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On a Lifting Theorem for the Structured Singular Value*

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We employ certain *lifting* ideas from Bercovici *et al.* (*Oper. Theory Adv. Appl.* **47** (1990), 195–220) in order to study the structured singular value. This can be used to study problems concerned with robust stability in control theory under various perturbation classes.

1. INTRODUCTION

The structured singular value introduced by Doyle and Safonov [5, 10] has proven to be an important tool in control theory. (See these works for the physical and engineering background about these ideas.) Unfortunately, it is very difficult to work with the structured singular value di-

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rectly. Therefore much research has been concentrated on using a certain upper bound which we will define below.

In several works [1, 8, 9, 11] it has been shown that in fact this upper bound actually gives a non-conservative measure of robust stability with respect to various perturbation measures. Motivated by this work, we would like to show how the lifting technique of [1] may be extended to operators on Hilbert space.

The basic idea is that the upper bound for the structured singular value defined in terms of certain scaling operators (see Section 2) may be interpreted as a structured singular value on a larger space with respect to an enhanced perturbation structure. This is made precise in Theorem 1 (the Lifting Theorem) below.

We now briefly sketch the contents of this paper. In Section 2, we define the structured singular value, its upper bound, and we derive some of its elementary properties. We follow the discussion of [1] here. In Section 3, we collect a number of lemmas which we will need in order to extend the finite dimensional lifting theorem of [1] to infinite dimensional operators. Then in Section 4, we state and prove our lifting theorem relating the structured singular value and its upper bound. Finally in Section 5, we apply this to systems with time-varying perturbations.

2. Background on Structured Singular Value

We would like to formally introduce the structured singular value now and give some of its basic properties. We base this discussion on [1]. Instead of working over diagonal sets of matrices as in [5], we can more generally work over an algebra of operators.

We now give the following mathematical definition to this setup. Let $\mathscr{E}$ be an arbitrary complex separable Hilbert space, and $\Delta \subset \mathscr{L}(\mathscr{E})$ (the space of bounded linear operators on $\mathscr{E}$), a subalgebra. For $A \in \mathscr{L}(\mathscr{E})$, $A \neq 0$, we define the *structured singular value*

$$\mu_\Delta(A) := [\inf\{\|X\| : X \in \Delta, -1 \in \sigma(AX)\}]^{-1}.$$

Moreover, we set

$$\hat{\mu}_\Delta(A) := \inf\{\|XAX^{-1}\| : X \in \Delta'\},$$

where Δ' is the commutant of Δ. Note that for $\Delta = \mathscr{L}(\mathscr{E})$, $\mu_\Delta(A) = \hat{\mu}_\Delta(A) = \|A\|$, while for $\Delta = \mathbf{C}I_{\mathscr{E}}$, $\mu_\Delta(A) = \hat{\mu}_\Delta(A) = \|A\|_{sp}$ (the spectral radius of A).

We now summarize some of the elementary properties of μ_Δ without proof. See [1, 5] for all the details.

LEMMA 1. *Notation as above.*

1. $\mu_\Delta(A) = \sup\{\|AX\|_{sp} : X \in \Delta, \|X\| \le 1\}$.
2. μ_Δ *is upper semicontinuous.*
3. *If* $\mathscr{E}$ *is finite dimensional, then* μ_Δ *is continuous.*
4. $\mu_\Delta(A) \le \hat{\mu}_\Delta(A)$.

3. PRELIMINARY RESULTS

In this section, we state and prove a series of technical lemmas which will allow us to extend the proof of the Lifting Theorem of [1] to the infinite dimensional case. As above, $\mathscr{E}$ will denote a complex separable Hilbert space. We begin with the following.

