JOURNAL OF FUNCTIONAL ANALYSIS 125, 438–451 (1994)

Equivariant Analytic Torsion for Compact Lie Group Actions

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Communicated by Richard B. Melrose

Received May 20, 1993

We define the equivariant analytic torsion for a compact Lie group action and study its dependence on the geometric data. © 1994 Academic Press, Inc.

1. INTRODUCTION

The Ray–Singer analytic torsion $\mathcal{T}(Z, F; g^{TZ}, h^F)$ is a real-valued spectral invariant of a closed connected Riemannian manifold Z and a flat complex vector bundle F on Z, which are equipped with a Riemannian metric g^{TZ} and a Hermitian metric h^F, respectively [10, 9]. When $\dim(Z)$ is odd, $\mathcal{T}(Z, F; g^{TZ}, h^F)$ is independent of the choices of g^{TZ} and h^F and hence gives a smooth topological invariant $\mathcal{T}(Z, F)$ of the pair (Z, F). In the original case considered by Ray and Singer, F admits a unitary structure and $\mathcal{T}(Z, F)$ equals the Reidemeister torsion of (Z, F), a homeomorphism invariant [5, 8].

It turned out to be fruitful to consider an equivariant extension of the analytic torsion, in which a finite group acts by isometries on (Z, F) [6, 7]. A natural question is then whether one can extend the definition of the analytic torsion to the case of the action of a compact Lie group G on (Z, F) by isometries. The right approach to this question was not clear. A hint is given by the recent work of the present author with J.-M. Bismut [3], in which the analytic torsion of a fiber bundle is defined as a differential form on the base. Morally speaking, one would apply the fiber bundle results to the following situation. Let BG be a classifying space for G and let EG be the contractible space upon which G acts freely, with $BG = EG/G$. As G acts on Z, there is a fibration with fiber Z, total space $EG \times_G Z$, and base BG. Thus one may hope to obtain a torsion invariant for the G-action on (Z, F) which lies in $H^*(BG; \mathbf{C})$.

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0022-1236/94 $6.00

Inspired by the fiber bundle results, in this note we give a direct construction of the equivariant analytic torsion for the action of G on (Z, F). We must make the further assumption that the G-action on the flat bundle F has vanishing moment. Then we define the equivariant torsion of $(Z, F; g^{TZ}, h^F)$ as a G-invariant formal power series on the Lie algebra g. As the vector space of such formal power series is a completion of $H^*(BG; \mathbf{C})$, our construction fits into the framework of equivariant cohomology. We show that if $\dim(Z)$ is odd then the nonconstant part of the formal power series expansion of the torsion is independent of g^{TZ} and h^F. Thus we obtain smooth topological invariants of the G-action on (Z, F). To show that these invariants are nontrivial, we compute them in the case of $U(1)$ acting on a circle.

We refer to [1, Chaps. 7, 8] for background on equivariant differential forms and equivariant local index theory techniques. We note that many of these techniques are due to Bismut [2]. As many of the propositions in this note are analogs of those in [3], we are sketchy in some of the proofs.

2. The Equivariant Analytic Torsion

We follow the notation of [1]. In particular, $[\cdot, \cdot]$ denotes a graded commutator. The Einstein summation convention is used freely. We let tr and tr_s denote traces and supertraces on finite-dimensional vector spaces. If W is a finite-dimensional vector bundle on a manifold Z, the trace of an endomorphism $T \in \mathrm{End}(W)$ is denoted by $\mathrm{tr}[T] \in C^\infty(Z)$.

Let G be a compact Lie group, with Lie algebra g. Let $\mathbf{C}[g]^G$ denote the space of G-invariant complex-valued formal power series on g.

Let Z be a closed connected oriented Riemannian manifold of dimension n upon which G acts by orientation-preserving isometries. The tangent bundle of Z is denoted by TZ, the Riemannian metric is denoted by g^{TZ}, a local orthonormal basis of TZ is denoted by $\{e_i\}_{i=1}^n$, and the dual basis is denoted by $\{\tau^i\}_{i=1}^n$. When convenient, we may assume that the $\{e_i\}_{i=1}^n$ have vanishing covariant derivative at a point. The complexified exterior bundle of Z is denoted by $\bigwedge^*(Z)$, and its space of smooth sections is denoted by $\Omega^*(Z)$. The Riemannian metric on Z induces an inner product $\langle \cdot, \cdot \rangle_{\Omega^*(Z)}$ on $\Omega^*(Z)$. Given $x \in g$, let X be the corresponding vector field on Z. Let $L_X: \Omega^*(Z) \to \Omega^*(Z)$ be Lie differentiation in the X-direction. Let $i_X: \Omega^*(Z) \to \Omega^{*-1}(Z)$ be interior multiplication by X and let $e_X: \Omega^*(Z) \to \Omega^{*+1}(Z)$ be exterior multiplication by the dual 1-form to X. Then e_X and i_X are adjoint operators. If Y is a vector field on Z, put

$$
\begin{aligned}
c(Y) &= e_Y - i_Y \\
\hat{c}(Y) &= e_Y + i_Y.
\end{aligned}
\tag{1}
$$

