Some Hidden Relations Involving the Ten Symmetry Classes of Plane Partitions

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Let B be a partially ordered product of three finite chains. For any group G of automorphisms of B, let $N_G(B, q)$ denote the rank generating function for G-invariant order ideals of B. If we regard B as a rectangular prism, $N_G(B, q)$ can be viewed as a generating function for plane partitions that fit inside B. Similarly, define $N'_G(B, q)$ to be the rank generating function for order ideals of the quotient poset B/G. We prove that $N_G(B, -1)$ and $N'_G(B, -1)$ count the number of plane partitions (i.e., order ideals of B) that are invariant under certain automorphisms and complementation operations on B. Consequently, one discovers that the number of plane partitions belonging to each of the ten symmetry classes identified by Stanley is of the form $N_G(B, \pm 1)$ or $N'_G(B, \pm 1)$ for some subgroup G of S_3, and conversely. We also discuss the occurrence of this phenomenon in general partially ordered sets, and use the theory of P-partitions to derive a criterion for one aspect of it. © 1994 Academic Press, Inc.

0. INTRODUCTION

The problem of determining the number of plane partitions invariant under various symmetry groups is among the most difficult problems in enumerative combinatorics. There are ten distinct symmetry classes of plane partitions, and hence essentially ten distinct enumerative problems, not counting the various q-analogues that some of the cases possess. A tremendous amount of work has been devoted to these problems (see References), starting with MacMahon at the beginning of this century. Considerable progress has been made in the past 15 years, particularly in the past few years, to the point that, with the exception of one q-analogue, all of the conjectures have been completely solved.

On the other hand, we still have no good explanation as to why these ten enumerative problems should have nice answers. Furthermore, although there are a number of elegant techniques that have been devel-

oped for these problems, so far there has been no unified method of proof. For example, the permanent-determinant method and non-intersecting path methods are only "half-methods" in the sense that they provide an elegant means of encoding the number of plane partitions in a given symmetry class as a determinant or Pfaffian. But these methods end there —we are still left with the evaluation of the determinants. Some of them are not too difficult to evaluate, but others among them are extraordinarily difficult. In any case, we are left with an unsatisfying miracle: Why should a collection of ten families of determinants and Pfaffians all have explicit closed formulas? Other approaches involving representation theory have been brought to bear on these problems and, where these have succeeded, they manage to "explain" the existence of the miraculous closed formulas as special instances of the Weyl denominator or dimension formulas. However, the use of representation theory has been successful in only five of the ten cases.

A second difficulty is with the formulas themselves. The ten symmetry classes can be conveniently divided into two families of sizes six and four. One family consists of the four symmetry classes of subgroups of S_3 that arise from ordinary permutations of three coordinate axes. The study of this family is quite old and can be traced back to MacMahon. All of the formulas for these cases (some known, some conjectured) were shown by Macdonald and Stanley [St1] to have a nice, uniform presentation. The second family consists of the six symmetry classes that involve the complementation operation first defined by Mills *et al.* [MRR2]. Until now, the known formulas for these cases have been *ad hoc*, but one of the results of the present paper is an extension of the Macdonald–Stanley format which shows that all ten of the formulas can be given a unified presentation.

This unified presentation is a consequence of what we call "the $q = -1$ phenomenon." We show that, for each of the enumerative problems in the six cases involving complementation, one obtains the number of such plane partitions by setting $q = -1$ in the q-analogue of one of the cases from the original family of four. (Some of the four cases have more than one q-analogue, which explains how six cases can be covered by four.)

Our proof of this phenomenon is unsatisfying. For most cases, it is simply a matter of comparing the known formulas. If there were nothing else to the proof, the result would still be interesting, but it would not require a paper of this length. But in fact there *is* something else: in one case, the q-analogue is still unproved; what we do amounts to proving this conjecture in the case $q = -1$. In another case, there is a q-analogue that is not only unproved, it is also *false*. Nevertheless we prove that it is true when $q = -1$.

Although our proof is unsatisfying, it raises a compelling possibility. If a nice (perhaps combinatorial) explanation of the $q = -1$ phenomenon

could be found, it would substantially reduce the number of "miraculous" closed formulas one is forced to accept. In a sequel to this paper, we will use representation theory to give nice explanations of the $q = -1$ phenomenon in at least two of the cases (cf. the discussion in Section 6). Kuperberg [private communication] has recently discovered some similar explanations.

The paper is organized as follows. In Section 1, we describe the main results in detail (Theorems 1.1 and 1.2). The proofs of these results are spread over Sections 2–5. In Section 2 we dispense with the simple cases that amount to repackaging of known results. In Section 3, we treat the case of cyclic symmetry. Although this is still a matter of repackaging, the details are somewhat more delicate. In Sections 4 and 5 we treat the nonroutine cases referred to above. Finally, in the last two sections, we consider the more general possibility of the $q = -1$ phenomenon occurring in the context of partially ordered sets, and derive a criterion for the phenomenon in one circumstance based on linear extensions of posets (Theorem 7.7).

1. THE MAIN RESULTS

A plane partition π is an array of nonnegative integers $[\pi_{ij}]_{i,j \geq 1}$, with finitely many nonzero entries, such that $\pi_{ij} \geq \max(\pi_{i,j+1}, \pi_{i+1,j})$. Alternatively, one may identify π with the set of lattice points $\{(i, j, k) \in \mathbf{Z}^3 : 1 \leq k \leq \pi_{ij}\}$; in this form, plane partitions are the (finite) order ideals of the poset $\mathbf{P}^3$. Note that the symmetric group S_3 acts on $\mathbf{P}^3$ by permuting coordinates, hence also on the set of plane partitions.

For any positive integer m, let $[m] = \{1, 2, \ldots, m\}$. Fix positive integers a, b, and c, and let $B = B(a, b, c)$ denote the subposet of $\mathbf{P}^3$ induced by $[a] \times [b] \times [c]$. The order ideals of B correspond to the plane partitions π with at most a (nonzero) rows, at most b (nonzero) columns, and the property that $\pi_{ij} \leq c$. Note that the mapping

$$(i, j, k) \mapsto (i, j, k)^c := (a + 1 - i, b + 1 - j, c + 1 - k)$$

defines an order-reversing involution on B, so B is a "complemented poset" in the sense to be defined in Section 6. In particular, following Mills, Robbins, Rumsey, and Stanley, we can define the complement of a plane partition π (regarded as an order ideal) by setting $\pi^c := \{(i, j, k) \in B : (i, j, k)^c \notin \pi\}$.

Let $K = \{\mathrm{id}, c\}$ denote the two-element symmetry group generated by complementation, and let $\Gamma \cong S_3 \times K$ denote the 12-element group gen-

erated by coordinate permutations and complementation. For any sub-group G of Γ, let $n_G(B)$ denote the number of G-invariant order ideals of B (i.e., the number of G-invariant plane partitions π such that $\pi \subseteq B$). Without loss of generality, we will always assume that, for a given symmetry group G, the parameters a, b, and c are chosen so that B itself is G-invariant. (For example, if G is the group of cyclic permutations of the coordinates, this requires $a = b = c$).

If G is a subgroup of S_3, then there are two natural q-analogues of the quantity $n_G(B)$ that have arisen previously in the enumeration of symmetry classes of plane partitions: the first is with respect to the size of the plane partition (as an order ideal of B); the second is with respect to the number of G-orbits of elements of B in the order ideal. Thus we define

$$N_G(B, q) := \sum_{\pi \in J(B)^G} q^{|\pi|},$$

$$N_G'(B, q) := \sum_{\pi \in J(B)^G} q^{|\pi/G|},$$

where both sums range over the set $J(B)^G$ of all G-invariant plane partitions π contained in B, and π/G denotes the set of G-orbits in π. The second sum can also be identified as the rank generating function for order ideals of the quotient poset B/G.

Note that $N_G'(B, q)$ is not defined for subgroups G of Γ that contain an order-reversing symmetry, since a G-invariant plane partition is not a union of G-orbits of B in such cases. On the other hand, one could in principle study the generating function $N_G(B, q)$ for arbitrary subgroups G of Γ, but if G contains an order-reversing symmetry, then any G-invariant order ideal must contain exactly $|B|/2$ elements. Thus for such groups, one has $N_G(B, q) = q^{|B|/2} n_G(B)$.

For each subgroup G of S_3, define the pair of rational functions

$$P_G(B, q) := \prod_{x \in B/G} \frac{1 - q^{|x|(r(x)+2)}}{1 - q^{|x|(r(x)+1)}},$$

$$P_G'(B, q) := \prod_{x \in B/G} \frac{1 - q^{r(x)+2}}{1 - q^{r(x)+1}},$$

where $r(\cdot)$ denotes the rank function on B/G (or B), and $|x|$ denotes the size of the G-orbit of x. It was Macdonald who first realized that $P_G(B, q)$

TABLE I

G	$N_G \overset{?}{=} P_G$	$N'_G \overset{?}{=} P'_G$
$\{1\}$	[Ma]	[Ma]
S_2	[A1–2], [M]	[A1–2], [M], [G]
C_3	[MRR1]	false
S_3	false	?

agreed with all of the known or conjectured product formulas for $N_G(B, q)$, and Stanley later realized that $P'_G(B, q)$ agreed with the known or conjectured product formulas for $N'_G(B, q)$. This is not to say that $N_G(B, q) = P_G(B, q)$ and $N'(B, q) = P'_G(B, q)$ for all G (see Table 1). In this table we have listed each instance where equality occurs, together with a reference to the first proof(s). Note that in the case $G = \{1\}$ there is just one case, not two, since the generating functions $N_G(B, q)$ and $N'_G(B, q)$ are obviously identical (and the same is true for the corresponding product formulas).

The "false" and "?" cases deserve further comment. First, in the case $G = C_3$, we have $N'_G(B, q) \neq P'_G(B, q)$. Furthermore, there probably is no simple formula for $N'_{C_3}(B, q)$, since for small instances of B, the generating function does not factor significantly over the rationals. On the other hand, it can be proved that $P'_{C_3}(B, q)$ is a polynomial[1] in this case (see Proposition 3.2), and based on calculations in special cases, it appears to have nonnegative coefficients. It would be interesting to find a parameter associated with cyclically symmetric plane partitions whose generating function is $P'_{C_3}(B, q)$.

In the case $G = S_3$, we have $N_G(B, q) \neq P_G(B, q)$ in an even stronger sense than in the previous case—not only does $N_{S_3}(B, q)$ appear not to factor significantly over the rationals, but $P_{S_3}(B, q)$ is also not a polynomial. Finally, in the "?" case, the fact that $N'_G(B, q) = P'_G(B, q)$ for $G = S_3$ is at this time only a conjecture, although the special case $q = 1$ has been proved recently [Ste2]. It is also unknown whether $P'_{S_3}(B, q)$ is a polynomial, but of course this would follow from the conjecture.

As we noted in the Introduction, it is now known that there are explicit product formulas for $n_G(B)$ for *every* subgroup of Γ, although the formulas for the six subgroups that involve complementation have been *ad hoc*, and have not enjoyed the degree of unity found in the four un-complemented cases. One of the objectives of the present paper is to provide a unification of the formulas for the ten cases.

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[1] In [St1] and later also [O], $P'_{C_3}(B, q)$ was erroneously reported not to be a polynomial.

To present our unification, it will be worthwhile to first classify in an organized way the ten conjugacy classes of subgroups of Γ. In order to discard all but the essential features, let us consider the task of classifying the subgroups of $G \times \mathbf{Z}_2$, where G is an arbitrary group, and $\mathbf{Z}_2 = \{\pm 1\}$ is the two-element group.

Let $g \mapsto \bar{g}$ denote the natural surjection $G \times \mathbf{Z}_2 \to G$. Extending this map to subgroups, we thus can label any subgroup H of $G \times \mathbf{Z}_2$ by a subgroup $\bar{H}$ of G. Conversely, given any subgroup $\bar{H}$ of G, there are three possibilities for the preimage H:

  (1) $H = \bar{H}$ (i.e., $H = \{(h, 1) : h \in \bar{H}\}$).

  (2) $H = \bar{H} \times \mathbf{Z}_2$ (i.e., $H = \{(h, \pm 1) : h \in \bar{H}\}$).

