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Joint Reductions, Tight Closure, and the Briançon – Skoda Theorem, II

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This paper is a sequel to [18] in which the author proved several Briançon–Skoda-type theorems using Hochster and Huneke's theory of tight closure for rings of prime characteristic. By "Briançon–Skoda theorem" we mean any result of the type $\overline{I^n} \subseteq (I^{n-k})^{\#}$, where I is an ideal, $n \geq k$ are integers, and where $^{\#}$ stands for any operator on ideals (such as identity, tight closure, or plus closure). We also mean any corresponding results for several ideals and their joint reductions. Examples of Briançon–Skoda theorems can be found in [16, 12, 13, 15, 8, 2]. For a review of history of the Briançon–Skoda theorems see [13].

Here we adopt the definitions and notation from [18], in particular we assume the definition of tight closure in prime characteristic p, the definition of joint reductions, and the definition of the analytic spread (denoted $l(\_)$). As in [18], an important theme here is uniformity: the integer k for which $\overline{I^n} \subseteq (I^{n-k})^{\#}$ for all $n \geq k$ (or the corresponding integers for several ideals and joint reductions) can often be chosen independently of the ideal I, and sometimes independently of the ring R. For example, k may be the number of generators of I, the analytic spread of I, or the dimension of R. This uniform theme was developed further by Huneke in [10] in proving the uniform Artin–Rees lemma. We use some of the techniques of [10]. We also use the main theorem of [18]:

($*$) *Let R be a Noetherian ring of characteristic p. Let $I_1, \ldots, I_k$ be ideals in R. Let $l_1, \ldots, l_k$ be positive integers and let $a_{ij} \in I_i$ for $1 \leq j \leq l_i$. Suppose that $(a_{11}, \ldots, a_{1l_1}, \ldots, a_{k1}, \ldots, a_{kl_k})$ is a joint reduction of*

$$\Big(\underbrace{I_1, \ldots, I_1}_{l_1 \text{ times}}, \ldots, \underbrace{I_k, \ldots, I_k}_{l_k \text{ times}}\Big).$$

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Then for any integers $n_i \geq 0$, $i = 1, \ldots, k$,

$$\overline{I_1^{l_1+n_1} \cdots I_k^{l_k+n_k}} \subseteq \left((a_{11}, \ldots, a_{1l_1})^{n_1+1} + \cdots + (a_{k1}, \ldots, a_{kl_k})^{n_k+1} \right)^*.$$

We point out that it was not necessary in [18] to assume in *any* of the results that certain elements do not lie in any minimal prime ideal of the ring. This is because an element is in the tight closure of an ideal if and only if it is in the tight closure of that ideal modulo every minimal prime (see Proposition 6.25 in [8]). So the proof of (*), as given in [18], only needs to be applied to domains.

In this paper we prove a version of the Briançon–Skoda theorem for F-rational rings, two versions for regular rings (Section 1), a version for plus closure (Section 3), and a few versions with "multipliers" (Section 2). An example of a version with multipliers is Lemma 7 which proves that for some fixed $c \in R$, $c\overline{I^n} \subseteq I^{n-\dim(R)}$ for all ideals I and all $n \geq \dim(R)$. Because of this "multiplier" c, we call theorems of Section 2 "almost Briançon–Skoda". If for an ideal its plus closure equals its tight closure, the result of Section 3 is an immediate consequence of (*).

1. Briançon–Skoda Theorem for F-Rational and Regular Rings

In [18] we used the definition of tight closure only for rings of characteristic p. Hochster and Huneke defined several notions of tight closure for Noetherian rings containing a field of characteristic zero. The definitions are quite involved, so we do not give them here. For equal characteristic zero case we use the notion which Hochster and Huneke label $*^{eq}$ (see (2.2.3), (3.2.1) in [7]). We mention that reduction to positive characteristic plays a major role in the definition of this notion of tight closure.

If R is any ring in which tight closure is defined and I is any ideal of R such that $I = I^*$, we say that I is *tightly closed*. If the ideal $(a_1, \ldots, a_k)$ has height at least k (could be the whole ring), we call $a_1, \ldots, a_k$ *parameters*. Rings in which every ideal generated by parameters is tightly closed are called *F-rational*. Hochster and Huneke proved that in a regular ring every ideal is tightly closed.

Now we prove a version of the Briançon–Skoda theorem for F-rational rings using reduction to characteristic p if the characteristic of the ring is zero. This reduction procedure also justifies the definition of tight closure for rings of characteristic zero.

Theorem 1. *Let R be a Noetherian algebra over a field K. Assume that R is F-rational. Let $I_1, \ldots, I_k$ be ideals of R and $(a_1, \ldots, a_k)$ a joint* *reduction of* $(I_1, \ldots, I_k)$. *Assume that* $\{a_1, \ldots, a_k\}$ *are parameters of* R *and that no* a_i *is contained in any minimal prime of* R. *Then*

$$\overline{I_1^{n_1} \cdots I_k^{n_k}} \subseteq (a_1^{n_1}, \ldots, a_k^{n_k}).$$

Proof. If R is of positive prime characteristic, this follows from (*). Thus we may assume that K has characteristic zero.

Suppose that the theorem is false for some k-tuple $\underline{n}$. So $n_1, \ldots, n_k > 0$. It is easy to see that $(a_1, \ldots, a_k)$ is a joint reduction of $(I_1, \ldots, I_k)$ if and only if $(a_1^{n_1}, \ldots, a_k^{n_k})$ is a joint reduction of $(I_1^{n_1}, \ldots, I_k^{n_k})$, so by replacing each I_i by $I_i^{n_i}$ and each a_i by $a_i^{n_i}$ we may assume that $\overline{I_1 \cdots I_k} \not\subseteq (a_1, \ldots, a_k)$.

Let $x \in \overline{I_1 \cdots I_k} \setminus (a_1, \ldots, a_k)$. We may also assume that x is not in any minimal prime of R, for if $\overline{I_1 \cdots I_k}$ is contained in the union of $(a_1, \ldots, a_k)$ and all the minimal primes of R, then by prime avoidance $\overline{I_1 \cdots I_k}$ must be contained in $(a_1, \ldots, a_k)$.

