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Brief Paper

State Space Based Dual-rate Self-tuning*

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Key Words—Adaptive control; self-tuning control; computer controlled system; stochastic system; estimation algorithm; state feedback.

Abstract—A state space based dual-rate self-tuning algorithm is proposed for the case where the output sampling rate is slower than the input update rate. A state space innovation model suitable for dual-rate self-tuning control is derived. The parameters of the innovation model together with the states are estimated by a recursive prediction error estimator. The control strategy has been based on a pole restriction principle. Simulation studies are presented to establish the practical usefulness of the algorithm.

1. *Introduction*

ADAPTIVE REGULATORS OFFER a practical solution to control problems where drift in the plant parameters or controller parameters may cause a sensitivity problem. Among the adaptive approaches the self-tuning regulators (STR) are becoming increasingly popular and have been tested in a number of process control applications (Sastry *et al.*, 1977; Kallstrom *et al.*, 1978; Mills *et al.*, 1991). In all of these applications, the sampling intervals in the input update and output measurements were equal. However, there are situations where the direct application of uniform rate STR poses implementation problems. Such situations may include the digital control of plants where the sampling period is required to be sufficiently small due to the fast plant dynamics, but the computational requirements per iteration demand a larger sampling interval. In other situations, such as chemical processing applications, the availability of output measurements may have a larger sampling interval compared to the control update (Lu *et al.*, 1990).

In these situations the dual-rate self-tuning presents a suitable solution to the control implementation problem. In addition to remedying the above, dual-rate sampling can also be used advantageously to prevent plants from becoming non-minimum in phase during the discretization (Kannaiah *et al.*, 1984). In a dual-rate control scheme the input update and output measurements are done at two different rates. Since STR's are based on certainty equivalence coupling of parameter estimation and control design, a dual-rate self-tuner usually performs data collection, parameter estimation, control design and control implementation at different rates.

A number of dual-rate self-tuning algorithms have been proposed in the literature (Araki and Hagiwara, 1986; Hang *et al.*, 1989, 1993; Lu *et al.*, 1990; Kannaiah *et al.*, 1984). Most of them are based on an input–output model. In this paper the application of a state space model for dual-rate self-tuning is considered, where the output sampling rate is slower than the input update rate. The output sampling interval is assumed to be an integral multiple of the input update rate. Such a situation may arise where the availability of output measurements has a large cycle time corresponding to the input. Even if the output measurement has an identical cycle time to the input, a slower model update corresponding to a larger sampling interval may be desirable in many cases. A reduced-order approximate model becomes more robust with a large sampling interval in the presence of unmodeled dynamics (Astrom and Wittenmark, 1989).

In this paper, a state space innovation model suitable for self-tuning control is derived for the proposed problem. The parameters of the innovation model together with the states are estimated by a recursive prediction error (RPE) estimator. The control strategy has been based on a pole restriction principle. A simulation study is carried on in order to establish the feasibility of the proposed algorithm.

The organization of the paper is as follows. Section 2 presents the development of the innovation model. Section 3 presents the joint parameter and state estimation algorithm of the innovation model using a recursive prediction error method. Section 4 deals with the control strategy and Section 5 reports on the simulation study.

2. *State space innovation model for the dual-rate plant*

Consider the stochastic dual-rate plant given by the following state space model:

$$\bar{\mathbf{x}}(k+1) = A\bar{\mathbf{x}}(k) + Bu(k) + \Gamma w(k); \quad k = 0, 1, 2, \ldots; \quad (1)$$

$$y(k) = C\bar{\mathbf{x}}(k) + v(k); \quad k = 0, J, 2J, \ldots; \quad (2)$$

where $\bar{\mathbf{x}}(k) \in \mathcal{R}^n$, $u(k) \in \mathcal{R}^r$, $y(k) \in \mathcal{R}^m$, $w(k) \in \mathcal{R}^q$ and $v(k) \in \mathcal{R}^m$ are the state, control input, measured output, internal disturbance/modeling error and measurement noise, respectively. The output sampling interval is assumed to be an integral multiple J of the input sampling interval, i.e. $y(k)$, $k = 0, J, 2J, \ldots$ are only available as the output. The system matrices A, B, C, and Γ are of proper dimensions. $w(k)$ and $v(k)$ are assumed to have zero mean, and individually and mutually uncorrelated (white) random sequences with variances Σ_w and Σ_v, respectively.

Now consider the optimal mean square state estimation of the plant. Since $w(k)$ and $v(k)$ are white and have zero mean, the following state estimation procedure can be adopted

$$\bar{\mathbf{x}}(k+1/k) = A\bar{\mathbf{x}}(k/k) + Bu(k), \quad (3)$$

$$\bar{\mathbf{x}}(k/k) = \bar{\mathbf{x}}(k/k-1) \quad \text{for } k \neq 0, J, 2J, \ldots, \quad (4)$$

$$\bar{\mathbf{x}}(k/k) = \bar{\mathbf{x}}(k/k-1) + K\varepsilon(k) \quad \text{for } k = 0, J, 2J, \ldots, \quad (5)$$

$$\varepsilon(k) = y(k) - C\bar{\mathbf{x}}(k/k-1) \quad \text{for } k = 0, J, 2J, \ldots, \quad (6)$$

where $\bar{\mathbf{x}}(i/j)$ denotes the estimate of $\bar{\mathbf{x}}(i)$ from the observations up to the jth instant and K is an $n \times m$ matrix to be chosen optimally in order to minimize the mean square error of estimation of the states.

Defining $\mathbf{x}(k) \triangleq \bar{\mathbf{x}}(k/k-1)$ and $L \triangleq AK$, equations (3) and (4), and (3) and (5), respectively, may be combined to give

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + Bu(k) \quad \text{for } k \neq 0, J, 2J, \ldots, \quad (7)$$

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + Bu(k) + L\varepsilon(k) \quad \text{for } k = 0, J, 2J, \ldots. \quad (8)$$

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Now equation (6) may be written as

$$y(k) = C\mathbf{x}(k) + \varepsilon(k) \quad \text{for } k = 0, J, 2J, \ldots. \tag{9}$$

Equations (7)–(9) give an alternative description of the dual-rate state variable model originally introduced by equations (1) and (2). We now show that if the gain matrix K is optimally chosen to minimize the mean square error then the sequence $\varepsilon(k)$ becomes an innovation sequence. Combining equations (2) and (6) one may obtain

$$\varepsilon(k) = C\tilde{\mathbf{x}}(k/k-1) + v(k) \quad \text{for } k = 0, J, 2J, \ldots, \tag{10}$$

where $\tilde{\mathbf{x}}(k/k-1) \triangleq \bar{\mathbf{x}}(k) - \bar{\mathbf{x}}(k/k-1)$ is the state estimation error. For a minimum mean square estimation, the estimation error must be orthogonal to the observations. Thus $\tilde{\mathbf{x}}(k/k-1)$ must be uncorrelated to $y(s)$, $s < k$. Further $v(k)$ being individually white, is uncorrelated to $y(s)$, $s < k$. Therefore, from equation (10) it may be concluded that $\varepsilon(k)$ is also uncorrelated to $y(s)$, $s < k$. In other words $\varepsilon(k)$ is an innovation sequence and the alternative plant description given by equations (7)–(9) is an innovation representation.