  (3) H is a "twist" of $\bar{H}$; i.e., there is an epimorphism $\theta : \bar{H} \to \mathbf{Z}_2$ such that

$$H = \left\{(h, 1) : h \in \bar{H}, \theta(h) = 1\right\} \cup \left\{(h, -1) : h \in \bar{H}, \theta(h) = -1\right\}.$$

Returning to Γ and $G = S_3$, we can arrange the ten conjugacy classes of subgroups of Γ into three families according to the above classification. First, there are the four conjugacy classes of subgroups of S_3 itself, namely, the trivial group, S_2, C_3, and S_3. In the second class there are also four, obtained by adjoining the operation of complementation to each of the subgroups in the first class; this yields K, $S_2 \times K$, $C_3 \times K$, and $\Gamma = S_3 \times K$. In the third class, note that only S_2 and S_3 afford homomorphisms onto $\mathbf{Z}_2$, so there are only two subgroups in this class; we shall denote them by S_2^* and S_3^*, respectively. The plane partitions invariant under S_2^* are those whose transpose coincides with their complement; the symmetry class S_3^* consists of those plane partitions invariant under both C_3 and S_2^*.

We are now ready to state the main results of this paper.

THEOREM 1.1. *If G is a subgroup of S_3, then*

$$n_{G \times K}(B) = N_G'(B, -1) = P_G'(B, -1).$$

A corollary of this result is the case $q = -1$ of the conjecture $N_{S_3}'(B, q) = P_{S_3}'(B, q)$.

THEOREM 1.2. *If G is a subgroup of S_3, and G^* is a twist of G, then*

$$n_{G^*}(B) = N_G(B, -1) = P_G(B, -1).$$

These two results cover the six symmetry classes that involve comple-
mentation. For the sake of completeness we should point out that the
known formulas for $n_G(B)$ in the four classes without complementation
are covered by the following result.

THEOREM 1.3.   *If G is a subgroup of S_3, then*

$$n_G(B) = P_G(B,1) = P'_G(B,1).$$

In summary, the number of plane partitions in each of the ten symmetry
classes is of the form $P_G(B, \pm 1)$ or $P'_G(B, \pm 1)$ for some $G \subseteq S_3$, and
conversely.

2. THE ROUTINE CASES

For positive integers a, let $\langle a \rangle = 1 - q^a$, and consider

$$\lim_{q \to -1} \frac{\langle a_1 \rangle \cdots \langle a_l \rangle}{\langle b_1 \rangle \cdots \langle b_l \rangle}$$

for arbitrary positive integers a_i and b_i $(1 \leq i \leq l)$. It is easy to see that
the limit is finite and nonzero if and only if there are equal numbers of
even integers among the a_i and b_i (or equivalently, equal numbers of odd
integers). Furthermore, since $\langle 2a + 1 \rangle / \langle 2b + 1 \rangle \to 1$ and $\langle 2a \rangle /$
$\langle 2b \rangle \to a/b$ as $q \to -1$, it follows that the limit can be obtained by
suppressing odd terms pairwise.

2.1. *The Trivial Group*

For clarity, let us write $N(a, b, c; q)$ and $P(a, b, c; q)$ in place of
$N_G(B, q) = N'_G(B, q)$ and $P_G(B, q) = P'_G(B, q)$ (resp.), where $B = [a] \times$
$[b] \times [c]$ and G is the trivial group. By the result of MacMahon (see
Table 1) it is known that $N(a, b, c; q) = P(a, b, c; q)$, so to prove Theo-
rem 1.1 in this case we need only to show that $P(a, b, c; -1) = n_K(B)$.

Note that if abc is odd, the complementation map $\pi \mapsto \pi^c$ changes the
parity of $|\pi|$, and thus proves $N(a, b, c; -1) = n_K(B) = 0$. Thus we may
henceforth assume that $B = [a] \times [b] \times [2c]$ for positive integers, a, b,

and c. In this case, it is easy to see that

$$P(a, b, 2c; q) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{\langle 2c + i + j - 1 \rangle}{\langle i + j - 1 \rangle}.$$

Odd terms occur in this expression when $i + j$ is even, so we have

$$P(a, b, 2c; - 1)$$

$$= \prod_{2i-1 \le a} \prod_{2j \le b} \frac{2c + 2i + 2j - 2}{2i + 2j - 2} \cdot \prod_{2i \le a} \prod_{2j-1 \le b} \frac{2c + 2i + 2j - 2}{2i + 2j - 2}$$

$$= \prod_{i \le (a+1)/2} \prod_{j \le b/2} \frac{c + i + j - 1}{i + j - 1} \cdot \prod_{i \le a/2} \prod_{j \le (b+1)/2} \frac{c + i + j - 1}{i + j - 1}$$

$$= P(\lfloor (a + 1)/2 \rfloor, \lfloor b/2 \rfloor, c; 1) \cdot P(\lfloor a/2 \rfloor, \lfloor (b + 1)/2 \rfloor, c; 1).$$

It is easy to see that this expression agrees with the known formulas for $n_K(B)$ due to Stanley (see Eqs. (3a)–(3c) of [St1]).

2.2. The Two-Element Group

First consider the $G = S_2$ case of Theorem 1.1. Note that, in order for B to be S_2-invariant, we may assume $B = [a]^2 \times [b]$. Since it is known that $N'_{S_2}(B, q) = P'_{S_2}(B, q)$ (see Table 1), it suffices merely to show that $P'_{S_2}(B, - 1) = n_{S_2 \times K}(B)$.

Using the elements $(i, j, k) \in B$ with $i \le j$ as orbit representatives for B/S_2, we obtain

$$P'_{S_2}(B; q) = \prod_{1 \le i \le j \le a} \prod_{k=1}^{b} \frac{\langle i + j + k - 1 \rangle}{\langle i + j + k - 2 \rangle} = \prod_{1 \le i \le j \le a} \frac{\langle b + i + j - 1 \rangle}{\langle i + j - 1 \rangle}.$$

$$(2.1)$$

Note that if b is odd, then the number of even terms in the numerator of this expression is the number of pairs (i, j) such that $1 \le i \le j \le a$ and $i + j$ is even, whereas the number of even terms in the denominator is the same except that $i + j$ should be odd. In this case, the map $(i, j) \mapsto (i, j - 1)$ defines an injection of the latter into the former. Since this injection is not a bijection (for example, the pair (a, a) is absent from the range), this proves that $- 1$ is a zero of $P'_{S_2}(B; q)$. Furthermore, it is easy to check that if $\pi = \pi_{ij}$ is a plane partition invariant under $S_2 \times K$, then $\pi_{a, 1} = b/2$. Thus $P'_{S_2}(B; - 1) = n_{S_2 \times K}(B) = 0$ if b is odd.

We still must treat the case of even b, so let us assume that $B = [a]^2 \times [2b]$. By cancelling the odd terms in (2.1), we obtain

$$
P'_{S_2}(B; -1) = \prod_{1 \leq 2i \leq 2j - 1 \leq a} \frac{2b + 2i + 2j - 2}{2i + 2j - 2}
$$

$$
\prod_{1 \leq 2i - 1 \leq 2j \leq a} \frac{2b + 2i + 2j - 2}{2i + 2j - 2}
$$

$$
= \prod_{1 \leq i < j \leq (a+1)/2} \frac{b + i + j - 1}{i + j - 1} \cdot \prod_{1 \leq i \leq j \leq a/2} \frac{b + i + j - 1}{i + j - 1}
$$

$$
= \prod_{1 \leq i \leq a/2} \prod_{1 \leq j \leq (a+1)/2} \frac{b + i + j - 1}{i + j - 1}
$$

$$
= P(\lfloor a/2 \rfloor, \lfloor (a + 1)/2 \rfloor, b; 1).
$$

It is easy to see that this expression agrees with the known formula for $n_{S_2 \times K}(B)$ due to Proctor (see the discussion of Case 7 in [St1]; for the proof, see [P1]).

We remark that since $|B/S_2| = \binom{a+1}{2} b$, the complementation map is parity-reversing (and thus proves $N'_{S_2}(B; -1) = 0$) only if $\binom{a+1}{2}$ and b are both odd.

Now consider the $G = S_2$ case of Theorem 1.2. Again, since it is already known that $N_{S_2}(B, q) = P_{S_2}(B, q)$ (see Table 1), we need only to show that $P_{S_2}(B, -1) = n_{S_2^*}(B)$. Assuming $B = [a]^2 \times [b]$, we have

$$
P_{S_2}(B, q) = \prod_{i=1}^{a} \prod_{k=1}^{b} \frac{\langle 2i + k - 1 \rangle}{\langle 2i + k - 2 \rangle} \cdot \prod_{1 \leq i < j \leq a} \prod_{k=1}^{b} \frac{\langle i + j + k - 1 \rangle_2}{\langle i + j + k - 2 \rangle_2}
$$

$$
= \prod_{i=1}^{a} \frac{\langle b + 2i - 1 \rangle}{\langle 2i - 1 \rangle} \cdot \prod_{1 \leq i < j \leq a} \frac{\langle b + i + j - 1 \rangle_2}{\langle i + j - 1 \rangle_2}, \qquad (2.2)
$$

where $\langle a \rangle_r := \langle ra \rangle = 1 - q^{ra}$. If b is odd, it is easy to see that this expression has a zero of order a at $q = -1$. One can also check that B contains no S_2^*-invariant plane partitions unless b is even. Thus we have $P_{S_2}(B, -1) = n_{S_2^*}(B) = 0$ for odd b. As in the previous case, the transpose-complementation map (i.e., the nontrivial element of S_2^*) does not succeed in proving that $N_{S_2}(B, -1) = 0$ in all of these cases; it succeeds only if $|B| = a^2 b$ is odd.

To treat the case of even b, we instead assume $B = [a]^2 \times [2b]$. Note that, by cancelling the odd terms in (2.2), we obtain

$$P_{S_2}(B, -1) = \prod_{1 \le i < j \le a} \frac{2b + i + j - 1}{i + j - 1}.$$

This expression is easily seen to be equivalent to the formula for $n_{S_2^*}(B)$ due to Proctor (see Corollary 4.1 of [P3]; cf. also Case 6 of [St1]).

3. CYCLIC SYMMETRY

We now prove that $G = C_3$ case of Theorem 1.1. The proof of this case cannot proceed as in the previous cases since $N'_{C_3}(B, q) \ne P'_{C_3}(B, q)$. However, since C_3-orbits are of sizes one and three, it follows that $|\pi| \equiv |\pi/C_3| \bmod 2$ for any C_3-invariant plane partition π, and hence $N_{C_3}(B, -1) = N'_{C_3}(B, -1)$. Since it is known that $N_{C_3}(B, q) = P_{C_3}(B, q)$ (see Table 1), it will therefore suffice to prove that $P'_{C_3}(B, -1) = P_{C_3}(B, -1) = n_{C_3 \times K}(B)$.

Assume $B = [a]^3$. Using the elements $(i, j, k) \in B$ with either (1) $i = j = k$, (2) $i \le j < k$, or (3) $i \ge j > k$ as orbit representatives for B/C_3, we obtain

$$P_{C_3}(B, q) = \prod_{i=1}^{a} \frac{\langle 3i - 1 \rangle}{\langle 3i - 2 \rangle} \cdot \prod_{1 \le i \le j < k \le a} \frac{\langle i + j + k - 1 \rangle_3}{\langle i + j + k - 2 \rangle_3}$$

$$\cdot \prod_{1 \le i < j \le k \le a} \frac{\langle i + j + k - 1 \rangle_3}{\langle i + j + k - 2 \rangle_3}$$

$$= \prod_{i=1}^{a} \frac{\langle 3i - 1 \rangle}{\langle 3i - 2 \rangle} \cdot \prod_{1 \le i \le j \le a} \frac{\langle a + i + j - 1 \rangle_3}{\langle i + 2j - 1 \rangle_3}$$

$$\cdot \prod_{1 \le i < j \le a} \frac{\langle a + i + j - 1 \rangle_3}{\langle i + 2j - 2 \rangle_3}.$$

The multiset decomposition

$$\{i + 2j - 1 : 1 \le i \le j \le a\} \cup \{i + 2j - 2 : 1 \le i < j \le a\}$$

$$= \{i + j - 1 : 2 \le 2i \le j \le 2a\}$$

$$= \{2i + j - 1 : 1 \le i \le j \le a\} \cup \{a + i + j - 1 : 1 \le i < j \le a\}$$

shows that (roughly) half the terms in the above expression can be

cancelled, yielding

$$P_{C_3}(B, q) = \prod_{i=1}^{a} \frac{\langle 3i - 1 \rangle}{\langle 3i - 2 \rangle} \cdot \prod_{1 \le i \le j \le a} \frac{\langle a + i + j - 1 \rangle_3}{\langle 2i + j - 1 \rangle_3}.$$

Essentially the same reasoning also shows that

$$P'_{C_3}(B, q) = \prod_{i=1}^{a} \frac{\langle 3i - 1 \rangle}{\langle 3i - 2 \rangle} \cdot \prod_{1 \le i \le j \le a} \frac{\langle a + i + j - 1 \rangle}{\langle 2i + j - 1 \rangle}.$$

In view of the similarity of these two expressions, let us define

$$Q_a(q) = \prod_{i=1}^{a} \frac{\langle 3i - 1 \rangle}{\langle 3i - 2 \rangle}, \qquad D_a(q) = \prod_{1 \le i \le j \le a} \frac{\langle a + i + j - 1 \rangle}{\langle 2i + j - 1 \rangle},$$

so that $P_{C_3}([a]^3, q) = Q_a(q)D_a(q^3)$ and $P'_{C_3}([a]^3, q) = Q_a(q)D_a(q)$.

