_g = (A+ZR^{-1}C_1)x_g \\ \qquad +(AQC_2'+B_1D_2' \\ \qquad +ZR^{-1}C_1QC_2')V^{-1/2}r \\ \qquad +(B_2+ZR^{-1}D_1)u \\ \quad =: A_gx_g+B_{1g}r+B_{2g}u, \\ \hat{v}_0 = C_0x_g+C_0QC_2'V^{-1/2}r+D_0u \\ \quad =: C_{0g}x_g+D_{01g}r+D_{02g}u \\ v_1 = R^{-1/2}C_1x_g \\ \qquad +R^{-1/2}C_1QC_2'V^{-1/2}r \\ \qquad +R^{-1/2}D_1u \\ \quad =: C_{1g}x_g+D_{11g}r+D_{12g}u \\ y = [x_g' \quad w']', \end{cases} \tag{51}$$

where $Z := AQC_1'-(AQC_2'+B_1D_2')V^{-1}C_2QC_1'$, and A, C_i, D_i, B_2 are given in (46). Let $G_{fi}(Q)$ denote the transfer matrix from (w, u) to $(\hat{v}_0, v_1, v)$ in (51). The next result gives the solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for the case of output feedback.

Theorem 5.2. Consider the plant $\mathcal{G}$ defined in (46). Suppose that Assumptions A1–A2 hold. Let $\mathcal{A}_\infty(\mathcal{G})$ be as defined in (9), and suppose that $\mathcal{A}_\infty(\mathcal{G})\neq\varnothing$. Then there exists a unique (real symmetric) matrix $Q\geq 0$ that satisfies the conditions (47)–(50). Let $\mathcal{G}_{fi}(Q)$ denote the auxiliary system defined in (51). Then the following statements hold:

(1) The set of admissible controllers $\mathcal{A}_\infty(\mathcal{G}_{fi}(Q))$ is nonempty, and the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance measure is given by

$$\nu(\mathcal{G}) = \operatorname{tr}(C_0QC_0') - \operatorname{tr}(C_0QC_2'V^{-1}C_2QC_0')+\nu(\mathcal{G}_{fi}(Q)), \tag{52}$$

where $v(\mathscr{G}_{fi}(Q))$ is the optimal mixed $\mathscr{H}_2/\mathscr{H}_\infty$ performance for the auxiliary full-information plant $\mathscr{G}_{fi}(Q)$.

(2) Given any $\alpha > v(\mathscr{G})$, there exists a static full information controller

$$\mathscr{K} := K_1 x_g + K_2 w, \tag{53}$$

such that $\mathscr{K} \in \mathscr{A}_{\infty,m}(\mathscr{G}_{fi}(Q))$ and

$$J(G_{fi}(Q), K) < \alpha - \operatorname{tr}(C_0QC_0') + \operatorname{tr}(C_0QC_2'V^{-1}C_2QC_0').$$

(3) For any full-information controller $\mathscr{K} = [\mathscr{K}_1\mathscr{K}_2] \in \mathscr{A}_{\infty,m}(\mathscr{G}_{fi}(Q))$, the dynamic output-feedback controller

$$\hat{\mathscr{C}} := \begin{cases} \sigma\xi = F\xi + Gy \\ u = H\xi + Jy, \end{cases} \tag{54}$$

where

$$\begin{aligned} F &:= A + ZR^{-1}C_1 + (B_2 + ZR^{-1}D_1) \\ &\quad \times (K_1 - K_2V^{-1/2}C_2) - (AQC_2' + B_1D_2' \\ &\quad + ZR^{-1}C_1QC_2')V^{-1/2}C_2, \\ G &:= (B_2 + ZR^{-1}D_1)K_2V^{-1/2} + (AQC_2' \\ &\quad + B_1D_2' + ZR^{-1}C_1QC_2')V^{-1}, \\ H &:= K_1 - K_2V^{-1/2}C_2, \\ J &:= K_2V^{-1/2}, \end{aligned} \tag{55}$$

satisfies

$$\mathscr{C} \in \mathscr{A}_\infty(\mathscr{G}) \quad \text{and} \quad J(G, \hat{C}) = \operatorname{tr}(C_0QC_0') - \operatorname{tr}(C_0QC_2'V^{-1}C_2QC_0') + J(G_{fi}(Q), K).$$

To solve the output-feedback mixed $\mathscr{H}_2/\mathscr{H}_\infty$ control problem using Theorem 5.2 the following steps should be followed.

(1) Check if $\mathscr{A}_\infty(\mathscr{G})$ is not empty. This can be done by applying the standard $\mathscr{H}_\infty$ theory for discrete-time systems and solving two discrete-time $\mathscr{H}_\infty$ Riccati equations as in Basar and Bernhard (1991), Iglesias and Glover (1991), Limebeer *et al.* (1989), Liu *et al.* (1991), and Stoorvogel (1990). If $\mathscr{A}_\infty(\mathscr{G})$ is not empty, let Q denote the unique solution to (47)–(50).

