A Hybrid State-Space Approach to Sampled-Data Feedback Control

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ABSTRACT

This paper develops a state-space theory for the study of linear shift-invariant finite-dimensional hybrid dynamical systems. By hybrid system, we mean an input-output operator relating hybrid signals, that is, signals which consist of two parts: a continuous-time part and a discrete-time part. We first introduce the hybrid state-space model and show the closedness properties under the fundamental operations. We next investigate the standard basic issues associated with this formalism: reachability, observability, and internal stability. We define the notion of H_2-norm of the hybrid system and show how to evaluate it. Since the analysis result can be applied to any hybrid system with the proposed hybrid state-space model, we apply the analysis result to the optimal digital control problem for standard four-block configuration with H_2-norm performance measure. The solution, with computational algorithm, is derived for the time-delay case.

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1. INTRODUCTION

This paper introduces a state-space theory for the study of linear shift-invariant finite-dimensional hybrid dynamical systems. By hybrid system, we mean an input-output operator relating hybrid signals, that is, signals which consist of two parts: a continuous-time part and a discrete-time part. Such systems are encountered in practice in every instance where a digital computer interacts with an analog environment. Obvious examples are in digital control, digital filtering, and signal processing. In particular, in [8, 11, 12] it has been shown that hybrid systems constitute a promising formalism to investigate the intersampling performance of sampled-data control loops. Although the modified z-transformation has been used for the analysis of the intersampling behavior, the results cannot be directly applied to the synthesis problems with the intersampling performance taken into account. Hence, we need a uniform system theory for hybrid systems, which is useful both for analysis and for synthesis.

The state-space framework for hybrid systems was introduced in [8, 11, 12], but it was not completely developed there, since the main focus was on analysis and synthesis of L_2-induced norm optimization for sampled-data control, and the hybrid system treated there has only continuous-time input and output, i.e., there was no discrete-time input or output. The main purpose of this paper is to develop a general hybrid system theory based on a hybrid state-space model. Since the (A, B, C, D)-form representation plays a key role in developing a nice and uniform system theory for continuous-time or discrete-time linear time-invariant systems, we here introduce a hybrid state-space (A, B, C, D) form for developing a hybrid system theory which is a natural extension of the conventional linear system theory. The model is a generalization of our previous one in [8, 11, 12], and it has hybrid state, hybrid input, and hybrid output vectors.

We first introduce the hybrid state-space model in Section 2.1, and Section 2.2 gives several interesting hybrid systems which can be represented by the model. Since the closedness properties for the fundamental operations, namely cascade, parallel, and feedback connections, must be satisfied in order to develop a uniform theory, they are investigated in Section 2.3. This implies that the hybrid model is general enough to represent any hybrid system consisting of several unit systems such as a continuous-time system, a discrete-time system, a sampler with a single sampling rate, or a hold with a periodic function. We note that the unit systems themselves have hybrid state-space forms.

There are two other interesting models for representing sampled-data systems. One is a lifting or function-space model proposed by Toivonen [18] and Yamamoto [22], and it has been used for the analysis and synthesis of sampled-data feedback systems. For example, Toivonen [18], Bamieh and Pearson [2], and Hayakawa et al. [9] have investigated the H_∞ control problem, and the tracking condition was discussed by Yamamoto [20], on the basis of this model. The lifting model is time invariant but is infinite dimensional, while our model is finite dimensional but is periodic with the sampling period. The relationship will be briefly remarked on in Section 3.1. Another model is a state-space model with discrete jumps, which was recently proposed by Sivashankar and Khargonekar [17]. The discrete jump represents the sampling operation in that model, whereas it appears explicit in our state-space model. The relation between the two models will be discussed briefly in Section 2.1.

In order to develop a hybrid state-space theory, we investigate the standard basic issues associated with this formalism: reachability, observability, and internal stability. This is done in Section 3, based on the general solution for the hybrid state-space model, which includes the intersampling behavior. The notions and the results are natural generalization of the previous ones, and they give the basis of the hybrid system theory.

Next we define the notion of H_2-norm of the hybrid system and show how to evaluate it. The norm is defined in the same manner as in [13]. In Section 4, we derive a discrete-time system which preserves the H_2-norm and give a computational algorithm for deriving the equivalent discrete-time system. Since the result can be applied to any hybrid system with the proposed hybrid state-space model, we apply it to the optimal digital control problem for the standard four-block configuration with H_2-norm performance measure.

In Section 5.1, we can easily show that the problem in the sampled-data setting can be reduced to an equivalent discrete-time H_2-norm optimization problem, since our hybrid state-space model covers the four-block configuration. We also consider the case where the continuous-time plant and/or the digital controller have time delays. Since computational time delay is inevitable in digital computation, this problem is important for implementation. Chen and Francis [4] have investigated the time-delay cases for H_2 optimal control by an operator-theoretic approach, and Nakajima [15] has discussed the robust stabilization problem for the delay case. In Section 2.2 we will show that the hybrid state-space model can represent any time lag followed by a hold device, and hence the results based on the hybrid state-space model can be directly applied to the sampled-data feedback control problem with time delay at the control input channel. The concrete solution for the H_2-norm optimization problem is given in Section 5.2.

Though the results are only slight extensions of the previous results in [13] and [4], we should emphasize that we can easily make a general formulation if we use the hybrid state-space model and that the synthesis result in any setting can be directly derived by applying the general analysis result in Section 4. This is an advantage in developing a general hybrid state-space theory.

2. HYBRID STATE-SPACE MODEL

2.1. *Model Representation*

In this subsection, we will introduce a hybrid state-space model with hybrid state, input, and output vectors, where "hybrid" means the signal is composed of two types of signals, namely analog and digital.

We first consider a sampled-data feedback control system shown in Figure 1 as a typical example of the hybrid system, where $P(s)$ and $K[z]$ denote a continuous-time plant and a discrete-time controller, respectively.[1] $u(t)$, $y(t)$, and $d(t)$ are analog signals of the control input, plant output, and disturbance input, respectively, while $z[k]$, $v[k]$, and $n[k]$ are discrete signals of the input and output of the controller and measurement noise, respectively. $n[k]$ reflects the quantization error in the A/D conversion. $\mathcal{S}_\tau$ denotes the sampling operator (or A/D convertor) with sampling period τ satisfying

$$\mathcal{S}_\tau y(t) = y(k\tau) =: y[k], \qquad k = 0, 1, 2, \dots . \tag{2.1}$$

[1] Throughout the paper, parentheses $(\cdot)$ around an independent variable indicate an analog function of continuous time or the Laplace transform of such a function, whereas square brackets $[\cdot]$ indicate a discrete sequence or the z-transform of such a sequence.

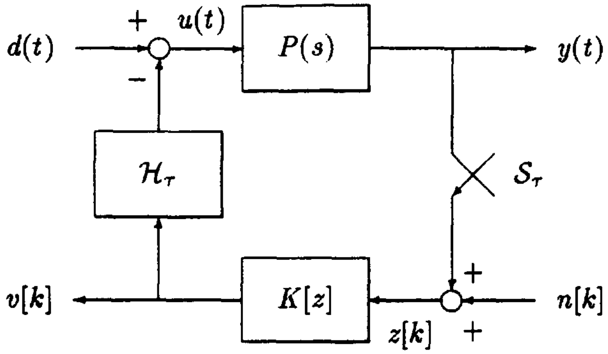

FIG. 1. Sampled-data feedback control system.

$\mathscr{H}_\tau$ denotes the hold operator (or D/A convertor) with τ-periodic hold function $H(t) = H(t + \tau)$ satisfying

$$\mathscr{H}_\tau v[k] = H(\sigma)v[k] =: v(k\tau + \sigma), \qquad 0 \leqslant \sigma < \tau. \tag{2.2}$$

Let the state-space representations of the continuous-time plant $P(s)$ and the discrete-time controller $K[z]$ be respectively given by

$$P(s): \quad \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t), \\ y(t) = C_p x_p(t); \end{cases} \tag{2.3}$$

$$K[z]: \quad \begin{cases} x_d[k+1] = A_k x_d[k] + B_k z[k], \\ \quad\; v[k] = C_k x_d[k] + D_k z[k]. \end{cases} \tag{2.4}$$

Then we can readily see that the state-space equation of the sampled data feedback control system in the period of $k\tau \leqslant t < (k+1)\tau$ can be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{x}_p(t) \\ x_d[k+1] \end{bmatrix} &= \begin{bmatrix} A_p - B_p H(t) D_k \mathscr{S}_\tau C_p & -B_p H(t) C_k \\ B_k \mathscr{S}_\tau C_p & A_k \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_d[k] \end{bmatrix} \\ &\quad + \begin{bmatrix} B_p & -B_p H(t) D_k \\ 0 & B_k \end{bmatrix} \begin{bmatrix} d(t) \\ n[k] \end{bmatrix}, \end{aligned} \tag{2.5}$$

$$\begin{bmatrix} y(t) \\ v[k] \end{bmatrix} = \begin{bmatrix} C_p & 0 \\ D_k \mathscr{S}_\tau C_p & C_k \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_d[k] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D_k \end{bmatrix} \begin{bmatrix} d(t) \\ n[k] \end{bmatrix}.$$

The above representation has the following features:

(1) There are three hybrid signals: for the state vector $(x_p(t), x_d[k])$, the input vector $(d(t), n[k])$, and the output vector $(y(t), v[k])$.

(2) We have two types of state transition equations. One is a differential equation for the analog state, and the other is a difference equation for the digital state.

(3) The expression contains the sample operator $\mathscr{S}_\tau$ and τ-periodic functions such as $B_p H(t) C_k$.

We now introduce a hybrid state-space model which adopts the above features and is a natural extension of the (A, B, C, D) form for continuous-time or discrete-time system. Consider a class of hybrid systems expressed as

$$\mathscr{T}_s:\quad \begin{cases} \mathscr{D}_h X_h[k] = \mathbf{A}(t) X_h[k] + \mathbf{B}(t) U_h[k], \\ \quad Y_h[k] = \mathbf{C}(t) X_h[k] + \mathbf{D}(t) U_h[k] \end{cases} \tag{2.6}$$

for $k\tau \leqslant t < (k+1)\tau$, where X_h, U_h, and Y_h are hybrid signals for the state, input, and output, respectively, defined by

$$X_h[k] := \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix}, \qquad U_h[k] := \begin{bmatrix} u_c(t) \\ u_d[k] \end{bmatrix}, \qquad Y_h[k] := \begin{bmatrix} y_c(t) \\ y_d[k] \end{bmatrix},$$

and $\mathscr{D}_h$ is an operator satisfying

$$\mathscr{D}_h \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} = \begin{bmatrix} \dot{x}_c(t) \\ x_d[k+1] \end{bmatrix}. \tag{2.7}$$

$\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$ are in a class $\mathscr{M}$ defined by

$$\mathscr{M} := \begin{bmatrix} \mathbf{C}_c \cup \mathbf{TS}_\tau & \mathbf{T}_\tau \\ \mathbf{CS}_\tau & \mathbf{C}_d \end{bmatrix}, \tag{2.8}$$

where

$\mathbf{C}_c$ = constant matrix operator (continuous signal $\rightarrow$ continuous signal),
$\mathbf{TS}_\tau$ = τ-periodic matrix operator $\times \mathscr{S}_\tau$ (continuous signal $\rightarrow$ continuous signal),
$\mathbf{T}_\tau$ = τ-periodic matrix operator (discrete signal $\rightarrow$ continuous signal),
$\mathbf{CS}_\tau$ = constant matrix operator $\times \mathscr{S}_\tau$ (continuous signal $\rightarrow$ discrete signal),
$\mathbf{C}_\mathrm{d}$ = constant matrix operator (discrete signal $\rightarrow$ discrete signal).