One can check directly that the number of even terms in the numerators and denominators of $D_{2a}(q)$ and $D_{2a+1}(q)$ are a^2 and $(a + 1)^2$, respectively. Hence, $D_a(-1)$ is finite (and nonzero). It is also easy to check that $Q_a(-1)$ is finite, so we may conclude that $P_{C_3}([a]^3, -1) = P'_{C_3}([a]^3, -1) = Q_a(-1)D_a(-1)$. Furthermore, since $Q_{2a+1}(q)$ has a zero at $q = -1$, it follows that $P'_{C_3}([2a + 1]^3, -1) = 0$. Note also that the complementation map reverses parity if $|B|$ is odd, which proves $n_{C_3 \times K}([2a + 1]^3) = 0$.

To finish the proof, we show that $P'_{C_3}([2a]^3, -1) = n_{C_3 \times K}([2a]^3)$. For this we have

$$Q_{2a}(-1) = \prod_{i=1}^{a} \frac{3i - 2}{3i - 1} = 1/Q_a(1),$$

and we claim that there is a similar relation involving $D_{2a}(q)$, namely,

$$D_{2a}(-1) = Q_a(1)[D_a(1)]^2. \tag{3.1}$$

Once this is established, it will follow that $P'_{C_3}([2a]^3, -1) = D_a(1)^2$. This agrees with the known formula for $n_{C_3 \times K}([2a]^3)$ due to Kuperberg [K].

Remark 3.1.   The numbers $D_a(1)$ for $a = 0, 1, 2, \ldots$ comprise the infamous sequence that begins $1, 2, 7, 42, 429, 7436, \ldots$ [R]. They were first considered by Andrews [A3], who proved that $D_a(1)$ is the number of descending plane partitions with parts $\le a$. He also conjectured that $D_a(q)$ was the generating function for these plane partitions, weighted according to the sum of the parts. This conjecture was later proved by Mills *et al.* [MRR1]. Note that this implies the non-obvious fact that $D_a(q)$ is a polynomial with (nonnegative) integer coefficients. Mills *et al.* later

conjectured that $D_a(1)^2 = n_{C_3 \times K}([2a]^3)$ and $D_a(1) = n_{S_3 \times K}([2a]^3)$ (see [MRR2] and [St1]); these conjectures were proved by Kuperberg (as cited above) and Andrews [A4], respectively.

Thus it remains only to verify (3.1). For this, we begin by noting that, since $D_{2a}(q)$ has a finite, nonzero limit at $q = -1$, we can evaluate the limit by means of the cancellation rule mentioned at the beginning of Section 2. Since the even terms in the numerator of $D_{2a}(q)$ occur for $i + j$ odd, their contribution amounts to

$$\prod_{1 \le 2i \le 2j-1 \le 2a} 2a + 2i + 2j - 2 \prod_{1 \le 2i-1 \le 2j \le 2a} 2a + 2i + 2j - 2$$

$$= \prod_{1 \le i, j \le a} 2a + 2i + 2j - 2.$$

Similarly, the contributions from the denominator amount to

$$\prod_{1 \le i \le 2j-1 \le 2a} 2i + 2j - 2 = \prod_{1 \le 2i \le 2j-1 \le 2a} 4i + 2j - 2$$

$$\times \prod_{1 \le 2i-1 \le 2j-1 \le 2a} 4i + 2j - 4,$$

so we have

$$D_{2a}(-1) = \frac{\prod_{1 \le i, j \le a} a + i + j - 1}{\prod_{1 \le i \le 2j-1 \le 2a} i + j - 1}$$

$$= \frac{\prod_{1 \le i, j \le a} a + i + j - 1}{\prod_{1 \le i < j \le a} 2i + j - 1} \cdot \frac{\prod_{1 \le i \le a+1} a + 2i - 1}{\prod_{1 \le i \le j \le a+1} 2i + j - 2}$$

$$= \frac{\prod_{1 \le i, j \le a} a + i + j - 1}{\prod_{1 \le i \le j \le a} 2i + j - 1} \cdot \frac{\prod_{1 \le i \le a} a + 2i - 1}{\prod_{1 \le i < j \le a+1} 2i + j - 2}$$

$$\cdot \prod_{i=1}^{a} \frac{3i - 1}{3i - 2}$$

$$= \prod_{i=1}^{a} \frac{3i - 1}{3i - 2} \cdot \left[\prod_{1 \le i \le j \le a} \frac{a + i + j - 1}{2i + j - 1} \right]^2 = Q_a(1) D_a(1)^2,$$

thus proving (3.1).

Although it was not needed to prove Theorem 1.1, the following fact is noteworthy.

PROPOSITION 3.2.   *We have $P'_{C_3}(B, q) \in \mathbf{Z}[q]$.*

Proof.   Let $\Phi_r(q) \in \mathbf{Z}[q]$ denote the rth cyclotomic polynomial, i.e., the minimal polynomial of a primitive rth root of unity over $\mathbf{Q}$. Let $A(q) \in \mathbf{Z}[q]$ be any polynomial whose zeros are roots of unity. It is easily shown that

$$\Phi_r(q^3) = \begin{cases} \Phi_{3r}(q) & \text{if } 3 | r \\ \Phi_{3r}(q)\Phi_r(q) & \text{if } 3 \nmid r. \end{cases}$$

It follows that if $B(q) \in \mathbf{Z}[q]$ divides $A(q^3)$, and no zeros of $B(q)$ are primitive rth roots of unity with $3 | r$, then $B(q)$ divides $A(q)$.

Now let $A(q) = D_a(q)$ (known to be a polynomial by Remark 3.1), and let $B(q)$ be the denominator obtained when $Q_a(q)$ is expressed as a quotient of relatively prime polynomials. It is clear from their definitions that the zeros of $D_a(q)$ and $P_{C_3}([a]^3, q)$ occur only at roots of unity. Therefore, since $P_{C_3}([a]^3, q) = Q_a(q)D_a(q^3)$ is known to be a polynomial with integer coefficients (Table 1), it follows that $B(q)$ divides $A(q^3)$. From the definition of $Q_a(q)$ it is clear that $B(q)$ is a divisor of $\prod_{i=1}^a (1 - q^{3i-2})$, so no zero of $B(q)$ is of order divisible by 3. Thus we have met the above hypotheses, so we conclude that $B(q)$ divides $D_a(q)$, and therefore $P_{C_3}([a]^3, q) = Q_a(q)D_a(q) \in \mathbf{Z}[q]$.   ∎

4. TOTAL SYMMETRY WITH COMPLEMENTATION

In this section, we prove Theorem 1.1 in the case $G = S_3$.

Consider the evaluation of $P'_{S_3}(B - 1)$, where $B = [a]^3$. By taking the elements (i, j, k) with $1 \leq i \leq j \leq k \leq a$ as orbit representatives for B/S_3 we obtain

$$P'_{S_3}(B, q) = \prod_{1 \leq i \leq j \leq k \leq a} \frac{\langle i + j + k - 1 \rangle}{\langle i + j + k - 2 \rangle} = \prod_{1 \leq i \leq j \leq a} \frac{\langle a + i + j - 1 \rangle}{\langle i + 2j - 2 \rangle}.$$

$$(4.1)$$

If a is odd, it is easy to show that this rational function has a zero of order $(a + 1)/2$ at $q = -1$, so we have $P'_{S_3}(B, -1) = 0$ in such cases. If $B = [2a]^3$, then the even terms in the numerator of (4.1) occur when $i + j$ is odd; in the denominator they occur when i is even. In both cases there are

a^2 such terms, so the limit is finite and nonzero (cf. the discussion at the beginning of Section 2). Suppressing the odd terms in the numerator yields

$$\prod_{2i \le 2j-1 \le 2a} 2a + 2i + 2j - 2 \cdot \prod_{2i-1 \le 2j \le 2a} 2a + 2i + 2j - 2$$

$$= \prod_{i,j \le a} 2a + 2i + 2j - 2.$$

Similarly, suppressing the odd terms in the denominator yields

$$\prod_{2i \le j \le 2a} 2i + 2j - 2 = \prod_{2i \le j \le a+i} 2i + 2j - 2 \cdot \prod_{a+i < j \le 2a} 2i + 2j - 2$$

$$= \prod_{i \le j \le a} 4i + 2j - 2 \cdot \prod_{i < j \le a} 2a + 2i + 2j - 2,$$

so we have

$$P'_{S_3}([2a]^3, -1) = \prod_{1 \le i \le j \le a} \frac{a + i + j - 1}{2i + j - 1} = D_a(1). \qquad (4.2)$$

By the result of Andrews [A4] (see Remark 3.1) we know that $D_a(1) = n_{S_3 \times K}([2a]^3)$. Since $n_{S_3 \times K}([2a + 1]^3)$ is obviously zero, we therefore have $P'_{S_3}(B, -1) = n_{S_3 \times K}(B)$ for all B.

The remainder (and bulk) of this section is devoted to proving that

$$N'_{S_3}(B, -1) = P'_{S_3}(B, -1) \qquad (4.3)$$

for all B. Before starting the proof, let us note that the number of S_3-orbits in $B = [a]^3$ is $\binom{a+2}{3}$. This quantity need not be odd when a is odd, so the complementation map cannot be used to prove that $N'_{S_3}([2a + 1]^3, -1) = 0$ for all a. We will have to establish this fact by less direct means.

The outline of the proof of (4.3) is as follows. First, we use non-intersecting path methods to construct a skew-symmetric matrix whose Pfaffian is $N'_{S_3}([a]^3, q)$. The matrices are obtained by elaborating the construction in [Ste2], where a Pfaffian is given for $n_{S_3}([a]^3)$. (Another Pfaffian for $N'_{S_3}([a]^3, q)$ has been given by Okada [O].) Once the Pfaffian is obtained, we specialize to $q = -1$, and use linear algebra to reduce the Pfaffian to a determinant that is known to be the number of descending plane partitions with parts $\le a$. Since it is relatively easy to prove that descending plane partitions are enumerated by this determinant (Theorem 3 of [A3]),

but relatively hard to prove that the determinant agrees with (4.2) (Theorem 10 of [A3]), the argument we give below can thus be viewed as a (nearly) self-contained proof that $N'_{S_3}([2a]^3, -1)$ is the number of descending plane partitions with parts $\leq a$.

4.1. Non-intersecting Paths

Let $\mathscr{D}_a$ be the directed graph with vertex set $\{(i, j): 0 \leq i \leq j < a\}$ and arcs directed from vertex u to vertex v if $v - u = (1, 0)$ or $(0, -1)$. In Section 1 of [Ste2] we proved that there is a one-to-one correspondence between S_3-invariant plane partitions in $B = [a]^3$ and sets of non-intersecting (directed) paths in $\mathscr{D}_a$ in which the initial points of the paths are of the form $(0, i)$, and the terminal points of the paths are of the form (j, j).

It will be necessary for what follows to create an enlarged graph $\mathscr{D}'_a$ by adding a new vertices of the form $(-1, 0), (-1, 1), \ldots, (-1, a - 1)$, with arcs directed from $(-1, i)$ to $(0, i)$. For example, the graph $\mathscr{D}'_5$ is displayed in Fig. 1. Note that, by deleting each first arc in a set of non-intersecting paths with initial points of the form $(-1, i)$, we obtain a set of such paths with initial points of the form $(0, i)$, and conversely.