Fix a set of generators for $I_1, \ldots, I_k$. Write each a_i as a linear combination of the chosen generators for I_i and call these equations $G_1, \ldots, G_k$. For each k-fold product of the chosen generators which is contained in $I_1 \cdots I_k$ write an equation of integral dependence over the ideal $a_1 I_2 \cdots I_k + \cdots + a_k I_1 \cdots I_{k-1}$. Let $F_1, \ldots, F_s$ be all these equations and let L be the largest degree of these F_i. Let F be an equation showing integral dependence of x over $I_1 \cdots I_k$ and let L' be the degree of F. Set $c = a_1^{L-1} \cdots a_k^{L-1} x^{L'}$. So $c \in R^0$.

Now let S be a finitely generated K-algebra such that there exists and algebra homomorphism $h\colon S \to R$ such that $a_1, \ldots, a_k, c, x$, the chosen generators of the I_i, and the G_i, F_i, and F are all elements of $h(S)$. Let $\_'$ stand for a fixed preimage of $\_$ in S. By modifying S further we may assume that $(a'_1, \ldots, a'_k)$ is a joint reduction of $(I'_1, \ldots, I'_k)$, that x' is integral over $I'_1 \cdots I'_k$, and that $a'_1, \ldots, a'_k$ and x' are all in S^0. If we can show that $x' \in (a'_1, \ldots, a'_k)^*$, then $x \in (a_1, \ldots, a_k)^* = (a_1, \ldots, a_k)$ (as R is F-rational), which proves the theorem. Thus it is enough to prove:

LEMMA 2. *Let* R *be a finitely generated algebra over a field* K *of characteristic zero. Let* $I_1, \ldots, I_k$ *be ideals of* R *and* $(a_1, \ldots, a_k)$ *a joint reduction of* $(I_1, \ldots, I_k)$. *Assume that no* a_i *is contained in any minimal prime of* R. *Then*

$$\overline{I_1 \cdots I_k} \subseteq (a_1, \ldots, a_k)^*.$$

Proof. We use the set-up from the proof of Theorem 1.

Write $R = K[X_1, \ldots, X_n]/J$. Lift $a_1, \ldots, a_k, c, x$, the chosen generators of the I_i, and the G_i, F_i, and F to $K[X_1, \ldots, X_n]$. In this way we obtain polynomials in $X_1, \ldots, X_n$ with coefficients in K. Let $u_1, \ldots, u_t$ be all these coefficients. Let $D = \mathbb{Z}[u_1, \dots, u_t] \subseteq K$. Let R_D be the canonical image of $D[X_1, \dots, X_n]$ in R. By construction all of the relevant elements and ideals live in this subring R_D of R, $a_i \in I_i R_D$, $x \in \overline{I_1 \cdots I_k R_D}$, and $x \notin (a_1, \dots, a_k)R_D$ for otherwise $x \in (a_1, \dots, a_k)$. By generic freeness (cf. [14, Theorem 24.1]) we may add another element to D to obtain that $(a_1, \dots, a_k)R_D$ and $R_D/(a_1, \dots, a_k)R_D$ are free D-modules.

As in the proof of (*) in [18], $cx^q \in (a_1^q, \dots, a_k^q)R_D$ for all sufficiently large integers q. So a fortiori $cx^q \in (a_1^q, \dots, a_k^q)R_D/mR_D$ for any maximal ideal m of D. But D is finitely generated over $\mathbb{Z}$, hence $m \cap \mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ and so R_D/mR_D has positive prime characteristic p. So

$$cx^q \in (a_1, \dots, a_k)^{[q]} R_D/mR_D \qquad \text{for all } q = p^e \gg 0.$$

This is true for every $m \in \operatorname{Max}(D)$. But $(D, R_D, (a_1, \dots, a_k)R_D)$ are descent data for R, $(a_1, \dots, a_k)R$, x, and c, so $x \in (a_1, \dots, a_k)R^*$. ∎

A similar, but a more involved procedure is used in the proof of

THEOREM 3. *Let R be a regular ring containing a field (of arbitrary characteristic). If $I_1, \dots, I_k$ are ideals of R, then*

$$\overline{I_1^{l_1+n_1} \cdots I_k^{l_k+n_k}} \subseteq (a_{11}, \dots, a_{1l_1})^{n_1+1} + \cdots + (a_{k1}, \dots, a_{kl_k})^{n_k+1}$$

for any joint reduction $(a_{11}, \dots, a_{1l_1}, \dots, a_{k1}, \dots, a_{kl_k})$ of

$$\Big(\underbrace{I_1, \dots, I_1}_{l_1 \text{ times}}, \dots, \underbrace{I_k, \dots, I_k}_{l_k \text{ times}}\Big)$$

and for any integers $n_i \geq 0$, $i = 1, \dots, k$.

Proof. The case of positive characteristic follows from (*) and the fact that in regular rings every ideal is tightly closed. So we may assume that R contains the field of rational numbers.

Let $J = I_1^{l_1+n_1} \cdots I_k^{l_k+n_k}$ and $K = (a_{11}, \dots, a_{1l_1})^{n_1+1} + \cdots + (a_{k1}, \dots, a_{kl_k})^{n_k+1}$. If $\bar{J} \not\subseteq K$, then $\bar{J} \not\subseteq K$ after localizing at some prime of R, so we may assume that R is local. Let $\hat{R}$ be the completion of R. After extending all the relevant ideals and elements to $\hat{R}$, all the hypotheses are still satisfied. If the conclusion holds in $\hat{R}$, then

$$\bar{J} \subseteq \overline{J\hat{R}} \cap R \subseteq K\hat{R} \cap R = K,$$

which contradicts the assumption. So the theorem is also false in $\hat{R}$ and we may assume that R is a complete Noetherian regular local ring containing a field. By Cohen's Structure theorem R is of the form

$k[[X_1, \ldots, X_d]]$, a power series ring in d variables over a field k. Let $\bar{k}$ be the algebraic closure of k. We extend all the relevant ideals and elements to $\bar{k}[[X_1, \ldots, X_d]]$ and still have a counterexample to the theorem as the extension is faithfully flat. So we may assume that R is a power series ring in d variables $X_1, \ldots, X_d$ over an algebraically closed field k.

