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NATURNAL CONVECTION OF A DARCIAN FLUID ABOUT A WAVY CONE

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ABSTRACT
        The results of a study of natural convection along an isother-
        mal wavy cone embedded in a fluid-saturated porous medium
        are presented. The boundary layer under consideration is
        cencerned with the regime where the Darcy-Rayleigh number
        Ra is very large. It is assumed that the surface waves have
        O(1) amplitude and wavelength. The boundary layer equa-
        tions are solved numerically using the Keller-box method.
        Detailed results of the effect of the sinusoidal waves on the
        heat flux on the wall are given.

Introduction

   Natural convection from wavy surfaces is of importance in several heat
transfer devices such as flat-plate solar collectors and flat-plate condensers
in refrigerators. The presence of roughness elements disturbs the flow past
surfaces and alters the heat transfer rate. Most of the reported surveys on
natural convection in a porous medium deal with cases in which surfaces
are uniform, see Nield and Bejan[1]. The interest of research in the case of
natural convection from wavy surfaces has surged during recent years, see
references [2-6].

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In the present paper, a theoretical study of the natual convection along a vertical cone with transverse curvature and sinusoidal undulations embedded in a fluid-saturated porous medium is reported. Of primary relevance to this work are the papers by Rees and Pop[5,6] in which the authors reported numerical solutions for the problem of natural convection along a vertical wavy surface in a porous medium. To the best of our knowledge, natural convection along a cone with transverse undulations in a porous medium has not been reported prior to the present analysis.

Analysis

Consider a vertical cone with transverse sinusoidal undulations embedded in a fluid-saturated porous medium, as illustrated in Fig.1. In particular, we assume that the surface profile is given by

$$\bar{y}_w = \bar{\sigma}(\bar{x}) = \bar{a}\,\sin\!\left(\frac{\pi\bar{x}}{\ell}\right) \tag{1}$$

where $\bar{a}$ is the amplitude of the transverse surface wave and ℓ is the characteristic length of the wave. The surface of the cone is held at a constant temperature T_w, which is greater than the temperature of the ambient fluid T_∞.
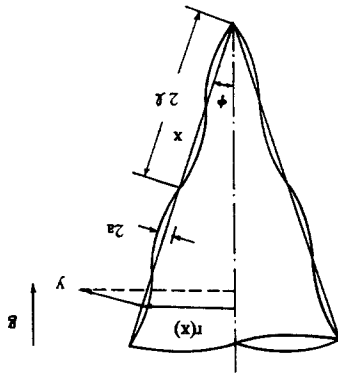

FIG.1
Physical Model and Coordinate System

The non-dimensional equations governing natural convection from the wavy cone can be written under the Darcy-Boussinesq approximation as

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0 \tag{2}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = Ra\left\{\frac{\partial\theta}{\partial y} + \frac{\partial\theta}{\partial x}\tan\Phi\right\} \tag{3}$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} \tag{4}$$

where (x, y) are the Cartesian coordinates, (u, v) are the velocity components along (x, y)-axes, Ra is the Darcy-Rayleigh number, Φ and r are the half angle and the local radius of the smooth surface of the cone, where r is given by

$$r = x\,\sin\Phi \tag{5}$$

The dimensionless variables are defined by

$$x = \frac{\bar{x}}{\ell} \;,\; y = \frac{\bar{y}}{\ell} \;,\; \psi = \frac{\bar{\psi}}{\alpha} \;,\; r = \frac{\bar{r}}{\ell} \;.\; \theta = \frac{T - T_\infty}{T_w - T_\infty} \;,$$

$$\sigma = \frac{\bar{\sigma}}{\ell} \;,\; a = \frac{\bar{a}}{\ell} \;,\; Ra = \frac{g\beta K(T_w - T_\infty)\ell\cos\Phi}{\alpha\nu} \tag{6}$$

where T is the temperature, g is the acceleration due to gravity, β is the expansion coefficient of the fluid, ν is the kinematic viscosity of the fluid, K and α are the permeability and thermal diffusivity of the fluid - saturated porous medium.