Let $\mathscr{P}_a$ denote the set of paths in $\mathscr{D}'_a$ with initial and terminal points of the form $(-1, i)$ and (j, j) for some i and j. Define the *area* bounded by a path P to be the number of vertices (i, j) such that there is a vertex of the form (i, k) on P with $i \geq 0$ and $j \leq k$. Let $|P|$ denote the area bounded by P. Note that $|P|$ can be obtained by assigning weights to the arcs as in Fig. 1 (horizontal arcs are given weight 0), and then adding the weights of the arcs of P.
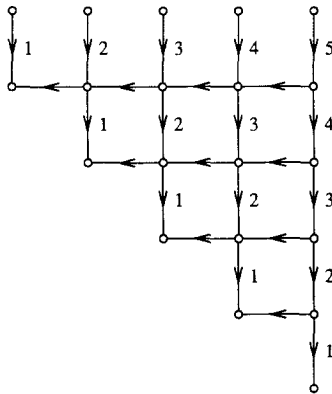

FIGURE 1

Let $\{P_1, \ldots, P_r\}$ be a set of non-intersecting paths in $\mathscr{D}'_a$, and assume that the paths are ordered so that the initial points $(-1, i_1), \ldots, (-1, i_r)$ of $P_1, \ldots, P_r$ satisfy $i_1 > \cdots > i_r$. The S_3-invariant plane partition π in $B = [a]^3$ that corresponds to this set of paths (according to the bijection in [Ste2]) can be described as follows: the vertex (j, k) of $\mathscr{D}_a$ is bounded by the path P_i if and only if $(i, i + j, i + k) \in \pi$. In particular, since $i \le i + j \le i + k$, it follows that the number of S_3-orbits of points in π corresponds to the sum of the areas bounded by the paths P_i.

Summarizing Lemma 1.2 of [Ste2] and the preceding analysis, we have

$$N'_{S_3}([a]^3, q) = \sum_{S \subset \mathscr{P}_a} q^{|S|}, \qquad (4.4)$$

where S ranges over all non-intersecting subsets of $\mathscr{P}_a$, and $|S|$ denotes the sum of the areas bounded by the paths in S.

4.2. Pfaffians

For $i \ge 0$, define $u_i(q) = \sum_{P \in \mathscr{P}_a} q^{|P|}$ to be the area generating function for all paths $P \in \mathscr{P}_a$ with initial point $(-1, i)$. For $j > i \ge 0$, define

$$u_{ij}(q) = \sum_{P, Q \in \mathscr{P}_a} q^{|P| + |Q|},$$

where the sum ranges over all non-intersecting pairs $P, Q \in \mathscr{P}_a$ such that the initial points of P and Q are $(-1, i)$ and $(-1, j)$. Note that u_i and u_{ij} are independent of a.

Given the path interpretation of $N'_{S_3}([a]^3, q)$ from (4.4), we can use Theorem 4.1 of [Ste1] to construct a skew-symmetric matrix whose Pfaffian is $N'_{S_3}([a]^3, q)$. (This has been done already for the case $q = 1$ in Section 2 of [Ste2].) In particular, by applying Theorem 4.1(b) of [Ste1], we obtain

LEMMA 4.1. *We have* $N'_{S_3}([2a + 1]^3, q) = \text{Pf}[u^*_{ij}]_{-1 \le i, j \le 2a}$, *where* $[u^*_{ij}]$ *is the unique skew-symmetric matrix satisfying*

$$u^*_{ij} = \begin{cases} (-1)^{i+j-1} + u_j(q) & \text{if } -1 = i < j, \\ (-1)^{i+j-1} + u_{ij}(q) & \text{if } 0 \le i < j. \end{cases}$$

Similarly, by applying Theorem 4.1(c) of [Ste1], we obtain

LEMMA 4.2.   *We have $N'_{S_3}([2a]^3, q) = \mathrm{Pf}[u_{ij}^{**}]_{-2 \le i, j < 2a}$, where $[u_{ij}^{**}]$ is the unique skew-symmetric matrix satisfying*

$$u_{ij}^{**} = \begin{cases} (-1)^{i+j-1} & \text{if } -2 = i < j, \\ (-1)^{i+j-1} + u_j(q) & \text{if } -1 = i < j, \\ (-1)^{i+j-1} + u_{ij}(q) & \text{if } 0 \le i < j. \end{cases}$$

It should be noted that the rows and columns we have used for these two matrices are in a permuted order relative to the ordering used in [Ste1], but is easy to check that this particular rearrangement does not affect the sign of the Pfaffian. It should also be noted that it will not suffice to merely evaluate the square roots of the determinants of the above matrices (at $q = -1$); we have no way of knowing *a priori* that $N'_{S_3}([a]^3, -1)$ is nonnegative. In particular, the sign of the outcome of our calculations depends on the branch of the Pfaffian chosen, so we should explicitly note that the branch used here and in [Ste1] is the one for which the direct sum of n copies of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has Pfaffian equal to one.

To explicitly describe the polynomials $u_i(q)$ and $u_{ij}(q)$, it will be convenient to introduce two notations. First, if z is any indeterminate and n is a nonnegative integer, define

$$(z; q)_n := (1 - z)(1 - zq) \cdots (1 - zq^{n-1}).$$

Second, for any Laurent polynomial $f(t)$, let $[t^n]f(t)$ denote the coefficient of t^n in $f(t)$. In particular, $[1]f(t)$ denotes the constant term.

LEMMA 4.3.   *For $j > i \ge 0$, we have*

(a) $u_i(q) = q^{i+1}(-q; q)_i.$

(b) $u_{ij}(q) = -q^{i+j+2}[1]\dfrac{t}{1 + t}[(qt; q)_i(qt^{-1}; q)_j - (qt^{-1}; q)_i(qt; q)_j].$

Proof.   (a) Let $c_{ij}(q)$ denote the generating function for paths in $\mathscr{D}'_a$ with initial and terminal points $(-1, i)$ and (j, j). Such paths can be specified by recording the series of weights on the vertical arcs of the path (cf. Fig. 1). The sequences so obtained consist of $j + 1$ strictly decreasing positive integers starting with $i + 1$ (and are characterized as such), so we have

$$c_{ij}(q) = q^{i+1}[t^j](-qt; q)_i = q^{i+1}[t^{-j}](-qt^{-1}; q)_i.$$

Therefore,

$$u_i(q) = \sum_j c_{ij}(q) = q^{i+1}(-q;q)_i.$$

For (b), note that, by the fundamental theorem on non-intersecting paths (e.g., Theorem 1 of [GV], or Theorem 1.2 of [Ste1]), we have

$$u_{ij}(q) = \sum_{0 \le k < l \le j} c_{ik}(q)c_{jl}(q) - c_{il}(q)c_{jk}(q)$$

$$= \sum_{0 \le k+l < j} c_{ik}(q)c_{j,j-l}(q) - \sum_{0 \le k+l < i} c_{i,i-l}(q)c_{jk}(q)$$

$$= q^{i+j+2} \sum_{0 \le k+l < j} [t^k](-qt;q)_i \cdot [t^{l-j}](-qt^{-1};q)_j$$

$$- q^{i+j+2} \sum_{0 \le k+l < i} [t^{l-i}](-qt^{-1};q)_i \cdot [t^k](-qt;q)_j$$

$$= q^{i+j+2} \sum_{0 \le r < j} [t^{r-j}](-qt;q)_i(-qt^{-1};q)_j - q^{i+j+2}$$

$$\times \sum_{0 \le r < i} [t^{r-i}](-qt^{-1};q)_i(-qt;q)_j$$

$$= q^{i+j+2}[1]\frac{t}{1-t}\big[(1-t^j)(-qt;q)_i(-qt^{-1};q)_j$$

$$- (1-t^i)(-qt^{-1};q)_i(-qt;q)_j\big].$$

To see that this does agree with the claimed formula, note that the transformation $t \to -t$ does not affect the constant term, and also that the Laurent polynomial

$$\frac{t}{1-t}\big[t^j(-qt;q)_i(-qt^{-1};q)_j - t^i(-qt^{-1};q)_i(-qt;q)_j\big]$$

has no constant term. ∎

Following [Ste2], let us define a skew-symmetric bilinear form on $\mathbf{Q}[t]$ by setting

$$(f,g) := [1]\frac{t}{1+t}[f(t)g(1/t) - f(1/t)g(t)]$$

for all $f, g \in \mathbf{Q}[t]$. Among the many useful properties of this form, let us explicitly note the following (cf. Lemma 3.1 of [Ste2]).

LEMMA 4.4. *For all $j > i \geq 0$ and $f, g \in \mathbf{Q}[t]$ we have*

(a) $(t^i, t^j) = (-1)^{i+j-1}$.

(b) $(f, 1) = f(-1) - f(0)$.

(c) $u_{ij}(q) = -q^2(q^i(qt; q)_i, q^j(qt; q)_j)$.

(d) $((t + t^2)f(t^2), (t + t^2)g(t^2)) = 0$.

Proof. Parts (a) and (b) are routine. Part (c) follows directly from Lemma 4.3. Part (d) is a simple consequence of (a). ∎

4.3. Linear Algebra

We now specialize to the case $q = -1$.

Let $\langle \, \cdot \, , \cdot \, \rangle$ denote the skew-symmetric bilinear form on $\mathbf{Q}[t]$ defined by the property that for $i, j \geq 0$, $\langle t^i, t^j \rangle$ is the i, j-entry of the matrices in Lemmas 4.1 and 4.2 at $q = -1$. By Lemma 4.4 we have

$$\langle t^i, t^j \rangle = (t^i, t^j) - \big((-1)^i(-t; -1)_i, (-1)^j(-t; -1)_j\big).$$

Since $(-t; -1)_{2i} = (1 - t^2)^i$ and $(-t; -1)_{2i+1} = (1 + t)(1 - t^2)^i$, it follows that

$$\langle f, g \rangle = (f, g) - \big(f_0(1 - t^2) - (1 + t)f_1(1 - t^2),$$
$$g_0(1 - t^2) - (1 + t)g_1(1 - t^2)\big) \qquad (4.5)$$

for all $f, g \in \mathbf{Q}[t]$, where $f(t) = f_0(t^2) + tf_1(t^2)$ and $g(t) = g_0(t^2) + tg_1(t^2)$.

Now extend $\langle \, \cdot \, , \cdot \, \rangle$ skew-symmetrically to $t^{-1}\mathbf{Q}[t]$ by defining

$$\langle t^{-1}, f \rangle = f(-1) - f(0) \qquad (4.6)$$

for all $f \in \mathbf{Q}[t]$. It is easy to see that $\langle t^{-1}, t^i \rangle = (-1)^i + u_i(-1)$ (cf. Lemma 4.3(a)), so Lemma 4.1 implies

$$N'_{S_3}([2a + 1]^3, -1) = \mathrm{Pf}[\langle t^i, t^j \rangle]_{-1 \leq i, j \leq 2a}.$$

We claim that $\langle \, \cdot \, , \cdot \, \rangle$ is degenerate on $t^{-1}\mathbf{Q}[t]$. More precisely, we claim that

$$\langle 2t^{-1} + 1, t^{-1} + f(t) \rangle = \langle 2t^{-1} + 1, f(t) \rangle = 0 \qquad (4.7)$$

for all $f \in \mathbf{Q}[t]$. Considering (4.6), this amounts to showing that $\langle 1, f \rangle = 2f(0) - 2f(-1)$. However, by (4.5) and Lemma 4.4(b), we have

$$\langle 1, f \rangle = (1, f) - \left(1, f_0(1 - t^2) - (1 + t)f_1(t^2)\right)$$

$$= (f(0) - f(-1)) - (f_0(1) - f_1(1) - f_0(0)) = 2f(0) - 2f(-1),$$
$$(4.8)$$

as desired. Consequently, we have $N'_{S_3}([2a + 1]^3, -1) = 0$.