Let X_{ij}, $i = i = 1, \ldots, k$, $j = 1, \ldots, l_i$, be indeterminates over R, one for each one of the a_{ij}. Let $I_i = (b_{i1}, \ldots, b_{im_i})$, and let Y_{ij} be indeterminates, one for each one of the b_{ij}. For each monomial in the b_{ij} which is an element of $I_1 \cdots I_k$ write an equation of integral dependence over $(a_{11}, \ldots, a_{1l_1})I_2 \cdots I_k + \cdots + (a_{k1}, \ldots, a_{kl_k})I_1 \cdots I_{k-1}$. Replace the a_{ij} in these equations by the X_{ij}, the b_{ij} by the Y_{ij}, and the coefficients of these polynomials which are not equal to 1 by distinct indeterminates Y_i. We then obtain a polynomial in the X_{ij}, Y_{ij}, and Y_i with coefficients in $\mathbb{Z}$. Let u be an element of $\bar{J}$ which is not contained in K. In an equation of integral dependence of u on J replace u by an indeterminate U, the b_{ij} by the Y_{ij}, and the coefficients which are not equal to 1 by additional distinct indeterminates Y_i. We thus obtain a polynomial in the Y_{ij}, Y_i, and U. Also consider the polynomials $X_{ij} - \sum_{k=1}^{m_i} R_{ijk} Y_{ik}$ in the variables X_{ij}, Y_{ij}, and R_{ijk}. Let Σ be the collection of all these polynomials in the X_{ij}, Y_{ij}, Y_i, R_{ijk}, and U with coefficients in $\mathbb{Z}$. By construction $R = k[[X_1, \ldots, X_d]]$ contains a solution $(a_{ij}, b_{ij}, y_i, r_{ijk}, u)$ of $\{F = 0 \mid F \in \Sigma\}$.

By the Artin Approximation theorem we may find a zero set $(\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{y}_i, \tilde{r}_{ijk}, \tilde{u})$ of Σ in $H = k[X_1, \ldots, X_d]^h_{(X_1, \ldots, X_d)}$, the Henselization of $k[X_1, \ldots, X_d]_{(X_1, \ldots, X_d)}$ (see [3]). Moreover, for any integer t we may find a solution which also satisfies

$$a_{ij} - \tilde{a}_{ij} \in (X_1, \ldots, X_d)^t R,$$

$$b_{ij} - \tilde{b}_{ij} \in (X_1, \ldots, X_d)^t R,$$

$$u - \tilde{u} \in (X_1, \ldots, X_d)^t R.$$

We choose t such that $u \notin K + (X_1, \ldots, X_d)^t R$.

Let $\tilde{I}_i = (\tilde{b}_{i1}, \ldots, \tilde{b}_{im_i})H$ and let $\tilde{J}$ and $\tilde{K}$ be ideals in H generated by the corresponding elements and ideals with $\tilde{}$ on top. By the Artin Approximation theorem all of the hypotheses still hold for H and for these new ideals and elements. Also, $\tilde{u}$ is contained in the integral closure of $\bar{\tilde{J}}$. If the conclusion of the theorem holds in H, we get

$$u = (u - \tilde{u}) + \tilde{u} \in (X_1, \ldots, X_d)^t R + \tilde{K}R \subseteq (X_1, \ldots, X_d)^t R + KR,$$

which contradicts the choice of t. Thus we still have a counterexample in H.

Now let B be the integral closure of $k[X_1, \ldots, X_d, \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{y}_i, \tilde{r}_{ijk}, \tilde{u}]_{i,j,k}$. We will show that the theorem is still false for B. As affine rings are excellent, B is a finitely generated k-algebra (for the definition of excellence see [14, p. 260]). Write B as $k[Z_1, \ldots, Z_h]/(F_1, \ldots, F_l)$. Let $n = (X_1, \ldots, X_d)R \cap B$. Since B is wedged between $k[X_1, \ldots, X_d]$ and R, it follows that n is a maximal ideal of B. As k is algebraically closed, by Hilbert's Nullstellensatz n is of the form $(Z_1 - r_1, \ldots, Z_h - r_h)$, where $F_i(\underline{r}) = 0$ for all i. We may change variables so that $n = (Z_1, \ldots, Z_d)$ and $F_i(\underline{0}) = 0$ for all i. Each element $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{y}_i, \tilde{r}_{ijk}, \tilde{u}$ of H and each ideal $\tilde{I}_i$, $\tilde{J}$, and $\tilde{K}$ of H also live in B.

By the Artin Approximation theorem B_n is a localization of a flat integral extension of $k[X_1, \ldots, X_d]_{(X_1, \ldots, X_d)}$. By integrality $\dim(B_n) \leq d$ and by flatness $\dim(B_n)$ is at least the height of the maximal ideal of $k[X_1, \ldots, X_d]_{(X_1, \ldots, X_d)}$. So $\dim(B_n) = d$. As $k[X_1, \ldots, X_d]_{(X_1, \ldots, X_d)} \subseteq B_n \subseteq R = k[[X_1, \ldots, X_d]]$, the completion $\widehat{B_n}$ of B_n maps onto R. By considering the dimensions the kernel of this map is necessarily a minimal prime of $\widehat{B_n}$ of dimension d. But B_n is normal and excellent, so $\widehat{B_n}$ is an integrally closed domain. Thus $\widehat{B_n} = R$. It follows that B_n is a regular local ring.

If the theorem holds in B_n, then it also holds in H. To prove this we need to use Lemma 2.4 and Examples (iv) and (v) on page 800 of [11] which say that if S is an excellent Noetherian local ring, then $\bar{p}\hat{S} = \overline{p\hat{S}}$ for any ideal p of S. Hence

$$
\begin{aligned}
\overline{\tilde{J}H} &= \overline{\tilde{J}HR} \cap H \\
&= \overline{\tilde{J}R} \cap H \\
&= \overline{\tilde{J}B_n R} \cap H \\
&= \overline{\tilde{J}B_n} R \cap H \\
&\subseteq \tilde{K}B_n R \cap H \\
&= \tilde{K}H,
\end{aligned}
$$

contradicting the assumption on H. So we still have a counterexample in B_n and hence also in B.

Consider the ideal of $k[Z_1, \ldots, Z_h]$ which is generated by all the minors of rank $r = \operatorname{ht}(F_1, \ldots, F_l)$ of the h by l matrix $(\partial F_i/\partial Z_j)$ (i.e., consider the Jacobian ideal). As B_n is a regular local ring, by the Jacobian criterion (see [14, Theorem 30.4]) there exists an r-minor Δ such that $\Delta(\underline{0}) \neq 0$.