It can be shown by direct substitution that the equivalent description of the plant given by equations (1) and (2) corresponding to the output sampling interval is given by

$$\mathbf{x}(k+J) = \underline{A}\mathbf{x}(k) + \sum_{i=1}^{J} \underline{B}_i u(k+i-1) + \underline{w}(k) \quad \text{for } k = 0, J, 2J, \ldots, \tag{11}$$

$$y(k) = C\mathbf{x}(k) + v(k) \quad \text{for } k = 0, J, 2J, \ldots, \tag{12}$$

where $\underline{A} = A^J$, $\underline{B}_i = A^{J-i}B$, and $\underline{w}$ is a zero mean uncorrelated sequence having variance

$$\sum_{i=0}^{J-1} \Omega_i \Sigma_w \Omega_i^{\mathrm{T}} \quad \text{with} \quad \Omega_i = A^i \Gamma.$$

The corresponding innovation model is also obtained by successive substitution of equations (7)–(9)

$$\mathbf{x}(k+J) = \underline{A}\mathbf{x}(k) + \sum_{i=1}^{J} \underline{B}_i u(k+i-1) + \underline{L}\varepsilon(k) \quad \text{for } k = 0, J, 2J, \ldots, \tag{13}$$

$$y(k) = C\mathbf{x}(k) + \varepsilon(k) \quad \text{for } k = 0, J, 2J, \ldots, \tag{14}$$

where $\underline{L} = A^{J-1}L$.

This description is another possible candidate for state and parameter estimation of the plant. However, a direct estimation of the plant parameters and states from the model described by equations (7)–(9) is more suitable from an adaptive control point of view. The control update requires the parameters and states of the plant model corresponding to the input sampling rate. The estimation of these quantities from that of the model described by equations (13) and (14) is computationally tedious.

The implicit assumption in the above approach is that there exists a one-to-one correspondence between the plant and noise matrices used in equations (7)–(9), and those in equations (13) and (14). The necessary and sufficient conditions for the validity of this assumption is stated through the following proposition.

Proposition. The matrices A, B, and L given in equations (7)–(9) are uniquely related to the matrices $\underline{A}$, $\underline{B}_i$ $(i = 1, \ldots, J)$ and $\underline{L}$ given by equations (11) and (13), if and only if (a) the pair $\{A, B\}$ is completely controllable and (b) A^{J-1} is nonsingular.

Proof. The proposition is proved in two parts. First, the uniqueness of $\underline{A}$, $\underline{B}_i$ $(i = 1, \ldots, J)$ and $\underline{L}$ for given A, B, and L can be clearly observed from the relationships following equations (12) and (14). Secondly, the uniqueness of A, B, and L for given $\underline{A}$, $\underline{B}_i$ $(i = 1, \ldots, J)$ and $\underline{L}$ is proved through contradiction as follows.

We rewrite the relationships following equations (12) and (14) as follows:

$$A^J = \underline{A}, \tag{15a}$$

$$A^{J-i}B = \underline{B}_i \quad \text{for } i = 1, \ldots, J, \tag{15b}$$

and

$$L = [A^{J-1}]^{-1}\underline{L}. \tag{15c}$$

The uniqueness of B for a given $\underline{B}_J$ can be observed by setting $i = J$ in equation (15b). To show the uniqueness of A, we assume that in addition to A there exists another matrix A_* that satisfies equations (15a) and (15b). Then it follows from equations (15a,b) that

$$A_*^i B = \underline{B}_{J-i} = A^i B \quad \text{for } i = 0, \ldots, J-1, \tag{16a}$$

$$A_*^i B = A_*^{NJ} A_*^{i-NJ} B = A^{NJ} A^{i-NJ} B = A^i B \quad \text{for } i = J, J+1, \ldots, \tag{16b}$$

where $N = \text{int}\,[i/J]$.

Equation (16a) is obtained from equation (15b), whereas equation (16b) is obtained by combining equations (15a) and (15b). Now from equations (16a,b) with $i = 1, 2, \ldots$, one gets

$$[A_*B \mid A_*^2B \mid A_*^3B \mid \cdots] = [AB \mid A^2B \mid A^3B \mid \cdots], \tag{17a}$$

or

$$A_*[B \mid A_*B \mid A_*^2B \mid \cdots] = A[B \mid AB \mid A^2B \mid \cdots], \tag{17b}$$

which applying equations (16a,b) further becomes

$$A_*[B \mid AB \mid A^2B \mid \cdots] = A[B \mid AB \mid A^2B \mid \cdots]. \tag{17c}$$

The controllability of the pair $\{A, B\}$ is a necessary and sufficient condition to impose that $A_* = A$. It can also be observed from (15c) that L is uniquely related to $\underline{L}$ if and only if, in addition, A^{J-1} is nonsingular.

Since, in general, the power of a matrix gradually makes it ill-conditioned, it is expected that as J increases the estimate of the noise matrix L may become erroneous. This sets an upper limit on J in order to get good estimate of L. This upper limit of J, however, will depend on the specific plant under consideration. This problem is observed in our simulation too, and is reported in Section 5.