Introducing the stream function ψ defined by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial y} \;;\; v = -\frac{1}{r}\frac{\partial\psi}{\partial x} \tag{7}$$

where the equation of continuity is satisfied identically and Eqs.(3) and (4) become

$$\frac{1}{r}\left\{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{r_x}{r}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\right)\right\} = Ra\left\{\frac{\partial\theta}{\partial y} + \frac{\partial\theta}{\partial x}\tan\Phi\right\} \tag{8}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{r}\left\{\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y}\right\} \tag{9}$$

where the subscript x denotes differentiation with respect to x. The boundary conditions of Eqs.(8) and (9) are

$$\psi = 0 \ , \quad \theta = 1 \quad on \quad y = \sigma(x) = a\,\sin(\pi x) \tag{10}$$

$$\frac{\partial \psi}{\partial y} = 0 \ , \quad \theta = 0 \quad as \quad y \to \infty \tag{11}$$

We now assume that the Darcy-Rayleigh number Ra is large so that natural convection takes place within a boundary layer whose cross-stream width is substantially smaller than the amplitude a of the wavy surface of the cone. Accordingly, we define new variables by substracting out the effect of the surface waves and then introduce the usual boundary layer variables defined as

$$x = \tilde{x} \ , \quad y = Ra^{-\frac{1}{2}}\tilde{y} + \sigma(\tilde{x}) \ , \quad \psi = Ra^{\frac{1}{2}}\tilde{\psi} \tag{12}$$

By substituting Eq.(12) into Eqs.(8) and (9), we get

$$\frac{1}{r}\left\{\left(1 + \sigma_{\tilde{x}}^2\right)\frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2} - 2\sigma_{\tilde{x}}Ra^{-\frac{1}{2}}\frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}\partial \tilde{y}} - \sigma_{\tilde{x}\tilde{x}}Ra^{-\frac{1}{2}}\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right.$$

$$\left. + Ra^{-\frac{1}{2}}\frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}^2} - \frac{r_{\tilde{x}}}{r}\left(1 - \sigma_{\tilde{x}}\right)Ra^{-\frac{1}{2}}\frac{\partial \tilde{\psi}}{\partial \tilde{y}} - \frac{r_{\tilde{x}}}{r}Ra^{-1}\frac{\partial \tilde{\psi}}{\partial \tilde{x}}\right\}$$

$$= \left(1 - \sigma_{\tilde{x}}\tan\Phi\right)\frac{\partial \theta}{\partial \tilde{y}} + Ra^{-\frac{1}{2}}\tan\Phi\frac{\partial \theta}{\partial \tilde{x}} \tag{13}$$

$$\left(1 + \sigma_{\tilde{x}}^2\right)\frac{\partial^2 \theta}{\partial \tilde{y}^2} - 2\sigma_{\tilde{x}}Ra^{-\frac{1}{2}}\frac{\partial^2 \theta}{\partial \tilde{x}\partial \tilde{y}} - \sigma_{\tilde{x}\tilde{x}}Ra^{-\frac{1}{2}}\frac{\partial \theta}{\partial \tilde{y}}$$

$$+ Ra^{-1}\frac{\partial^2 \theta}{\partial \tilde{x}^2} = \frac{1}{r}\left\{\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\frac{\partial \theta}{\partial \tilde{x}} - \frac{\partial \tilde{\psi}}{\partial \tilde{x}}\frac{\partial \theta}{\partial \tilde{y}}\right\} \tag{14}$$

and the boundary conditions are now

$$\tilde{y} = 0 \ : \ \tilde{\psi} = 0 \ , \ \ \theta = 1 \tag{15}$$

$$\tilde{y} = \infty \ : \ \frac{\partial \tilde{\psi}}{\partial \tilde{y}} = 0 \ , \ \ \theta = 0 \tag{16}$$