Now lift the elements $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ $\tilde{y}_i$, $\tilde{r}_{ijk}$, and $\tilde{u}$ to polynomials in $Z_1, \ldots, Z_h$ over k and collect the coefficients of these and of the F_i into a (finite) set Λ. Also lift all of the polynomials in Σ evaluated at $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ $\tilde{y}_i$, $\tilde{r}_{ijk}$, and $\tilde{u}$

to elements of the ideal $(F_1, \ldots, F_l)$. Add to Λ all the coefficients appearing in these equations. Also add $\Delta(\underline{0})^{-1}$ to Λ. Then Λ is a finite set. Later we may need to add one more element of k to Λ. Let $L = \mathbb{Q}(\Lambda)$ and $D = \mathbb{Z}[\Lambda]$. So D is a finitely generated $\mathbb{Z}$-algebra and $D \subseteq L$.

The preimages of the elements $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{y}_i$, $\tilde{r}_{ijk}$, $\tilde{u}$, F_i, and Δ in $k[Z_1, \ldots, Z_h]$ by construction actually live in $D[Z_1, \ldots, Z_h]$. Define $C_D = D[Z_1, \ldots, Z_h]/(F_1, \ldots, F_l)$ and $N_D = (Z_1, \ldots, Z_h)C_D$. Let the subscript $_{-D}$ denote the images of $\tilde{u}$, $\tilde{K}$, and $\tilde{J}$ in C_D. By construction $u_D \notin K_D + N_D^l$.

By the theorem on generic freeness (see [14, Theorem 24.1]) C_D and $C_D/K_D + N_D^l$ become free D-modules after inverting a single element of D. So we add the inverse of this element to Λ and assume that C_D and $C_D/K_D + N_D^l$ are free D-modules.

Let m be a maximal ideal of D. As $\mathbb{Z}$ is a Jacobson ring and D is finitely generated over $\mathbb{Z}$, $m \cap \mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$ (see [4, Exercise 5.25]). Also, D/m is a finitely generated extension field of $\mathbb{Z}/m \cap \mathbb{Z}$, so it is a finite field and hence perfect. Thus we may apply the Jacobian criterion [14, Theorem 30.4 to $C_{D/m} = (C_D/mC_D)_N$. As $\Delta(\underline{0})$ is a unit in $D[Z_1, \ldots, Z_h]$, it follows that $C_{D/m}$ is a regular ring. We mark all elements and ideals in $C_{D/m}$ by the subscript $_{-D/m}$. So $C_{D/m}$ has positive characteristic and so the theorem is true for $C_{D/m}$. So $\overline{J_{D/m}} \subseteq K_{D/m}$ and $u_{D/m} \in K_{D/m}$. By assumption $u_D \notin K_D + N_D^l$. So as $C_D/K_D + N_D^l$ is a free D-module, we write $C_D/K_D + N_D^l = \Sigma_\alpha D_\alpha$, where each D_α equals D. Then we may write u_D as $\Sigma_\alpha r_\alpha \in \Sigma_\alpha D_\alpha$, where r_α is zero for all but finitely many α and is nonzero for some α. Hence $u_{D/m}$ equals $\Sigma_\alpha r_{\alpha, D/m}$. But $u_{D/m} \in K_{D/m}$, so $r_\alpha D/m = 0$ for all α and all maximal ideals m of D. So $r_\alpha \in \bigcap_{m \in \operatorname{Max}(D)} m$. But this intersection is zero as D is a Jacobson domain. Hence $u_D \in K_D + N_D^l$, a contradiction. ∎

A corollary of this is Hochster and Huneke's version for regular local rings containing a field, saying that $\overline{I^{n+l(I)}} \subseteq I^n$ for all ideals I and for all integers n. Aberbach and Huneke improved this to:

THEOREM 4 (Aberbach and Huneke [2]). *Let (R, m) be a regular local ring containing a field. Let $I \subseteq R$ be any ideal having analytic spread l, and let J be any reduction of I. Set $h = \operatorname{bight}(I)$. Then for all $n \geq 0$ we have that*

$$\overline{I^{l+n}} \subseteq J^{n+1}\left(J^{l-h}\right)^{un},$$

where the superscript ^{un} denotes the intersection of the isolated primary components of an ideal.

The methods of the proof of Aberbach and Huneke's result combined with the methods from [18] give the following generalization for several ideals and joint reductions:

THEOREM 5. *Let (R, m) be a regular local ring containing a field, the I_i and a_{ij} as in* (*). *Then for all $\underline{n}$ we have that*

$$\overline{\underline{I}^{\underline{l}+\underline{n}}} \subseteq \sum_i (a_{i1}, \ldots, a_{il_i})^{n_i+1}\left((a_{i1}, \ldots, a_{il_i})^{l_i-n_i}\right)^{un}.$$

2. TWO ALMOST BRIANÇON–SKODA THEOREMS

Here we show how in some rings certain elements and ideals multiply integral closures into reductions and joint reductions. In general we cannot omit these multipliers, so we call these results "almost Briançon–Skoda." Corollary 10 is a "true" Briançon–Skoda theorem.

We will need the following theorem of Lipman and Sathaye:

THEOREM 6 [12, Theorem 2]. *Let R be a Noetherian regular domain and S a finitely generated R-algebra containing S which is also a domain. Suppose that the quotient field of S is a finite separable field extension of the quotient field of R. Then the 0th Fitting ideal $J_{S/R}$ of the S-module of Kähler differentials$\Omega_{S/R}$ multiplies the integral closure of S into S.*

We use this result, Rees and Sally's constructions from [15], and Huneke's constructions from [10]:

LEMMA 7. *Let (R, m) be a complete Noetherian local domain containing the field of rational numbers. Let $c \in R$ such that R_c is regular. Then there exists an integer h such that $c^h\overline{I^n} \subseteq I^{n-l(I)}$ for all ideals I in R and all $n \geq l(I)$, $l(I)$ being the analytic spread of I. Hence $c^h\overline{I^n} \subseteq I^{n-\dim(R)}$ for all ideals I in R and all $n \geq \dim(R)$.*

Proof. Let $d = \dim(R)$ and $n = \mu(m)$. Choose a minimal set of generators $x_1, \ldots, x_n$ of m in the following way. Let $x_1 \in m \setminus m^2$. After we have chosen $x_1, \ldots, x_{i-1}$, let $x_i \in m \setminus m^2 \cup W$, where W is the union of the primes minimal over $R/(x_{i_1}, \ldots, x_{i_k})$, where $k < d$ and the i_j vary between 1 and $i - 1$. If $i \leq n$, x_i exists by prime avoidance. In this way we obtain $x_1, \ldots, x_n$ such that $m = (x_1, \ldots, x_n)$ and such that $(x_{i_1}, \ldots, x_{i_d})$ is m-primary whenever $1 \leq i_1 < i_2 < \cdots < i_d \leq n$.