3. *Parameter and state estimation*

In this section a recursive prediction error (RPE) algorithm for joint parameter identification and state estimation of the innovation model is derived. In this study, an observable canonical state space form is considered (El-Sherief and Sinha, 1979) which puts the following restriction on the structures of A and C:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

where

$$A_{ii} = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{n_i-1} \\ 0 & \\ \hline a_{ii}(1) \cdots a_{ii}(n_i) \end{array}\right], \qquad A_{ij} = \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \hline a_{ij}(1) & \cdots & a_{ij}(n_j) \end{array}\right]$$

for $i \neq j$, and

$$C = \begin{bmatrix} \mathbf{e}^1 \\ \mathbf{e}^{n_1+1} \\ \vdots \\ \mathbf{e}^{n_1+n_2+\cdots+n_{m-1}+1} \end{bmatrix}. \tag{18}$$

$\mathbf{e}^i$ are the ith unit row vectors of appropriate dimensions, I_n is an $n \times n$ identity matrix and the integers n_i are called the observability indices of the system with

$$n_1 + n_2 + \cdots + n_m = n.$$

B does not have any special structure. It is assumed that the observability indices n_i are estimated a priori. The problem is of estimating the unknown parameters in A, B and L. Consider the innovation model given by equations (7)–(9) and the following criterion function

$$V(\hat{\boldsymbol{\theta}}) = \tfrac{1}{2}\mathrm{E}\{\hat{\varepsilon}(k)^{\mathrm{T}}\hat{\Sigma}_e^{-1}\hat{\varepsilon}(k)\}, \tag{19}$$

where $\theta = [\bar{\theta}_1, \ldots, \bar{\theta}_m]$,

$$\bar{\theta}_j = [a^j \mid b^{n_1+\cdots+n_{j-1}+1} \cdots b^{n_1+\cdots+n_j} \mid l^{n_1+\cdots+n_{j-1}+1} \cdots l^{n_1+\cdots+n_j}],$$

$$\Sigma_\varepsilon \equiv \text{covariance matrix of } \varepsilon(t),$$

$$\text{with } a^j = [a_{j1}(1) \cdots a_{j1}(n_1) \mid a_{j2}(1) \cdots \mid \cdots \mid \cdots a_{jm}(n_m)],$$

$$b^j \equiv j\text{th row of } B, \quad l^j \equiv j\text{th row of } L,$$

$\{\cdot\}$ denotes the estimate and $E\{\cdot\}$ denotes the expectation. Since the sequence $\varepsilon(k)$ is white with covarinace Σ_ε, the identification problem may be formulated as a minimization of $V(\hat{\theta})$. θ is the unknown parameter row vector of dimension n_θ.

Define

$$\Psi(k) = -\{d\hat{\varepsilon}(k)/d\hat{\theta}\} = \{d\hat{y}(k)/d\hat{\theta}\} \quad (\dim: m \times n_\theta). \quad (20)$$

$\Psi(k)$ provides a descent direction for the minimization of $V(\theta)$. In order to compute $\Psi(k)$ the following variables need to be defined.

$$\phi(k) = \{d\hat{x}(k)/d\theta\}|_{\theta=\hat{\theta}},$$

and

$$M(k) = \begin{cases} \partial[A(\theta)\hat{x}(k) + B(\theta)u(k) + L(\theta)\hat{\varepsilon}(k)]/\partial\theta|_{\theta=\hat{\theta}} \\ \qquad \text{for } k = 0, J, 2J, \ldots \\ \partial[A(\theta)\hat{x}(k) + B(\theta)u(k)]/\partial\theta|_{\theta=\hat{\theta}} \\ \qquad \text{for } k \neq 0, J, 2J, \ldots. \end{cases} \quad (21)$$

With the above definitions and following the approach of Ljung and Soderström (1983) an RPE algorithm for the recursive minimization of $V(\hat{\theta})$ is then given as follows.

For $k = J, 2J, 3J, \ldots$

(i) Compute the predicted output y and its parameter derivative Ψ

$$\hat{y}(k) = C\hat{x}(k) \quad (\dim: m \times 1), \quad (22)$$

$$\Psi(k) = C\phi(k) \quad (\dim: m \times n_\theta). \quad (23)$$

(ii) Compute the prediction error and the innovation covariance matrix

$$\hat{\varepsilon}(k) = y(k) - \hat{y}(k) \quad (\dim: m \times 1), \quad (24)$$

$$\Sigma_\varepsilon(k) = \Sigma_\varepsilon(k-J) + \frac{J}{k}[\hat{\varepsilon}(k)\hat{\varepsilon}(k)^T - \Sigma_\varepsilon(k-J)] \quad (\dim: m \times m). \quad (25)$$

(iii) Compute the parameter adaptation gain

$$\Lambda(k) = [\lambda\Sigma_\varepsilon(k) + \Psi(k)P(k-J)\Psi(k)^T]^{-1}\Psi(k)P(k-J) \quad (\dim: m \times n_\theta). \quad (26)$$

(iv) Update the parameters

$$\theta^*(k) = \theta(k-J) + \hat{\varepsilon}(k)^T\Lambda(k) \quad (\dim: 1 \times n_\theta),$$

$$\theta(k) = \begin{cases} \theta^*(k) & \text{if } \theta^*(k) \in D_s, \\ \theta(k-J) & \text{otherwise.} \end{cases} \quad (27)$$

(v) Update the parameter covariance matrix

$$P(k) = \begin{cases} \frac{1}{\lambda}[P(k-J) - P(k-J)\Psi(k)^T[\lambda\Sigma_\varepsilon(k) \\ \quad + \Psi(k)P(k-J)\Psi(k)^T]^{-1}\Psi(k)P(k-J)] \\ \quad \text{if } \theta^*(k) \in D_s, \\ P(k-J) \quad \text{otherwise,} \end{cases} \quad (\dim: n_\theta \times n_\theta). \quad (28)$$

(vi) Obtain the next state estimate

$$\hat{x}(k+1) = A(\theta)\hat{x}(k) + B(\theta)u(k) + L(\theta)\hat{\varepsilon}(k). \quad (29)$$

(vii) Compute $\phi(k+1)$

$$\phi(k+1) = [A(\theta) - L(\theta)C]\phi(k) + M(k) \quad (\dim: n \times n_\theta). \quad (30)$$

While for $k \neq J, 2J, 3J, \ldots$,

(i) Obtain the next state estimate

$$\hat{x}(k+1) = A(\theta)\hat{x}(k) + B(\theta)u(k). \quad (31)$$

(ii) Compute $\phi(k+1)$

$$\phi(k+1) = A(\theta)\phi(k) + M(k) \quad (\dim: n \times n_\theta). \quad (32)$$

In the above

$$M(k) = \begin{bmatrix} \mu_1(k) & & \\ & \ddots & \\ & & \mu_m(k) \end{bmatrix} \quad (\dim: n \times n_\theta) \quad (33)$$

where $\mu_j(k) =$

$$\begin{cases} \begin{bmatrix} 0 & u^T(k) & 0 & 0 & \hat{\varepsilon}^T(k) & 0 & 0 \\ \vdots & & \ddots & & & \ddots & \\ \hat{x}^T(k) & 0 & 0 & u^T(k) & 0 & 0 & \hat{\varepsilon}^T(k) \end{bmatrix} \\ \qquad \text{for } k = 0, J, \ldots \\ \begin{bmatrix} 0 & u^T(k) & 0 & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & & & & \\ \hat{x}^T(k) & 0 & 0 & u^T(k) & 0 & 0 & 0 \end{bmatrix} \\ \qquad \text{for } k \neq 0, J, \ldots \end{cases}$$

$$[\dim: n_j \times (n + 2n_j m)] \quad (34)$$

where the partitions in the above correspond to the partitions in $\bar{\theta}_j$ given in equation (19).