LEMMA 2. *Let* $\mathscr{X}$ *be a finite dimensional linear subspace in* $\mathscr{L}(\mathscr{E})$ *and let* $Q \in \mathscr{L}(\mathscr{E})$ *be such that for every* $X \in \mathscr{X}$, *there exists* $h_n \in \mathscr{E}$, $\|h_n\| = 1$, $Qh_n \to 0$ *and* $\langle Xh_n, h_n\rangle \to 0$. *Then there exists finite rank operators* K_n, $0 \le K_n \le I$, $\operatorname{Tr} K_n = 1$ *such that* $\|QK_n\|_1 \to 0$ *and* $\operatorname{Tr}(XK_n) \to 0$ *for all* $X \in \mathscr{X}$ *where* $\|\cdot\|_1$ *denotes the trace class norm.*

Proof. For each integer $n \ge 1$ consider the set S_n consisting of those linear functionals ϕ on $\mathscr{X}$ which can be written as

$$\phi(X) = \operatorname{Tr}(XK), \qquad X \in \mathscr{X},$$

for some finite rank operator K such that $0 \le K \le I$, $\operatorname{Tr} K = 1$, and $\|QK\|_1 < 1/n$. Clearly S_n is a convex subset of $\mathscr{X}^*$. We claim that the closure of S_n contains zero. Indeed, if it did not, the bipolar theorem would imply the existence of $X \in \mathscr{X}$ such that $\Re\phi(X) \ge 1$ for all $\phi \in S_n$. This, however, is contrary to the hypothesis. Thus we can find $\phi_n \in S_n$ such that $\|\phi_n\| < 1/n$. If we write $\phi_n(X) = \operatorname{Tr}(XK_n)$ with $0 \le K_n \le I$, $\operatorname{Tr} K_n = 1$, and $\|QK_n\|_1 < 1/n$, the sequence K_n satisfies the requirements of the lemma. ∎

LEMMA 3. *Let A be a finite dimensional* $\mathbf{C}^*$*-algebra. Then A has only finitely many equivalence classes of cyclic representations.*

Proof. As seen in [12, Chap. I, Sect. 11), A is isomorphic to a sum of the form $\oplus_{j=1}^{N} \mathscr{L}(\mathbf{C}^{n_j})$. For each $j = 1, 2, \ldots, N$ there is an irreducible representation φ_j of A on $\mathbf{C}^{n_j}$, and every representation of A is uniquely determined up to unitary equivalence by a sequence $(m_1, m_2, \ldots, m_N)$ of cardinal numbers. The representation corresponding to $(m_1, m_2, \ldots, m_N)$ is simply

$$\varphi_1^{(m_1)} \oplus \varphi_2^{(m_2)} \oplus \cdots \oplus \varphi_N^{(m_N)},$$

where $\varphi^{(m)}$ denotes the direct sum of m copies of φ. Now, if φ has a cyclic vector, then $\varphi_j^{(m_j)}$ must also have a cyclic vector, and thus $\varphi_j^{(m_j)} \mid \mathcal{L}(\mathbf{C}^{n_j})$ has a cyclic vector. Since $\varphi_j^{(m_j)}$ acts on a space of dimension $m_j n_j$, the existence of this cyclic vector supplies

$$m_j n_j \leq \dim \mathcal{L}(\mathbf{C}^{n_j}) = n_j^2;$$

hence, $m_j \leq n_j$. Clearly there are only $\prod_{j=1}^{N} (n_j + 1)$ N-tuples with $(m_1, m_2, \ldots, m_N)$ satisfying these inequalities. ▮

LEMMA 4. *Let $Y_j \in \mathcal{L}(\mathcal{E})$ and $h_j \in \mathcal{E}$ be sequences which satisfy*

(i) $\sup_j \operatorname{rank} Y_j < \infty$, $\sup_j \|Y_j\| < \infty$;
(ii) $\lim_{j\to\infty} \|(Y_j - I)h_j\| = 0$;
(iii) $\|h_j\| = 1 \ \forall j$.

Then $\liminf_{n\to\infty} \|Y_j\|_{\text{sp}} \geq 1$.