By Cohen's Structure theorem $R = k[[Y_1, \ldots, Y_n]]/P$ for some prime P in $k[[Y_1, \ldots, Y_n]]$, where k is the coefficient field of R and each Y_i maps to x_i. Note that the height of P equals $\dim(k[[Y_1, \ldots, Y_n]]) - \dim(R) = n -$

d. Let J be the ideal which is generated by ht(P)-minors of the Jacobian matrix $(\partial(f_1,\dots,f_p)/\partial(Y_1,\dots,Y_n))$, where $P=(f_1,\dots,f_p)$. As R_c is regular, by the Jacobian criterion for power series rings over a field of characteristic 0, c is contained in the radical of J (see [14, p. 240]). So there exists an integer h such that c^h lies in J. Thus it suffices to show that $J\overline{I^n}\subseteq I^{n-l(I)}$ for all ideals I in R and all $n\geq l(I)$.

Let Δ be an $(n-d)$-minor of the Jacobian matrix coming from the columns for $Y_{i_1},\dots,Y_{i_{n-d}}$. As J is generated by such minors, it is enough to prove that $\Delta\overline{I^n}\subseteq I^{n-l(I)}$ for all ideals I in R and all $n\geq l(I)$. Let $j_1,\dots,j_d$ be positive integers such that $\{i_1,\dots,i_{n-d}\}\cup\{j_1,\dots,j_d\}=\{1,\dots,n\}$. By Cohen's Structure theorem R is finite over the power series ring $A=k[[x_{j_1},\dots,x_{j_d}]]$ in d variables over k.

Let I be an ideal of R and $l=l(I)$. As the residue field of R is infinite there exists a reduction $(a_1,\dots,a_l)$ of I. Let $S=R[t,a_1t^{-1},\dots,a_lt^{-1}]$ where t is an indeterminate over R. Then S is a finitely generated $A[t]$-algebra, and the quotient field of S is a finite separable extension of the quotient field of $A[t]$. As $A[t]$ is a regular domain, by Lipman and Sathaye's theorem $K\overline{S}\subseteq S$, where K is the 0th Fitting ideal of the S-module of Kähler differentials $\Omega_{S/A[t]}$. Write S as $A[t][Y_{i_1},\dots,Y_{i_{n-d}},Z_1,\dots,Z_l]/Q$, where Z_i maps to a_it^{-1} and Y_{i_j} maps to x_{i_j}. Let each a_i be the image of some $A_i\in A[t][Y_{i_1},\dots,Y_{i_{n-d}}]$. Then $F_i=Z_it-A_i$ is an element of Q. So a representation of $\Omega_{S/A[t]}$ (obtained from the second fundamental sequence for modules of differentials) is of the form

$$\left[\begin{array}{ccc|ccc}
\frac{\partial f_1}{\partial Y_{i_1}} & \cdots & \frac{\partial f_1}{\partial Y_{i_{n-d}}} & & & \\
\vdots & & \vdots & & 0 & \\
\frac{\partial f_p}{\partial Y_{i_1}} & \cdots & \frac{\partial f_p}{\partial Y_{i_{n-d}}} & & & \\
\hline
\frac{\partial F_1}{\partial Y_{i_1}} & \cdots & \frac{\partial F_1}{\partial Y_{i_{n-d}}} & t & & 0 \\
\vdots & & \vdots & & \ddots & \\
\frac{\partial F_l}{\partial Y_{i_1}} & \cdots & \frac{\partial F_l}{\partial Y_{i_{n-d}}} & 0 & & t
\end{array}\right]$$

with possibly more rows. So one of the elements of K is the determinant of the displayed matrix, namely $\Delta t^l\in K$. By Lipman and Sathaye's theorem then Δt^l multiplies the integral closure of S into S. But the integral closure of S contains $R \oplus \overline{(a_1, \dots, a_l)}t \oplus \overline{(a_1, \dots, a_l)^2}t^2 \oplus \cdots$. Thus Δ multiplies $\overline{I^n} = \overline{(a_1, \dots, a_l)^n}$ into $(a_1, \dots, a_l)^{n-l}$ for all $n \geq l$ and any reduction $(a_1, \dots, a_l)$ of I. So $\Delta\overline{I^n} \subseteq I^{n-l}$ for all $n \geq l$. But I was arbitrary and l is bounded by d for all ideals I. ∎

Note that the proof shows that h can be chosen so that it works for all such c. Namely any such c lies in $\sqrt{J}$. Let h be such that $\sqrt{J}^h \subseteq J$. Then this h clearly works.

THEOREM 8. *Let R be an excellent reduced Noetherian local ring containing the field of rational numbers. Let $c \in R$ such that R_c is regular. Then there exists an integer h such that $c^h\overline{I^n} \subseteq I^{n-l(I)}$ for all ideals I in R and all $n \geq l(I)$. Hence $c^h\overline{I^n} \subseteq I^{n-\dim(R)}$ for all ideals I in R and all $n \geq \dim(R)$.*
Moreover, h can be chosen to be independent of c.

Proof. As R is excellent, the completion $\hat{R}$ of R is reduced and $\hat{R}_c$ is regular. Let $\operatorname{Min}(\hat{R}) = \{P_1, \dots, P_s\}$. Then $(\hat{R}/P_i)_c$ is regular (or 0) for each i. Let I be an ideal of R. As $l(I\hat{R} + P_i/P_i) \leq l(I\hat{R}) = l(I)$ for each i, by Lemma 7 there exists an integer h' such that

$$c^{h'}\overline{I^n\hat{R}} \subseteq \bigcap_{i=1}^{s} \left(I^{n-l(I)}\hat{R} + P_i\right).$$

By the remark following Lemma 7, h' may be chosen to be independent of c.

As $\hat{R}$ embeds into the finitely generated $\hat{R}$-module $\hat{R}/P_1 \times \cdots \times \hat{R}/P_s$, and equality holds after localization at $\{1, c, c^2, \dots\}$, there exists an integer h'' such that $c^{h''}$ multiplies $\hat{R}/P_1 \times \cdots \times \hat{R}/P_s$ into $\hat{R}$. This h'' may also be chosen to be independent of c. Let J be $\hat{R}\colon(\hat{R}/P_1 \times \cdots \times \hat{R}/P_s)$ (an ideal in $\hat{R}$). Then h'' such that $J^{h''} \subseteq J$ works.