We use the simple notation

$$\mathscr{T}_s = \{\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)\} \in \mathscr{H}_{\mathrm{sys}}$$

for representing the hybrid system with the state-space form above, if $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$ are all in $\mathscr{M}$. In this case, a concrete representation of the hybrid state-space model is given as follows:

$$\mathscr{T}_s: \begin{cases} \begin{bmatrix} \dot{x}_c(t) \\ x_d[k+1] \end{bmatrix} = \begin{bmatrix} A_c + A_{cs}(t)\mathscr{S}_\tau & A_{cd}(t) \\ A_{ds}\mathscr{S}_\tau & A_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} \\ \qquad + \begin{bmatrix} B_c + B_{cs}(t)\mathscr{S}_\tau & B_{cd}(t) \\ B_{ds}\mathscr{S}_\tau & B_d \end{bmatrix} \begin{bmatrix} u_c(t) \\ u_d[k] \end{bmatrix}, \\ \begin{bmatrix} y_c(t) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} C_c + C_{cs}(t)\mathscr{S}_\tau & C_{cd}(t) \\ C_{ds}\mathscr{S}_\tau & C_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} \\ \qquad + \begin{bmatrix} D_c + D_{cs}(t)\mathscr{S}_\tau & D_{cd}(t) \\ D_{ds}\mathscr{S}_\tau & D_d \end{bmatrix} \begin{bmatrix} u_c(t) \\ u_d[k] \end{bmatrix} \end{cases} \tag{2.9}$$

for $k\tau \leqslant t < (k+1)\tau$, where $x_c(t) \in \Re^{n_c}$ and $x_d[k] \in \Re^{n_d}$ denote the analog and discrete state variables, respectively; $u_c(t) \in \Re^{m_c}$ and $u_d[k] \in \Re^{m_d}$ are the piecewise continuous and discrete inputs, respectively; $y_c(t) \in \Re^{p_c}$ and $y_d[k] \in \Re^{p_d}$ are the continuous and discrete outputs, respectively; A_c, A_{cd}, and A_d in $\mathbf{A}(t)$ are constant matrices of appropriate dimensions; and $A_{cs}(t)$ and $A_{cd}(t)$ in $\mathbf{A}(t)$ are τ-periodic matrices of appropriate dimensions. We have the same description for the matrices in $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$.

Figure 2 illustrates the reason why we consider the class $\mathscr{M}$ for the operator from a hybrid signal to a hybrid signal. $M_c \in \mathbf{C}_c$ and $M_d \in \mathbf{C}_d$ represent the analog-to-analog and digital-to-digital operators, respectively. The operator $M_{cd}\mathscr{H}_\tau$ from the digital signal $w_d[k]$ to the analog signal $z_c(t)$ belongs to $\mathbf{T}_\tau$, while the operator $M_{ds}\mathscr{S}_\tau$ from the analog signal $w_c(t)$ to the digital signal $z_d[k]$ is clearly in $\mathbf{CS}_\tau$. In general, we need two more operators. One is an operator from analog signal to analog signal through sampler and hold, denoted by $\mathscr{H}_\tau M_{cs}\mathscr{S}_\tau$ in Figure 2. We can see that $\mathscr{H}_\tau M_{cs}\mathscr{S}_\tau$ belongs to $\mathbf{TS}_\tau$. $B_p H(t) D_k \mathscr{S}_\tau C_p = B_p H(t) D_k C_p \mathscr{S}_\tau$ in (2.5) corresponds to this term, so it cannot be neglected for the general expression. Finally, we have an operator from digital signal to digital signal through hold and sampler, denoted by $\mathscr{S}_\tau M_{sd}\mathscr{H}_\tau$ in Figure 2. However, $\mathscr{S}_\tau M_{sd}\mathscr{H}_\tau$ is just a constant operator, i.e., $\mathscr{S}_\tau M_{sd}\mathscr{H}_\tau \in \mathbf{C}_d$. Hence, we need not take account of this kind of operation independently.

REMARK. Recently, another state-space model for sampled-data systems was proposed by Sivashankar and Khargonekar [17], where they introduced the notion of discrete jump of the state at each sampling instant. The model

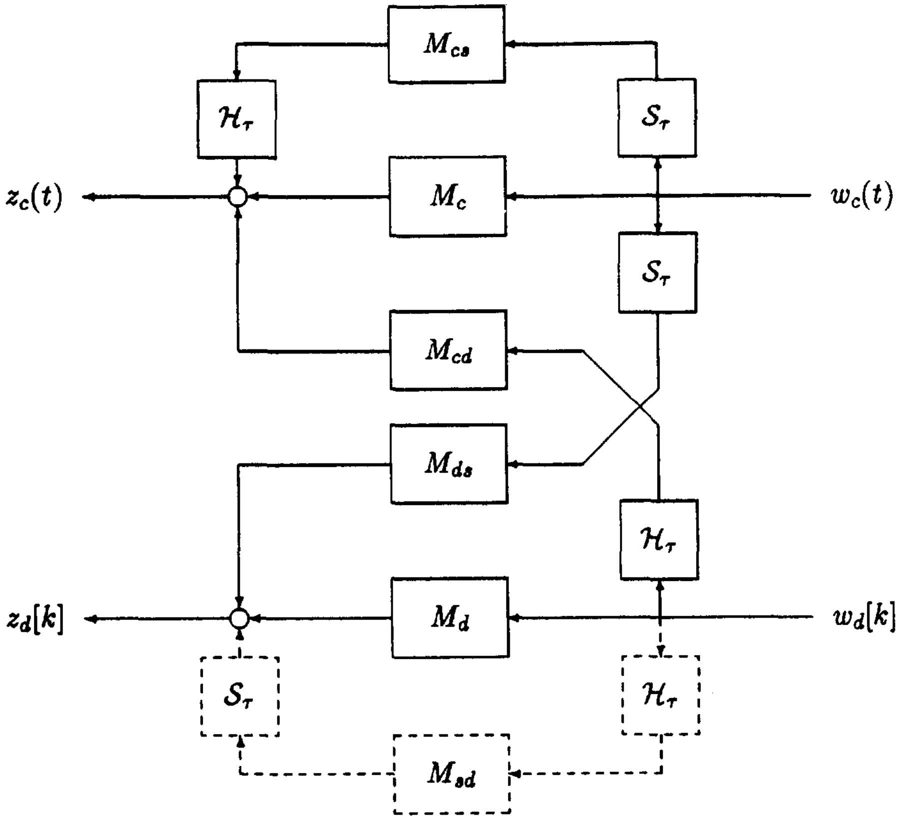

FIG. 2. Operation for hybrid signals.

is expressed as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \qquad t \neq k\tau, \\
x(k\tau^+) &= A_{dd}(k\tau) + B_{dd}u_d(k\tau), \\
y(t) &= Cx(t), \\
y_d(k\tau) &= C_{dd}(k\tau).
\end{aligned} \tag{2.10}
$$

We can show that our hybrid state-space model can be converted to the above model if we augment the state appropriately. For example, we consider a hybrid system

$$\begin{bmatrix} \dot{x}_c(t) \\ x_d[k+1] \end{bmatrix} = \begin{bmatrix} A_c + A_{cs}\mathscr{S}_\tau & A_{cd} \\ A_{ds}\mathscr{S}_\tau & A_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix}.$$

Then we have the form (2.10) with

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_s(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} A_c & A_{cs} & A_{cd} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_s(t) \\ x_d(t) \end{bmatrix},$$

$$\begin{bmatrix} x_c(k\tau^+) \\ x_s(k\tau^+) \\ x_d(k\tau^+) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ A_{ds} & 0 & A_d \end{bmatrix} \begin{bmatrix} x_c(k\tau) \\ x_s(k\tau) \\ x_d(k\tau) \end{bmatrix},$$

where we need an additional state $x_s(t)$ for representing the A_{cs} term.

2.2. *Examples of Hybrid Systems*

Using the hybrid state-space representation, the cascade, parallel, and feedback operations of hybrid systems are defined in the same manner as in the continuous-time case, as will be discussed in the next subsection. In this subsection, we will show several examples of the hybrid state-space representations.

We first confirm that the sampled-data feedback control system shown in Figure 1 can be represented by the hybrid state-space model (2.9). Let us define X_h, U_h, and Y_h by

$$X_h[k] = \begin{bmatrix} x_p(t) \\ x_d[k] \end{bmatrix}, \qquad U_h[k] = \begin{bmatrix} d(t) \\ n[k] \end{bmatrix}, \qquad Y_h[k] = \begin{bmatrix} y(t) \\ v[k] \end{bmatrix}.$$

Then it is easy to see that the sampled-data feedback control system shown in

Figure 1 can be represented by a hybrid state-space model with

$$\mathbf{A}(t) = \begin{bmatrix} A_p - B_p H(t) D_k C_p \mathscr{S}_\tau & -B_p H(t) C_k \\ B_k C_p \mathscr{S}_\tau & A_k \end{bmatrix},$$

$$\mathbf{B}(t) = \begin{bmatrix} B_p & -B_p H(t) D_k \\ 0 & B_k \end{bmatrix}, \tag{2.11}$$

$$\mathbf{C}(t) = \begin{bmatrix} C_p & 0 \\ D_k C_p \mathscr{S}_\tau & C_k \end{bmatrix}, \quad \mathbf{D}(t) = \begin{bmatrix} 0 & 0 \\ 0 & D_k \end{bmatrix}.$$

Since the model (2.9) is general, it contains pure continuous-time and discrete-time systems as special cases, and it also covers systems without continuous-time or discrete-time states. For example, the sampler and the hold themselves can be represented by the model, and it is easy to show that an open-loop system with sampler, discrete-time system $K[z]$ given by (2.4), and hold $H(t)$, as depicted in Figure 3, has a hybrid state-space model

$$\mathbf{A}(t) = \begin{bmatrix} - & - \\ - & A_k \end{bmatrix}, \qquad \mathbf{B}(t) = \begin{bmatrix} - & - \\ B_k \times \mathscr{S}_\tau & - \end{bmatrix},$$

$$\mathbf{C}(t) = \begin{bmatrix} - & H(t) \times C_k \\ - & - \end{bmatrix}, \quad \mathbf{D}(t) = \begin{bmatrix} H(t) \times D_k \times \mathscr{S}_\tau & - \\ - & - \end{bmatrix}.$$

Moreover, we can show that any time delay followed by a hold $\mathscr{H}_\tau$ as shown in Figure 4 can be represented by the hybrid state-space model.