Then we have

$$c(Y_1)\,c(Y_2)+c(Y_2)\,c(Y_1)=-2\langle Y_1, Y_2\rangle_{TZ},$$
$$\hat{c}(Y_1)\,\hat{c}(Y_2)+\hat{c}(Y_2)\,\hat{c}(Y_1)=2\langle Y_1, Y_2\rangle_{TZ}, \tag{2}$$
$$c(Y_1)\,\hat{c}(Y_2)+\hat{c}(Y_2)\,c(Y_1)=0.$$

Thus c and $\hat{c}$ generate two graded-commuting Clifford algebras.

The space of equivariant differential forms $\Omega_G(Z)$ is $((\mathbf{C}[g]\otimes\Omega^*(Z)))^G$, the G-invariant $\Omega^*(Z)$-valued formal power series on g. It is convenient to write an equivariant differential form α as if it were actually an $\Omega^*(Z)$-valued function on g. Thus we write α as α_x, where $x\in g$ and $\alpha_x\in\Omega^*(Z)$. The equivariant differential d_g is then given by $(d_g\alpha)_x=d\alpha_x-i_X\alpha_x$.

Let ∇^{TZ} be the Levi-Civita connection on TZ. We also let ∇^{TZ} denote the induced connection on $\bigwedge^*(Z)$. Let R^{TZ} be the curvature 2-form. Define $\hat{R}^{TZ}\in\Omega^2(Z;\operatorname{End}(\bigwedge^*Z))$ by

$$\hat{R}^{TZ}=\tfrac{1}{4}\langle e_j, R^{TZ}e_k\rangle_{TZ}\,\hat{c}^j\hat{c}^k. \tag{3}$$

The Riemannian moment $\mu_x^{TZ}\in\operatorname{End}(TZ)$ of $x\in g$ acts on $Y\in TZ$ by [1, Example 7.8]

$$\mu_x^{TZ}Y=-\nabla_Y^{TZ}X. \tag{4}$$

Define $\hat{\mu}_x^{TZ}\in C^\infty(Z;\operatorname{End}(\bigwedge^*Z))$ by

$$\hat{\mu}_x^{TZ}=\tfrac{1}{4}\langle e_j, \mu_x^{TZ}e_k\rangle_{TZ}\,\hat{c}^j\hat{c}^k. \tag{5}$$

The equivariant curvature is given by

$$R_x^{TZ}=R^{TZ}+\mu_x^{TZ}. \tag{6}$$

Define Pf: $so(n)\to\mathbf{R}$ to be the Pfaffian if n is even and zero if n is odd. Then the equivariant Euler class $\chi\in\Omega_G(Z)$ is given by

$$\chi_x=\operatorname{Pf}\left(\frac{R_x^{TZ}}{2\pi}\right). \tag{7}$$

It is equivariantly closed.

Let F be a flat G-equivariant complex vector bundle on Z. Let ∇^F denote the flat connection on F. Let $\Omega^*(Z;F)$ denote the space of smooth F-valued differential forms on Z. It is a $\mathbf{Z}$-graded vector space, with the number operator N acting as multiplication by j on $\Omega^j(Z;F)$. The action of $x\in g$ on $\Omega^*(Z;F)$ is denoted by L_X. We extend i_X, e_X, $c(X)$, and $\hat{c}(X)$ to act on $\Omega^*(Z;F)$. If $\mathcal{O}$ is a trace-class operator on the L^2-completion of $\Omega^*(Z;F)$, we let $\operatorname{Tr}_s[\mathcal{O}]\in\mathbf{C}$ denote the supertrace of $\mathcal{O}$ with respect to the $\mathbf{Z}_2$-grading on $\Omega^*(Z;F)$.

Let h^F be a G-invariant Hermitian metric on F. We do not require that h^F be covariantly constant with respect to ∇^F. Let $(\nabla^F)^*$ be the adjoint connection to ∇^F, with respect to h^F. Define a Hermitian connection on F by

$$\nabla^{F,u} = \tfrac{1}{2}((\nabla^F)^* + \nabla^F) \tag{8}$$

and define $\psi \in \Omega^1(Z; \mathrm{End}(F))$ by

$$\psi = (\nabla^F)^* - \nabla^F. \tag{9}$$

Then

$$\nabla^{F,u} = \nabla^F + \frac{\psi}{2}. \tag{10}$$

The curvature of $\nabla^{F,u}$ is

$$R^{F,u} = -\frac{\psi^2}{4}. \tag{11}$$

We let $\nabla^{TZ \otimes F,u}$ denote the tensor product of the connections ∇^{TZ} and $\nabla^{F,u}$. Define $\mathscr{R} \in \Omega^2(Z; \mathrm{End}(\wedge^*(Z) \otimes F))$ by

$$\mathscr{R} = (\hat{R}^{TZ} \otimes I_F) + (I_{\wedge^*(Z)} \otimes R^{F,u}). \tag{12}$$

The Hodge duality operator on $\Omega^*(Z)$ extends to a linear operator $*$ on $\Omega^*(Z; F)$. There is an inner product on $\Omega^*(Z; F)$ given by

$$\langle \omega_1, \omega_2 \rangle_{\Omega^*(Z;F)} = \int_Z \langle \omega_1(z) \wedge * \omega_2(z) \rangle_{h^F}. \tag{13}$$

Let $d_F\colon \Omega^*(Z; F) \to \Omega^*(Z; F)$ denote exterior differentiation and let d_F^* be its adjoint.