In equation (27), D_s is a stability region given by the condition that all eigenvalues of $A^{J-1}(A - LC)$ lie within the unit circle. This can be tested by using the Jury criteria (Astrom and Wittenmark, 1989). This condition ensures that the RPE algorithm remains stable. It may be emphasized here that the above condition does not imply that the dynamic system should be stable. λ is a forgetting factor with $0.95 \leq \lambda \leq 1.0$. λ may be generated by the following equation

$$\lambda(k) = \lambda_\infty - \lambda_r[\lambda_\infty - \lambda(k-1)]; \quad \lambda(0) = \lambda_0, \quad (35)$$

where λ_0 and λ_∞ are the initial and final values of λ, and λ_r controls the rate at which the transition takes place. Although from a theoretical point of view, λ_∞ should be 1 to ensure convergence to the constant plant parameters, in order to keep the algorithm open to possible change in plant parameters λ_∞ is normally chosen as less than 1. To start the iteration $P(0)$ may be set to I or $10I$, elements of $\theta(0)$ may be set to the best guess or to small values ($\cong 0.1$) and elements of $\Lambda(0)$, $\phi(0)$, $M(0)$ and $\Psi(0)$ may be set to zero.

The computation of $P(k)$ by equation (28) is not numerically robust. It is sensitive to round-off errors that can accumulate and may fail to keep $P(k)$ non-negative definite. Various modifications have been proposed to reduce this problem. Interested readers are referred to Bierman (1977), Anderson and Moore (1979) and Ljung and Soderström (1983). In our simulation, equation (28) is used and no numerical problem has been encountered.

The convergence of the proposed recursive algorithm can be analysed by considering the associated differential equation (de) (Ljung and Soderström, 1983). Although convergence to the true parameter value is not guaranteed, it can be shown that one of the possible convergence points of the algorithm is the true parameter θ.

4. *Pole restriction control*

The advantage of state variable based control is the efficient (computationally simple) implementation of advance control strategy using the following state feedback

$$u(k) = -F(k)u(k). \quad (36)$$

The most popular control strategies using the state space model are the pole placement (PP) and linear quadratic (LQ) control. In the former, state feedback control is designed from closed loop pole specification while in the latter, control is designed by minimizing a quadratic cost function. PP and LQ have their own advantages and disadvantages. A control strategy that can be derived to incorporate the advantages of PP and LQ is the pole restriction control (PR) (Hang *et al.*,

1991). Although other control strategies including the PP and LQ can be employed, in this paper we considered the PR control.

In this approach, the closed loop poles are placed within a specified disk in the Z-plane using a modified LQ design. The disk is specified from the maximum permitted overshoot and settling time (or time constant) requirements of the system output. The design is based on the overshoot and settling time requirements, but the input amplitude usually remains small due to the less stringent requirement on the closed loop pole locations. In addition, once such a controller is obtained for a certain estimated plant model, the controller parameters need not be updated as long as the estimted closed loop poles remain within the specified disk. The basis of the pole restriction algorithm is the following result due to Furuta and Kim (1987).

Construct the matrices

$$\bar{A} = \frac{(A - \alpha I)}{\beta} \quad \text{and} \quad \bar{B} = \frac{B}{\beta}, \tag{37}$$

and solve for P in the matrix Riccati equation (MRE) given as follows

$$P - \bar{A}^{\mathrm{T}} P \bar{A} + \bar{A}^{\mathrm{T}} P \bar{B}(R + \bar{B}^{\mathrm{T}} P \bar{B})^{-1} \bar{B}^{\mathrm{T}} P \bar{A} - Q = 0, \tag{38}$$

with $R > 0$ and $Q = H^{\mathrm{T}} H$ where the pair (A, H) is observable. Then the feedback rule

$$u(k) = -F\mathbf{x}(k),$$

with

$$F = (R + \bar{B}^{\mathrm{T}} P \bar{B})^{-1} \bar{B}^{\mathrm{T}} P \bar{A}, \tag{39}$$

ensures that all the closed loop poles of the system given by equation (1) lie inside a disk in the Z-plane having center at α and radius β.

Solution of the MRE in each iteration, however, makes the algorithm computationally very demanding. In order to reduce the computational requirements, the state feedback control design is called only when the eigenvalues of $\hat{A} - \hat{B}F$ move outside the specified disk. This can be done by checking whether the characteristic polynomial of $\bar{A} - \bar{B}F$ satisfies the Jury stability criteria.

Following the approach of Ahmed (1993), the self-tuning control using the pole restriction may be summarized as follows: In the kth iteration

(a) if k is not an integer multiple of J, estimate the state vector $\mathbf{x}$ and Ψ using equations (31) and (32), set $F(k) = F(k-1)$ and go to (e), otherwise

(b) estimate the plant matrices A, B, L and the state vector $\mathbf{x}$ using equations (22)–(30). Obtain $\bar{A}$ and $\bar{B}$ using equation (37)

(c) if the characteristic polynomial of $\bar{A} - \bar{B}F$ satisfies Jury stability criteria set $F(k) = F(k-1)$ and go to (e), otherwise

(d) solve for the symmetric positive definite matrix P in the discrete-time algebraic MRE given by equation (38) and set

$$F(k) = (R + \bar{B}^{\mathrm{T}} P \bar{B})^{-1} \bar{B}^{\mathrm{T}} P \bar{A} \tag{40}$$

(e) apply the control $u(k) = -F(k)\hat{\mathbf{x}}(k)$.