Now let $h = h' + h''$. Then

$$\begin{aligned} c^h\overline{I^n\hat{R}} &\subseteq c^{h''} \bigcap_{i=1}^{s} \left(I^{n-l(I)}\hat{R} + P_i\right) \\ &= c^{h''}\left(I^{n-l(I)}\left(\hat{R}/P_1 \times \cdots \times \hat{R}/P_s\right) \cap \hat{R}\right) \\ &\subseteq I^{n-l(I)}\hat{R}. \end{aligned}$$

Hence $c^h\overline{I^{n}}^{1} \subseteq c^h\overline{I^n\hat{R}} \cap R \subseteq I^{n-l(I)}\hat{R} \cap R = I^{n-l(I)}$. ∎

THEOREM 9. *Let (R, m) be a complete Noetherian local domain. Let $d = \dim(R)$. Assume that R contains $\mathbb{Q}$, has isolated singularity, and is Cohen–Macaulay. Then there exists an m-primary ideal J such that for any d m-primary ideals $I_1, \ldots, I_d$,*

$$\overline{JI_1 \cdots I_d} \subseteq \text{any joint reduction of } (I_1, \ldots, I_d).$$

Proof. Let $I_i = (a_{i1}, \ldots, a_{il_i})$, each a_{ij} regular. Let $R_g = R[Y_1, \ldots, Y_N]_{mR[Y_1, \ldots, Y_N]}$ for N sufficiently large (the subscript $_g$ is Rees' and stands for "general" extension). Let $x_i = a_{i1}Y_{\cdot} + \cdots + a_{il_i}Y_{\cdot}$, where the $Y_{\cdot}$ are all distinct. As R (and hence R_g) is Cohen–Macaulay, by using prime avoidance we may assume in addition that each a_{ij} is a regular element in $R_g/(x_1, \ldots, x_{i-1})$.

Now let φ: $\{1, \ldots, d-1\} \to \mathbb{N}$ be a function such that $1 \leq \varphi(i) \leq l_i$. For $i \leq d-1$ let

$$y_i = \frac{x_i - a_{i\varphi(i)}Y_{\cdot}}{a_{i\varphi(i)}} \in \text{quotient field of } R_g,$$

where $Y_{\cdot}$ is such that $a_{i\varphi(i)}Y_{\cdot}$ is a summand of x_i. Let $S = R_g[y_1, \ldots, y_{d-1}]$. By the choice of the a_{ij}, the x_i form a regular sequence, so

$$S \cong \frac{R_g[X_1, \ldots, X_{d-1}]}{(a_{1\varphi(1)}X_1 - a_{1\varphi(1)}y_1, \ldots, a_{d-1\varphi(d-1)}X_{d-1} - a_{d-1\varphi(d-1)}y_{d-1})},$$

where X_i is identified with y_i for each i. Thus S is isomorphic to $R_g/(x_1, \ldots, x_{d-1})$, where this new R_g is formed by adjoining $N + d - 1$ indeterminates to R. Hence S is independent of φ.

By Cohen's Structure theorem there exists a regular local ring $A \subseteq R$ such that R is finite over A and $A = k[[Z_1, \ldots, Z_d]]$, a power series ring in d variables over the coefficient field k of R. Write $R = A[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$. As $\mathrm{ht}(f_1, \ldots, f_m) = \dim(A[\underline{T}]) - \dim(R) = n$, by Krull's Principal Ideal theorem we get that $m \geq n$. We also have a finite extension

$$A_g = A[Y_1, \ldots, Y_N]_{(\underline{Z})A[Y_1, \ldots, Y_N]} \subseteq R_g,$$

A_g is a regular local ring and $R_g = A_g[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$. Let $\tilde{a}_{i\varphi(i)}$ be a lifting of $a_{i\varphi(i)}$ and $\tilde{x}_i$ a lifting of x_i to $A_g[T_1, \ldots, T_n]$. The Jacobian matrix for S over A_g has the form

$$\left[\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial T_1} & \cdots & \frac{\partial f_1}{\partial T_n} & & & \\ \vdots & & \vdots & & 0 & \\ \frac{\partial f_m}{\partial T_1} & \cdots & \frac{\partial f_m}{\partial T_n} & & & \\ \hline & & & \bar{a}_{1\varphi(1)} & & 0 \\ & \frac{\partial\left(\bar{a}_{i\varphi(i)}X_i - \bar{x}_i\right)}{\partial T_j} & & & \ddots & \\ & & & 0 & & \bar{a}_{d-1\varphi(d-1)} \end{array}\right].$$

Hence as $m \geq n$, $J_{S/A_g} = J_{R_g/A_g} J_{S/R_g}$. By the Lipman–Sathaye theorem, $J_{S/A_g}\bar{S} \subseteq S$. As $J_{S/R_g} = (a_{1\varphi(1)} \cdots a_{d-1\varphi(d-1)})$ and $J_{R_g/A_g} = J_{R/A}R_g$, it follows that

$$a_{1\varphi(1)} \cdots a_{d-1\varphi(d-1)} J_{R/A} \subseteq S : \bar{S}.$$

As S is independent of φ, we let φ vary to obtain $I_1 \cdots I_{d-1} J_{R/A} \subseteq S : \bar{S}$.

The regular local ring A above is any power series ring contained in R such that R is finite over A. Now we will choose several such A so that the resulting $J_{R/A}$'s will generate an m-primary ideal. As in the proof of the previous theorem choose $x_1, \ldots, x_{n+d}$ such that $m = (x_1, \ldots, x_{n+d})$ and such that $(x_{i_1}, \ldots, x_{i_d})$ is m-primary whenever $1 \leq i_1 < \cdots < i_d \leq n+d$. Write $R = k[[T_1, \ldots, T_{n+d}]]/(f_1, \ldots, f_m)$, where the image of T_i in R is x_i. This is possible by Cohen's Structure theorem. Note that $\mathrm{ht}(f_1, \ldots, f_m) = n$. By the Jacobian criterion for power series rings over a field of characteristic 0 (see [14, p. 240]) and the fact that R has isolated singularity, the n by n minors of the $(n+d)$ by m matrix $(\partial f_i/\partial T_j)$ generate an m-primary ideal J. Now let I be the d-tuple $(i_1, \ldots, i_d)$, where $1 \leq i_1 < \cdots < i_d \leq n+d$. Then by Cohen's Structure theorem $A_I = k[[x_{i_1}, \ldots, x_{i_d}]]$ is a power series ring contained in R such that R is finite over A_I. As J_{R/A_I} is the ideal generated by the n by n minors of the n by m matrix $(\partial f_i/\partial T_j)_{i \neq i_1, \ldots, i_d}$, we see that $J = \sum J_{R/A_I}$ as I varies over admissible d-tuples.