The moment of $x \in g$ relative to ∇^F is defined to be [1, Definition 7.5]

$$\mu_x^F = L_X - (d_F i_X + i_X d_F). \tag{14}$$

Assumption 1. The flat bundle F is such that μ_x^F vanishes for all $x \in g$.

DEFINITION 1. For $x \in g$ and $t > 0$, define operators on $\Omega^*(Z; F)$ by

$$D_{x,t} = \sqrt{t}\, d_F - \frac{i_X}{4\sqrt{t}} \tag{15}$$

$$D'_{x,t} = \sqrt{t}\, d_F^* + \frac{e_X}{4\sqrt{t}} \tag{16}$$

$$H_{x,t} = -(D'_{x,t} - D_{x,t})^2. \tag{17}$$

LEMMA 1. *We have*

$$(D_{x,t})^2 = (D'_{x,t})^2 = -\frac{1}{4} L_X \tag{18}$$

$$[L_X, D_{x,t}] = [L_X, D'_{x,t}] = 0 \tag{19}$$

$$H_{x,t} = (D'_{x,t} + D_{x,t})^2 + L_X \tag{20}$$

$$[D_{x,t}, H_{x,t}] = [D'_{x,t}, H_{x,t}] = 0 \tag{21}$$

$$[N, D_{x,t}] = 2t \frac{d}{dt} D_{x,t} \tag{22}$$

$$[N, D'_{x,t}] = -2t \frac{d}{dt} D'_{x,t}. \tag{23}$$

Proof. The proof follows from a simple calculation. ∎

Equation (20) shows that $H_{x,t}$ is essentially the same as the Bismut Laplacian [1, Definition 8.9]. It is precisely the same if h^F is covariantly constant with respect to ∇^F.

Define a connection $\mathscr{D}_{x,t}$ on $\bigwedge^*(Z) \otimes F$ by saying that for Y a vector field on Z and $s \in \Omega^*(Z; F)$,

$$(\mathscr{D}_{x,t})_Y\, s = \nabla_Y^{TZ \otimes F, u} s - \frac{\langle X, Y \rangle_{TZ}}{4t}\, s. \tag{24}$$

Let $\Delta_{x,t}$ be the corresponding rough Laplacian on $\Omega^*(Z; F)$. Let $K \in C^\infty(Z)$ be the scalar curvature.

PROPOSITION 1. *We have the Lichnerowicz-type formula*

$$H_{x,t} = t\left(\Delta_{x,t} + \frac{K}{4} + \frac{1}{2} c^i c^j \mathscr{R}(e_i, e_j)\right) + \hat{\mu}_x^{TZ}$$
$$+ t\left(\frac{1}{4}\psi_j^2 + \frac{1}{8}\hat{c}^j \hat{c}^k [\psi_j, \psi_k] - \frac{1}{2} c^j \hat{c}^k \nabla_{e_j}^{TZ \otimes F, u} \psi_k\right). \tag{25}$$

Proof. If h^F is covariantly constant with respect to ∇^F then Eq. (25) follows from [1, Proposition 8.12]. If G is trivial then Eq. (25) is equivalent to [4, Theorem 4.13]. The proof in the general case is done by combining the proofs of the two above special cases. We omit the details. ∎

We see from Proposition 1 that $H_{x,t}$ is an elliptic operator. The heat kernel $e^{-H_{x,t}}$ is a trace-class operator whose supertrace can be formally expanded in x, to give an element of $\mathbf{C}[g]^G$. From [1, Proposition 8.11], for all $t > 0$, $\mathrm{Tr}_s[e^{-H_{x,t}}]$ equals $\mathrm{ind}_G(e^{-x}, d_F + d_F^*)$, the equivariant index of $d_F + d_F^*$: $\Omega^{\text{even}}(Z; F) \to \Omega^{\text{odd}}(Z; F)$ evaluated at the group element e^{-x}. The homotopy invariance of cohomology implies that $\operatorname{ind}_G(e^{-x}, d_F + d_F^*)$ equals $\operatorname{rk}(F)\,\chi(Z)$ for all $x \in g$. Thus $\operatorname{Tr}_s[e^{-H_{x,t}}]$ is independent of both t and x.