4.1. *Tracking reference input.* It is straightforward to show that when tracking of the step reference input y_r is desired, the control based on the above principle for a square plant (i.e. $m = r$) becomes

$$u(k) = -F(k)\mathbf{x}(k) + G(k)y_r(k), \tag{41}$$

where

$$G^{-1}(k) = C[I_n - A + BF(k)]^{-1}B.$$

5. *Simulation studies*

In order to test the proposed algorithms and verify their feasibility, simulation studies are carried out on the following two-input two-output plant with $J = 5$

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0.10 & 0.90 & 0.05 \\ 0.15 & 0.10 & 0.90 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.60 & 0.10 \\ 0.18 & 0.10 \\ 0.10 & 0.20 \end{bmatrix} \times u(k) + w(k) \quad k = 0, 1, 2, \ldots, \tag{42}$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k) + v(k) \quad k = 0, 5, 10, \ldots,$$

$$\mathrm{cov}\{w(k)\} = 0.001I, \quad \mathrm{cov}\{v(k)\} = 0.0001I.$$

The plant is an unstable one with open loop poles at -0.1, 0.83 and 1.06. The starting values of all the parameter estimates are taken as 0.05, except a_{22} and a_{33}. a_{22} and a_{33} are both taken at 0.6 which imply the initial assumption that the plant outputs are nearly decoupled with an average time constant of 0.6. The following starting values are used

$$P(0) = I, \quad \lambda_0 = 0.95, \quad \lambda_r = 0.95, \quad \lambda_\infty = 0.99,$$
$$\mathbf{x} = [0.0 \quad 0.0 \quad 0.0]^{\mathrm{T}}.$$

The simulation is started by superimposing zero mean random signals of variance 0.1 on the control inputs for the first 200 (input) iterations. This provided 'rich input' for the estimator so that good estimates of the system parameters and states are obtained quickly.

For the pole restriction control, the performance specifications at the output are assumed to be (a) a damping coefficient of at least 0.5 (which corresponds to, at most, a 16% overshoot) and (b) a 2% settling time of, at most, twenty (input) samples. The damping coefficient constraint confines the closed loop poles within the heart shaped region shown in Fig. 1, while the settling constraint confines the closed loop poles within the dotted circle. These, together, require that all the closed loop poles lie within the hatched region. For the proposed algorithm the hatched region is approximated by the inner solid circle in the Z-plane as shown in Fig. 1. The circle has a center at 0.3 and a radius 0.5. Q and R matrices are taken as $0.01I$ and I, respectively. The relatively large value of R assigns a high penalty to input amplitude demanding it to be smooth.

Figure 2(a) and (b) shows the simulation results with the proposed control. The set point for output 1 is varied between -2.5 and 2.5 while the set point for output 2 is varied between -5.0 and 5.0. The input amplitude is restricted between ± 20. The computation of the state feedback matrix $F(k)$ (and therefore solution of ARE) is required only seven times, at output iterations of 1, 2, 4, 9, 61, 63 and 182. The closed loop poles at the end of 200 (output) iterations are found to be at -0.1130, 0.8065 and 0.6069, which also corresponds to a 2% settling time of approximately eight (input) samples.

Figure 3(a) shows the normalized parameter estimation error (PEE) defined as

$$\mathrm{PEE} = \frac{\|\underline{\boldsymbol{\theta}} - \hat{\underline{\boldsymbol{\theta}}}\|^2}{\|\underline{\boldsymbol{\theta}}\|^2} \tag{43}$$

where $\underline{\boldsymbol{\theta}}$ contains the parameters of $\boldsymbol{\theta}$ that belong to A and B. After 200 output iterations the final estimated values of A and B are found to be

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0.0711 & 0.9306 & 0.0467 \\ 0.1290 & 0.1300 & 0.8916 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 0.5825 & 0.0929 \\ 0.1857 & 09.0987 \\ 0.0932 & 0.1973 \end{bmatrix}.$$

In order to investigate the estimation accuracy of the elements of $\underline{L}$, the theoretical values of $\underline{L}$ and L are obtained as follows. Variance of $\underline{w}$ (see equations (11) and (12)) is computed using the expression following equation (12). Next, the steady state Kalman gain $\underline{K}$ for the plant model corresponding to the output sampling rate is obtained by solving a matrix Riccati equation. Then $\underline{L}$ and L, respectively, were obtained as $\underline{L} = \underline{A}\underline{K}$ and $L = A^{-(J-1)}\underline{L}$. They are found as

$$L = \begin{bmatrix} 0.7346 & 0.1817 \\ 0.7595 & 0.2135 \\ 0.2310 & 0.8992 \end{bmatrix} \quad \underline{L} = \begin{bmatrix} 0.8128 & 0.3316 \\ 0.8445 & 0.3686 \\ 0.8185 & 0.8102 \end{bmatrix}.$$

The values in L, however, are very unreliable as the matrix A^{J-1} is near singular. The condition number of this matrix is found to be 2.8×10^4. The elements of L are also found to be extremely sensitive to slight changes in the elements of $\underline{L}$. The estimate of L and $\underline{L}$ after 200 output iterations are found