If we formally let Ra approach to infinity, Eqs.(13) and (14) reduce to the following boundary layer equations for the problem under consideration,

$$\frac{1 + \sigma_{\tilde{x}}^2}{r} \frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2} = \left(1 - \sigma_{\tilde{x}} \tan\Phi\right) \frac{\partial \theta}{\partial \tilde{y}} \tag{17}$$

$$\left(1 + \sigma_{\tilde{x}}^2\right) \frac{\partial^2 \theta}{\partial \tilde{y}^2} = \frac{1}{r} \left\{ \frac{\partial \tilde{\psi}}{\partial \tilde{y}} \frac{\partial \theta}{\partial \tilde{x}} - \frac{\partial \tilde{\psi}}{\partial \tilde{x}} \frac{\partial \theta}{\partial \tilde{y}} \right\} \tag{18}$$

and the boundary conditions are those given by Eqs.(15) and (16). It should be mentioned that there was tacitly assumed that the surface waves have $O(1)$ amplitude and wavelength.

We shall now proceed to obtain numerical solutions of Eqs.(17) and (18). Thus, we assume that the solution is written as

$$\tilde{\psi} = r\xi^{\frac{1}{2}} f(\xi, \eta) \ , \ \ \theta = g(\xi, \eta) \tag{19}$$

where

$$\xi = \tilde{x} \ , \ \ \eta = \frac{\tilde{y}}{\xi^{\frac{1}{2}} \left(1 + \sigma_\xi^2\right)} \tag{20}$$

and r is defined by Eq.(5). Introducing Eqs.(19) and (20) into Eqs.(17) and (18) gives

$$\frac{\partial^2 f}{\partial \eta^2} = \left\{ 1 - \sigma_\xi \tan\Phi \right\} \frac{\partial g}{\partial \eta} \tag{21}$$

$$\frac{\partial^2 g}{\partial \eta^2} + \frac{3}{2} f \frac{\partial g}{\partial \eta} = \xi \left\{ \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial g}{\partial \eta} \right\} \tag{22}$$

which are subject to the boundary conditions

$$\eta = 0 \quad : \quad f(\xi, 0) = 0 \quad , \quad g(\xi, 0) = 1 \tag{23}$$

$$\eta = \infty \quad : \quad \frac{\partial f(\xi, \infty)}{\partial \eta} = 0 \quad , \quad g(\xi, \infty) = 0 \tag{24}$$

It is worthy of mentioning that in the case of a smooth cone ($a = 0$), Eqs.(21) and (22) reduce to ordinary differential equations, which are identical with those found by Cheng et. al. [7].

An important physical quantity for this problem is the local Nusselt number defined as

$$Nu = \frac{\bar{x} q_w}{k(T_w - T_\infty)} \tag{25}$$

where k is the thermal conductivity of the fluid-saturated porous medium and q_w is the heat flux at the wall, which is defined by

$$q_w = -k\vec{n} \cdot \nabla T \tag{26}$$

and

$$\vec{n} \left\{ -\frac{\sigma_{\bar{x}}}{\left(1 + \sigma_{\bar{x}}^2\right)^{1/2}}, \frac{1}{\left(1 + \sigma_{\bar{x}}^2\right)^{1/2}}, \right\} \tag{27}$$

is the unit vector normal to the wavy surface of the cone. Using Eqs.(6) and (12), one can show that Nu can be expressed as

$$\frac{Nu}{Ra^{\frac{1}{2}}} = \frac{\xi^{\frac{1}{2}}}{\left(1 + \sigma_\xi^2\right)^{\frac{1}{2}}} \left\{ -\frac{\partial g}{\partial \eta} \right\}_{\eta=0} \tag{28}$$

Result and Discussions

The boundary layer equations, Eqs.(21) and (22), along with the boundary conditions, Eqs.(23) and (24), were solved numerically by means of the Keller-box method (see references[8,9,10]) for some representative values of the wavy amplitude a and half angle Φ. The variation of Nusselt number