From small-time heat-kernel asymptotics, it follows that $\operatorname{Tr}_s[Ne^{-H_{x,t}}]$ has a small-t asymptotic expansion of the form $t^{-n/2} \cdot$ (a power series in t). In fact, by adapting the arguments of [4, Theorem 7.10] and [3, Theorem 3.21], one can show that the t^0-term in the asymptotic expansion is given by

$$t^0\text{-term of } \operatorname{Tr}_s[Ne^{-H_{x,t}}] = \begin{cases} n\operatorname{rk}(F)\,\chi(Z)/2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{26}$$

Put

$$\chi'(Z; F) = \sum_{j=0}^{n} (-1)^j j \operatorname{rk}(H^j(Z; F)). \tag{27}$$

Following [3, Theorem 3.21], the $t \to \infty$ asymptotics of $\operatorname{Tr}_s[Ne^{-H_{x,t}}]$ are given by

$$\operatorname{Tr}_s[Ne^{-H_{x,t}}] = \chi'(Z; F) + O\left(\frac{1}{\sqrt{t}}\right). \tag{28}$$

DEFINITION 2. The equivariant analytic torsion $\mathscr{T} \in \mathbf{C}[g]^G$ is such that for all $x \in g$,

$$\mathscr{T}_x = -\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(\operatorname{Tr}_s[Ne^{-H_{x,t}}] - \chi'(Z; F))\, dt. \tag{29}$$

It follows from Eqs. (26) and (28) that the expression being differentiated in (29) is well-defined for $\operatorname{Re}(s) \gg 0$, and its meromorphic extension to the complex plane is holomorphic near $s = 0$. Clearly, $\mathscr{T}_0$ coincides with the Ray–Singer analytic torsion [10, 9].

PROPOSITION 2. *Let $\{h^F(\varepsilon)\}_{\varepsilon \in \mathbf{R}}$ be a smooth 1-parameter family of G-invariant Hermitian metrics on F. Define $(h^F)^{-1}(dh^F/d\varepsilon) \in C^\infty(Z; \operatorname{End}(F))$, with an obvious notation. Then*

$$\frac{d}{d\varepsilon} \operatorname{Tr}_s[Ne^{-H_{x,t}}] = t\frac{d}{dt} \operatorname{Tr}_s\left[(h^F)^{-1} \frac{dh^F}{d\varepsilon} e^{-H_{x,t}}\right]. \tag{30}$$

Proof. We abbreviate $(h^F)^{-1}(dh^F/d\varepsilon)$ by V. Clearly,

$$\frac{d}{d\varepsilon} D_{x,t} = 0.$$

Let $\bar{F}^*$ be the antidual bundle to F. A Hermitian metric h^F induces an isomorphism $\hat{h}^F\colon F \to \bar{F}^*$. Define an operator $\bar{D}^*_{x,t}$ on $\Omega(Z; \bar{F}^*)$ by

$$\bar{D}^*_{x,t} = \sqrt{t}\, d_{\bar{F}^*} - \frac{i_X}{4\sqrt{t}}.$$

Then

$$D'_{x,t} = (\hat{h}^F)^{-1}\, \bar{D}^*_{x,t} \hat{h}^F$$

and so

$$\frac{d}{d\varepsilon} D'_{x,t} = \left[D'_{x,t}, (\hat{h}^F)^{-1} \frac{d\hat{h}^F}{d\varepsilon} \right] = [D'_{x,t}, V].$$

As the supertrace of a graded commutator vanishes, we may equally well assume that

$$\frac{d}{d\varepsilon} D_{x,t} = -\frac{1}{2} [D_{x,t}, V], \qquad \frac{d}{d\varepsilon} D'_{x,t} = \frac{1}{2} [D'_{x,t}, V]. \tag{31}$$

Then

$$\begin{aligned}
\frac{d}{d\varepsilon} \mathrm{Tr}_s[N e^{-H_{x,t}}] &= -\int_0^1 \mathrm{Tr}_s\left[N e^{-uH_{x,t}} \frac{dH_{x,t}}{d\varepsilon} e^{-(1-u)H_{x,t}} \right] du \\
&= \frac{1}{2} \int_0^1 \mathrm{Tr}_s[N e^{-uH_{x,t}} \\
&\quad \times [(D'_{x,t} - D_{x,t}), [(D'_{x,t} + D_{x,t}), V]]\, e^{-(1-u)H_{x,t}}]\, du \\
&= \frac{1}{2} \int_0^1 \mathrm{Tr}_s[[N, (D'_{x,t} - D_{x,t})]\, e^{-uH_{x,t}} \\
&\quad \times [(D'_{x,t} + D_{x,t}), V]\, e^{-(1-u)H_{x,t}}]\, du \\
&= -t \int_0^1 \mathrm{Tr}_s\left[\frac{d(D'_{x,t} + D_{x,t})}{dt} e^{-uH_{x,t}} \right. \\
&\quad \left. \times [(D'_{x,t} + D_{x,t}), V]\, e^{-(1-u)H_{x,t}} \right] du \\
&= -t \int_0^1 \mathrm{Tr}_s\left[\left[\frac{d(D'_{x,t} + D_{x,t})}{dt}, (D'_{x,t} + D_{x,t}) \right] e^{-uH_{x,t}} \right. \\
&\quad \left. \times V e^{-(1-u)H_{x,t}} \right] du \\
&= -t \int_0^1 \mathrm{Tr}_s\left[V e^{-(1-u)H_{x,t}} \frac{dH_{x,t}}{dt} e^{-uH_{x,t}} \right] du \\
&= t \frac{d}{dt} \mathrm{Tr}_s[V e^{-H_{x,t}}]. \quad \blacksquare
\end{aligned} \tag{32), (33}$$