For the even cases, let us further extend $\langle \cdot, \cdot \rangle$ skew-symmetrically to $t^{-2}\mathbf{Q}[t]$ by defining

$$\langle t^{-2}, f(t) \rangle = -f(-1)$$

for all $f \in t^{-1}\mathbf{Q}[t]$. By Lemma 4.2, it follows that

$$N'_{S_3}\left([2a]^3, -1\right) = \mathrm{Pf}[\langle t^i, t^j \rangle]_{-2 \le i, j < 2a}.$$

It will be convenient to eliminate the first two rows of this Pfaffian. To achieve this, consider the linear transformation $\psi \colon t^{-2}\mathbf{Q}[t] \to t^{-2}\mathbf{Q}[t]$ defined by

$$\psi(f(t)) = f(t) + \tfrac{1}{2}[t^{-1}]f(t)$$

for all $f \in t^{-2}\mathbf{Q}[t]$. This is a lower triangular, unit-diagonal transformation with respect to the ordered basis $t^{-2}, t^{-1}, 1, t, \ldots$, so it follows that

$$N'_{S_3}\left([2a]^3, -1\right) = \mathrm{Pf}[\langle \psi(t^i), \psi(t^j) \rangle]_{-2 \le i, j < 2a}.$$

However, $\langle \psi(t^{-2}), \psi(t^{-1}) \rangle = \langle t^{-2}, t^{-1} + 1/2 \rangle = 1/2$, and

$$\langle \psi(f(t)), \psi(t^{-1}) \rangle = \langle f(t), t^{-1} + 1/2 \rangle = 0$$

for all $f \in \mathbf{Q}[t]$ by (4.7), so, by a Laplace-type expansion along the first two rows and columns of the Pfaffian, we obtain

$$2N'_{S_3}\left([2a]^3, -1\right) = \mathrm{Pf}[\langle \psi(t^i), \psi(t^j) \rangle]_{0 \le i, j < 2a} = \mathrm{Pf}[\langle t^i, t^j \rangle]_{0 \le i, j < 2a}.$$

Now define another linear map $\varphi \colon \mathbf{Q}[t] \to \mathbf{Q}[t]$ by setting $\varphi(tf(t^2)) = tf(t^2)$ and

$$\varphi(f(t^2)) = (1 + 1/t)(f(t^2) - f(0)) + f(0)$$

$$= f(t^2) + t^{-1}(f(t^2) - f(0))$$

for all $f \in \mathbf{Q}[t]$. It is easy to see that φ is an upper-triangular, unit-diagonal transformation with respect to the ordered basis $1, t, t^2, \ldots,$ so we have

$$2N'_{S_3}\big([2a]^3, -1\big) = \mathrm{Pf}\big[\langle \varphi(t^i), \varphi(t^j)\rangle\big]_{0 \le i, j < 2a}.$$

We claim that the image of $\mathbf{Q}[t^2]$ under φ is isotropic; i.e.,

$$\langle \varphi(f(t^2)), \varphi(g(t^2))\rangle = 0$$

for all $f, g \in \mathbf{Q}[t]$. To prove this, it suffices by linearity and skew-symmetry to consider two cases: (1) $f(0) = g(0) = 0$, and (2) $f(t) = 1, g(0) = 0$. In the first case, we have $\varphi(f(t^2)) = (1 + 1/t)f(t^2)$ and $\varphi(g(t^2)) = (1 + 1/t)g(t^2)$, so (4.5) implies

$$
\begin{aligned}
\langle \varphi(f(t^2)), \varphi(g(t^2))\rangle &= \big((t^2 + t)f^*(t^2), (t^2 + t)g^*(t^2)\big) \\
&\quad + \big((t^2 + t)f^*(1 - t^2), (t^2 + t)g^*(1 - t^2)\big),
\end{aligned}
$$

where $f^*(t) = f(t)/t$ and $g^*(t) = g(t)/t$. Both terms in this expression are zero, by Lemma 4.4(d). In the second case, note that (4.8) implies

$$\langle \varphi(f(t^2)), \varphi(g(t^2))\rangle = \langle 1, (t^2 + t)g^*(t^2)\rangle = 0,$$

so the claim follows.

Thus (4.9) is the Pfaffian of a matrix $[b_{ij}]$ with the property that $b_{2i, 2j} = 0$ for all i, j. It is easy to show that for such matrices one has

$$\mathrm{Pf}\big[b_{ij}\big]_{0 \le i, j < 2a} = \det\big[b_{2i, 2j+1}\big]_{0 \le i, j < a}, \qquad (4.10)$$

so we have

$$2N'_{S_3}\big([2a]^3, -1\big) = \det\big[\langle \varphi(t^{2i}), \varphi(t^{2j+1})\rangle\big]_{0 \le i, j < a}.$$

More generally, it follows that

$$2N'_{S_3}\big([2a]^3, -1\big) = \det\big[\langle \varphi(f_i(t^2)), \varphi(tf_j(t^2))\rangle\big]_{0 \le i, j < a},$$

where $f_0, f_1, f_2, \ldots,$ is any sequence of monic polynomials with $\deg(f_i) = i$.

To simplify matters, let us introduce another bilinear form on $\mathbf{Q}[t]$ by defining

$$\langle f, g\rangle' := \langle \varphi(f(t^2)), \varphi(tg(t^2))\rangle$$

for all $f, g \in \mathbf{Q}[t]$. If $f(0) = 0$, then we have $\varphi(f(t^2)) = (1 + 1/t)f(t^2)$,

so, in such cases,

$$\langle f, g \rangle' = \langle (1 + 1/t)f(t^2), tg(t^2) \rangle = ((t + t^2)f^*(t^2), tg(t^2))$$
$$- ((t + t^2)f^*(1 - t^2), (1 + t)g(1 - t^2)),$$

where $f^*(t) = f(t)/t$. Using the definition of $(\cdot, \cdot)$, we obtain

$$\langle f, g \rangle' = [1]\frac{t}{1+t}[(1+t)f^*(t^2)g(t^{-2}) - (1 + 1/t)f^*(t^{-2})g(t^2)]$$

$$- [1]\frac{t}{1+t}[(1+t)^2 f^*(1 - t^2)g(1 - t^{-2})$$

$$- (1 + 1/t)^2 f^*(1 - t^{-2})g(1 - t^2)]$$

$$= [1]tf^*(t^2)g(t^{-2}) - [1]f^*(t^{-2})g(t^2)$$

$$- [1](t + t^2)f^*(1 - t^2)g(1 - t^{-2})$$

$$+ [1](1 + 1/t)f^*(1 - t^{-2})g(1 - t^2)$$

$$= -[1]f^*(t^{-2})g(t^2) + [1](1 - t^2)f^*(1 - t^2)g(1 - t^{-2})$$

$$= [1]f(1 - t)g(1 - 1/t) - [1]tf(1/t)g(t).$$

Note that the third equality is obtained by deleting the odd powers of t.

In the case $f(t) = 1$, (4.8) implies $\langle 1, g \rangle' = \langle 1, tg(t^2) \rangle = 2g(1)$, whereas the above expression yields $g(1)$. Thus, to summarize the above calculations, we have proved

LEMMA 4.5. *If* $f_0, f_1, f_2, \ldots \in \mathbf{Q}[t]$ *are monic and* $\deg(f_i) = i$, *then*

$$N'_{S_3}([2a]^3, -1) = \det[c_{ij}]_{0 \leq i, j < a},$$

where $c_{ij} = [1]f_i(1 - t)f_j(1 - 1/t) - tf_i(1/t)f_j(t)$.

By choosing $f_i(t) = (t - 1)^i$, we obtain

$$c_{ij} = [1](-t)^{i-j} + [1](-t)^{-(i-1)}(t - 1)^{i+j} = \delta_{ij} + (-1)^{i+j}\binom{i + j}{i - 1}.$$

If we rescale the ith row and column of $[c_{ij}]$ by $(-1)^i$, we obtain

$$N'_{S_3}([2a]^3, -1) = \det\left[\delta_{ij} + \binom{i + j}{i - 1}\right]_{0 \leq i, j < a}.$$

The first row of this matrix is $[1, 0, 0, \ldots]$; eliminating this row and the first

column therefore yields

$$N'_{S_3}([2a]^3, -1) = \det\left[\delta_{ij} + \binom{i + j + 2}{i}\right]_{0 \le i, j < a - 1}.$$

By Theorem 10 of [A3] this determinant is known to be $D_a(1)$ (cf. (4.2)), so the proof of (4.3) is now complete.

5. TOTAL SYMMETRY WITH A TWIST

In this section, we prove Theorem 1.2 in the case $G = S_3$.
Assume $B = [a]^3$. Using the notation introduced in Section 2.2, we have

$$
\begin{aligned}
P_{S_3}(B, q) &= \prod_{i=1}^{a} \frac{\langle 3i - 1\rangle}{\langle 3i - 2\rangle} \cdot \prod_{1 \le i \ne j \le a} \frac{\langle 2i + j - 1\rangle_3}{\langle 2i + j - 2\rangle_3} \\
&\qquad \cdot \prod_{1 \le i < j < k \le a} \frac{\langle i + j + k - 1\rangle_6}{\langle i + j + k - 2\rangle_6} \\
&= \prod_{i=1}^{a} \frac{\langle 3i - 1\rangle\langle 3i - 2\rangle_3}{\langle 3i - 2\rangle\langle 3i - 1\rangle_3} \cdot \prod_{i,j=1}^{a} \frac{\langle 2i + j - 1\rangle_3}{\langle 2i + j - 2\rangle_3} \\
&\qquad \cdot \prod_{1 \le i < j \le a} \frac{\langle a + i + j - 1\rangle_6}{\langle i + 2j - 1\rangle_6} \\
&= \prod_{i=1}^{a} \frac{\langle 3i - 1\rangle\langle 3i - 2\rangle_3\langle a + 2i - 1\rangle_3}{\langle 3i - 2\rangle\langle 3i - 1\rangle_3\langle 2i - 1\rangle_3} \\
&\qquad \cdot \prod_{1 \le i < j \le a} \frac{\langle a + i + j - 1\rangle_6}{\langle i + 2j - 1\rangle_6}. \qquad\qquad (5.1)
\end{aligned}
$$

If a is odd, it is easy to see that this rational function has a zero of order a at $q = -1$, so we have $P_{S_3}(B, -1) = 0$ in such cases. In the even cases, one can show that the first of the two factors in (5.1) approaches 1 in the limit $q \to -1$, so we have

$$
P_{S_3}([2a]^3, -1) = \prod_{1 \le i < j \le 2a} \frac{2a + i + j - 1}{i + 2j - 1} = \prod_{j=1}^{2a-1} \frac{(2j + 1)!(2a + 2j)!}{(3j + 1)!(2a + j)!}. \tag{5.2}
$$

On the other hand, there is an explicit formula for $n_{S_3^*}(B)$ due to Mills *et al.* [MRR3]. By Theorem 5.3 of [Ste2] (an alternative proof of an equivalent formula) it is known that

$$2^{2a-1} n_{S_3^*}([2a]^3) = n_{S_3}([2a-1]^3) = P'_{S_3}([2a-1]^3, 1).$$

Furthermore, by (4.1) we have

$$P'_{S_3}([2a-1]^3, 1) = \prod_{1 \le i \le j < 2a} \frac{2a + i + j - 2}{i + 2j - 2}$$

$$= \prod_{j=0}^{2a-2} \frac{(2j)!(2a+2j)!}{(3j+1)!(2a+j-1)!}. \qquad (5.3)$$

Using (5.2) and (5.3), it is now a routine matter to check that

$$P'_{S_3}([2a-1]^3, 1) = 2^{2a-1} P_{S_3}([2a]^3, -1),$$

thus proving $P_{S_3}([2a]^3, -1) = n_{S_3^*}([2a]^3)$.

Now consider the evaluation of $N_{S_3}(B, -1)$. If a is odd, then $|B|$ is odd, so the complementation map provides a sign-reversing involution that proves $N_{S_3}(B, -1) = n_{S_3^*}(B) = 0$, and this agrees with the above analysis of $P_{S_3}(B, -1)$.

The remainder (and bulk) of this section is devoted to proving that

$$N_{S_3}([2a]^3, -1) = n_{S_3^*}([2a]^3).$$

The proof will be similar to the proof of (4.3). We use non-intersecting path methods to construct a skew-symmetric matrix whose Pfaffian is $N_{S_3}([2a]^3, q)$, and then use linear algebra to show that, for $q = -1$, the Pfaffian can be reduced to a determinant whose value is known to be $n_{S_3^*}([2a]^3)$.

5.1. Non-intersecting Paths

Let $\mathscr{D}_a$ and $\mathscr{D}'_a$ denote the directed graphs we defined in Section 4.1, and let $\mathscr{P}_a$ continue to denote the set of all paths in $\mathscr{D}'_a$ whose initial and terminal points are of the form $(-1, i)$ and (j, j). Recall that there is a one-to-one correspondence between S_3-invariant plane partitions in $B = [a]^3$ and non-intersecting subsets of $\mathscr{P}_a$.