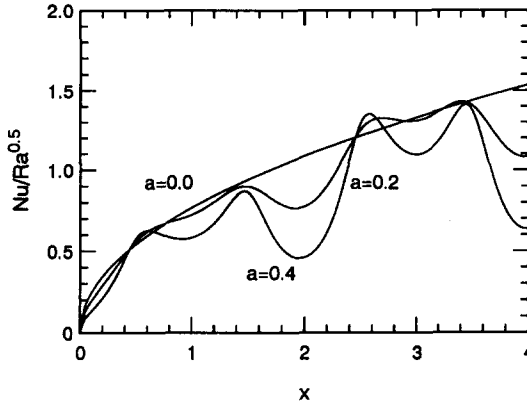with ξ for $a=0$ (smooth cone), 0.2, 0.4 and $\Phi=10°$, $15°$, $20°$, $30°$ are shown inFigs.2 to 4.

FIG.2

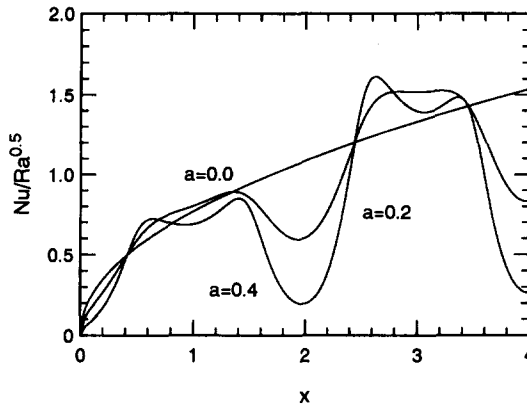Variation of Local Nusselt Number with x for $\Phi=15°$

FIG.3

Variation of Local Nusselt Number with x for $\Phi=30°$

Fig.2 shows the Nusselt number ratio defined on the left-side of Eq.(28) as a function of ξ. The effect of the amplitude a, on the average, is seen

to decrease the heat transfer rate as compared with the limiting case of a smooth cone. The range of x in the figure is from 0 to 4 which corresponds to two complete cycles of the undulations as shown in Fig.1. The raise and fall of the curves is seen to follow the change of the surface contour. Consider one complete cycle, i.e., for x from 0 to 2. The rate of heat transfer is seen to increase for the first quarter of the cycle (from $x \approx 0$ to $x \approx 0.5$) and then decreases in the second quarter (from $x \approx 0.5$ to $x \approx 1$). From $x \approx 1$. to $x \approx 1.5$ (i.e., the third quarter), the heat transfer rate increases whereas in the last quarter (from $x \approx 1.5$ to $x \approx 2$), a larger decrease in the heat transfer rate resulted. The cycle will repeat itself as x is advanced by another cycle. The results represent the nonlinear coupling of the change in fluid velocity and orientation of the gravitation. Fig.3 shows the same conclusion can be obtained for anothe half cone angle, namely, $\Phi = 30°$. Fig.4 show the effect of the cone angle Φ on the rate of heat transfer. As expected, the effects are more pronounced for larger half cone angles.
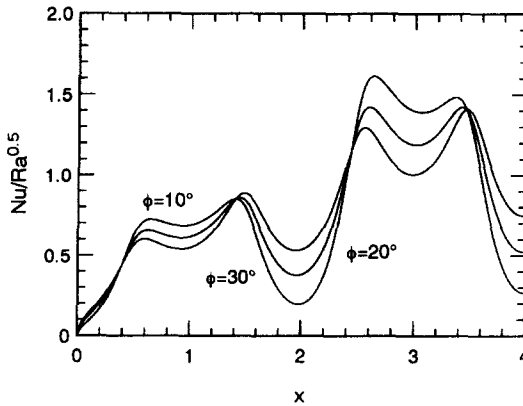

FIG.4

Variation of Local Nusselt Number with x for a=0.4

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