By Hodge theory, we can identify the **Z**-graded vector space $H^*(Z; F)$ with $\operatorname{Ker}(d_F^* - d_F)$. Then $H^*(Z; F)$ inherits a Hermitian inner product $h^{H^*(Z; F)}$. Let

$$P: \Omega^*(Z; F) \to \operatorname{Ker}(d_F^* - d_F) \tag{34}$$

be the orthogonal projection operator.

PROPOSITION 3. *As operators on* $H^*(Z; F)$,

$$(h^{H^*(Z; F)})^{-1} \frac{dh^{H^*(Z; F)}}{d\varepsilon} = P(h^F)^{-1} \frac{dh^F}{d\varepsilon} P. \tag{35}$$

Proof. It is enough to prove the validity of (35) when $\varepsilon = 0$. Let $\{v_i(\varepsilon)\}$ be a 1-parameter family of bases of $\operatorname{Ker}(d_F^* - d_F)$ which is orthonormal when $\varepsilon = 0$ and whose de Rham cohomology classes are independent of ε. Then $dv_i/d\varepsilon \in \operatorname{im}(d_F)$ and

$$\begin{aligned}
\frac{d\langle v_i, v_j\rangle_{\Omega^*(Z; F)}}{d\varepsilon} &= \left\langle \frac{dv_i}{d\varepsilon}, v_j \right\rangle_{\Omega^*(Z; F)} + \left\langle v_i, \frac{dv_j}{d\varepsilon} \right\rangle_{\Omega^*(Z; F)} + \frac{dh^{\Omega^*(Z; F)}}{d\varepsilon}(v_i, v_j) \\
&= \frac{dh^{\Omega^*(Z; F)}}{d\varepsilon}(v_i, v_j) \\
&= \left\langle v_i, (h^F)^{-1} \frac{dh^F}{d\varepsilon} v_j \right\rangle_{\Omega^*(Z; F)} \\
&= \left\langle v_i, P(h^F)^{-1} \frac{dh^F}{d\varepsilon} P v_j \right\rangle_{\Omega^*(Z; F)}.
\end{aligned} \tag{36}$$

However,

$$\frac{d\langle v_i, v_j\rangle_{\Omega^*(Z; F)}}{d\varepsilon} = \frac{dh^{H^*(Z; F)}}{d\varepsilon}(v_i, v_j). \tag{37}$$

Combining (36) and (37) gives the validity of (35) when $\varepsilon = 0$. ∎

PROPOSITION 4. *We have*

$$\lim_{t \to \infty} \operatorname{Tr}_s \left[(h^F)^{-1} \frac{dh^F}{d\varepsilon} e^{-H_{x,t}} \right] = \operatorname{Tr}_s \left[P(h^F)^{-1} \frac{dh^F}{d\varepsilon} P \right]. \tag{38}$$

Proof. The heuristic idea of the proof is that as $t \to \infty$, the $1/\sqrt{t}$ terms of (15) and (16) become irrelevant. Then $H_{x,t}$ approaches $-t(d_F^* - d_F)^2$ and $e^{-H_{x,t}}$ approaches P. The special feature of the present situation is that there are no terms of order t^0 in (15) or (16). The details of the proof are as in [3, Theorem 2.13] and we omit them. ∎

PROPOSITION 5. *The expression* $\mathrm{Tr}_s[(h^F)^{-1}(dh^F/d\varepsilon)\, e^{-H_{x,t}}]$ *has a limit as* $t \to 0$ *given by*

$$\lim_{t \to 0} \mathrm{Tr}_s\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-H_{x,t}}\right] = \int_Z \chi_x \cdot \mathrm{tr}\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{\psi^2/4}\right]. \quad (39)$$

Proof. The proof follows from local index theory techniques as in [2, Section 2] or [1, Section 8.3]. That is, doing an appropriate rescaling and using Proposition 1, one finds

$$\lim_{t \to 0} \mathrm{Tr}_s\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-H_{x,t}}\right] = \int_Z \chi_x \cdot \mathrm{tr}\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-R^{F,u}}\right]. \quad (40)$$