Suppose that the time delay L can be rewritten as

$$L := l\tau + \mu$$

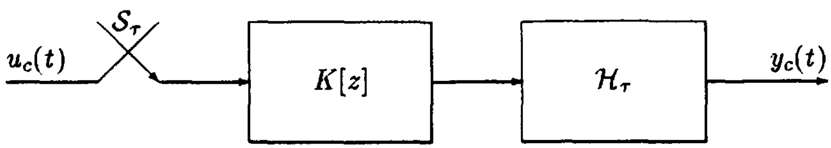

FIG. 3. An open-loop sampled-data system.

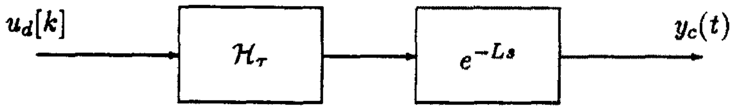

FIG. 4. A time delay with hold.

where l is a nonnegative integer and $\mu \in (0, \tau]$. Since $H(t)$ is τ-periodic, we can easily see that the output $y_c(t)$ in the kth sampling period $[k\tau \leqslant t < (k+1)\tau]$ can be expressed as

$$y_c(t) = \begin{cases} H(t-\mu)u_d[k-l-1], & k\tau \leqslant t < k\tau + \mu, \\ H(t-\mu)u_d[k-l], & k\tau + \mu \leqslant t < (k+1)\tau. \end{cases}$$

Let us now define two τ-periodic functions $H_{\mu 1}(t)$ and $H_{\mu 2}(t)$ as

$$H_{\mu 1}(t) := \begin{cases} 0, & k\tau \leqslant t < k\tau + \mu, \\ H(t-\mu), & k\tau + \mu \leqslant t < (k+1)\tau, \end{cases} \tag{2.12}$$

$$H_{\mu 2}(t) := \begin{cases} H(t-\mu), & k\tau \leqslant t < k\tau + \mu, \\ 0, & k\tau + \mu \leqslant t < (k+1)\tau. \end{cases} \tag{2.13}$$

Then we have the following hybrid state-space model of the system with input $u_d[k]$ and output $y_c(t)$ for the case $l = 0$:

$$x_d[k+1] = 0 \cdot x_d[k] + u_d[k],$$

$$y_c(t) = H_{\mu 2}(t)x_d[k] + H_{\mu 1}(t)u_d[k].$$

Since the form for the case $l \geqslant 1$ can be readily derived in the same manner, it is omitted.

This property is quite useful for real control situations, since the computational time delay $0 < L \leqslant \tau$ is inevitable in the implementation of any digital controller. This notion can also be applied to the continuous-time system with any time delay at the control input channel. We note that the delay may not be synchronized with the sampling period τ. An H_2 optimal control problem for such situations will be investigated in Section 5, based on the H_2-norm analysis for the general hybrid system in Section 4.

2.3. *Connections in $\mathscr{H}_{\text{sys}}$*

In order to show that any system composed of several subsystems in $\mathscr{H}_{\text{sys}}$ can be described by a hybrid state-space model (2.9), it is enough to show that the hybrid state-space representation is closed under four fundamental operations, namely scalar multiplication and parallel, cascade, and feedback connections.

It is obvious that scalar multiplication is closed in $\mathscr{H}_{\text{sys}}$. More precisely, we have

$$\alpha\Sigma = \{\mathbf{A}(t), \mathbf{B}(t), \alpha\mathbf{C}(t), \alpha\mathbf{D}(t)\} \in \mathscr{H}_{\text{sys}} \tag{2.14}$$

for any scalar α and $\Sigma = \{\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)\} \in \mathscr{H}_{\text{sys}}$. Therefore, we need only show the closedness of the parallel, cascade, and feedback connections.

Before doing so, we introduce three operations—sum, product, and inverse—in the class $\mathscr{M}$. Let $P_i(t)$ in $\mathscr{M}$ be given by

$$P_i(t) = \begin{bmatrix} P_{ic} + P_{ics}(t)S_\tau & P_{icd}(t) \\ P_{ids}S_\tau & P_{id} \end{bmatrix} \in \mathscr{M} \qquad (i = 1, 2). \tag{2.15}$$

Then we can define:

(1) *Sum.* If addition in $\mathscr{M}$ is defined by

$$P_1(t) \oplus_M P_2(t)$$
$$= \begin{bmatrix} (P_{1c} + P_{2c}) + \{P_{1cs}(t) + P_{2cs}(t)\}S_\tau & P_{1cd}(t) + P_{2cd}(t) \\ (P_{1ds} + P_{2ds})S_\tau & P_{1d} + P_{2d} \end{bmatrix} \tag{2.16}$$

then we have

$$P_1(t) \oplus_M P_2(t) \in \mathscr{M} \tag{2.17}$$

and

$$\{P_1(t) \oplus_M P_2(t)\}U = P_1(t)U + P_2(t)U \qquad \forall U = \begin{bmatrix} u_c(t) \\ u_d[k] \end{bmatrix}. \tag{2.18}$$

(2) *Product.* If multiplication in $\mathscr{M}$ is defined by

$$P_1(t) \otimes_m P_2(t) = \begin{bmatrix} M_c + M_{cs}(t)S_\tau & M_{cd}(t) \\ M_{ds}S_\tau & M_d \end{bmatrix}, \tag{2.19}$$

where

$$
\begin{aligned}
M_c &:= P_{1c} P_{2c}, \\
M_{cs}(t) &:= P_{1c} P_{2cs}(t) + P_{1cs}(t) P_{2c} + P_{1cs}(t) P_{2cs}(0) + P_{1cd}(t) P_{2ds}, \\
M_{cd}(t) &:= P_{1c} P_{2cd}(t) + P_{1cs}(t) P_{2cd}(0) + P_{1cd}(t) P_{2d}, \\
M_{ds} &:= P_{1ds} P_{2c} + P_{1ds} P_{2cs}(0) + P_{1d} P_{2ds}, \\
M_d &:= P_{1ds} P_{2cd}(0) + P_{1d} P_{2d},
\end{aligned}
\tag{2.20}
$$

then we have

$$P_1(t) \otimes_M P_2(t) \in \mathscr{M} \tag{2.21}$$

and

$$\{P_1(t) \otimes_M P_2(t)\} U = P_1(t)\{P_2(t) U\} \qquad \forall U = \begin{bmatrix} u_c(t) \\ u_d[k] \end{bmatrix}. \tag{2.22}$$

(3) *Inverse.* Suppose P_{1c} and $P_1(0)$ are nonsingular and $P_1^{-1}(0)$ is expressed as

$$P_1^{-1}(0) = \begin{bmatrix} P_1(0) + P_{1cs}(0) & P_{1cd}(0) \\ P_{1ds} & P_{1d} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{P}_{cc} & \hat{P}_{cd} \\ \hat{P}_{dc} & \hat{P}_{dd} \end{bmatrix}. \tag{2.23}$$

Then there exists $P_1^{-1}(t) \in \mathscr{M}$ which satisfies

$$P_1^{-1}(t)(P_1(t) U) = U \qquad \forall U = \begin{bmatrix} u_c(t) \\ u_d[k] \end{bmatrix}, \tag{2.24}$$

and it is given by

$$P_1^{-1}(t) = \begin{bmatrix} F_c + F_{cs}(t) S_\tau & F_{cd}(t) \\ F_{ds} S_\tau & F_d \end{bmatrix}, \tag{2.25}$$

where

$$
\begin{aligned}
F_c &:= P_{1c}^{-1}, \\
F_{cs}(t) &:= -P_{1c}^{-1}\left\{P_{1cs}(t)\hat{P}_{cc} + P_{1cd}(t)\hat{P}_{dc}\right\}, \\
F_{cd}(t) &:= -P_{1c}^{-1}\left\{P_{1cs}(t)\hat{P}_{cd} + P_{1cd}(t)\hat{P}_{dd}\right\}, \\
F_{ds} &:= \hat{P}_{dc}, \\
F_d &:= \hat{P}_{dd}.
\end{aligned}
\tag{2.26}
$$

We are now ready to show the closedness of the parallel, cascade, and feedback connections.

PROPOSITION 2.1. *Suppose* $\Sigma_i = \{\mathbf{A}_i(t), \mathbf{B}_i(t), \mathbf{C}_i(t), \mathbf{D}_i(t)\} \in \mathscr{H}_{\text{sys}}$ $(i = 1, 2)$. *The parallel connection of* Σ_1 *and* Σ_2, $\Sigma_p = \Sigma_1 + \Sigma_2$, *belongs to* $\mathscr{H}_{\text{sys}}$, *and its hybrid state-space model is given by*

$$\Sigma_p = \left\{\mathbf{A}_p(t), \mathbf{B}_p(t), \mathbf{C}_p(t), \mathbf{D}_p(t)\right\} \tag{2.27}$$

where

$$
\begin{aligned}
\mathbf{A}_p(t) &= R\begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} R^{-1}, \qquad & \mathbf{B}_p(t) &= R\begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, \\
\mathbf{C}_p(t) &= \begin{bmatrix} C_1(t) & C_2(t) \end{bmatrix} R^{-1}, & \mathbf{D}_p(t) &= D_1(t) \oplus_M D_2(t),
\end{aligned}
\tag{2.28}
$$

and

$$R = \begin{bmatrix} I_{n_{c1}} & 0 & 0 & 0 \\ 0 & 0 & I_{n_{d1}} & 0 \\ 0 & I_{n_{c2}} & 0 & 0 \\ 0 & 0 & 0 & I_{n_{d2}} \end{bmatrix}. \tag{2.29}$$