Proof. Since $\lim(h_j - Y_j h_j) = 0$, the projection k_j of h_j onto the range of Y_j will satisfy $\|k_j\| \to 1$ and $k_j - Y_j k_j \to 0$. We can therefore assume that $h_j \in$ range Y_j. Since these spaces have uniformly bounded dimension, we may as well assume that everything occurs in a finite dimensional space. That is, we may assume that $\dim \mathcal{E} < \infty$. In this case, we may also assume that the Y_j converge in norm to an operator Y and that the h_j converge in norm to a unit vector h. Clearly $Yh = h$, so that $1 \in \sigma(Y)$. Since $\dim \mathcal{E} < \infty$, the spectral radius is continuous on $\mathcal{L}(\mathcal{E})$, and so we conclude that $\liminf_{n\to\infty} \|Y_j\|_{\text{sp}} \geq 1$, as required. ▮

We will now state two additional results which we will need to prove the Lifting Theorem in the next section. The first result is on a *relative Toeplitz–Hausdorff theorem* from [3].

THEOREM 1. *For all $T, Q \in \mathcal{L}(\mathcal{E})$, the set*

$$W_Q(T) = \{\lambda = \lim_{n\to\infty} \langle Th_n, h_n\rangle : h_n \in H, \|h_n\| = 1, \lim_{n\to\infty} \|Qh_n\| = 0\}$$

is a compact convex set.

The second result concerns the continuity of the spectrum on closed similarity orbits whose proof may be found in [4].

THEOREM 2. *Let $T \in \mathcal{L}(\mathcal{E})$ and let $D_j \in \mathcal{L}(\mathcal{E})$ be a sequence of invertible operators such that*

$$T_0 = \lim_{j\to\infty} D_j T D_j^{-1}.$$

If the set $\{D_j, D_j^{-1} : j = 1, 2, \ldots\}$ *is contained in a finite dimensional subspace, then* $\|T_0\|_{sp} = \|T\|_{sp}$.

4. The Lifting Theorem

In this section we will formulate and prove the Lifting Theorem which turns out to be very useful in analyzing the structured singular value. In the finite dimensional case, i.e., $\mathscr{E}$ being finite-dimensional, it was employed in [1] to show that $\mu_\Delta(A) = \hat{\mu}_\Delta(A)$ when the relevant diagonal algebra has three or fewer blocks, and so it gave an alternative proof of a result due to Doyle [5].

Let HS($\mathscr{E}$) denote the space of all Hilbert–Schmidt operators on $\mathscr{E}$ equipped with the Hilbert space structure

$$\langle T_1, T_2\rangle := \mathrm{Tr}(T_2^* T_1),$$

where Tr denotes the trace. Define the operator $L_A : \mathrm{HS}(\mathscr{E}) \rightarrow \mathrm{HS}(\mathscr{E})$ by $L_A := AX$. Now we set

$$\overline{\mu}_\Delta(A) := \mu_{\tilde{\Delta}}(L_A),$$

where

$$\tilde{\Delta} := \{L_X : X \in \Delta'\}'.$$

In what follows, we will assume that Δ' is finite dimensional $*$-algebra, but that $\mathscr{E}$ is arbitrary. We can now state the following.

Theorem 3 (Lifting Theorem). *Let* Δ' *be a finite dimensional* $*$*-algebra. Then*

$$\hat{\mu}_\Delta(A) := \overline{\mu}_\Delta(A).$$

Proof. The proof follows exactly the proof of Theorem 3 in [1] with several modifications necessary since $\mathscr{E}$ may be infinite dimensional. The lemmas that we proved above were designed to exactly push the proof through in this case. For the convenience of the reader, we will give all the necessary details.

First note as in [1], one can easily show that

$$\mu_\Delta(A) \leq \overline{\mu}_\Delta(A) \leq \hat{\mu}_\Delta(A).$$

If $\hat{\mu}_\Delta(A) = 0$ there is nothing to prove. Therefore we may assume without loss of generality that $\hat{\mu}_\Delta(A) = 1$ and we must show that $\overline{\mu}_\Delta(A) \geq 1$.