The proposition now follows from combining (11) and (40). ∎

PROPOSITION 6. *We have*

$$\frac{d\mathcal{T}_x}{d\varepsilon} = -\mathrm{tr}_s\left[(h^{H^*(Z;F)})^{-1}\frac{dh^{H^*(Z;F)}}{d\varepsilon}\right] + \int_Z \chi_x \cdot \mathrm{tr}\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{\psi^2/4}\right]. \quad (41)$$

Proof. Put $W = \lim_{t\to\infty} \mathrm{Tr}_s\,[(h^F)^{-1}(dh^F/d\varepsilon)\, e^{-H_{x,t}}]$. We can switch the order of differentiation and integration to obtain

$$\begin{aligned}\frac{d\mathcal{T}_x}{d\varepsilon} &= -\frac{d}{d\varepsilon}\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{Tr}_s[Ne^{-H_{x,t}}] - \chi'(Z;F))\,dt \\ &= -\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^s\frac{d}{dt}\mathrm{Tr}_s\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-H_{x,t}}\right]dt \\ &= -\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^s\frac{d}{dt}\left(\mathrm{Tr}_s\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-H_{x,t}}\right] - W\right)dt. \quad (42)\end{aligned}$$

One then integrates by parts, and as in [6, pp. 437–438], one obtains

$$\begin{aligned}\frac{d\mathcal{T}_x}{d\varepsilon} = &-\lim_{t\to\infty}\mathrm{Tr}_s\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-H_{x,t}}\right] \\ &+ \left(\text{the } t^0 \text{ term in the small-}t\text{ expansion of } \mathrm{Tr}_s\left[(h^F)^{-1}\frac{dh^F}{d\varepsilon}e^{-H_{x,t}}\right]\right).\end{aligned} \quad (43)$$

The proposition now follows from combining (43) and Propositions 3–5. ∎

Now fix h^F and let $\{g^{TZ}(\varepsilon)\}_{\varepsilon\in\mathbf{R}}$ be a smooth 1-parameter family of G-invariant Riemannian metrics on Z. Let $*(\varepsilon)$ be the corresponding

Hodge duality operators on $\Omega(Z; F)$. There is a canonically defined class $\hat{\chi} \in \Omega_G(Z)/\mathrm{im}(d_g)$, constructed from g^{TZ} and $dg^{TZ}/d\varepsilon$, such that

$$\frac{d\chi}{d\varepsilon} = d_g \hat{\chi}. \tag{44}$$

Define $c(h^F) \in \Omega^*(Z)$ by

$$c(h^F) = \sum_{j=0}^{\infty} \frac{\mathrm{Tr}[\psi^{2j+1}]}{(2j+1)\, 4^j j!}. \tag{45}$$

It is easy to check that $c(h^F)$ is closed and that its de Rham cohomology class is independent of h^F. As h^F is G-invariant, $c(h^F)$ is equivariantly closed as an element of $\Omega_G(Z)$.

PROPOSITION 7. *We have*

$$\frac{d\mathcal{T}_x}{d\varepsilon} = -\mathrm{tr}_s\left[(h^{H^*(Z;F)})^{-1}\frac{dh^{H^*(Z;F)}}{d\varepsilon}\right] + \int_Z \hat{\chi}_x \cdot c(h^F). \tag{46}$$

Proof. The analogs of Propositions 2, 3, and 4 hold, with $(h^F)^{-1}(dh^F/d\varepsilon)$ replaced by $*^{-1}(d*/d\varepsilon)$. The proofs are virtually the same as before. In order to compute the $t \to 0$ limit of $\mathrm{Tr}_s[*^{-1}(d*/d\varepsilon)\, e^{-H_{x,t}}]$, one can introduce an auxiliary variable σ as in [4, Section 4]. Upon rescaling σ as in [4, Section 4] and the other variables as in [1, Section 8.3], one uses Proposition 1 to show

$$\lim_{t \to 0} \mathrm{Tr}_s\left[*^{-1}\frac{d*}{d\varepsilon}e^{-H_{x,t}}\right] = \int_Z \mathcal{R}_x^k \cdot \mathrm{tr}[(\nabla^{F,u}\psi_k)\, e^{\psi^2/4}], \tag{47}$$

where $\mathcal{R}_x^k$ is a certain tensor constructed from g^{TZ} and $dg^{TZ}/d\varepsilon$. (The last term in (25) is responsible for the $(\nabla^{F,u}\psi_k)$ term in (47).) Then

$$\begin{aligned} \lim_{t \to 0} \mathrm{Tr}_s\left[*^{-1}\frac{d*}{d\varepsilon}e^{-H_{x,t}}\right] &= \int_Z \mathcal{R}_x^k \cdot \mathrm{tr}[\tau^j(\nabla_{e_j}^{TZ\otimes F,u}\psi_k)\, e^{\psi^2/4}] \\ &= \int_Z \mathcal{R}_x^k \cdot \mathrm{tr}[\tau^j(\nabla_{e_k}^{TZ\otimes F,u}\psi_j)\, e^{\psi^2/4}] \\ &= \int_Z \mathcal{R}_x^k \cdot \nabla_{e_k}^{TZ} c(h^F) \\ &= -\int_Z \nabla_{e_k}^{TZ}\mathcal{R}_x^k \cdot c(h^F). \end{aligned} \tag{48}$$