Let $\{P_1, \ldots, P_r\}$ denote the set of paths corresponding to some S_3-invariant plane partition π, numbered so that the initial points $(-1, i_1), \ldots, (-1, i_r)$ of $P_1, \ldots, P_r$ satisfy $i_1 > \cdots > i_r$. Recall that if the vertex (j, k) (where $j \ge 0$) is in the area bounded by the path P_i, then
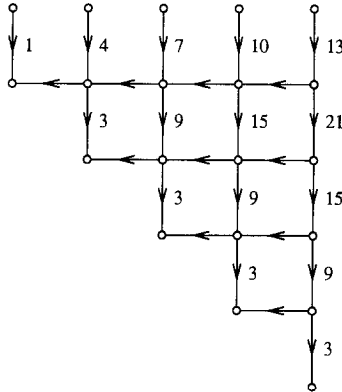
FIGURE 2

$(i, i + j, i + k) \in \pi$, and conversely. Therefore, let us modify the definition of the "area" bounded by a path P to be a weighted sum over the vertices of $\mathscr{D}_a$ bounded by P. The weight of a vertex should be the size of the S_3-orbit of the corresponding element of B. Thus, the vertex $(0, 0)$ has weight 1, the vertices of the form (j, j) or $(0, j)$ (where $j > 0$) have weight 3, and the vertices of the form (j, k) (where $k > j > 0$) have weight 6. As a counterpart to (4.4), we therefore have

$$N_{S_3}\!\left([a]^3, q\right) = \sum_{S \subset \mathscr{P}_a} q^{|S|^*}, \qquad (5.4)$$

where S ranges over all non-intersecting subsets of $\mathscr{P}_a$, and $|S|^*$ denotes the sum of the weighted areas bounded by the paths in S. If we assign weights to the arcs of $\mathscr{D}_a'$ as in Fig. 2 (with horizontal arcs having weight 0), then the weighted area bounded by a path $P \in \mathscr{P}_a$ can be obtained as the sum of the weights of the arcs of P.

5.2. Pfaffians

Proceeding as in Section 4.2, let $v_i(q) = \sum_{P \in \mathscr{P}_a} q^{|P|^*}$ denote the generating function for the weighted area of all paths in $\mathscr{P}_a$ with initial points $(-1, i)$, and for $j > i \geq 0$ define

$$v_{ij}(q) = \sum_{P, Q \in \mathscr{P}_a} q^{|P|^* + |Q|^*},$$

where the sum ranges over all non-intersecting pairs $P, Q \in \mathscr{P}_a$ such that the initial points of P and Q are $(-1, i)$ and $(-1, j)$.

Given the path interpretation of $N_{S_3}(B, q)$ in (5.4), the following is a direct consequence of Theorem 4.1(c) of [Ste1] (cf. Lemmas 4.1 and 4.2).

LEMMA 5.1.   *We have* $N_{S_3}([2a]^3, q) = \text{Pf}[v_{ij}^*]_{-2 \leq i, j < 2a}$, *where* $[v_{ij}^*]$ *is the unique skew-symmetric matrix satisfying*

$$v_{ij}^* = \begin{cases} (-1)^{i+j-1} & \text{if } -2 = i < j, \\ (-1)^{i+j-1} + v_j(q) & \text{if } -1 = i < j, \\ (-1)^{i+j-1} + v_{ij}(q) & \text{if } 0 \leq i < j. \end{cases}$$

The following result provides a simple formula for $v_i(q)$, and expresses $v_{ij}(q)$ in terms of the skew-symmetric form $(\cdot, \cdot)$ we introduced in Section 4.2.

LEMMA 5.2.   *For* $j > i \geq 0$, *we have*

(a) $v_i(q) = q^{3i+1}(-q^3; q^6)_i$.

(b) $v_{ij}(q) = -q^2(q^{3i}(q^3 t; q^6)_i, q^{3j}(q^3 t; q^6)_j)$.

Proof.   Using the weights in Fig. 2, let $c_{ij}(q)$ denote the generating function for paths in $\mathcal{D}'_a$ with initial and terminal points $(-1, i)$ and (j, j). It is easily shown that

$$c_{ij}(q) = q^{3i+1}[t^j](-q^3 t; q^6)_i.$$

The remainder of the proof is now essentially identical to that of Lemma 4.3.   ∎

5.3. Linear Algebra

Now specialize to the case $q = -1$.

Let us define a skew-symmetric bilinear form $[\cdot, \cdot]$ on $t^{-2}\mathbf{Q}[t]$ by setting

$$[f, g] := (f(t), g(t)) - (f(-1-t), g(-1-t)) \qquad (5.5)$$

for all $f, g \in \mathbf{Q}[t]$,

$$[t^{-1}, f] := f(-1) - f(0)$$

for all $f \in \mathbf{Q}[t]$, and

$$[t^{-2}, f] := -f(-1)$$

for all $f \in t^{-1}\mathbf{Q}[t]$. Using Lemma 5.2, it is easy to check that $[t^i, t^j]$ is the matrix of Lemma 5.1 at $q = -1$, so we have

$$N_{S_3}([2a]^3, -1) = \mathrm{Pf}([t^i, t^j])_{-2 \leq i, j < 2a}. \tag{5.6}$$

To eliminate the first two rows and columns of this Pfaffian, let us define a linear map $\psi : t^{-2}\mathbf{Q}[t] \to t^{-2}\mathbf{Q}[t]$ by setting

$$\psi(bt^{-2} + ct^{-1} + f(t)) = bt^{-2} + (c + f(-1))t^{-1} + f(t)$$

for all $f \in \mathbf{Q}[t]$ and $b, c \in \mathbf{Q}$. Note that this mapping has an upper-triangular, unit-diagonal matrix with respect to the ordered basis $t^{-2}, t^{-1}, 1, t, \ldots$. The Pfaffian in (5.6) will therefore be unaffected if we replace $[t^i, t^j]$ with $[\psi(t^i), \psi(t^j)]$. However, we have $[\psi(t^{-2}), \psi(t^{-1})] = [t^{-2}, t^{-1}] = 1$, and

$$\left[\psi(t^{-2}), \psi(f(t))\right] = \left[t^{-2}, f(-1)t^{-1} + f(t)\right] = f(-1) - f(-1) = 0$$

for all $f \in \mathbf{Q}[t]$. Thus, by a Laplace-type expansion of the Pfaffian, we can delete the first two rows and columns of $[\psi(t^i), \psi(t^j)]$, obtaining

$$N_{S_3}([2a]^3, -1) = \mathrm{Pf}\left([\psi(t^i), \psi(t^j)]\right)_{0 \leq i, j < 2a}.$$

The same effect can be achieved by defining a new bilinear form on $\mathbf{Q}[t]$ so that

$$\begin{aligned}
[f, g]' &:= [\psi(f), \psi(g)] = \left[f(-1)t^{-1} + f(t), g(-1)t^{-1} + g(t)\right] \\
&= [f, g] + f(-1)\left[t^{-1}, g(t)\right] - g(-1)\left[t^{-1}, f(t)\right] \\
&= [f, g] + f(0)g(-1) - f(-1)g(0) \tag{5.7}
\end{aligned}$$

for all $f, g \in \mathbf{Q}[t]$. In these terms, we have

$$N_{S_3}([2a]^3, -1) = \mathrm{Pf}([t^i, t^j]')_{0 \leq i, j < 2a},$$

and more generally,

$$N_{S_3}([2a]^3, -1) = \mathrm{Pf}\left([f_i, f_j]'\right)_{0 \leq i, j < 2a}, \tag{5.8}$$

where $f_0, f_1, \ldots$ is any monic basis of $\mathbf{Q}[t]$ with $\deg(f_i) = i$.

Now consider the involution on $\mathbf{Q}[t]$ defined by $f(t) \mapsto f^{\dagger}(t) := f(-1 - t)$. Let $\mathbf{Q}[t]^{\pm} = \{f \in \mathbf{Q}[t] : f(t) = \pm f(-1 - t)\}$ denote the two eigenspaces

for $^+$ on $\mathbf{Q}[t]$. By (5.5), it is easy to see that

$$[f^+, g^+] = -[f, g]$$

for all $f, g \in \mathbf{Q}[t]$, and the same is true for $[f, g]'$. It follows that $[f, g] = [f, g]' = 0$ for $f, g \in \mathbf{Q}[t]^+$ or $f, g \in \mathbf{Q}[t]^-$; i.e., $\mathbf{Q}[t]^{\pm}$ are both isotropic subspaces with respect to both forms. Since the mapping $f \mapsto f(t - 1/2)$ defines linear isomorphisms between $\mathbf{Q}[t]^{\pm}$ and the subspaces of even and odd polynomials in $\mathbf{Q}[t]$, it follows that there exist bases $f_i(t)$ of $\mathbf{Q}[t]$ such that $\deg(f_i) = i$ and $f_0, f_2, \ldots$ (resp., $f_1, f_3, \ldots$) span $\mathbf{Q}[t]^+$ (resp., $\mathbf{Q}[t]^-$). In such cases, we have $[f_{2i}, f_{2j}]' = 0$, so (4.10) and (5.8) imply

$$N_{S_3}([2a]^3, -1) = \det\big([f_{2i}, f_{2j+1}]'\big)_{0 \le i, j < a},$$

assuming that f_i is monic. Furthermore, since the mapping $f(t) \mapsto (t + 1/2)f(t)$ defines a linear isomorphism from $\mathbf{Q}[t]^+$ onto $\mathbf{Q}[t]^-$, it follows that once we have chosen f_{2i}, we can then choose $f_{2i+1}(t) = (t + 1/2)f_{2i}(t)$, obtaining

$$N_{S_3}([2a]^3, -1) = \det\big([f_{2i}(t), (t + 1/2)f_{2j}(t)]'\big)_{0 \le i, j < a},$$

where $f_0, f_2, \ldots$ is any monic basis of $\mathbf{Q}[t]^+$ with $\deg(f_{2i}) = 2i$.

We now introduce a linear map $\varphi : \mathbf{Q}[t]^+ \to \mathbf{Q}[t]^+$ defined by

$$\varphi(f)(t) := \frac{t(t + 1)}{(t - 1)(t + 2)}(f(t) - f(1)) + f(1).$$

Note that if $f(1) = 0$ then $f(-2) = 0$, since $f(t) = f(-1 - t)$, so this map is indeed well-defined. It is easy to see that $f \mapsto \varphi(f)$ preserves the degree and leading coefficient of f, so we have

$$N_{S_3}([2a]^3, -1) = \det\big([\varphi(f_{2i})(t), (t + 1/2)f_{2j}(t)]'\big)_{0 \le i, j < a}, \quad (5.9)$$

where $f_0, f_2, \ldots$ are as above.

LEMMA 5.3. *For all* $f, g \in \mathbf{Q}[t]^+$, *we have*

$$[\varphi(f)(t), (t + 1/2)g(t)]' = [1]f(t)g(1/t).$$

Proof. First note that if $f^+ \in \mathbf{Q}[t]^+$ and $f^- \in \mathbf{Q}[t]^-$, then (5.7) simplifies to

$$[f^+, f^-]' = [f^+, f^-] - 2f^+(0)f^-(0) = 2(f^+, f^-) - 2f^+(0)f^-(0), \quad (5.10)$$

using (5.5).

By linearity, it suffices to consider two cases: $f(t) = 1$, and $f(1) = 0$. In the first case, we have $\varphi(f) = 1$, and therefore

$$[\varphi(f)(t), (t + 1/2)g(t)]' = 2(1, (t + 1/2)g(t)) - g(0) = g(0)$$

by successive applications of (5.10) and Lemma 4.4(b). This agrees with $[1]f(t)g(1/t)$.

In the second case, we have $\varphi(f)(t) = t(t + 1)f(t)/(t - 1)(t + 2)$. Using (5.10) and the definition of $(\cdot, \cdot)$, we obtain

$$[\varphi(f)(t), (t + 1/2)g(t)]'$$

$$= 2\left(\frac{t(t + 1)}{(t - 1)(t + 2)}f(t), (t + 1/2)g(t)\right)$$

$$= [1]\frac{2t}{1 + t}\left[\frac{t(t + 1)(1/2 + 1/t)}{(t - 1)(t + 2)}f(t)g(1/t)\right.$$

$$\left. - \frac{(1 + 1/t)(1/2 + t)/t}{(1/t - 1)(1/t + 2)}f(1/t)g(t)\right]$$

$$= [1]\frac{t}{t - 1}f(t)g(1/t) - [1]\frac{t}{1 - t}f(1/t)g(t)$$

$$= [1]\frac{t}{t - 1}f(t)g(1/t) - [1]\frac{1}{t - 1}f(t)g(1/t) = [1]f(t)g(1/t).$$

Note that both summands in the third equality are Laurent polynomials (since $f(1) = 0$), so we justified in substituting $t \to 1/t$ in the following step. ∎

As a monic basis for $\mathbf{Q}[t]^+$, let us choose $f_{2i}(t) = t^i(t + 1)^i$. Since

$$[1]f_{2i}(t)f_{2j}(1/t) = [1]t^{i-j}(1 + t)^i(1 + 1/t)^j = \binom{i + j}{2j - i},$$

it follows that

$$N_{S_3}([2a]^3, -1) = \det\left[\binom{i + j}{2j - i}\right]_{0 \le i, j < a},$$

by (5.9) and Lemma 5.3. Using non-intersecting path methods, Mills *et al.* have shown that this determinant is the number of S_3^*-invariant plane partitions in $[2a]^3$ (see Theorem 3 of [MRR3]). Hence, the proof of Theorem 1.2 is now complete.