Choose invertible operators $X_j \in \Delta'$ such that $\|X_j A X_j^{-1}\| \to \hat{\mu}_\Delta(A)$. Since Δ' is a finite dimensional algebra, we may assume without loss of generality that $X_j A X_j^{-1} \to A_0$, $\|A_0\| = 1$. (This follows since A_j belongs to the finite dimensional space $\Delta' A \Delta'$.) Obviously $\|XA_0X^{-1}\| \geq \|A_0\|$ for all $X, X^{-1} \in \Delta'$.

Now for $X \in \Delta'$, $\|X\| < 1$, we have that

$$\|(I - X)A_0(I + X + X^2 + \cdots)\| \geq 1.$$

Hence for every $X \in \Delta'$ and for sufficiently small $\varepsilon = \varepsilon_j > 0$, there exists $h = h_j$ with $\|h\| = 1$, such that

$$\|(I - \varepsilon X)A_0(I + \varepsilon X + \varepsilon^2 X^2 + \cdots)h\|^2 \geq 1 - \varepsilon^2; \tag{1}$$

that is,

$$\langle A_0^* A_0 h, h\rangle + 2\varepsilon \Re \langle A_0^*(A_0 X - XA_0)h, h\rangle + O(\varepsilon^2) \geq 1 - \varepsilon^2$$

or, equivalently,

$$2\varepsilon \Re \langle A_0^*(A_0 X - XA_0)h, h\rangle + O(\varepsilon^2) \geq \langle (I - A_0^* A_0)h, h\rangle - \varepsilon^2. \tag{2}$$

Dividing by ε_j and letting $\varepsilon_j \to 0$ as $j \to \infty$, we see from (1), (2) that we have a sequence $\|h_j\| = 1$, such that

$$\langle (I - A_0^* A_0)h_j, h_j\rangle \to 0, \tag{3}$$

$$\liminf_{j \to \infty} \Re \langle A_0^*(A_0 X - XA_0)h_j, h_j\rangle \geq 0. \tag{4}$$

Hence we can conclude that

$$\liminf_{j \to \infty} \Re \langle (X - A_0^* X A_0)h_j, h_j\rangle \geq 0. \tag{5}$$

Set

$$Q := I - A_0^* A_0, \qquad T := X - A_0^* X A_0.$$

Then from (3), (5), we see that

$$Qh_j \to 0, \qquad \liminf_{j \to \infty} \Re \langle Th_j, h_j\rangle \geq 0. \tag{6}$$

Applying the above argument to ζX for any $\zeta \in \partial\mathbf{D}$ (the unit circle), we see that there exists a sequence $h_j^{(\zeta)}$, $\|h_j^{(\zeta)}\| = 1$, such that

$$Qh_j^{(\zeta)} \to 0, \qquad \liminf_{j\to\infty} \Re\,\zeta\langle Th_j, h_j\rangle \geq 0. \tag{7}$$

Now from Theorem 1, relative to the operators Q and T, we see that $W_Q(T)$ is a compact, convex set. (This result plays the role of the Toeplitz–Hausdorff theorem in [1].) Hence if $0 \notin W_Q(T)$, then there would exist $\zeta \in \partial\mathbf{D}$ such that

$$\liminf_{j\to\infty} \Re\,\zeta\langle Th_j, h_j\rangle < 0$$

for every sequence $\{h_j\}$ of unit vectors such that $\lim_{j\to\infty} \|Qh_j\| = 0$, thus contradicting (7).

Thus, we have shown that for each $X \in \Delta'$, there exists a sequence h_j, $\|h_j\| = 1$, such that

$$(I - A_0^* A_0)h_j \to 0, \qquad \langle (X - A_0^* X A_0)h_j, h_j\rangle \to 0. \tag{8}$$

Note that the sequence h_j depends on X. The reason that we now have to "Lift" to the space of Hilbert–Schmidt operators is to find a sequence independent of X.