One computes that $-\nabla^{TZ}_{e_k}\mathscr{R}^k_x = \hat{\chi}_x$. Thus the analog of Proposition 5 is

$$\lim_{t\to 0} \mathrm{Tr}_s\left[*^{-1}\frac{d*}{d\varepsilon} e^{-H_{x,t}}\right] = \int_Z \hat{\chi}_x \cdot c(h^F). \tag{49}$$

The proposition now follows as in the proof of Proposition 6. ∎

If g^{TZ} and g'^{TZ} are two G-invariant Riemannian metrics on Z, let $\{g(\varepsilon)\}_{\varepsilon\in[0,1]}$ be a smooth 1-parameter family of G-invariant Riemannian metrics on Z such that $g(0)=g^{TZ}$ and $g(1)=g'^{TZ}$. Define $\tilde{\chi}(g^{TZ}, g'^{TZ}) \in \Omega_G(Z)/\mathrm{im}(d_g)$ by

$$\tilde{\chi}_x(g^{TZ}, g'^{TZ}) = \int_0^1 \hat{\chi}_x(\varepsilon)\, d\varepsilon. \tag{50}$$

One can check that $\tilde{\chi}(g^{TZ}, g'^{TZ})$ depends only on g^{TZ} and g'^{TZ}, and not on the 1-parameter family chosen. By construction,

$$d_g\tilde{\chi}(g^{TZ}, g'^{TZ}) = \chi(g'^{TZ}) - \chi(g^{TZ}). \tag{51}$$

Similarly, if h^F and h'^F are two G-invariant Hermitian metrics on F, let $\{h(\varepsilon)\}_{\varepsilon\in[0,1]}$ be a smooth 1-parameter family of G-invariant Hermitian metrics on F such that $h(0)=h^F$ and $h(1)=h'^F$. Define $\tilde{c}(h^F, h'^F) \in \Omega^*(Z)/\mathrm{im}(d)$ by

$$\tilde{c}(h^F, h'^F) = \int_0^1 \mathrm{tr}\left[h(\varepsilon)^{-1}\frac{dh}{d\varepsilon} e^{\psi^2(\varepsilon)/4}\right]. \tag{52}$$

One can check that $\tilde{c}(h^F, h'^F)$ depends only on h^F and h'^F, and not on the 1-parameter family chosen. By construction,

$$d\tilde{c}(h^F, h'^F) = c(h'^F) - c(h^F). \tag{53}$$

Let $h^{H^*(Z;F)}$ and $h'^{H^*(Z;F)}$ be the Hermitian metrics on $H^*(Z;F)$ induced by (g^{TZ}, h^F) and (g'^{TZ}, h'^F), respectively. Let the corresponding volume forms on $\mathrm{H}^p(Z;F)$ be $\mathrm{vol}(h^{\mathrm{H}^p(Z;F)})$ and $\mathrm{vol}(h'^{\mathrm{H}^p(Z;F)})$.

Proposition 8. *We have*

$$\begin{aligned}\mathscr{T}_x(g'^{TZ}, h'^F) - \mathscr{T}_x(g^{TZ}, h^F) = &\int_Z \tilde{\chi}_x(g^{TZ}, g'^{TZ})\cdot c(h^F)\\ &+ \int_Z \chi_x(g'^{TZ})\cdot \tilde{c}(h^F, h'^F)\\ &- \sum_{p=0}^n (-1)^p \ln\left(\frac{\mathrm{vol}(h'^{H^p(Z;F)})}{\mathrm{vol}(h^{H^p(Z;F)})}\right).\end{aligned} \tag{54}$$

Proof. It follows from Propositions 6 and 7 and Eqs. (50)–(53) that the difference between the two sides of (54) is independent of g'^{TZ} and h'^F. As both sides vanish when $g'^{TZ} = g^{TZ}$ and $h'^F = h^F$, the proposition follows. ∎

If G is trivial then Eq. (54) is equivalent to [4, Theorem 0.1].

COROLLARY 1. *If* $\dim(Z)$ *is odd then* $\mathcal{T}_x - \mathcal{T}_0$ *is independent of* g^{TZ} *and* h^F *and is thus a smooth topological invariant of the G-pair* (Z, F).

Proof. If $\dim(Z)$ is odd then $\chi = \tilde{\chi} = 0$, and so the corollary follows from Proposition 8. ∎

If $\dim(Z)$ is odd and $H^*(Z; F) = 0$ then it follows from Proposition 8 that $\mathcal{T}_0$ is also independent of g^{TZ} and h^F. However, this is simply a consequence of [10, 9], as $\mathcal{T}_0$ is the same as the Ray–Singer analytic torsion.