PROPOSITION 2.2. *Suppose* $\Sigma_i = \{\mathbf{A}_i(t), \mathbf{B}_i(t), \mathbf{C}_i(t), \mathbf{D}_i(t)\} \in \mathscr{H}_{\text{sys}}$ $(i = 1, 2)$. *The cascade connection of* Σ_1 *and* Σ_2, $\Sigma_c = \Sigma_1 \cdot \Sigma_2$, *belongs to* $\mathscr{H}_{\text{sys}}$, *and its hybrid state-space model is given by*

$$\Sigma_c = \left\{\mathbf{A}_c(t), \mathbf{B}_c(t), \mathbf{C}_c(t), \mathbf{D}_c(t)\right\}, \tag{2.30}$$

where

$$\mathbf{A}_c(t) = R\begin{bmatrix} A_1(t) & B_1(t) \otimes_M C_2(t) \\ 0 & A_2(t) \end{bmatrix} R^{-1},$$

$$\mathbf{B}_c(t) = R\begin{bmatrix} B_1(t) \otimes_M D_2(t) \\ B_2(t) \end{bmatrix}, \tag{2.31}$$

$$\mathbf{C}_c(t) = \begin{bmatrix} C_1(t) & D_1(t) \otimes_M C_2(t) \end{bmatrix} R^{-1},$$

$$\mathbf{D}_c(t) = D_1(t) \otimes_M D_2(t).$$

PROPOSITION 2.3. *Suppose*

$$\det\{I \oplus \mathbf{D}_1(0) \otimes \mathbf{D}_2(0)\}^{-1} \neq 0, \qquad \det(I + D_{1c}D_{2c})^{-1} \neq 0$$

hold for $\Sigma_i = \{\mathbf{A}_i(t), \mathbf{B}_i(t), \mathbf{C}_i(t), \mathbf{D}_i(t)\} \in \mathscr{H}_{\text{sys}}$ $(i = 1, 2)$. *There exists a feedback connection of* Σ_1 *and* Σ_2, $\Sigma_f = (I + \Sigma_2 \cdot \Sigma_1)^{-1} \cdot \Sigma_2$, *which belongs to* $\mathscr{H}_{\text{sys}}$, *and its hybrid state-space model is given by*

$$\Sigma_f = \{\mathbf{A}_f(t), \mathbf{B}_f(t), \mathbf{C}_f(t), \mathbf{D}_f(t)\}, \tag{2.32}$$

where

$$\mathbf{A}_f(t) = R\begin{bmatrix} A_1(t) & B_1(t) \otimes_M C_2(t) \\ 0 & A_2(t) \end{bmatrix} R^{-1} \oplus_M (-R)\begin{bmatrix} B_1(t) \otimes_M D_2(t) \\ B_2(t) \end{bmatrix}$$

$$\otimes_M \{I \oplus_M D_1(t) \otimes_M D_2(t)\}^{-1} \otimes_M [C_1(t) \quad D_1(t) \otimes_M C_2(t)] R^{-1},$$

$$\mathbf{B}_f(t) = R\begin{bmatrix} B_1(t) \otimes_M D_2(t) \\ B_2(t) \end{bmatrix} \otimes_M (I \oplus_M D_1(t) \otimes_M D_2(t))^{-1}, \tag{2.33}$$

$$\mathbf{C}_f(t) = [0 \quad C_2(t)] R^{-1} \oplus_M - D_2(t) \otimes_M \{I \oplus_M D_1(t) \otimes_M D_2(t)\}^{-1}$$

$$\otimes_M [C_1(t) \quad D_1(t) \otimes_M C_2(t)] R^{-1},$$

$$\mathbf{D}_f(t) = D_2(t) \otimes \{I \oplus_M D_1(t) \otimes_M D_2(t)\}^{-1}.$$

These results imply that the hybrid state-space model is closed under the fundamental operations of scalar multiplication and parallel, cascade, and feedback connections. Note that the structure of the matrices in (2.28), (2.31), and (2.33) is essentially the same as for the usual state-space expression (A, B, C, D), except for the presence of a permutation matrix R defined in (2.29) for adjusting the order of analog and discrete signals. This closedness property with similar matrix structure is useful in the development of CAD systems for sampled-data systems [16].

REMARK. We have several interesting subclasses. If we consider the class of hybrid systems in which all the functions in the hybrid model (2.9) are constant, it turns out to comprise sampled-data systems with zero-order holds, i.e., $H(t) \equiv I$. We can readily see that this subclass is also closed under the above fundamental operations. Another interesting subclass is that of hybrid systems with only continuous-time input and output, i.e., $m_d = 0$ and $p_d = 0$, which has been already investigated in [8, 11, 12].

3. STABILITY, REACHABILITY AND OBSERVABILITY

3.1. *Intersampling Behavior*

Before discussing the stability, reachability, and observability of the system $\mathscr{T}_s$ with the hybrid state-space model (2.9), we will derive the solution of (2.9).

We first solve the differential equation for the analog state $x_c(t)$ in (2.9). Then, we can readily see that the hybrid state variable at sampling instance is given by

$$\begin{bmatrix} x_c((k+1)\tau) \\ x_d[k+1] \end{bmatrix}$$

$$= \begin{bmatrix} e^{A_c\tau} + \int_0^\tau e^{A_c(\tau-\xi)} A_{cs}(\xi)\, d\xi & \int_0^\tau e^{A_c(\tau-\xi)} A_{cd}(\xi)\, d\xi \\ A_{ds} & A_d \end{bmatrix} \begin{bmatrix} x_c(k\tau) \\ x_d[k] \end{bmatrix}$$

$$+ \begin{bmatrix} \int_0^\tau e^{A_c(\tau-\xi)} B_c u_c(k\tau+\xi)\, d\xi \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} \int_0^\tau e^{A_c(\tau-\xi)}B_{cs}(\xi)\,d\xi & \int_0^\tau e^{A_c(\tau-\xi)}B_{cd}(\xi)\,d\xi \\ B_{ds} & B_d \end{bmatrix}\begin{bmatrix} u_c(k\tau) \\ u_d[k] \end{bmatrix}, \tag{3.1}$$

and the intersampling behavior of the analog state $x_c(k\tau+\sigma)$ for $0 \leqslant \sigma \leqslant \tau$, $k = 0, 1, 2, \dots$, is expressed as

$$\begin{aligned} &x_c(k\tau+\sigma) \\ &\quad= \begin{bmatrix} e^{A_c\sigma} + \int_0^\sigma e^{A_c(\sigma-\xi)}A_{cs}(\xi)\,d\xi & \int_0^\sigma e^{A_c(\sigma-\xi)}A_{cd}(\xi)\,d\xi \end{bmatrix}\begin{bmatrix} x_c(k\tau) \\ x_d[k] \end{bmatrix} \\ &\qquad+ \int_0^\sigma e^{A_c(\sigma-\xi)}B_c u_c(k\tau+\xi)\,d\xi \\ &\qquad+ \begin{bmatrix} \int_0^\sigma e^{A_c(\sigma-\xi)}B_{cs}(\xi)\,d\xi & \int_0^\sigma e^{A_c(\sigma-\xi)}B_{cd}(\xi)\,d\xi \end{bmatrix}\begin{bmatrix} u_c(k\tau) \\ u_d[k] \end{bmatrix}. \end{aligned} \tag{3.2}$$

Therefore, we have the intersampling behavior of the continuous-time output $y_c(k\tau+\sigma)$ for $0 \leqslant \sigma \leqslant \tau$, $k = 0, 1, 2, \dots$, and the discrete-time output $y_d[k]$ for $k = 0, 1, 2, \dots$, as follows:

$$\begin{aligned} \begin{bmatrix} y_c(k\tau+\sigma) \\ y_d[k] \end{bmatrix} &= \begin{bmatrix} C_c x_c(k\tau+\sigma) \\ 0 \end{bmatrix} + \begin{bmatrix} C_{cs}(\sigma) & C_{cd}(\sigma) \\ C_{ds} & C_d \end{bmatrix}\begin{bmatrix} x_c(k\tau) \\ x_d[k] \end{bmatrix} \\ &\quad+ \begin{bmatrix} D_c u_c(k\tau+\sigma) \\ 0 \end{bmatrix} + \begin{bmatrix} D_{cs}(\sigma) & D_{cd}(\sigma) \\ D_{ds} & D_d \end{bmatrix}\begin{bmatrix} u_c(k\tau) \\ u_d[k] \end{bmatrix} \\ &= \begin{bmatrix} \mathscr{C}_{cc}(\sigma) & \mathscr{C}_{cd}(\sigma) \\ C_{ds} & C_d \end{bmatrix}\begin{bmatrix} x_c(k\tau) \\ x_d[k] \end{bmatrix} \\ &\quad+ \begin{bmatrix} D_c u_c(k\tau+\sigma) \\ 0 \end{bmatrix} + \begin{bmatrix} D_{cs}(\sigma) & D_{cd}(\sigma) \\ D_{ds} & D_d \end{bmatrix}\begin{bmatrix} u_c(k\tau) \\ u_d[k] \end{bmatrix}, \end{aligned} \tag{3.3}$$

where

$$\mathscr{C}_{cc}(t) := C_c\left(e^{A_c t} + \int_0^t e^{A_c(t-\xi)}A_{cs}(\xi)\,d\xi\right) + C_{cs}(t), \tag{3.4}$$

$$\mathscr{C}_{cd}(t) := C_c\int_0^t e^{A_c(t-\xi)}A_{cd}(\xi)\,d\xi + C_{cd}(t). \tag{3.5}$$

The above solution for the hybrid state-space representation (2.9) will be used for the analysis of stability, reachability, and observability in the succeeding subsections. We note that the above solution gives a unified treatment of the simulation for any type of sampled-data system, which may have many components consisting of both continuous-time and discrete-time systems.

REMARK. If we consider continuous-time signals such as $u_c(k\tau + \sigma)$ and $y_c(k\tau + \sigma)$ in the above solutions as functions over the interval $k\tau \leqslant t < (k+1)\tau$, then the appropriate arrangements of the above equations lead to a lifting model in [18] and a function-space model in [22].

3.2. *Stability*

We introduce the notion of exponential stability for the hybrid system, which must take into account the fact that some state variables evolve in continuous time while others evolve in discrete time.

DEFINITION 3.1. The hybrid system $\mathscr{T}_s$ given by (2.9) is *uniformly exponentially stable* if there exist positive constants α_c, β_c, α_d, and β_d such that

$$\|x_c(t)\| \leqslant \beta_c e^{-\alpha_c t}\|X_h[0]\| \qquad \forall t \geqslant 0, \tag{3.6}$$

$$\|x_d[k]\| \leqslant \beta_d e^{-\alpha_d k}\|X_h[0]\| \qquad k = 0, 1, \ldots, \tag{3.7}$$

holds for every initial condition $X_h[0] = \begin{bmatrix} x_c^T(0) & x_d^T[0] \end{bmatrix}^T$ whenever $u_c(t) \equiv 0$ and $u_d[k] \equiv 0$.

In Definition 3.1 the uniformity refers to the fact that the positive constants α_c, β_c, α_d, and β_d are independent of the initial condition $X_h[0] = \begin{bmatrix} x_c^T(0) & x_d^T[0] \end{bmatrix}^T$.