Thus we apply Lemma 2 to obtain K_j, such that

$$\|QK_j\|_1 \to 0, \qquad \mathrm{Tr}((X - A_0^* X A_0)K_j) \to 0. \tag{9}$$

We can write $K_j = H_j^2$, where H_j is a positive Hilbert–Schmidt operator for $j \geq 1$. Hence we get from (9) that

$$\|L_X L_{A_0} H_j\|_{\mathrm{HS}}^2 - \|L_X H_j\|_{\mathrm{HS}}^2 \to 0 \qquad \forall X \in \Delta'. \tag{10}$$

Passing to appropriate subsequences of the j's, we can now apply Lemma 3 to obtain partial isometries U_j and V_j on $\mathcal{H} = \mathrm{HS}(\mathcal{E})$ commuting with L_X, $X \in \Delta'$, such that the initial space of U_j is $\{L_X H_j : x \in \Delta'\}$, the initial space of V_j is $\{L_X A_0 H_j : x \in \Delta'\}$, and

$$U_j L_X H_j = L_X H_{j_0}, \qquad V_j L_X A_0 H = L_X A_0 H_{j_0} \qquad \forall X \in \Delta'$$

with some fixed j_0. Without loss of generality, we can assume that the limits $H = \lim_{j\to\infty} U_j H_j$ and $K = \lim_{j\to\infty} V_j L_{A_0} H_j$ exist in $\mathcal{H}_{j_0} := \{L_X H_{j_0} : X \in \Delta'\}$ and $\mathcal{K}_{j_0} := \{L_X L_{A_0} H_{j_0} : X \in \Delta'\}$, respectively. Then (10) implies that

$$\|L_X K\|_{\mathrm{HS}}^2 = \|L_X H\|_{\mathrm{HS}}^2, \qquad \forall X \in \Delta'.$$

Therefore there exists a partial isometry W with initial space H_{j_0} and final space K_{j_0} such that

$$WL_X K = L_X H \qquad \forall X \in \Delta'.$$

Hence $R_j := U_j^* W V_j$ are partial isometries, commuting with L_X, $X \in \Delta'$, and rank $R_j = U_j^* W V_j \leq$ rank $W < \infty$. Also

$$\|(R_j L_{A_0} - I)H_j\|_{\mathrm{HS}} = \|WV_j L_{A_0} H_j - U_j H_j\|_{\mathrm{HS}} \to \|WK - H\|_{\mathrm{HS}} = 0.$$

We now apply Lemma 4 (with HS($\mathscr{C}$) in place of $\mathscr{C}$ and $R_j L_{A_0}$ in place of Y_j) to deduce $\liminf_{j\to\infty}\|R_j L_{A_0}\|_{\mathrm{sp}} \geq 1$. Since R_j commutes with L_X, $X \in \Delta'$, we have

$$L_{X_k} R_j L_A L_{X_k^{-1}} \to R_j L_{A_0} \qquad \text{in norm}$$

for $k \to \infty$. We can then apply Theorem 2 to deduce that $\|R_j L_{A_0}\|_{\mathrm{sp}} = \|R_j L_A\|_{\mathrm{sp}}$. Consequently, we have

$$\liminf_{j\to\infty} \|L_A R_j\|_{\mathrm{sp}} = \liminf_{j\to\infty} \|R_j L_A\|_{\mathrm{sp}} = \liminf_{j\to\infty} \|R_j L_{A_0}\| \geq 1.$$

Thus,

$$\bar{\mu}_\Delta(A) = \mu_{\hat{\Delta}}(L_A) \geq \liminf_{j\to\infty} \|L_A X_j\|_{\mathrm{sp}} \geq 1 = \hat{\mu}_\Delta(A),$$

which completes the proof of the theorem. ∎

Remarks. (i) Theorem 3 was proven in [1] when dim $\mathscr{C} < \infty$. Recently, Fan has announced another proof of the Lifting Theorem in this case [6].