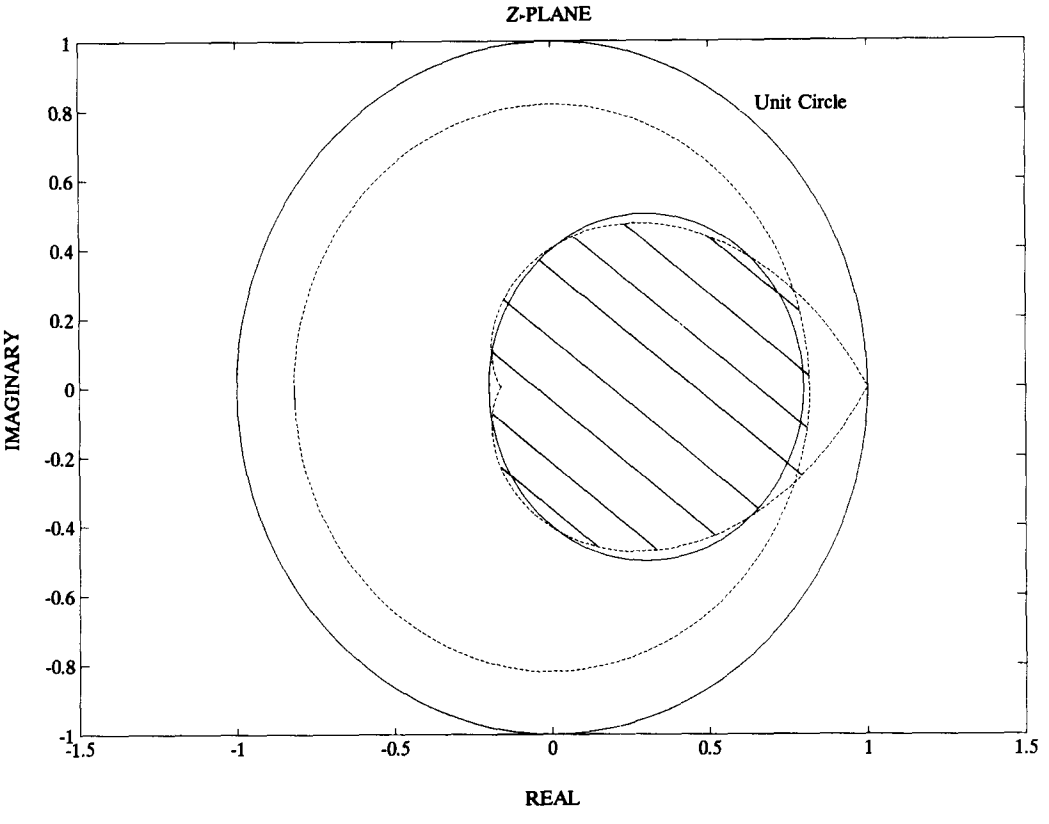


FIG. 1. Hitched region approximated by a circle for the pole restriction control.

to be

$$\hat{L} = \begin{bmatrix} 0.8050 & 0.0608 \\ 0.6278 & 0.3207 \\ 0.1532 & 0.4792 \end{bmatrix},$$

$$\hat{\underline{L}} = \hat{A}^{J-1}\hat{L} = \begin{bmatrix} 0.6282 & 0.3703 \\ 0.6504 & 0.3945 \\ 0.5885 & 0.5749 \end{bmatrix}.$$

It can be observed that $\hat{L}$ is in large error with L. This is from the fact that due to near singularity of A^{J-1} a small error in $\hat{\underline{L}}$ is reflected as a large error in $\hat{L}$. In the line of this explanation, the elements of $\hat{\underline{L}}$ can be observed to be in good agreement with that of $\underline{L}$.

The normalized state estimation error (SEE) defined as

$$\text{SEE} = \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|^2}{\|\mathbf{x}\|^2},$$

is shown in Fig. 3(b). It can be seen that after 40 (output) iterations, the estimated states are very close to the true states. The initial large error in the state estimation is due to the poor parameter estimates and the addition of perturbed input during the 'tuning period'.

Figure 4(a) and (b) shows the input and output where control inputs are updated only at output sampling instants. This corresponds to uniform-rate self-tuning. For this simulation an RPE algorithm was used for identifying the representation corresponding to the output sampling rate given by equations (13) and (14). A pole restriction (PR) control was used with the specification that the damping coefficient be at least 0.5 and have a 2% settling time of at most 4 (output) samples. These requirements are approximated by enforcing the closed loop poles to lie within a circle in the Z-plane having a center at 0.1 and a radius 0.3. It can be observed that this plant cannot be satisfactorily controlled by employing uniform-rate sampling corresponding to the output availability.

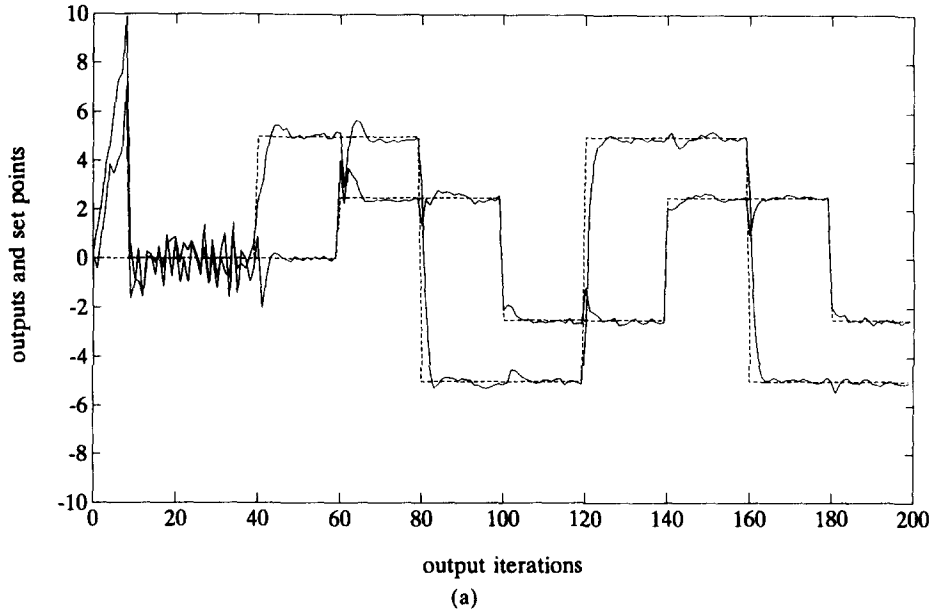
It has been mentioned in Section 2 that there exists a plant dependent upper limit of J beyond which the estimation of parameters and control will become ineffective. In order to observe the effect of varying J on the proposed self-tuning algorithm, the value of J has been gradually increased in the simulation. It is observed that both parameter estimation and control deteriorates as J is increased. The parameter estimates and control have been found satisfactory (final PEE < 0.05 and setpoint tracking being visually good) until J has been less than 10. When J became 10, final PEE became 0.7 and the output failed to follow the set points. With $J = 10$ the condition number of A^{J-1} rose to 4.9×10^9.