PROPOSITION 3.1. *The hybrid system $\mathscr{T}_s$ given by* (2.9) *is uniformly exponentially stable if and only if all the eigenvalues of the matrix $\mathscr{A}_d$ defined* *by*

$$\mathscr{A}_d = \begin{bmatrix} e^{A_c\tau} + \int_0^{\tau} e^{A_c(\tau-\xi)} A_{cs}(\xi)\, d\xi & \int_0^{\tau} e^{A_c(\tau-\xi)} A_{cd}(\xi)\, d\xi \\ A_{ds} & A_d \end{bmatrix} \tag{3.8}$$

belong to the interior of the unit disk.

Proof. The proof is the same as that for hybrid model in [12], so it is omitted. ∎

Since the above stability condition can be applied to any hybrid system in $\mathscr{H}_{\mathrm{sys}}$, we can easily derive the stability condition for the sampled-data feedback control system shown in Figure 1 with zero-order hold [i.e., $H(t) \equiv I$] and strictly proper controller (i.e., $D_k = 0$) as follows:

$$\mathscr{A}_d = \begin{bmatrix} e^{A_p\tau} & \int_0^{\tau} e^{A_p\xi} B_p\, d\xi\, C_k \\ -B_k C_p & A_k \end{bmatrix} \tag{3.9}$$

has no eigenvalues inside the unit disk. Also note that the proposition covers the stability conditions for both pure continuous-time system and pure discrete-time systems as special cases:

(1) A continuous-time system $\dot{x}_c(t) = A_c x_c(t)$ is exponentially stable iff all the eigenvalues of $e^{A_c\tau}$ are inside the unit disk, i.e., all the eigenvalues of A_c belong to the open left-hand half complex plane.

(2) A discrete-time system $x_d[k+1] = A_d x_d[k]$ is exponentially stable iff all the eigenvalues of A_d are inside the unit disk.

3.3. *Reachability and Observability*

Reachability generally refers to the ability to drive the state vector of a system from the origin to a specified final condition in a finite time. It is also generally understood that all the components of the state vector achieve their desired values at the same final time. However, since in (2.9) some state variables evolve in continuous time and others in discrete time, we introduce the following notion.

DEFINITION 3.2. For the hybrid system $\mathscr{T}_s$ given by (2.9), the state $[x_{cf}^T, x_{df}^T]^T$ is *sample time reachable* (STR) if there exist a nonnegetive integer k, $u_c(t)$ defined on $[0, k\tau]$, and $u_d[l]$, $0 \leqslant l \leqslant k$, such that the system $\mathscr{T}_s$, starting with initial conditions $x_c(0) = 0$, $x_d[0] = 0$ and driven by $u_c(t)$, $u_d(l)$, achieves the final condition $\left[x_c^T(k\tau), x_d^T[k] \right] = \left[x_{cf}^T, x_{df}^T \right]$. The system $\mathscr{T}_s$ given by (2.9) is *sample time reachable* if all the states are STR.

Sample time reachability of $\mathscr{T}_s$ is readily checked using the following result.

PROPOSITION 3.2. *Consider the discrete pair* $(\mathscr{A}_d, \mathbf{B}_d)$, *where* $\mathscr{A}_d$ *is defined in* (3.8) *and* $\mathbf{B}_d$ *is defined as*

$$\mathbf{B}_d = \begin{bmatrix} \int_0^\tau e^{A_c(\tau - \xi)} B_{cs}(\xi)\, d\xi & \int_0^\tau e^{A_c \xi} B_c B_c^T e^{A_c^T \xi}\, d\xi & \int_0^\tau e^{A_c \xi} B_{cd}(\xi)\, d\xi \\ B_{ds} & 0 & B_d \end{bmatrix}. \tag{3.10}$$

Then the hybrid system $\mathscr{T}_s$ *given by* (2.9) *is sample time reachable if and only if the pair* $(\mathscr{A}_d, \mathbf{B}_d)$ *is reachable.*

Proof. The proof is similar to that for the hybrid model in [12], so it is omitted. ∎

Observability generally refers to the ability to determine the initial condition of a homogeneous system from the time history of its output. Since the initial time occurs simultaneously for all components of the state vector, there is no difficulty in following the usual path to extend the concept of observability to the hybrid system $\mathscr{T}_s$ given by (2.9).

DEFINITION 3.3. For the hybrid system $\mathscr{T}_s$ given by (2.9), the vector $\left[x_{c0}^T, x_{d0}^T \right]^T$ is *unobservable* if, using as initial condition $\left[x_c^T(0), x_d^T[0] \right] = \left[x_{c0}^T, x_{d0}^T \right]$ *and input* $u_c(t) = 0$ $(t \geqslant 0)$, $u_d[k] = 0$ $(k = 0, 1, 2, \dots)$, *we have* $y_c(t) = 0$ *for almost all* $t \geqslant 0$ *and* $y_d[k] = 0$ *for all* $k = 0, 1, 2, \dots$. *The system* $\mathscr{T}_s$ *given by* (2.9) *is* observable *if the only unobservable state is the origin.*

Observability of $\mathscr{T}_s$ is also easily checked with the following result.

PROPOSITION 3.3. *Define a positive semidefinite matrix M as*

$$M = \int_0^T \begin{bmatrix} \mathscr{C}_{cc}^T(\sigma) \\ \mathscr{C}_{cd}^T(\sigma) \end{bmatrix} [\mathscr{C}_{cc}(\sigma) \quad \mathscr{C}_{cd}(\sigma)]\, d\sigma + \begin{bmatrix} C_{ds}^T \\ C_d^T \end{bmatrix} [C_{ds} \quad C_d], \tag{3.11}$$

and factor it as

$$M = \mathbf{C}_d^T \mathbf{C}_d, \tag{3.12}$$

where $\mathscr{C}_{cc}(\sigma)$ and $\mathscr{C}_{cd}(\sigma)$ are defined by (3.4) and (3.5), respectively. Then the hybrid system $\mathscr{T}_s$ given by (2.9) is observable if and only if the pair $(\mathscr{A}_d, \mathbf{C}_d)$ is observable.

Proof. The proof is similar to that for the hybrid model in [12], so it is omitted. ■

4. H_2-NORM ANALYSIS

In this section we first define an H_2-norm of the hybrid state-space model $\mathscr{T}_s$ given by (2.9), and then show how to compute the H_2-norm.

Let L_2^* be the space of piecewise continuous square-integrable functions. The L_2-norm of $z_c(t)$ in L_2^* is defined as

$$\| z_c(t) \|_2 = \left(\int_0^\infty z_c^T(t) z_c(t)\, dt \right)^{1/2}. \tag{4.1}$$

l_2 denotes the space of square-summarable sequences, and the l_2-norm of $z_d[k]$ in l_2 is defined as

$$\| z_d[k] \|_2 = \left(\sum_{k=0}^{\infty} z_d^T[k] z_d[k] \right)^{1/2}. \tag{4.2}$$

Combining these two, we define a type of L_2-norm of a hybrid signal

$$\begin{bmatrix} z_c(t) \\ z_d[k] \end{bmatrix}$$

as follows:

DEFINITION 4.1.

$$\left\| \begin{bmatrix} z_c(t) \\ z_d[k] \end{bmatrix} \right\|_2 := \sqrt{\| z_c(t) \|_2^2 + \| z_d[k] \|_2^2}. \tag{4.3}$$

Since $\mathscr{T}_s$ is a τ-periodic system, we cannot apply the definition of the H_2-norm for continuous-time systems based on the frequency-domain characterization. However, we can define the correspondences for the hybrid system based on the time-domain interpretation. Here, we extend the definition by Khargonekar and Sivashankar [13], where they introduced a type of H_2-norm, so it can be applied for any periodic system in general.[2]

DEFINITION 4.2. The H_2-norm of the hybrid system $\mathscr{T}_s$ given by (2.9) is defined by

$$\|\mathscr{T}_s\|_2 := \left(\frac{1}{\tau} \int_0^\tau \sum_{i=1}^{m_c} \left\| \mathscr{T}_s \begin{bmatrix} \delta(t-\nu)e_i \\ 0 \end{bmatrix} \right\|_2^2 d\nu + \sum_{j=1}^{m_d} \left\| \mathscr{T}_s \begin{bmatrix} 0 \\ \delta_{k0} e_j \end{bmatrix} \right\|_2^2 \right)^{1/2}$$

$$= \left(\frac{1}{\tau} \int_0^\tau \sum_{i=1}^{m_c} \| z_{hci} \|_2^2 \, d\nu + \sum_{j=1}^{m_d} \| z_{hdj} \|_2^2 \right)^{1/2}, \tag{4.4}$$

where $\delta(t)$ and δ_{ij} denote the Dirac and Kronecker delta functions, respectively. e_i is the unit vector whose ith element is 1 and others are all zero, and z_{hci} and z_{hdj} denote the corresponding hybrid outputs for $w_c(t) = \delta(t - \nu)e_i$ and $w_d[k] = \delta_{k0} e_j$, respectively.

We can easily see that $B_{cs}(t)$, B_{ds}, D_c, $D_{cs}(t)$, and D_{ds} in the hybrid state-space model (2.9) must be zero to assure the boundedness of the above H_2-norm. Also we note that the continuous-time input $w_c(t)$ cannot be directly sampled in this case, and hence the Dirac delta function can be allowed to be an input for the evaluation of the H_2-norm by a reasonable slight extension of the input space.

Therefore, we here consider a subclass of the hybrid systems with hybrid

[2] Of course, we can also extend another definition of the H_2 norm by Chen and Francis [4] in a similar way.

state-space representation[3]

$$\begin{bmatrix} \dot{x}_c(t) \\ x_d[k+1] \end{bmatrix} = \begin{bmatrix} A_c + A_{cs}(t)\mathscr{S}_\tau & A_{cd}(t) \\ A_{ds}\mathscr{S}_\tau & A_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} + \begin{bmatrix} B_c & B_{cd}(t) \\ 0 & B_d \end{bmatrix} \begin{bmatrix} w_c(t) \\ w_d[k] \end{bmatrix},$$
$$\begin{bmatrix} z_c(t) \\ z_d[k] \end{bmatrix} = \begin{bmatrix} C_c + C_{cs}(t)\mathscr{S}_\tau & C_{cd}(t) \\ C_{ds}\mathscr{S}_\tau & C_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} + \begin{bmatrix} 0 & D_{cd}(t) \\ 0 & D_d \end{bmatrix} \begin{bmatrix} w_c(t) \\ w_d[k] \end{bmatrix}, \tag{4.5}$$

where $k\tau \leqslant t < (k+1)\tau$.