(2) Construct the auxiliary full-information plant $\mathscr{G}_{fi}(Q)$ defined in (51). Let $\epsilon > 0$ be given. Solve the convex program (27), corresponding to $\mathscr{G}_{fi}(Q)$, to compute a full-information gain $K = [K_1\ K_2] \in \mathscr{A}_{\infty,m}(\mathscr{G}_{fi}(Q))$ such that $J(G_{fi}(Q), K) \le v(\mathscr{G}_{fi}(Q)) + \epsilon$. With this full-information gain, construct the output-feedback controller $\mathscr{C}$ in (54). Then $\mathscr{C}$ belongs to $\mathscr{A}_\infty(\mathscr{G})$, and satisfies $J(G, \hat{C}) \le v(\mathscr{G}) + \epsilon$.

Proof. It follows from Stoorvogel (1990), that the plant $\mathscr{G}$ given by (46) can be represented as the feedback interconnection shown in Fig. 5, where $Q \ge 0$ satisfies equations (47)–(50), and

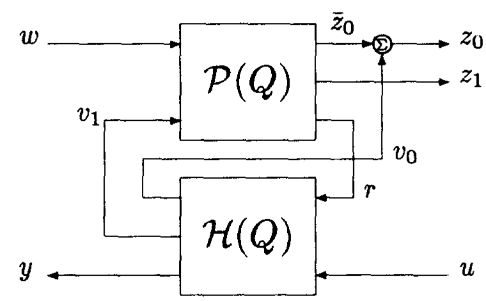

FIG. 5. Equivalent representation of the transfer matrix G.

the plants $\mathscr{P}(Q)$ and $\mathscr{H}(Q)$ are defined by

$$\mathscr{P}(Q) := \begin{cases} \sigma x_p = (A - (AQC_2' + B_1D_2')V^{-1}C_2)x_p \\ \quad + (B_1 - (AQC_2' + B_1D_2')V^{-1}D_2)w \\ \quad - ZR^{-1/2}v_1 =: A_px_p \\ \quad + B_{1p}w + B_{2p}v_1 \\ \bar{z}_0 = C_0x_p =: C_{0p}x_p \\ z_1 = (C_1 - C_1QC_2'V^{-1}C_2)x_p \\ \quad - C_1QC_2'V^{-1}D_2w + R^{1/2}v_1 \\ \quad =: C_{1p}x_p + D_{1p}w + D_{2p}v_1 \\ r = V^{-1/2}C_2x_p + V^{-1/2}D_2w \\ \quad =: C_{2p}x_p + D_{21p}w, \end{cases} \tag{56}$$

and

$$\mathscr{H}(Q) := \begin{cases} \sigma x_g = A_gx_g + B_{1g}r + B_{2g}u \\ v_0 = C_{0g}x_g + D_{02g}u \\ v_1 = C_{1g}x_g + D_{11g}r + D_{12g}u \\ y = C_2x_g + V^{1/2}r, \end{cases} \tag{57}$$

where the matrices in (57) are those introduced in the definition of $\mathscr{G}_{fi}(Q)$ given by (51). In (Stoorvogel, 1990), it has also been shown that $\mathscr{C} \in \mathscr{A}(\mathscr{G})$ if and only if $\mathscr{C}$ internally stabilizes the interconnection of $\mathscr{P}(Q)$ and $\mathscr{H}(Q)$. Routine algebra also shows that Q is the controllability gramian of the pair $(A_p, [B_{1p}\ B_{2p}])$, and that $\mathscr{P}(Q)$ satisfies the hypotheses of Lemma 5.1.

First, we show that

$$v(\mathscr{G}_{fi}(Q)) \le v(\mathscr{G}) - \operatorname{tr}(C_0QC_0') + \operatorname{tr}(C_0QC_2'V^{-1}C_2QC_0), \tag{58}$$

and that part 2 of the theorem holds. Suppose $\alpha > v(\mathscr{G})$. From the definition of $v(\mathscr{G})$ it follows that there exists a controller $\mathscr{C} \in \mathscr{A}_\infty(\mathscr{G})$ such that $J(G, C) < \alpha$. Apply the controller $\mathscr{C}$ to the interconnection of Fig. 5. Now using Lemma 5.1, since $\mathscr{C} \in \mathscr{A}_\infty(\mathscr{G})$ we get that $\mathscr{C} \in \mathscr{A}_\infty(\mathscr{H}(Q))$. Moreover,

$$J(G, C) = J(H(Q), C) + \operatorname{tr}(C_0QC_0') + 2\operatorname{tr}(D_0D_cC_2QC_0'), \tag{59}$$

where $D_c = C(\infty)$ denotes the direct feedthrough term of the controller $\mathscr{C}$.