PROPOSITION 9. *If* $\dim(Z)$ *is even and* h^F *is covariantly constant with respect to* ∇^F, *then* $\mathcal{T} = 0$.

Proof. This follows from a Hodge duality argument, as in [10] and [3, Theorem 3.26]. ∎

We now compute the equivariant analytic torsion for $U(1)$ acting on a circle by an r-fold covering. First, let $r > 1$ be a positive integer, let $\zeta \neq 1$ be an rth root of unity, and let $\rho: \mathbf{Z} \to \mathrm{Aut}(\mathbf{C})$ be the corresponding representation. Put $F = \mathbf{R} \times_\rho \mathbf{C}$. Then F is a $U(1)$-equivariant flat complex line bundle over S^1 with vanishing moment. There is a $U(1)$-invariant Hermitian metric on F induced from the standard inner product on $\mathbf{C}$. Define the jth polylogarithm function of ζ by

$$\mathrm{Li}_j(\zeta) = \sum_{m=1}^{\infty} \frac{\zeta^m}{m^j}.$$

PROPOSITION 10. *For* $iy \in u(1)$, *we have*

$$\mathcal{T}_{iy} = 2 \sum_{j \text{ even}} \binom{2j}{j} \mathrm{Re}(\mathrm{Li}_{j+1}(\zeta)) \left(\frac{ry}{8\pi}\right)^j - 2i \sum_{j \text{ odd}} \binom{2j}{j} \mathrm{Im}(\mathrm{Li}_{j+1}(\zeta)) \left(\frac{ry}{8\pi}\right)^j. \tag{55}$$

Proof. The calculation proceeds by means of the Poisson summation formula, as in [3, Theorem 4.13], and we omit the details. ∎

Now let $U(1)$ act on S^1 in the standard way and let F be the trivial flat complex line bundle on S^1.

PROPOSITION 11. *For* $iy \in u(1)$, *we have*

$$\mathcal{T}_{iy} - \mathcal{T}_0 = 2 \sum_{k=1}^{\infty} \binom{4k}{2k} \mathrm{Li}_{2k+1}(1) \left(\frac{y}{8\pi}\right)^{2k}. \tag{56}$$

Proof. The proof is similar to that of Proposition 10. ∎

Although Eq. (56) is derived using standard metrics on S^1 and F, by Corollary 1 the result is independent of the metrics chosen. Similarly, the result of Proposition 10 is independent of the metrics chosen.

Note 1. Recall that if D is a G-invariant Dirac-type operator on Z and $H_{x,t}$ is the corresponding Bismut Laplacian, then [1, Section 8.3]

$$\mathrm{Tr}_s[e^{-H_{x,t}}] = \mathrm{Tr}_s[e^{-x}e^{-H_{0,t}}] = \mathrm{ind}_G(e^{-x}, D). \tag{57}$$

One may ask if there is a similar relationship for the equivariant analytic torsion. Namely, as in [6, Section X], put

$$\tilde{\mathcal{T}}(e^{-x}) = -\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(\mathrm{Tr}_s[Ne^{-x}e^{-H_{0,t}}] - \chi'(Z;F))\,dt. \tag{58}$$

Then the question is whether $\mathcal{T}_x$ equals $\tilde{\mathcal{T}}(e^{-x})$. One finds by explicit calculation that this is not the case even for a circle.

Note 2. The numerical coefficients in (55) differ from those of [3, Theorem 4.13] because of a slightly different definition of the analytic torsion. Essentially, instead of using the exponential function in Definition 2, Ref. [3] uses the function $(1+2x)\,e^x$.

Note 3. The analytic torsion $\mathcal{T}_x$ is localized around the identity element e of G, in that it is the germ of a function defined around e. We see no reason why it should extend analytically to a function on the connected component G_0 of G, especially in view of the arbitrariness of definition mentioned in Note 2.

Given $\gamma \in G$, one can define an analytic torsion which is localized around γ as follows. Put

$$\chi'(\gamma, Z; F) = \sum_{j=0}^{n} (-1)^j j\,\mathrm{tr}[\gamma|_{H^j(Z;F)}]. \tag{59}$$

Let g_γ be the Lie algebra of the centralizer G_γ of γ. Define $\mathcal{T}^{(\gamma)} \in \mathbf{C}[g_\gamma]^{G_\gamma}$ by saying that for $x \in g_\gamma$,

$$\mathcal{T}_x^{(\gamma)} = -\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(\mathrm{Tr}_s[N\gamma e^{-H_{x,t}}] - \chi'(\gamma, Z; F))\,dt. \tag{60}$$

If G is finite, one recovers the equivariant torsion of [6, 7].

Note 4. It should be possible to define an equivariant Reidemeister torsion by means of a decomposition of Z as a G-CW complex. In particular, given a G-Morse function on Z, one obtains such a decomposition, and it should be possible to study the relationship between the analytic and cellular torsion invariants, along the lines of [4].

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