Therefore $I_1 \cdots I_{d-1} J \subseteq S : \bar{S}$ and J is an m-primary ideal. It follows by Theorems 2.6 and 3.5 of [15] that $\overline{I_1 \cdots I_{d-1} I_d J}$ is contained in every joint reduction of $(I_1, \ldots, I_d)$ extended to R_g. But R_g is faithfully flat over R, so $\overline{I_1 \cdots I_{d-1} I_d J}$ is contained in every joint reduction of $(I_1, \ldots, I_d)$. ∎

We assumed in this theorem that the ring R be Cohen–Macaulay. This was first used in showing that $S = R_g[y_1, \ldots, y_{d-1}]$ is isomorphic to $R_g[X_1, \ldots, X_{d-1}]$ modulo a $(d-1)$-generated ideal. Because of this we may use Theorem 3.5 of [15] to obtain that

$$\overline{I_d[(S:\bar{S}) \cap R]} \subseteq (x_1, \ldots, x_{d-1}, a_{d1}Z_1 + \cdots + a_{dd}Z_d) \\ S[Z_1, \ldots, Z_d]_{mS[\underline{Z}]},$$

where $Z_1, \ldots, Z_d$ are indeterminates over R_g. By Theorem 1.6 of [15] the intersection of the latter module with R is contained in "almost every" joint reduction of $(I_1, \ldots, I_d)$ (we do not define "almost every" here as we do not use it). Rees and Sally gave an example in [15, Sect. 2] showing that this intersection need not be contained in every joint reduction. However, if R is Cohen–Macaulay, the intersection is contained in every joint reduction (see Theorem 2.6 of [15]). So the Cohen–Macaulay assumption appears to be necessary in the previous theorem, at least for this line of proof.

COROLLARY 10. *Let (R, m) be as in Theorem 9. Then there exist integers $k_1, \ldots, k_d$ such that for all $n_1 \geq k_1, \ldots, n_d \geq k_d$, for all m-primary ideals $I_1, \ldots, I_d$, and for every joint reduction $(a_1, \ldots, a_d)$ of $(I_1, \ldots, I_d)$,*

$$\overline{I_1^{n_1} \cdots I_d^{n_d}} \subseteq \left(a_1^{n_1 - k_1}, \ldots, a_d^{n_d - k_d}\right).$$

Proof. Let J be as in Theorem 9. Let k be any non-negative integer such that $m^k \subseteq J$. As in the proof of Theorem 1 we see that $(a_1^{n_1}, \ldots, a_d^{n_d})$ is a joint reduction of $(q_1^{n_1}, \ldots, q_d^{n_d})$. So we may apply the theorem for any non-negative integers $k_1, \ldots, k_d$ such that $k_1 + \cdots + k_d \geq k$. ∎

A similar argument as in Theorem 8 extending Lemma 7 extends Theorem 9 to the following:

COROLLARY 11. *Let (R, m) be a d-dimensional excellent, local, Cohen–Macaulay, reduced ring containing the rationals and with isolated singularity. Then there exists an integer h such that for each $c \in m$ and for any m-primary ideals $I_1, \ldots, I_d$,*

$$c^h \overline{I_1 \cdots I_d} \subseteq \text{any joint reduction of } (I_1, \ldots, I_d).$$

Thus there exists an m-primary ideal J such that $J\overline{I_1 \cdots I_d}$ is contained in the ideal generated by any joint reduction of $(I_1, \ldots, I_d)$.

3. Briançon–Skoda Theorem and Plus Closure

Let (R, m) be a Noetherian local domain of positive characteristic. Hochster and Huneke denoted by R^+ (read "R plus") the integral closure of R in an algebraic closure of the quotient field of R. In [6, Lemma 6.25], they proved that for every ideal I in R, $IR^+ \cap R \subseteq I^*$. This ideal $IR^+ \cap R$ is called *the plus closure of* I and is denoted I^+. For R^+, the context will determine whether R is thought of as an ideal or as a ring. It is an open question whether $I^+ = I^*$ for all ideals I. Hochster and Huneke proved in [5] that if I is generated by at most 3 parameters in a domain with test elements, then $I^+ = I^*$. Smith proved in [17] that $I^+ = I^*$ if I is generated by part of a system of parameters of any length in a locally excellent domain. This was extended further by Aberbach in [1].

In general, however, all that is known is that $I^+ \subseteq I^*$. So a Briançon–Skoda-type theorem, such as (*), with tight closure replaced by plus closure, is an improvement. See Section 7 of [5] for some examples of such Briançon–Skoda theorems. Below we give a version for joint reductions.

Theorem 12. *Let (R, m) be a complete Noetherian local domain of dimension 2. Assume that R has positive characteristic and an infinite residue field. Let I and J be ideals of R and (a, b) a joint reduction of (I, J) such that one of the following conditions is satisfied*:

(i) *(a, b) is m-primary*,

(ii) *the analytic spread of either I or of J is 0 or 1*,

(iii) *a is part of a minimal reduction of I and b is part of a minimal reduction of J.*

Then $\overline{I^k J^l} \subseteq (a^k, b^l)^+$ for all integers k, l.

Proof. Note that it is enough to prove the theorem for $k = l = 1$.

(i) By (*), $\overline{IJ} \subseteq (a, b)^*$. Thus by Smith's result quoted earlier, as (a, b) is m-primary, $\overline{IJ} \subseteq (a, b)^+$.

(ii) Let S be the integral closure of R in the quotient field of R. As R is complete, S is a finite R-module and hence a complete Noetherian local domain of dimension two. It is still true that (a, b) is a joint reduction of (IS, JS). If $\overline{IJS}$ is contained in $(a, b)S^+$, then $\overline{IJ} \subseteq \overline{IJS} \cap R \subseteq (a, b)S^+ \cap R = (a, b)R^+ \cap R = (a, b)^+$ as $R^+ = S^+$.