We now start to compute the H_2-norm. Let $\mathscr{C}_{c1}(t)$, $\mathscr{C}_{d2}$, and $\mathscr{D}_c(t)$ be

$$\begin{bmatrix} \mathscr{C}_{c1}(t) \\ \mathscr{C}_{d2} \end{bmatrix} := \begin{bmatrix} \mathscr{C}_{cc}(t) & \mathscr{C}_{cd}(t) \\ C_{ds} & C_d \end{bmatrix}, \tag{4.6}$$

$$\mathscr{D}_c(t) := C_c \int_0^t e^{A_c(t-\xi)} B_{cd}(\xi)\, d\xi + D_{cd}(t), \tag{4.7}$$

where $\mathscr{C}_{cc}(t)$ and $\mathscr{C}_{cd}(t)$ are defined in (3.4) and (3.5), respectively. Also define

$$\mathscr{W}(t) := \frac{1}{\tau} C_c \int_0^t e^{A_c \xi} B_c B_c^T e^{A_c^T \xi}\, d\xi\, C_c^T. \tag{4.8}$$

Then a direct calculation based on the definition of H_2-norm above with the solution (3.3) for the hybrid state-space model (4.5) leads to

$$\|\mathscr{T}_s\|_2^2 = \operatorname{trace}\left\{\int_0^\tau \{\mathscr{C}_{c1}(t)\mathscr{X}_c\mathscr{C}_{c1}^T(t) + \mathscr{W}(t)\}\, dt\right\}$$
$$+ \operatorname{trace}\left\{\int_0^\tau \{\mathscr{C}_{c1}(t)\mathscr{X}_d\mathscr{C}_{c1}^T(t) + \mathscr{D}_c(t)\mathscr{D}_c^T(t)\}\, dt\right\}$$
$$+ \operatorname{trace}\{\mathscr{C}_{d2}\mathscr{X}_c\mathscr{C}_{d2}^T\} + \operatorname{trace}\{\mathscr{C}_{d2}\mathscr{X}_d\mathscr{C}_{d2}^T + D_d D_d^T\}, \tag{4.9}$$

[3] We use w and z rather than u and y for representing the input and output of the system to accommodate the synthesis problem which will be investigated in the next section.

where $\mathscr{X}_c$ and $\mathscr{X}_d$ denote the solutions of the following Lyapunov equations:

$$\mathscr{X}_c = \mathscr{A}_d \mathscr{X}_c \mathscr{A}_d^T + \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.10}$$

$$\mathscr{X}_d = \mathscr{A}_d \mathscr{X}_d \mathscr{A}_d^T + \mathscr{B}_{d2} \mathscr{B}_{d2}^T, \tag{4.11}$$

and $\hat{W}$ and $\mathscr{B}_{d2}$ are defined as follows:

$$\hat{W} := \frac{1}{\tau} \int_0^\tau e^{A_c \xi} B_c B_c^T e^{A_c^T \xi} \, d\xi, \tag{4.12}$$

$$\mathscr{B}_{d2} := \begin{bmatrix} \mathscr{B}_{cd} \\ B_d \end{bmatrix} = \begin{bmatrix} \int_0^\tau e^{A_c(\tau - \xi)} B_{cd}(\xi) \, d\xi \\ B_d \end{bmatrix}. \tag{4.13}$$

Let $\mathscr{X}_s$ be defined by

$$\mathscr{X}_s := \mathscr{X}_c + \mathscr{X}_d \tag{4.14}$$

or equivalently, as the solution of the following Lyapunov equation:

$$\mathscr{X}_s = \mathscr{A}_d \mathscr{X}_s \mathscr{A}_d^T + \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix} + \mathscr{B}_{d2} \mathscr{B}_{d2}^T. \tag{4.15}$$

Then we have another expression for the H_2-norm:

$$\begin{aligned} \|\mathscr{T}_s\|_2^2 = \operatorname{trace}\left\{ \int_0^\tau \left\{ \mathscr{C}_{c1}(t) \mathscr{X}_s \mathscr{C}_{c1}^T(t) + \mathscr{D}_c(t) \mathscr{D}_c^T(t) \right\} dt \right\} \\ + \operatorname{trace}\left\{ \int_0^\tau \mathscr{W}(t) \, dt \right\} + \operatorname{trace}\left\{ \mathscr{C}_{d2} \mathscr{X}_s \mathscr{C}_{d2}^T + D_d D_d^T \right\}. \end{aligned} \tag{4.16}$$

The above investigation leads to an equivalent discrete-time system which preserves the H_2-norm.

THEOREM 4.1. *Consider a uniformly exponentially stable hybrid system $\mathscr{T}_s$ given by* (4.5). *The H_2-norm of $\mathscr{T}_s$ defined in Definition* 4.2 *is given by*

$$\|\mathscr{T}_s\|_2 = \left\| \hat{T}_d[z] \right\|_2, \tag{4.17}$$

where the realization of $\mathscr{T}_d[z]$ *is expressed as*[4]

$$\hat{T}_d[z] := \left[\begin{array}{c|c} \mathscr{A}_d & \hat{\mathbf{B}}_d \\ \hline \hat{\mathbf{C}}_d & \hat{\mathbf{D}}_d \end{array}\right] = \begin{array}{c} \begin{array}{cccc} n_c & n_d & n_c & m_d \end{array} \\ \left[\begin{array}{cc|cc} \mathscr{A}_{d11} & \mathscr{A}_{d12} & \hat{B}_{d11} & \mathscr{B}_{cd} \\ A_{ds} & A_d & 0 & B_d \\ \hline \hat{C}_{d11} & \hat{C}_{d12} & \hat{D}_{d11} & \hat{D}_{d12} \\ C_{ds} & C_d & 0 & D_d \end{array}\right] \end{array} \begin{array}{l} n_c \\ n_d \\ n_c + n_d \\ p_d \end{array} \tag{4.18}$$

Here $\mathscr{A}_d$ *and* $\mathscr{B}_{cd}$ *are defined by* (3.8) *and* (4.13), *respectively. The other matrices in* (4.18), $\hat{B}_{d11}$, $\hat{C}_{d11}$, $\hat{C}_{d12}$, $\hat{D}_{d11}$, *and* $\hat{D}_{d12}$, *are given as follows:*

Step 1. $\hat{B}_{d11} \in \Re^{n_c \times n_c}$ *is obtained by the following factorization:*

$$\hat{B}_{d11}\hat{B}_{d11}^T = \hat{W}. \tag{4.19}$$

Step 2. *Define* $\hat{M}$ *and factor* $\hat{M}$ *into partitioned matrices as*

$$\hat{M} := \int_0^\tau \mathscr{C}_{c1}^T(t)\mathscr{C}_{c1}(t)\,dt = \begin{bmatrix} \hat{C}_{d11}^T \\ \hat{C}_{d12}^T \end{bmatrix}\begin{bmatrix} \hat{C}_{d11} & \hat{C}_{d12} \end{bmatrix}, \tag{4.20}$$

where $\hat{C}_{d11} \in \Re^{(n_c+n_d)\times n_c}$ *and* $\hat{C}_{d12} \in \Re^{(n_c+n_d)\times n_d}$.

Step 3. *Let* $\hat{\Delta}_1$ *be an* $n_c \times n_c$ *real matrix satisfying*

$$\hat{\Delta}_1\Delta_1^T = \frac{1}{\tau}\int_0^\tau\int_0^t e^{A_c\xi}B_cB_c^Te^{A_c^T\xi}\,d\xi\,dt\,C_c^TC_c.$$

---

[4] We use

$$\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right]$$

for representing a discrete-time system $x_d[k+1] = A_dx_d[k] + B_du_d[k]$, $y_d[k] = C_dx_d[k] + D_du_d[k]$.

Then we obtain $\hat{D}_{d11} \in \Re^{(n_c+n_d)\times n_c}$ *as follows:*

$$\hat{D}_{d11} = \begin{array}{cc} n_c & \\ \begin{bmatrix} \hat{\Delta}_1 \\ 0 \end{bmatrix} & \begin{array}{l} n_c \\ n_d \end{array} \end{array}$$

Step 4. $\hat{D}_{d12} \in \Re^{(n_c+n_d)\times m_d}$ *is any arbitrary matrix satisfying*

$$\operatorname{trace}\left\{\hat{D}_{d12}^T \hat{D}_{d12}\right\} = \operatorname{trace}\left\{\int_0^T \mathscr{D}_c^T(t)\mathscr{D}_c(t)\,dt\right\}.$$

For example, if $m_d \leqslant (n_c + n_d)$, *let* $\hat{\Delta}_2$ *be an* $m_d \times m_d$ *real matrix such that*

$$\hat{\Delta}_2^T \hat{\Delta}_2 = \int_0^T \mathscr{D}_c^T(t)\mathscr{D}_c(t)\,dt.$$

Then we obtain $\hat{D}_{d12} \in \Re^{(n_c+n_d)\times m_d}$ *as follows:*

$$\hat{D}_{d12} := \begin{array}{cc} m_d & \\ \begin{bmatrix} \hat{\Delta}_2 \\ 0 \end{bmatrix} & \begin{array}{l} m_d \\ n_c + n_d - m_d \end{array} \end{array}$$

Proof. The proof is in two steps. The first step is to show that the controllability gramian of $\hat{T}_d[z]$ is given by $\mathscr{X}_s$, and the second is to compute the H_2-norm of $\hat{T}_2[z]$ with $\mathscr{X}_s$.

Step 1. The controllability gramian of $\hat{T}_d[z]$ is defined to be the unique solution $\hat{L}_d$ of the Lyapunov equation

$$\hat{L}_d = \mathscr{A}_d \hat{L}_d \mathscr{A}_d^T + \hat{\mathbf{B}}_d \hat{\mathbf{B}}_d^T, \tag{4.21}$$

and it is easy to see from (4.15) that $\hat{L}_d = \mathscr{X}_s$ holds.

Step 2. Using the standard definition of the H_2-norm for discrete-time systems, the square of the H_2-norm of $\hat{T}_d[z]$ is expressed as

$$\left\|\hat{T}_2[z]\right\|_2^2 = \operatorname{trace}\left\{\hat{\mathbf{C}}_d \hat{L}_d \hat{\mathbf{C}}_d^T + \hat{\mathbf{D}}_d \hat{\mathbf{D}}_d^T\right\}.$$

Tedious calculations yield the following relation:

$$\operatorname{trace}\left\{\hat{\mathbf{C}}_d \hat{L}_d \hat{\mathbf{C}}_d^T\right\} = \operatorname{trace}\left\{\int_0^\tau \mathscr{C}_{c1}(t)\hat{L}_d \mathscr{C}_{c1}^T(t)\,dt + \mathscr{C}_{d2}\hat{L}_d \mathscr{C}_{d2}^T\right\}$$

We can also obtain

$$\operatorname{trace}\left\{\hat{\mathbf{D}}_d \hat{\mathbf{D}}_d^T\right\} = \operatorname{trace}\left\{\int_0^\tau \mathscr{W}(t)\,dt\right\}$$

$$+ \operatorname{trace}\left\{\int_0^\tau \mathscr{D}_c^T(t)\mathscr{D}_c(t)\,dt\right\} + \operatorname{trace}\left\{D_d D_d^T\right\}.$$

From step 1, the controllability gramians of $\hat{T}_d[z]$ and $\mathscr{X}_s$ are coincident. Consequently, we have

$$\left\|\hat{T}_d[z]\right\|_2^2 = \|\mathscr{T}_s\|_2^2 = \operatorname{trace}\left\{\int_0^\tau \{\mathscr{C}_{c1}(t)\mathscr{X}_s\mathscr{C}_{c1}^T(t) + \mathscr{D}_c(t)\mathscr{D}_c^T(t)\}\,dt\right\}$$

$$+ \operatorname{trace}\left\{\int_0^\tau \mathscr{W}(t)\,dt\right\} + \operatorname{trace}\left\{\mathscr{C}_{d2}\mathscr{X}_s\mathscr{C}_{d2}^T + D_d D_d^T\right\}$$

This completes the proof. ■

REMARK. If the hybrid system (4.5) is sample time reachable, we can see that the controllability gramian $\hat{L}_d$ of the equivalent discrete-time system $\hat{T}_d[z]$ given by the solution of the Lyapunov equation (4.15) or (4.21) is positive definite. The dual is also true. In other words, if the hybrid system (4.5) is observable, we can see that the observability gramian of the equivalent discrete-time system $\hat{T}_d[z]$ is positive definite.