Define the full-information controller

$$C^* = C[C_2 \quad V^{1/2}],$$

and apply the controller $\mathscr{C}^*$ to the full-information plant $\mathscr{G}_{fi}(Q)$ defined in (51). Clearly, $\mathscr{C}^* \in \mathscr{A}_\infty(\mathscr{G}_{fi}(Q))$. Further, an easy calculation shows that

$$J(G_{fi}(Q), C^*) = J(H(Q), C) + 2\mathrm{tr}\,(D_0 D_c C_2 Q C_0') + \mathrm{tr}\,(C_0 Q C_2' V^{-1} C_2 Q C_0'). \quad (60)$$

From (59)–(60), we obtain

$$J(G_{fi}(Q), C^*) = J(G, C) - \mathrm{tr}\,(C_0 Q C_0') + \mathrm{tr}\,(C_0 Q C_2' V^{-1} C_2 Q C_0).$$

Now we can apply Theorem 4.1 to the full-information plant $\mathscr{G}_{fi}$ to conclude that there exists a static controller $\mathscr{K} \in \mathscr{A}_{\infty,m}(\mathscr{G}_{fi}(Q))$ such that

$$v(G_{fi}(Q)) \le J(G_{fi}(Q), K) \le J(G_{fi}(Q), C^*) < \alpha - \mathrm{tr}\,(C_0 Q C_0') + \mathrm{tr}\,(C_0 Q C_2' V^{-1} C_2 Q C_0).$$

This shows that part 2 of the theorem holds. Moreover, taking limit as $\alpha \to v(\mathscr{G})$, we may also conclude that (58) holds.

We now show that part 3 of the theorem holds, and that

$$v(G_{fi}(Q)) \ge v(G) - \mathrm{tr}\,(C_0 Q C_0') + \mathrm{tr}\,(C_0 Q C_2' V^{-1} C_2 Q C_0). \quad (61)$$

Here the key is to synthesize the given full-information control law $u = K_1 x_g + K_2 r$ using a dynamic feedback controller which uses only the output y for the system $\mathscr{H}(Q)$. In this construction, we essentially invert the transfer function from r to y in $\mathscr{H}(Q)$.

Define the dynamic controller

$$\hat{\mathscr{C}} := \begin{cases} \sigma x_c = A_g x_c + B_{2g} u + B_{1g} V^{-1/2}(y - C_2 x_c) \\ u = K_1 x_c + K_2 V^{-1/2}(y - C_2 x_c), \end{cases} \quad (62)$$

where K_1 and K_2 are such that $\mathscr{K} := [\mathscr{K}_1 \ \mathscr{K}_2] \in \mathscr{A}_\infty(\mathscr{G}_{fi}(Q))$, and A_g, B_g and V are defined in (51) and (47), respectively. Note that the controller (62) is an "observer based controller" for the auxiliary plant $\mathscr{H}(Q)$. Using this fact, and the stabilizing property of Q, it is easy to see that $\hat{\mathscr{C}} \in \mathscr{A}(\mathscr{H}(Q))$, and

$$T_{v_1 r}(H(Q), \hat{C}) = T_{v_1 r}(G_{fi}(Q), K).$$

Thus, $\hat{\mathscr{C}} \in \mathscr{A}_\infty(\mathscr{H}(Q))$.

Applying the controller $\hat{\mathscr{C}}$ to the interconnection of Fig. 5, it follows from Lemma 5.1 that $\hat{\mathscr{C}} \in \mathscr{A}_\infty(\mathscr{G})$. Furthermore, calculations as in first part of the proof imply that

$$v(\mathscr{G}) \le J(G, \hat{C}) = \mathrm{tr}\,(C_0 Q C_0') - \mathrm{tr}\,(C_0 Q C_2' V^{-1} C_2 Q C_0') + J(G_{fi}(Q), K).$$

This shows that part 3 holds. Moreover, by taking infimum over all full-information controllers $\mathscr{K} \in \mathscr{A}_\infty(\mathscr{G}_{fi}(Q))$, we may also conclude that (61) is satisfied. Now equations (58) and (61) together prove part 1 of the theorem. ■

6. CONCLUSIONS

In this paper we have considered a (sub-optimal) mixed $\mathscr{H}_2/\mathscr{H}_\infty$ control problem for discrete-time systems. This synthesis problem is well motivated since it represents a problem of (LQG) disturbance attenuation, as measured by the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ performance measure, subject to a robust stability constraint.

We have shown that when the state of the plant, or the state and the exogenous input, is available for feedback, *memoryless feedback gains* offer the best possible performance. The optimal mixed $\mathscr{H}_2/\mathscr{H}_\infty$ performance, with state or full information feedback, was shown to be given by the value of a *finite-dimensional convex program*. This means that there are efficient numerical methods to compute the optimal performance, and a nearly optimal feedback gain. The reader may find an excellent description of some of these algorithms in Boyd and Barratt (1990). An ellipsoid algorithm has been developed in Rotea (1991) for a problem similar to the one considered in this paper.

In the case of output-feedback, it is shown that mixed $\mathscr{H}_2/\mathscr{H}_\infty$ controllers can be chosen to be a combination of an $\mathscr{H}_\infty$ *filter*, and a *full-information gain* for the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ synthesis problem of a suitably constructed auxiliary plant. Thus, the output-feedback problem is no more difficult than the full-information problem. This appears to be the first complete solution to the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ problem with dynamic output-feedback for discrete-time system.

Finally, the results in this paper may be combined with those of Rotea and Khargonekar (1991b) to solve mixed $\mathscr{H}_2/\mathscr{H}_\infty$ problems that involve time domain ℓ_∞ constraints.

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