6. *Conclusions*

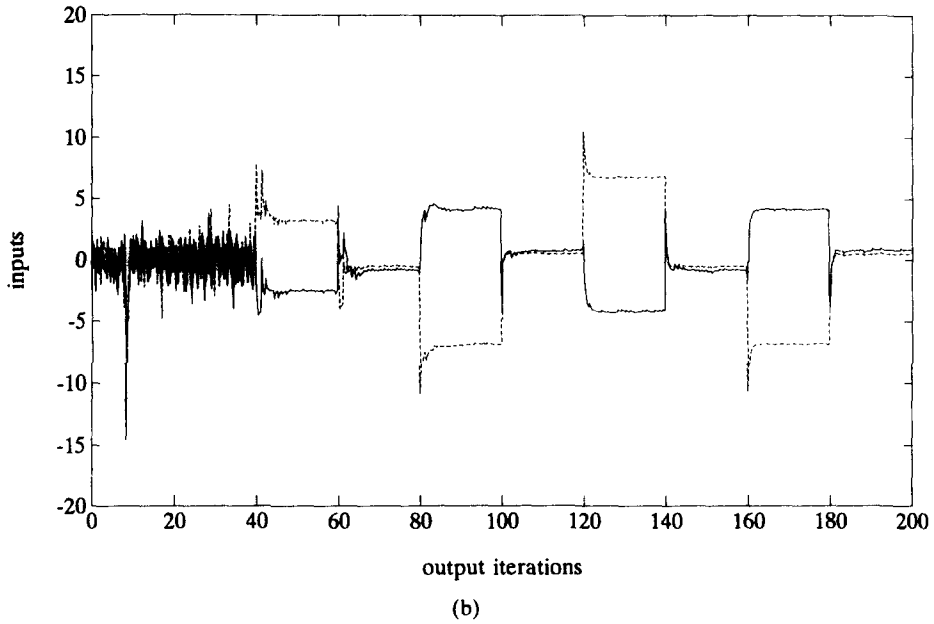
In this paper a state space based dual-rate self-tuning algorithm is proposed. The algorithm is applicable when the output sampling rate is slower than the input update rate. A state space innovation model suitable for dual-rate self-tuning control is derived. The plant matrices and the innovation gain matrix together with the states are estimated by a recursive prediction error estimator. The control is achieved through state feedback employing a pole restriction principle.

Results of a simulation study have been provided to establish the feasibility of the algorithm. It is shown that the proposed algorithm is well capable of controlling a plant having dual sampling rate at the input and output. It is also shown that a dual-rate self-tuner may provide tighter control compared to its uniform-rate counterpart.

Although the state space based dual-rate adaptive control is demonstrated using an observable canonical state space form, the developments are equally applicable to a controllable canonical state space form. Direct identification of such a model and estimation of the associated states can also be done by a recursive prediction error (RPE) algorithm. This approach may reduce the computational requirements in some cases, if a pole placement control is adopted. However, the corresponding RPE algorithm will have a higher degree of nonlinearity which may pose some problems in the convergence to the global minimum.



(a)



(b)

FIG. 2(a) and (b). Dual-rate self-tuning control using the proposed method.

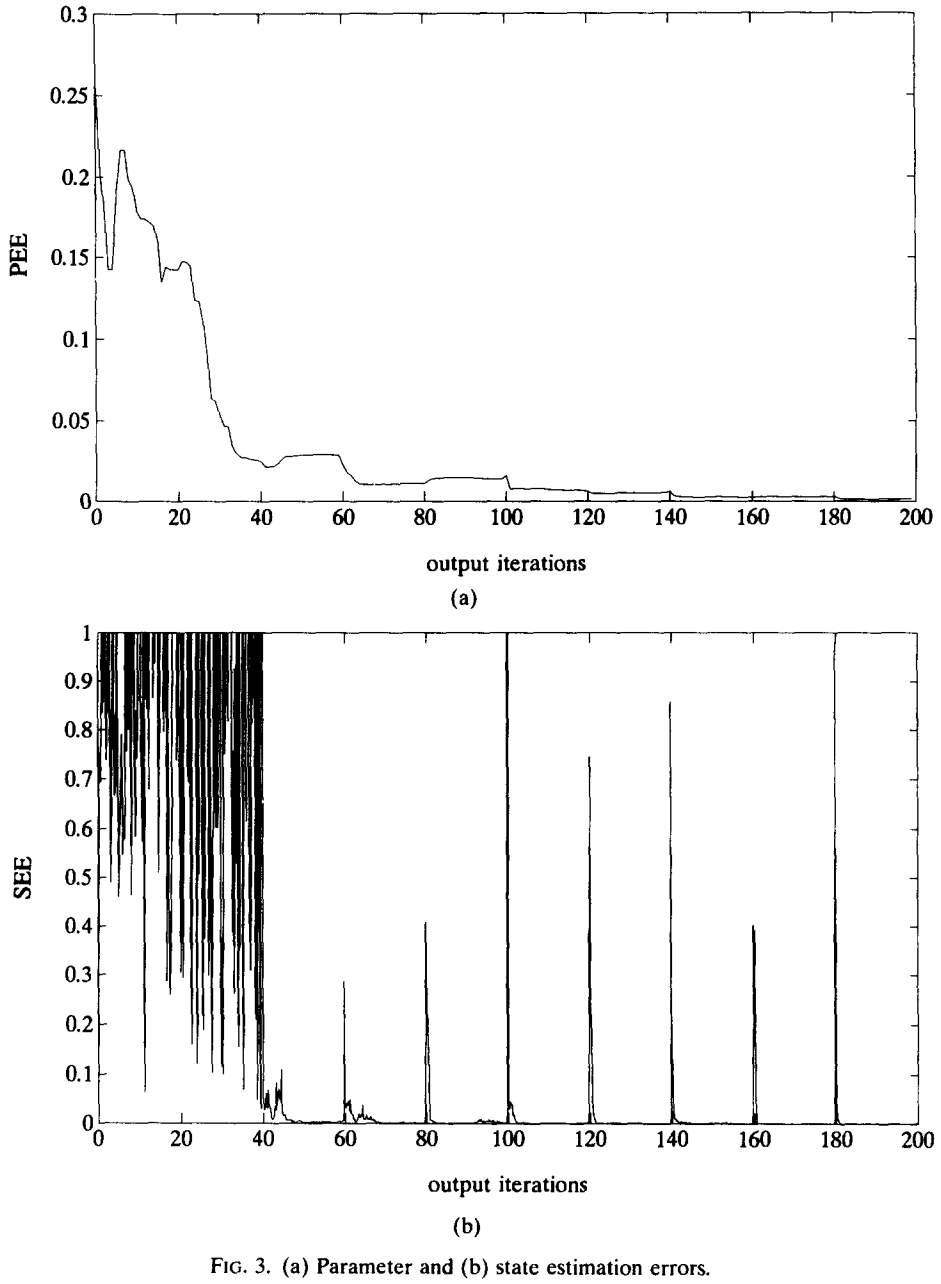
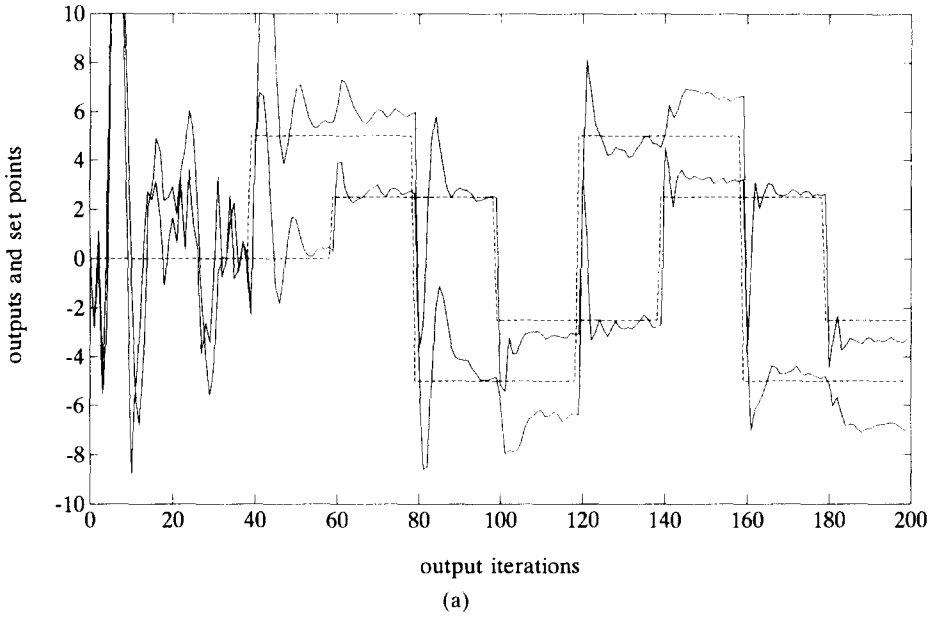
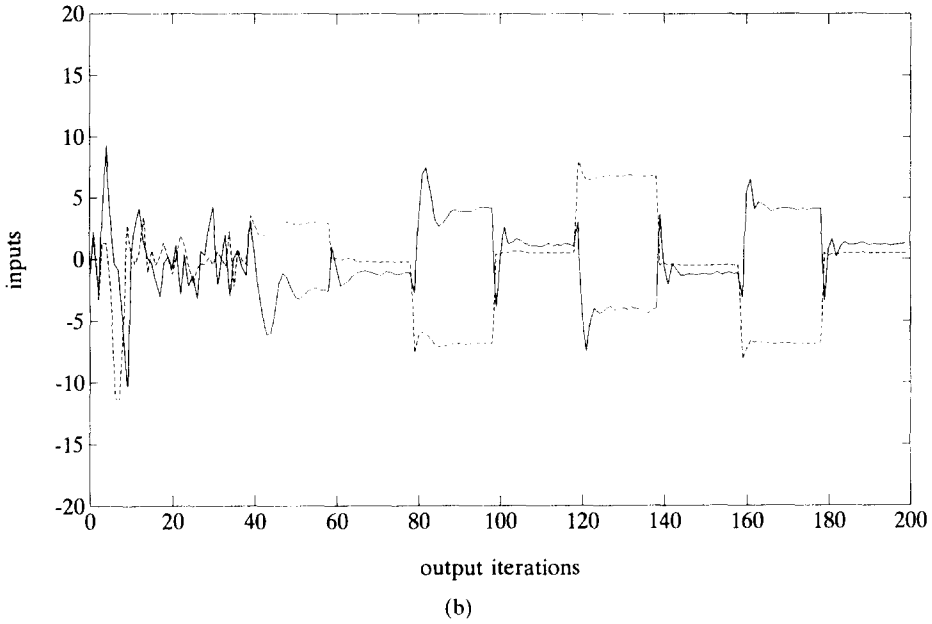