(ii) Using the well-known fact that μ_Δ is continuous for $\mathscr{C}$ finite dimensional, the theorem implies that $\hat{\mu}_\Delta$ is continuous in this case as well [1].

5. Time-Varying Perturbations

Theorem 3 is valid for any finite-dimensional C^*-algebra, Δ'. We have already remarked how this result was used in [1] to show that $\hat{\mu}_\Delta = \mu_\Delta$ for block algebras with three or fewer blocks. This result is also strongly connected to some recent work on robust stability with respect to time-varying perturbations [11].

Let ℓ_+^2 be the space of square summable one-sided sequences in $\mathbf{C}$, let $\mathscr{C}$ denote the set of all bounded linear operators on ℓ_+^2. Further, let $A : \ell_+^2(\mathbf{C}^n) \to \ell_+^2(\mathbf{C}^n)$ be an arbitrary bounded linear operator. In control theory terminology, A defines a (possibly) time-varying system. Time-invariant systems are defined by Toeplitz operators. (Here $\ell_+^2(\mathbf{C}^n)$ denotes the space of square summable sequences in $\mathbf{C}^n$, i.e., the space of finite energy vector-valued signals with n components.) Then we want to interpret $\hat{\mu}_\Delta(A)$ as a structured singular value on an extended space with an enhanced perturbation structure. Note $\mathscr{C}$ in this case is the Hilbert space $\ell_+^2(\mathbf{C}^n)$.

Define the algebra of perturbations

$$\Delta := \{\begin{bmatrix} \delta_1 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \delta_n \end{bmatrix} : \delta_i \in \mathscr{C},\ i = 1, \dots, n\}.$$

Then the commutant of Δ is the finite dimensional C^*-algebra,

$$\Delta' := \{\begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} : d_i \in \mathbf{C},\ i = 1, \dots, n\}.$$

Note that a constant $d \in \mathbf{C}$ defines an operator on ℓ_+^2 via multiplication.

From Theorem 1, it follows that the $\hat{\mu}$ given by the infimum of $\|XAX^{-1}\|$ over all constant X–scales equals $\overline{\mu}_\Delta(A)$. The question is, what robust stability analysis problem does $\overline{\mu}_\Delta(A)$ correspond to?

The Lifting Theorem now gives the following interpretation of $\hat{\mu}_\Delta(A)$. We regard the operator A as acting on the set of Hilbert–Schmidt operators $\mathscr{H} := \mathrm{HS}(\ell_+^2(\mathbf{C}^n))$ via left multiplication L_A. The Hilbert space $\mathscr{H}$ has a very simple representation. Indeed, we see that an element $h \in \mathscr{H}$ admits the (time-domain) representation

$$h = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{bmatrix}, \qquad (11)$$

where $h_{ij} : \ell_+^2 \to \ell_+^2$ for $1 \le i, j \le n$, and $\mathrm{Tr}(h^*h) < \infty$.

We would now like to give the following convenient representation for an element $h \in \mathcal{H}$. Let r_j denote the jth row of h in (11) and let r_j' denote the transpose (i.e., write the row as a column) for $1 \leq j \leq n$. Then we represent

$$h \cong \begin{bmatrix} r_1' \\ r_2' \\ \vdots \\ r_n' \end{bmatrix}.$$

Then

$$\tilde{\Delta}_1 \cong \left[\left\{\begin{bmatrix} \Delta_1 & 0 & 0 & 0 \\ 0 & \Delta_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta_n \end{bmatrix} : \Delta_j \in \mathcal{C}_n \right\}\right],$$

where $\mathcal{C}_n$ denotes the algebra of $n \times n$ matrices with elements in $\mathcal{C}$. $\tilde{\Delta}$ is a space of time-varying perturbations and we have from the Lifting Theorem that

$$\hat{\mu}_\Delta(A) = \mu_{\tilde{\Delta}}(L_A).$$

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