The H_2-norm of $\mathscr{T}_s$ can be calculated by a simple computational algorithm with three exponentiations for the case where all τ-periodic matrices in (4.5) are constant. The algorithm can be derived by using the fundamental results for the relation between the integrals and the matrix exponentiations in [19].

5. H_2 OPTIMAL CONTROL

Consider the sampled-data feedback control system shown in Figure 5, where $w(t) \in \mathfrak{R}^{m_1}$, $u(t) \in \mathfrak{R}^{m_2}$, $z(t) \in \mathfrak{R}^{p_1}$, and $y(t) \in \mathfrak{R}^{p_2}$ denote the exogenous input, control input, controlled output, and measured output

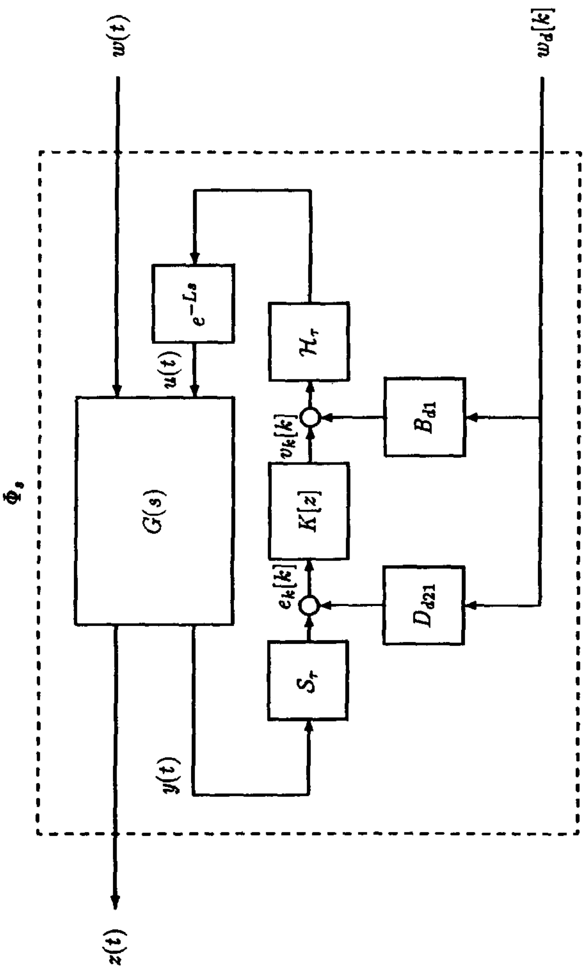

FIG. 5. Standard sampled-data configuration with time delay.

respectively, all of which are analog signals, while $w_d[k] \in \Re^{m_d}$, $e_k[k] \in \Re^{p_2}$, and $v_k[k] \in \Re^{m_2}$ denote the discrete-time exogenous input, input to controller, and output from controller, respectively, all of which are discrete signals. $G(s)$, $K[z]$, $\mathscr{H}_\tau(t)$, and $\mathscr{S}_\tau$ denote the continuous-time generalized plant (including frequency-shaped weighting matrices and antialiasing filters), the discrete-time controller to be designed, the given hold function, and the sampling operator with the sampling period $\tau > 0$, respectively. The state-space representations of $G(s)$[5] and $K[z]$ are respectively given by

$$G(s): \quad \begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) = C_1 x(t) + D_{12} u(t), \\ y(t) = C_2 x(t), \end{cases} \tag{5.1}$$

$$K[z]: \quad \begin{cases} x_k[k+1] = A_k x_k[k] + B_k e_k[k], \\ \quad\; v_k[k] = C_k x_k[k] + D_k e_k[k], \end{cases} \tag{5.2}$$

where $x(t) \in \Re^n$ and $x_k[k] \in \Re^{n_k}$.

The time delay e^{-Ls} followed by the hold represents the pure time lag in the continuous-time plant and/or the computational time delay in the digital controller, which may not be synchronized with the sampling period τ. We notice that $0 < \mu < \tau$ is inevitable for the digital implementation of the controller $K[z]$. If we compute the second equation in (5.2) rather than the first one in each sampling interval, then $v_k[k]$ and hence the control input can be determined after this calculation. In this case, the computational time delay μ is equal to the time for computing the second equation. Chen and Francis [4] have investigated a similar problem by an operator-theoretic approach, where the generalized plant itself is assumed to be stable and there is no discrete-time input $w_d[k]$. We note that our setting with one time delay after hold covers the case discussed in [4], where each element in the generalized plant has independent time delay, since the time delays in the portions of $w(t)$ and $z(t)$ can be removed without changing the H_2-norm, and that in the portion of $y(t)$ can be combined with that in the control input portion. Hence, our setting is general enough to treat many problems occurring in the real world.

Suppose that the time delay L can be expressed as

$$L := l\tau + \mu,$$

[5] The assumption $D_{21} = 0$ is posed to assure the L_2-stability of Φ_s [11]. $D_{11} = 0$ is also required for the H_2 problem, as in the continuous-time case.

where l is a nonnegative integer and $\mu \in (0, \tau]$. Then we can easily see that the control input $u(t)$ is given by

$$u(t) = \begin{cases} H(t-\mu)(v_k[k-l-1] + B_{d1}w_d[k]), & k\tau \leqslant t < k\tau + \mu, \\ H(t-\mu)(v_k[k-1] + B_{d1}w_d[k]), & k\tau + \mu \leqslant t < (k+1)\tau. \end{cases}$$

The inputs of the controller are determined by

$$e_k[k] = \mathscr{S}_\tau y(t) + D_{d21}w_d[k]. \tag{5.3}$$

D_{d21} denotes the discrete-time measurement noise such as the quantization error in A/D conversion, and B_{d11} reflects the discrete-time roundoff error accumulated in the digital computation. Since it is well known that the quantization errors can be treated as white noises with zero means (see e.g., [10]) and the deterministic H_2 control problem is essentially equivalent to a stochastic problem [6], the problem for reducing the effect of the quantization errors can be posed as an H_2 control problem.

Hence, the setting above is more general than the previous ones, and it completely models the real situation except for indirect considerations of the roundoff errors in the digital computation.

The main purpose of this subsection is to show that we can directly obtain the synthesis results by just applying the analysis result to the hybrid state-space model for the above closed-loop system with time delay. This is one of the merits in developing a general hybrid state-space theory.

We make the following reasonable assumptions to assure the existence of the stabilizing controller:

(A1) (A, B_2) is stabilizable, and (A, C_2) is detectable.

(A2) The sampling period τ is chosen so that (A_s, B_{s2}) is stabilizable and (A_s, C_{s2}) is detectable, where

$$A_s := e^{A\tau}, \qquad B_{s2} := \int_0^\tau e^{A(\tau-\xi)}B_2 H(\xi)\, d\xi, \qquad C_{s2} := C_2. \tag{5.4}$$

First, we will show that the system Φ_s with such a time-delay can be represented by the hybrid state-space model. Let us define two τ-periodic functions $H_{\mu 1}$ and $H_{\mu 2}$ as in (2.12) and (2.13), respectively. Then the direct calculation leads to the hybrid state-space representation (4.5) for the closed-loop system Φ_s as follows:

Case 1. $l = 0$:

$$A_c = A, \qquad A_{cs}(t) = B_2 H_{\mu 1}(t) D_k C_s,$$

$$A_{cd}(t) = B_2 \begin{bmatrix} H_{\mu 1}(t) C_k & H_{\mu 2}(t) \end{bmatrix},$$

$$A_{ds} = \begin{bmatrix} B_k \\ D_k \end{bmatrix} C_2, \qquad A_d = \begin{bmatrix} A_k & 0 \\ C_k & 0 \end{bmatrix},$$

$$B_c = B_1, \qquad B_{cd}(t) = B_2 H_{\mu 2}(t)(D_k D_{d21} + B_{d1}),$$

$$B_d = \begin{bmatrix} B_k D_{d21} \\ D_k D_{d21} + B_{d1} \end{bmatrix}, \qquad C_c = C_1, \qquad C_{cs}(t) = D_{12} H_{\mu 1}(t) D_k C_2,$$

$$C_{cd}(t) = D_{12} \begin{bmatrix} H_{\mu 1}(t) C_k & H_{\mu 2}(t) \end{bmatrix},$$

$$D_{cd}(t) = D_{12} H_{\mu 2}(t)(D_k D_{d21} + B_{d1}).$$

Case 2. $l \geqslant 1$:

$$A_c = A, \qquad A_{cs}(t) = 0,$$

$$A_{cd}(t) = B_2 \begin{bmatrix} 0 & \cdots & 0 & H_{\mu 1}(t) & H_{\mu 2}(t) \end{bmatrix},$$

$$B_c = B_1, \qquad B_{cd}(t) = 0,$$

$$A_{ds} = \begin{bmatrix} B_k \\ D_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} C_2, \qquad A_d = \begin{bmatrix} A_k & 0 & \cdots & \cdots & 0 \\ C_k & 0 & & & \vdots \\ 0 & I_{m_2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{m_2} & 0 \end{bmatrix},$$

$$B_d = \begin{bmatrix} B_k D_{d21} \\ D_k D_{d21} + B_{d1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C_c = C_1, \qquad C_{cs}(t) = 0,$$

$$C_{cd} = D_{12} \begin{bmatrix} 0 & \cdots & 0 & H_{\mu 1}(t) & H_{\mu 2}(t) \end{bmatrix}, \qquad D_{cd}(t) = 0.$$

We can therefore apply the analysis results in Theorem 4.1 to the system with time delay.