But does assumption (ii) remain true in S If $l(I)$ is zero, there is nothing to show. So we assume that $l(I) = 1$. Then there exists a nonzero element $f \in I$ such that $\overline{(f)} = \bar{I}$. Then by considering the valuations of the quotient field of R which contain R we obtain that $fS = IS = \overline{IS}$. Hence $l(IS) = 1$. So (ii) remains true after passing to S.

Thus we may rename S as R and assume that R is integrally closed so that all principal ideals are integrally closed. It follows that $I = (f)$, so $a = rf$ for some $r \in R$. If r is a unit in R, then $\overline{IJ} = \overline{fJ} \subseteq \overline{(f)} = (f) = (a)$, which is contained in $(a, b)^+$ and so we are done. Now suppose that $r \in m$. Then

$$\begin{aligned}(f)\bar{J} &= \overline{fJ}\\ &= \overline{IJ}\\ &= \overline{aJ + bI}\\ &= \overline{rfJ + (bf)}\\ &= (f)\left(\overline{rJ + (b)}\right)\\ &\subseteq (f)\left(\overline{mJ + (b)}\right)\\ &\subseteq (f)\bar{J}.\end{aligned}$$

So $\bar{J} = \overline{mJ + (b)}$ and J is integral over $mJ + (b)$, which by Nakayama's lemma means that J is integral over (b). Thus $(b) \subseteq \bar{J} = \overline{(b)} = (b)$ and $\overline{IJ} = \overline{Ib} = b\bar{I} \subseteq (b) \subseteq (a, b)^+$.

(iii) By assumption there exist $a_2 \in I$ and $b_2 \in J$ such that $\overline{(a, a_2)} = \bar{I}$ and $\overline{(b, b_2)} = \bar{J}$. Then (a, b) is a joint reduction of $((a, a_2), (b, b_2))$. As $\overline{IJ} = \overline{(a, a_2)(b, b_2)}$, we may assume that $I = (a, a_2)$ and $J = (b, b_2)$.

Let $x \in \overline{IJ} = \overline{aJ + bI}$. Write $x^l + \alpha_1 x^{l-1} + \cdots + \alpha_l = 0$ with $\alpha_i \in (aJ + bI)^i$. Let

$$F = X^l + p_1 X^{l-1} + \cdots + p_l \in \mathbb{Z}_p[X, A, A_2, B, B_2, Y_1, \ldots, Y_N]$$

for N sufficiently large, where $X, A, A_2, B, B_2, Y_1, \ldots, Y_N$ are indeterminates over $\mathbb{Z}_p$ and where each p_i is the sum of all monomials in $(A(B, B_2) + B(A, A_2))^i$ with each monomial having a different Y_j as a coefficient. In other words, F is the generic polynomial lifted from the equation of integrality of x over $aJ + bI$, where x is lifted to X, a is lifted to A, b to B, a_2 to A_2, b_2 to B_2, and the various coefficients to the variables Y_i.

Similarly we lift an equation of integral dependence of $a_2 b_2$ over $aJ + bI$ to a generic polynomial in $\mathbb{Z}_p[X, A, A_2, B, B_2, Y_1, \ldots, Y_N]$, possibly adding more indeterminates Y_i. We call this polynomial G. Now we let

$$S = \frac{\mathbb{Z}_p[X, A, A_2, B, B_2, Y_1, \ldots, Y_N]}{(F, G)}.$$

By construction (A, B) is a joint reduction of $((A, A_2)S, (B, B_2)S)$ and X is integral over $(A, A_2)(B, B_2)S$.

It is easy to see that $\{A, B, F, G\}$ is a regular sequence in $\mathbb{Z}_p[X, A, A_2, B, B_2, Y_1, \dots, Y_N]$, a polynomial ring over the field $\mathbb{Z}_p$, so $\{F, G, A, B\}$ is also a regular sequence. Hence S is Cohen–Macaulay and the height of $(A, B)S$ is two.

Let $S \to R$ be the obvious ring homomorphism and let K be its kernel. So K is a prime ideal in S. Let P be a minimal prime ideal in S which is contained in K and let $T = S/P$. As $\mathrm{ht}((A, B)T) = 2$, by (*) and by Smith's result from [17], the image of X in T lies in $(A, B)(T_Q)^+ \cap T$ for every prime ideal Q in T. By [9, Lemma 6.5] then the image of X lies in $\bigcap_Q (A, B)(T^+)_{T \setminus Q} \cap T$. So for each prime ideal Q in T there exists an element $s \in T \setminus Q$ such that $sX \in ((A, B)T)^+$. Thus X lies in $((A, B)T)^+$. By Lemma 6.5 in [9] then the image of X in S/K lies in

$$(A, B)(S/K)^+ \cap S/K \subseteq (a, b)R^+ \cap R = (a, b)^+. \quad \blacksquare$$

It may be possible to prove Theorem 12 for any ideals I and J by a similar constuction as in (iii). Namely, if $I = (a_1, \dots, a_n)$ with $a_1 = a$ and $J = (b_1, \dots, b_m)$ with $b_1 = b$, we can construct a polynomial F in $\mathbb{Z}_p[X, A_1, \dots, A_n, B_1, \dots, B_m, Y_1, \dots, Y_N]$ which lifts an equation of integral dependence of x on $aJ + bI$, and for each pair (i, j) with $i, j > 1$, we can construct a generic polynomial F_{ij} in the same polynomial ring with possibly more indeterminates Y_i which lifts an equation of integral dependence of $a_i b_j$ over $aJ + bI$ (similar to the construction of G earlier). Then we set

$$S = \frac{\mathbb{Z}_p[X, A_1, \dots, A_n, B_1, \dots, B_m, Y_1, \dots, Y_N]}{(F) + \left(\{F_{ij} | i, j > 1\}\right)},$$

and let P be a minimal prime ideal of S contained in the kernel K of the obvious ring homomorphism $S \to R$. With this set-up, whenever $(A_1, B_1)S/P$ has height 2, the proof of (iii) goes through.

The computer programs "Macaulay" and "Maple" cannot handle even the case $n = 2$, $m = 3$, and all degrees of integral equations equal to 2, as the number of variables is quite large. But the generic nature of these equations makes such a generalization at least plausible. Another plausible generalization is the corresponding theorem for complete domains of arbitrary dimension. This approach may be helped by an understanding of Gröbner bases of ideals generated by generic polynomials.

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