6. Posets with Good Complements

By a complemented poset we mean a triple $P = (X, \leq, c)$ consisting of a finite set X, a partial order $\leq$ on X, and an order-reversing involution $c\colon X \to X$. By abuse of notation we will sometimes identify P with the underlying set X.

Let $J(P)$ denote the lattice of order ideals of P. Note that the involution c can be lifted to $J(P)$ in an obvious way by defining $I \mapsto I^c :=$ $\{x \in X : x^c \notin I\}$ for all $I \in J(P)$. Let $N(P, q)$ denote the rank generating function of $J(P)$, i.e.,

$$N(P, q) := \sum_{I \in J(P)} q^{|I|},$$

and let $\mathrm{sc}(P)$ denote the number of self-complementary (i.e., $I = I^c$) order ideals of P. The part of the "$q = -1$ phenomenon" covered by Theorem 1.1 amounts to the assertion that

$$N(P, -1) = \mathrm{sc}(P) \tag{6.1}$$

for any of the (complemented) posets P of the form B/G, where G is a subgroup of S_3, and B is a product of three chains. Of course, our proof that these posets have this nice property is unsatisfactory in that it gives no insight into *why* the result is true. In an attempt to improve our understanding, it is natural to look for general classes of posets for which (6.1) holds. Of course, one obvious class of such posets is that for which $|X|$ is odd. In this case, the involution $I \mapsto I^c$ changes the parity of $|I|$, and thus proves $N(P, -1) = \mathrm{sc}(P) = 0$.

At the extreme, one might desire a general classification of all posets that satisfy (6.1) (or, even more generally, one could replace $N(P, q)$ by the rank generating function of any ranked, complemented poset, and look for instances where setting $q = -1$ yields the number of self-complementary elements), but such a task is probably so unwieldy as to be hopeless. A more realistic goal is to look for restricted circumstances where (6.1) holds. In this and the following section we will study one such circumstance in detail.

Assume that $P = (X, \leq, c)$ is a complemented poset. For each $k > 0$, define $P \times [k]$ to be the complemented poset obtained by partially ordering $X \times [k]$ via the product of the order on P and the natural total order on $[k]$, using $(x, i) \mapsto (x^c, k + 1 - i)$ as the order-reversing involution. Let $N_k(P, q) = N(P \times [k], q)$ denote the rank generating function for

$J(P \times [k])$, and let $sc_k(P) = sc(P \times [k])$ denote the number of self-complementary order ideals of $P \times [k]$. We will say that c is a *good complement* for P if

$$N_k(P, -1) = sc_k(P)$$

for all $k > 0$; i.e., a good complement is one for which $P \times [k]$ satisfies (6.1) for all k.

By Theorem 1.1 (in the cases with $|G| \leq 2$) we know that the posets $P = [a] \times [b]$ and $P = [a]^2/S_2 \cong J([2] \times [a-1])$ have good complements. In fact, there is a general class of posets with good complements, arising from representations of Lie algebras, that includes these two examples as special cases. To explain, let V be an irreducible representation of highest weight ω for some simple Lie algebra $\mathfrak{g}$. The weights of V are partially ordered by the rule that $\mu > \nu$ whenever $\mu - \nu$ is a sum of positive roots. If ω is a minuscule weight, then this poset is a distributive lattice, and thus of the form $J(P_\omega)$ for some poset P_ω. (For a proof and further details see [P2].) The posets P_ω are known as *minuscule posets*.

A uniform, representation-theoretic proof of the following result will appear in a sequel to this paper.

THEOREM 6.1. *Minuscule posets have good complements.*

The poset $P = [a] \times [b]$ is indeed minuscule; it is associated with the ath exterior power of the defining representation of sl_n ($n = a + b$). Similarly, the poset $P = J([2] \times [n])$ is the minuscule poset associated with the spin representation of so_{2n+4}. For a list of all minuscule posets, see Proposition 4.2 of [P2] or Exercise 4.25(f) of [St2]. We should remark that Theorem 6.1 immediately suggests the conjecture that Gaussian posets[2] have good complements. However, there are no known connected Gaussian posets that are not minuscule, and it is unknown whether all Gaussian posets are self-dual.

In the following section, we will derive a criterion (see Theorem 7.7) that makes it feasible, at least for small posets, to check whether a given order-reversing involution is a good complement. We have implemented a computer program that uses this criterion to find connected posets with good complements. We found that the numbers of such posets up to isomorphism on 4, 5, 6, and 7 vertices are 2, 4, 5, and 21, respectively. The smallest examples not covered by Theorem 6.1 are illustrated in Fig. 3. Dotted lines have been drawn to indicate the action of the order-reversing involution in cases that would otherwise be ambiguous.

Given that a product of two chains has a good complement, it is natural to ask whether a product of three or more chains might have this property.

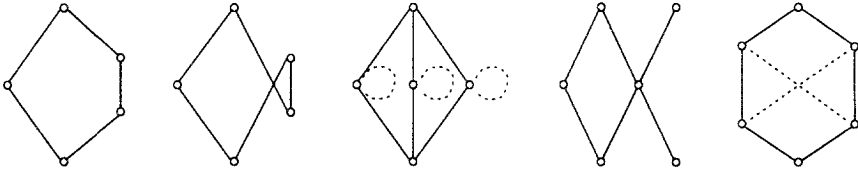[2]For the definition of a Gaussian poset, see Exercise 4.25 of [St2].

FIGURE 3

In fact, by explicit computations (and Theorem 7.7), one can show that the poset $[2]^3$ (i.e., the boolean algebra generated by three points) does have a good complement, but that neither $[2]^4$ nor $[2]^2 \times [3]$ has this property.

7. SELF-COMPLEMENTARY P-PARTITIONS

If $P = (X, \leq)$ is an arbitrary finite poset (not necessarily complemented), then the order ideals of $P \times [k]$ can also be regarded as the set of all P-partitions with parts $\leq k$, i.e., the set of order-reversing maps $f : P \to \{0, 1, \ldots, k\}$. The order ideal corresponding to the P-partition f is given by $\{(x, i) : 1 \leq i \leq f(x)\}$. Define $|f| = \sum_{x \in X} f(x)$, so that $|f|$ is the size of the order ideal corresponding to f. In these terms, the quantity $N_k(P, q)$ can be viewed as the generating function for P-partitions with parts $\leq k$, weighted by the sum of the parts.

Assume $|X| = n$, and let $X = \{x_1, \ldots, x_n\}$ be a natural labeling of the elements of X, i.e., a linear ordering with the property that

$$x_i \leq x_j \qquad \text{implies } i \leq j. \tag{7.1}$$

Let

$$\mathscr{L}(P) = \left\{ w \in S_n : x_{w_i} \leq x_{w_j} \text{ implies } i \leq j \right\}$$

denote the set of linear extensions of P. For $w \in S_n$, let $D(w) = \{i \in [n-1] : w_i > w_{i+1}\}$ denote the descent set of w, and define $d(w) := |D(w)|$ and $\mathrm{maj}(w) := \sum_{i \in D(w)} i$. The following result is Exercise 4.24(b) of [St2]. We include a sketch of the proof since the details will be important for what follows.

LEMMA 7.1. *For any naturally labeled poset P on n elements, we have*

$$N_k(P, q) = \sum_{w \in \mathscr{L}(P)} q^{\mathrm{maj}(w)} \begin{bmatrix} n + k - d(w) \\ n \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *denotes the Gaussian polynomial.*

Proof. For any P-partition f, write $f(i)$ as an abbreviation for $f(x_i)$. By the fundamental lemma on P-partitions (Lemma 4.5.3 of [St2]) one knows that the set of P-partitions with parts $\leq k$ is the disjoint union of the sets

$$
\mathscr{A}_k(w) = \{ f : k \geq f(w_1) \geq \cdots \geq f(w_n) \geq 0;
$$
$$
i \in D(w) \Rightarrow f(w_i) > f(w_{i+1}) \},
$$

where w ranges over all of $\mathscr{L}(P)$.

In the special case $w = \mathrm{id}$, it is clear that

$$
\sum_{f \in \mathscr{A}_k(w)} q^{|f|} = \begin{bmatrix} n + k \\ n \end{bmatrix}_q . \tag{7.2}
$$

In the general case, let e_w denote the unique minimal element of $\mathscr{A}_k(w)$ with respect to $| \cdot |$. It is easy to show that $|e_w| = \mathrm{maj}(w)$, and that $f \mapsto f - e_w$ defines a bijection between $\mathscr{A}_k(w)$ and $\mathscr{A}_{k - d(w)}(\mathrm{id})$. Hence (7.2) implies

$$
\sum_{f \in \mathscr{A}_k(w)} q^{|f|} = q^{\mathrm{maj}(w)} \begin{bmatrix} n + k - d(w) \\ n \end{bmatrix}_q ,
$$

and the result follows. ∎

We now seek a counterpart to Lemma 7.1 for the quantity $\mathrm{sc}_k(P)$ we defined in Section 6. Thus, let us assume that $P = (X, \leq, c)$ is a complemented partial order of an n-set X. The order-reversing involution on $P \times [k]$, when translated into the language of P-partitions, corresponds to the involution $f \mapsto f^c$, where $f^c(x) := k - f(x^c)$. In these terms, $\mathrm{sc}_k(P)$ is the number of self-complementary P-partitions (i.e., $f = f^c$) with parts $\leq k$.

Let us define a labeling $X = \{x_1, \ldots, x_n\}$ of P to be *c-compatible* if it is natural (i.e., satisfies (7.1)), and $(x_i)^c = x_{n+1-i}$ for $1 \leq i \leq n$.

LEMMA 7.2. *A complemented poset $P = (X, \leq, c)$ has a c-compatible labeling if and only if $|\{x \in X : x = x^c\}| \leq 1$.*

Proof. We claim that X can be partitioned into three sets, L, I, and U, where L is an order ideal of P, $I = \{x \in X : x = x^c\}$, and $U = \{x \in X : x^c \in L\}$ (an order filter of P). The proof is by induction on $|X|$, the case $|X| \leq 2$ being trivial. We begin by choosing a minimal element of P such that $x \neq x^c$. If no such element exists, then all minimal elements of P are c-invariant and thus belong to I. However, in this case, the fact that $x \mapsto x^c$ is order-reversing forces $X = I$. Otherwise, we may delete x and x^c from P, obtaining a smaller complemented poset P'. By induction, we

may therefore partition $X - \{x, x^c\}$ into sets L', I, and U', where L' is an order ideal of P' and U' is the filter complementary to L'. In this case, it is easy to see that $L = L' \cup \{x\}$ and $U = U' \cup \{x^c\}$ are a complementary ideal/filter pair for P, so the claim follows.

To prove the lemma, note that if P has a c-compatible labeling, then there is clearly at most one c-invariant element, namely x_i, where $i = (n + 1)/2$. For the converse, partition X into L, I, and U as above, where $|I| \le 1$. We obtain a c-compatible ordering of X by first choosing a linear extension of L, followed by the single element of I (if it exists), and then followed by the ordering of U that is complementary to the one chosen for L. ∎

For the moment, let us fix a particular c-compatible labeling of P (and assume that P has one). For any linear extension $w \in \mathscr{L}(P)$, define the complement $w^c \in \mathscr{L}(P)$ by setting $(w^c)_{n+1-i} := n + 1 - w_i$, and let $\mathscr{L}^0(P) := \{w \in \mathscr{L}(P) : w = w^c\}$ denote the set of self-complementary linear extensions. Note that $i \in D(w)$ if and only if $n - i \in D(w^c)$.

LEMMA 7.3. *If $P = (X, \le, c)$ is a complemented poset of size n with a c-compatible labeling, then*

$$\mathrm{sc}_k(P) = \sum_{w \in \mathscr{L}^0(P)} \begin{bmatrix} n + k - d(w) \\ n \end{bmatrix}_{-1}.$$

Proof. Using notation from the proof of Lemma 7.1, note that if $f \in \mathscr{A}_k(w)$, then $f^c \in \mathscr{A}_k(w^c)$. Therefore, self-complementary P-partitions belong only to the sets $\mathscr{A}_k(w)$ for $w \in \mathscr{L}^0(P)$, so we need only to prove that

$$|\{f \in \mathscr{A}_k(w) : f = f^c\}| = \begin{bmatrix} n + k - d(w) \\ n \end{bmatrix}_{-1} \qquad (7.3)$$

for each such w. Thus let us fix some $w \in \mathscr{L}^0(P)$, and suppose $f = f^c \in \mathscr{A}_k(w)$. As before, we write $f(i)$ in place of $f(x_i)$.