THEOREM 5.1. *Consider a sampled-data feedback system Φ_s, and suppose assumptions* (A1) *and* (A2) *hold. Then a discrete-time compensator $K[z]$ stabilizes Φ_s and optimizes $\|\Phi_s\|_2$ if and only if $K[z]$ stabilizes the fictitious discrete-time plant $\tilde{G}_\mu[z]$ and optimizes the H_2-norm of the fictitious discrete-time closed-loop system $\Phi_{\mu d}[z] := \Sigma_d(\tilde{G}_\mu[z], K[z])$, where $\tilde{G}_\mu[z]$ is composed of $G_\mu[z]$, z^{-l}, and B_d as depicted in Figure* 6. *The realization of $G_\mu[z]$ is expressed as*

$$G_\mu[z] := \left[\begin{array}{cc|cc|c} A_s & A_\mu & B_{s1} & 0 & B_{\mu 2} \\ 0 & 0 & 0 & 0 & I \\ \hline C_{\mu 11} & C_{\mu 12} & D_{s11} & 0 & D_{\mu 12} \\ \hline C_{s2} & 0 & 0 & D_{d21} & 0 \end{array}\right] \tag{5.5}$$

Here, A_s and C_{s2} are defined by (5.4); *B_{s1} and D_{s11} are defined by* (5.6) *and* (5.7), *respectively. The other matrices in* (5.5), *A_μ, $B_{\mu 2}$, B_{s1}, D_{s11}, $C_{\mu 11}$, $C_{\mu 12}$, and $D_{\mu 12}$, are obtained as follows*:

Step 1. *Define A_μ and $B_{\mu 2}$ as follows*:

$$A_\mu := \int_0^\tau e^{A(\tau-\mu)} B_2 H_{\mu 2}(\xi)\, d\xi,$$

$$B_{\mu 2} := \int_0^\tau e^{A(\tau-\mu)} B_2 H_{\mu 1}(\xi)\, d\xi.$$

Step 2. *$B_{s1} \in \Re^{n\times n}$ is obtained by the following factorization*:

$$B_{s1}B_{s1}^T := \frac{1}{\tau}\int_0^\tau e^{A\xi}B_1B_1^Te^{A^T\xi}\, d\xi. \tag{5.6}$$

Step 3. *Let Δ_1 be an $n \times n$ real matrix satisfying*

$$\Delta_1\Delta_1^T = \frac{1}{\tau}\int_0^\tau\int_0^t e^{A\xi}B_1B_1^Te^{A^T\xi}\, d\xi\, dt\, C_1^TC_1.$$

Then we obtain $D_{s11} \in \Re^{n\times n}$ as follows:

$$D_{s11} = \begin{array}{c} n \\ \left[\begin{array}{c} \Delta_1 \\ 0 \end{array}\right] \end{array} \begin{array}{l} n \\ m_2 \end{array} \tag{5.7}$$

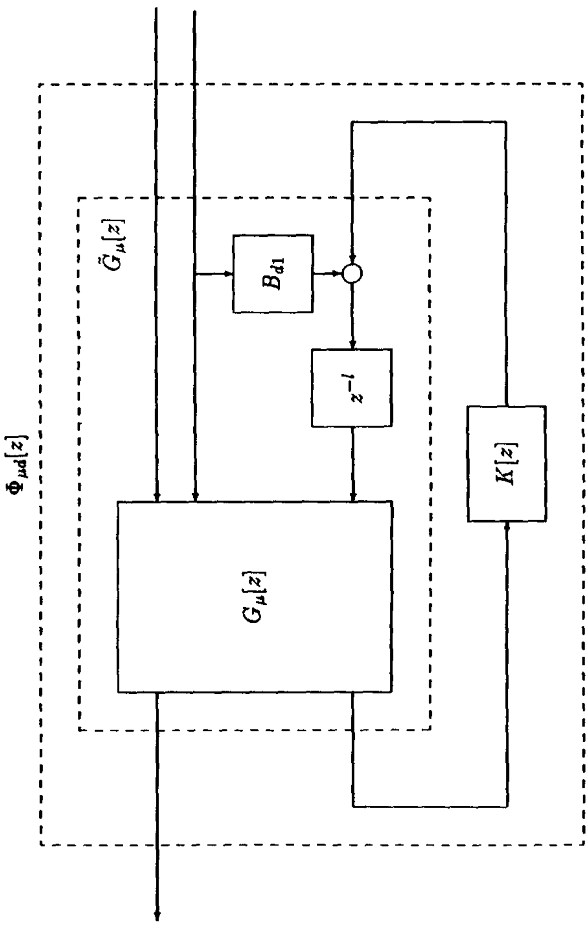

FIG. 6. Equivalent discrete-time H_2 problem.

Step 4. *Define*

$$M_\mu := \int_0^T \begin{bmatrix} C_1^T(t) \\ C_{\mu 2}^T(t) \\ C_{\mu 3}^T(t) \end{bmatrix} \begin{bmatrix} C_1(t) & C_{\mu 2}(t) & C_{\mu 3}(t) \end{bmatrix} dt, \tag{5.8}$$

where $C_1(t)$, $C_{\mu 2}(t)$, *and* $C_{\mu 3}(t)$ *are defined by*

$$C_1(t) := C_1 e^{At}, \tag{5.9}$$

$$C_{\mu 2}(t) := C_1 \int_0^t e^{A(t-\xi)} B_2 H_{\mu 1}(\xi)\, d\xi + D_{12} H_{\mu 1}(t), \tag{5.10}$$

$$C_{\mu 3}(t) := C_1 \int_0^t e^{A(t-\xi)} B_2 H_{\mu 2}(\xi)\, d\xi + D_{12} H_{\mu 2}(t). \tag{5.11}$$

Factor M_μ *into partitioned matrices as*

$$M_\mu = \begin{bmatrix} C_{\mu 11}^T \\ D_{\mu 12}^T \\ C_{\mu 12}^T \end{bmatrix} \begin{bmatrix} C_{\mu 11} & D_{\mu 12} & C_{\mu 12} \end{bmatrix}, \tag{5.12}$$

where $C_{\mu 11} \in \Re^{(n+m_2)\times n}$, $C_{\mu 12} \in \Re^{(n+m_2)\times m_2}$, *and* $D_{\mu 11} \in \Re^{(n+m_2)\times m_2}$.

Proof. For the sake of brevity, we only show the case of $l = 0$, since the proof for the case of $l \geqslant 1$ is similar.

$\mathscr{C}_{c1}(t)$ in Theorem 4.1 is given by

$$\mathscr{C}_{c1}(t) = \begin{bmatrix} C_1(t) & C_2(t) & C_3(t) \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ D_k C_2 & C_k & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Thus $\hat{M}$ in Theorem 4.1 is given by

$$\hat{M} = \int_0^T \mathscr{C}_{c1}^T(t) \mathscr{C}_{c1}(t)\, dt = \begin{bmatrix} I & 0 & 0 \\ D_k C_2 & C_k & 0 \\ 0 & 0 & I \end{bmatrix}^T M_\mu \begin{bmatrix} I & 0 & 0 \\ D_k C_2 & C_k & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Also, $\mathscr{D}_c(t)$ in Theorem 4.1 is given by

$$\mathscr{D}_c(t) = C_2(t)(D_k D_{d21} + B_{d1}).$$

Therefore, applying Theorem 4.1 to Φ_s, we have a discrete-time system which preserves the H_2-norm. The state-space representation of the equivalent discrete-time system is given by

$$\left[\begin{array}{ccc|cc} A_s + B_{\mu 2} D_k C_2 & B_{\mu 2} C_k & A_\mu & B_{s1} & B_{\mu 2}(D_k D_{d21} + B_{d1}) \\ B_k C_2 & A_k & 0 & 0 & B_k D_{d21} \\ D_k C_2 & C_k & 0 & 0 & D_k D_{d21} + B_{d1} \\ \hline C_{\mu 11} + D_{\mu 12} D_k C_2 & D_{\mu 12} C_k & C_{\mu 12} & D_{s11} & D_{\mu 12}(D_k D_{d21} + B_{d1}) \end{array}\right].$$

It is easy to recognize that the above discrete-time system is exactly the $\Phi_{\mu d}[z]$ generated when $K[z]$ regulates the fictitious discrete-time plant $\tilde{G}_\mu[z]$. ■

The computational algorithm for deriving the equivalent system is given as follows:

COMPUTATIONAL ALGORITHM (H_2 synthesis for the time-delay case).

Step 0. *Computation of A_s, A_μ, and $B_{\mu 2}$.*
Step 1. *Computation of B_{s1}.*
Step 2. *Computation of M_μ.* Let

$$\exp\left(\begin{bmatrix} A_N & C_N^T C_N \\ 0 & A_N \end{bmatrix}(\tau - \mu)\right) = \begin{bmatrix} * & \Gamma_{N1} \\ 0 & \Phi_{N1} \end{bmatrix},$$

$$\exp\left(\begin{bmatrix} A_N & C_N^T C_N \\ 0 & A_N \end{bmatrix}\mu\right) = \begin{bmatrix} * & \Gamma_{N2} \\ 0 & \Phi_{N2} \end{bmatrix}.$$

Then we have

$$M_{\mu 1} = \Phi_{N1}^T \Gamma_{N1}, \qquad M_{\mu 2} = \Phi_{N2}^T \Gamma_{N2},$$

where

$$\hat{A}_N = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \qquad \hat{C}_N = \begin{bmatrix} C_1 & D_{12} \end{bmatrix}.$$

M_μ is given by

$$M_\mu = B_{N1}^T M_{\mu 1} B_{N1} + B_{N2}^T M_{\mu 2} B_{N2},$$

where

$$B_{N1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \qquad B_{N2} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Factor M_μ as in (5.12).

REMARK. The technique used above can also be applied to the H_∞ problem as discussed in [7]; there is also an independent treatment of the H_∞ problem with time delay by a lifting approach [14].

6. CONCLUSIONS

We have developed a hybrid system theory based on a hybrid state-space model. We first showed the closedness properties of the proposed model for the fundamental operations. We next investigated the standard basic issues of internal stability, reachability, and observability. The notions and the results are natural generalizations of the previous ones, and they give the foundation of the hybrid system theory. We also defined the notion of H_2-norm of the hybrid system and showed how to evaluate it. The analysis result can be applied to any hybrid system with the proposed hybrid state-space model; we applied it to the optimal-digital-control problem for the standard four-block configuration with H_2-norm performance measure. The solutions with computational algorithms have been derived for both non-time-delay and time-delay cases.

Though the results are only slight extensions of the previous results, we should emphasize that we can easily make a general formulation if we use the hybrid state-space model and that the synthesis result in any setting can be directly derived by applying the general analysis result. This is an advantage for developing a general hybrid state-space theory. It is also of note that the unified treatment is desirable for developing a CAD package [16].

Interesting future areas of research related to this paper are

(1) application to the H_∞-type control problem with hybrid input and output signals,

(2) the investigation of the relation between the state-space methods and the frequency-domain approaches [20, 21, 1].

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