FIG. 3. (a) Parameter and (b) state estimation errors.



(a)



(b)

FIG. 4(a) and (b). Uniform-rate self-tuning control.

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References

Ahmed, M. A. (1993). State-space based adaptive control with pole restriction. In *IEE Proc. D (Control Theory and Appl.)*, **140,** (6), 380–392.

Anderson, D. B. O. and J. B. Moore (1979). *Optimal Filtering.* Prentice-Hall, New York.

Araki, M. and T. Hagiwara (1986). Pole assignment by multirate sampled data output feedback. *Int. J. Control,* **44,** 1661–1673.

Astrom, K. J. and B. Wittenmark (1989). *Computer Controlled Systems: Theory and Design.* Prentice-Hall, Englewood Cliffs, NJ.

Bierman, G. J. (1977). *Factorization Methods for Discrete Sequential Estimation.* Academic Press, San Diego, CA.

El-Sherief, H. and N. K. Sinha (1979). Bootstrap estimation of parameters and states of linear multivariable systems. *IEEE Trans. Autom. Control,* **AC-24,** 340–342.

Furuta, K. and S. B. Kim (1987). Pole assignment in a specified disk. *IEEE Trans. Autom. Control,* **AC-32,** 423–427.

Hang, C. C., K. W. Lim and B. W. Chong (1989). A dual rate adaptive digital Smith predictor. *Automatica,* **25,** 1–16.

Hang, C. C., K. W. Lim and W. K. Ho (1991). Generalized minimum variance stochastic self-tuning controller with pole restriction. In *IEE Proc. D (Control Theory and Appl.)*, **138,** pp. 25–32.

Hang, C. C., Y. S. Cai and K. W. Lim (1993). A dual rate self tuning pole placement controller. *IEEE Trans. Ind. Electronics,* **IE-40,** 116–129.

Kallstrom, C. G., K. J. Astrom, N. E. Thorell, J. Errikson and L. Sten (1978). Adaptive autopilots for steering large tankers. Presented at IFAC Congress, Helsinki.

Kannaiah, J., O. P. Malik and G. S. Hope (1984). Microprocessor-based universal regulator using dual-rate sampling. *IEEE Trans. Ind. Electronics,* **IE-31,** 306–312.

Ljung, L. and T. Soderström (1983). *Theory and Practice of Recursive Identification.* MIT Press, Cambridge, MA.

Lu, W., D. G. Fisher and S. L. Shah (1990). Multirate constrained adaptive control. *Int. J. Control,* **51,** 1439–1456.

Mills, P. M., P. L. Lee and P. McIntosh (1991). A practical study of adaptive control of an alumina calciner. *Automatica,* **27,** 441–448.

Sastry, V. A., D. E. Seborg and R. K. Wood (1977). Self-tuning regulator applied to a binary distillation column. *Automatica,* **13,** 417–424.