Case 1. Assume n is odd, and let $r = (n - 1)/2$. In this case, $w_{r+1} = r + 1$ and $f(r + 1) = k/2$ are forced, so there are no such f unless k is even. Assuming that k is indeed even, the constraints characterizing membership in $\mathscr{A}_k(w)$ (given that $f = f^c$) are

$$k \ge f(w_1) \ge \cdots \ge f(w_r) \ge k/2,$$
$$i \in D(w) \Rightarrow f(w_i) > f(w_{i+1}) \qquad (1 \le i \le r). \qquad (7.4)$$

Since $w = w^c$, it follows that exactly half the elements of $D(w)$ are in the range $1 \le i \le r$, so the number of such f satisfying (7.4) is also the

number of sequences $f_1, \ldots, f_r$ such that $k - d(w)/2 \geq f_1 \geq \cdots \geq f_r \geq k/2$; i.e., $\binom{r + k/2 - d(w)/2}{r}$. Thus agrees with (7.3), since $\begin{bmatrix} 2a + 1 \\ 2r + 1 \end{bmatrix}_{-1} = \binom{a}{r}$. Similarly, the fact that there are no solutions if k is odd agrees with (7.3) since $\begin{bmatrix} 2a \\ 2r + 1 \end{bmatrix}_{-1} = 0$.

Case 2. Assume n and k are even, and let $r = n/2$. In this case, the constraints characterizing membership in $\mathscr{A}_k(w)$ (given that $f = f^c$) are identical to (7.4). In particular, note that if $r \in D(w)$, then the constraint $f(w_r) > f(w_{r+1})$ can be replaced by the constraint $f(w_r) > k/2$. Furthermore, since $w = w^c$, it follows that the number of strict inequalities that occur in (7.4) is either $(d(w) + 1)/2$ or $d(w)/2$, according to whether $r \in D(w)$. Hence, the number of solutions for f in this case is $\binom{r + k/2 - \lceil d(w)/2 \rceil}{r}$. This agrees with (7.3), since $\begin{bmatrix} 2a + 1 \\ 2r \end{bmatrix}_{-1} = \begin{bmatrix} 2a \\ 2r \end{bmatrix}_{-1} = \binom{a}{r}$.

Case 3. Assume n is even and k is odd, and let $r = n/2$. In this case, the presence or absence of $r \in D(w)$ is immaterial since $f(w_r) = f(w_{r+1})$ (and $f = f^c$) can only happen if $f(w_r) = f(w_{r+1}) = k/2$. Therefore, the conditions for membership in $\mathscr{A}_k(w)$ are

$$k \geq f(w_1) \geq \cdots \geq f(w_r) \geq (k + 1)/2,$$

$$i \in D(w) \Rightarrow f(w_i) > f(w_{i+1}) \qquad (1 \leq i < r).$$

The number of strict inequalities that occur in these constraints is either $(d(w) - 1)/2$ or $d(w)/2$, according to whether $r \in D(w)$. Hence, the number of solutions for f in this case is $\binom{r + (k - 1)/2 - \lfloor d(w)/2 \rfloor}{r}$. Again, it is easy to check that this agrees with (7.3). ∎

The problem of enumerating self-complementary P-partitions in a general complemented poset is reducible to the case in which there exists a c-compatible labeling. Indeed, if f is a self-complementary P-partition with parts $\leq k$, then $f(x) = k/2$ for every $x \in I$, where $I = \{x \in X : x = x^c\}$. We therefore define the *contraction* of P to be the complemented poset P_* obtained by identifying all the elements of I (if any exist). In other words, if I is empty, then $P_* = P$; otherwise, we delete I from X, replacing the deleted elements by a single element, say x_*, with the property that if $x < y$ (resp. $x > y$) for some $x \in I$ and $y \in X - I$, then $x_* < y$ (resp., $x_* > y$). To maintain transitivity, it may also be necessary to add the relation $y < z$ if $y < x_* < z$. In any case, the fact that this construction does yield a partial order relies only on the fact that I is an antichain of P. If f is a self-complementary P-partition with parts $\leq k$, we may define the contraction f_* by setting $f_*(x) = f(x)$ for $x \notin I$, and (if $I \neq \varnothing$) $f_*(x_*) = k/2$ = the common value of f on I. It is clear that

$sc_k(P) = sc_k(P_*)$, since $f \mapsto f_*$ defines a bijection between self-complementary P-partitions and P_*-partitions. Thus we can replace Lemma 7.3 with the following more general result.

LEMMA 7.4. *Let* $P = (X, \leq, c)$ *be a complemented poset with contraction* P_*. *If* P_* *is given a c-compatible labeling, then*

$$sc_k(P) = \sum_{w \in \mathscr{L}^P(P_*)} \left[\begin{matrix} m + k - d(w) \\ m \end{matrix} \right]_{-1},$$

where $m = |P_*|$.

From this result it follows that $sc_{2k}(P)$ and $sc_{2k+1}(P)$ are polynomial functions of k. A more precise statement is as follows.

COROLLARY 7.5. *If* $P = (X, \leq, c)$ *is a complemented poset, then*

(a) $sc_{2k}(P)$ *is a polynomial of degree* $(n - i)/2$,

(b) $sc_{2k+1}(P)$ *is either identically* 0 *(if* $i > 0$*), or a polynomial of degree* $n/2$ *(if* $i = 0$*), where* $n = |X|$ *and* $i = |\{x \in X : x = x^c\}|$.

Proof. Let $m = |P_*|$. If m is even, then $i = 0$ and both assertions follow directly from Lemma 7.4 and the fact that for fixed a, $\left[\begin{matrix} a + 2k \\ m \end{matrix} \right]_{-1}$ is a polynomial of degree $m/2$ with a positive leading coefficient. If m is odd then $i > 0$ and $d(w)$ is even for every $w \in \mathscr{L}^0(P_*)$. Part (a) thus follows from the fact that $\left[\begin{matrix} 2a + 2k + 1 \\ m \end{matrix} \right]_{-1}$ is a polynomial of degree $(m - 1)/2$ with a positive leading coefficient, and part (b) is a consequence of the fact that $\left[\begin{matrix} 2a + 2k \\ m \end{matrix} \right]_{-1} = 0$. ∎

For example, let P be the second poset in Fig. 3. This complemented poset has the property that $P = P_*$, and it has two self-complementary linear extensions—one with no descent and one with two descents. Thus by Lemma 7.4, we have $sc_{2k+1}(P) = 0$ and

$$sc_{2k}(P) = \left[\begin{matrix} 2k + 5 \\ 5 \end{matrix} \right]_{-1} + \left[\begin{matrix} 2k + 3 \\ 5 \end{matrix} \right]_{-1} = (k + 1)^2.$$

Remark 7.6. Assuming that $P = (X, \leq, c)$ is a complemented poset, let us partition X into three parts L, I, and U, as in the proof of Lemma 7.2. If f is any self-complementary P-partition with parts $\leq k$, then f is completely determined by its restriction to L. Let us translate f by defining $\bar{f}(x) := f(x) - k/2$, and assume that the elements of L are labeled $x_1, x_2, \ldots, x_r$. If we disregard certain integrality conditions, the

inequalities that characterize $\bar{f}$ are of the form

  (a) $\bar{f}(x_i) \geq \bar{f}(x_j)$ if $x_i \leq x_j$,
  (b) $\bar{f}(x_i) + \bar{f}(x_j) \leq 0$ if $x_i \leq (x_j)^c$,
  (c) $\bar{f}(x_i) \leq 0$ if $x_i \leq x$ for some $x \in I$,

together with the bounds $|\bar{f}(x_i)| \leq k/2$. If we regard $\bar{f}$ as a linear functional on the real vector space freely generated by L, then conditions (a)–(c) can be viewed as asserting that $\bar{f}(\alpha) \geq 0$ for all α in some subset of the root system B_r (or C_r). This shows that the notion of a self-complementary P-partition is an example of Reiner's root-system analogue of P-partitions [Re].

The following result gives the promised characterization of good complements.

THEOREM 7.7. *Let $P = (X, \leq, c)$ be a complemented poset with contraction P_*. Assuming that P and P_* are naturally labeled, and that the labeling of P_* is c-compatible, then c is a good complement if and only if*

$$\sum_{w \in \mathscr{L}(P)} (-1)^{\mathrm{maj}(w)} t^{d(w)} = (t; -1)_{n-m} \sum_{w \in \mathscr{L}^0(P_*)} t^{d(w)},$$

where $n = |P|$ and $m = |P_*|$.

Proof. Let $F(P, t, q) = \sum_{k \geq 0} N_k(P, q) t^k$. Since

$$\sum_{k \geq 0} \begin{bmatrix} n + k \\ n \end{bmatrix}_q t^k = 1/(t; q)_{n+1}$$

(e.g., see Example I.2.3 of [M]), Lemma 7.1 implies that

$$F(P, t, q) = \frac{1}{(t; q)_{n+1}} \sum_{w \in \mathscr{L}(P)} q^{\mathrm{maj}(w)} t^{d(w)}.$$

Similarly, if $G(P, t) = \sum_{k \geq 0} \mathrm{sc}_k(P) t^k$, then Lemma 7.4 implies

$$G(P, t) = \frac{1}{(t; -1)_{m+1}} \sum_{w \in \mathscr{L}^0(P_*)} t^{d(w)}.$$

By definition, c is a good complement if and only if $G(P, t) = F(P, t, -1)$. Since $n = m$ or m is odd, it follows that $(t; -1)_{n+1}/(t; -1)_{m+1} = (t; -1)_{n-m}$. Now compare the two expressions for F and G. ∎

REFERENCES

[A1]   G. E. Andrews, Plane partitions (I): The MacMahon conjecture, in "Studies in Foundations and Combinatorics: Vol. 1" (G.-C. Rota, Ed.), *Advances in Mathematics Supplementary Studies*, Academic Press, New York, 1978, pp. 131–150.

[A2]   G. E. Andrews, Plane partitions (II): The equivalence of the Bender–Knuth and MacMahon conjectures, *Pacific J. Math.* **72** (1977), 283–291.

[A3]   G. E. Andrews, Plane partitions (III): The weak Macdonald conjecture, *Invent. Math.* **53** (1979), 193–225.

[A4]   G. E. Andrews, Plane partitions V: The T.S.S.C.P.P. conjecture, *J. Combin. Theory Ser. A* **66** (1994), 28–39.

[GV]   I. M. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.

[G]   B. Gordon, A proof of the Bender–Knuth conjecture, *Pacific J. Math.* **108** (1983), 99–113.

[K]   G. Kuperberg, Symmetries of plane partitions and the permanent-determinant method, *J. Combin. Theory Ser. A*, to appear.

[M]   I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, London/New York, 1979.

[Ma]   P. A. MacMahon, "Combinatory Analysis," Vol. II, Cambridge, 1918. Reprinted by Chelsea, New York, 1960.

[MRR1] W. H. Mills, D. P. Robbins, and H. Rumsey, Proof of the Macdonald conjecture, *Invent. Math.* **66** (1982), 73–87.

[MRR2] W. H. Mills, D. P. Robbins, and H. Rumsey, Self-complementary totally symmetric plane partitions, *J. Combin. Theory Ser. A* **42** (1986), 277–292.

[MRR3] W. H. Mills, D. P. Robbins, and H. Rumsey, Enumeration of a symmetry class of plane partitions, *Discrete Math.* **67** (1987), 43–55.

[O]   S. Okada, On the generating functions for certain classes of plane partitions, *J. Combin. Theory Ser. A* **51** (1989), 1–23.

[P1]   R. A. Proctor, Shifted plane partitions of trapezoidal shape, *Proc. Amer. Math. Soc.* **89** (1983), 553–559.

[P2]   R. A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, *European J. Combin.* **5** (1984), 331–350.

[P3]   R. A. Proctor, Odd symplectic groups, *Invent. Math.* **92** (1988), 307–332.

[Re]   V. Reiner, Quotients of Coxeter complexes and *P*-partitions. *Mem. Amer. Math. Soc.* **95** (460) (1992).

[R]   D. P. Robbins, The story of $1, 2, 7, 42, 429, 7436, \ldots$, *Math. Intelligence* **13**(2) (1991), 12–19.

[St1]   R. P. Stanley, Symmetries of plane partitions, *J. Combin. Theory Ser. A* **43** (1986), 103–113.

[St2]   R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth & Brooks/Cole, Monterey, 1986.

[Ste1] J. R. Stembridge, Nonintersecting paths, pfaffians and plane partitions, *Adv. in Math.* **83** (1990), 96–131.

[Ste2] J. R. Stembridge, The enumeration of totally symmetric plane partitions, *Adv. in Math.*, to appear.