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UNSTEADY ACOUSTIC PROPAGATION THROUGH DUCTS WITH LOCALIZED VIBRATION

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1. INTRODUCTION

The present work is an exploratory study on the acoustic generation in a compressible flow through ducts using one of the well-established approaches in the field of transient gas dynamics. The basic problem under consideration consists of a constant-area duct through which a compressible fluid flows. Due to the fact that certain machines are adjacent, or that the duct is fastened to the machine, the vibration of the machine causes vibration of the duct wall. As a result, the duct wall generates acoustic waves in the fluid, especially when the thickness of the wall is thin. The problem points to a localized vibration of the duct wall, as shown in Figure 1. The length of the duct is long compared with the hydraulic diameter of the duct. Even though many papers are available in the literature on the sound propagation through the fluid in ducts [1–11], the problem of a long duct with a localized vibration is a topic which has not been treated.

Vibration of the wall as a result of the vibration of the machine on, e.g., both sides of the duct can be modelled as a localized perturbation change of the cross-sectional area, as

$$F = F_0 + (\epsilon F_0)\,\mathrm{e}^{-\delta(x-x_0)^2}, \tag{1}$$

where F_0 is the duct area without disturbance, ϵF_0 is the maximum amplitude of the change in cross-sectional area due to vibration of the duct wall, x_0 is the location of the wall vibration, ω ($=2\pi f$) is the angular frequency of vibration, and δ is the parameter indicating the degree of localization, respectively.

In handling this type of problem, the unsteady one-dimensional model, in which the length of the duct is long compared with the cross-sectional area, is well established in the field of unsteady gas dynamics and is the only practical method for the analysis of such problems. It will be seen that as a result of applying this model to the present problem, an analytical solution relating all the physical parameters can now be obtained. An analytical solution is always preferred in any physical or engineering analysis even though certain unavoidable assumptions must be made to achieve this goal. The present work, therefore, does serve a useful purpose.

Similar to other complicated problems, the problem must be studied from different angles before any definitive conclusions can be finalized. The present work, therefore, offers very useful results from the one-dimensional point of view—an approach that is well established in the field of gas dynamics. The one-dimensional approach, by its very nature, supplies only information on the variation of the flow and acoustic properties averaged over the cross-sectional area of the duct as a function of the axial distance along the axial direction. It is hoped that, by presenting the analytical solution using the present model, further works, both analytical and experimental, can be stimulated in the literature.

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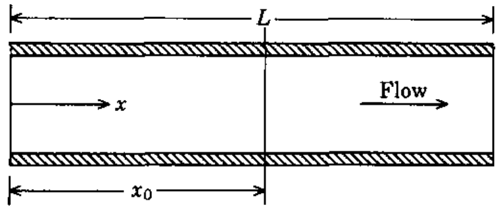

Figure 1. The physical duct model.

2. ANALYSIS

2.1. *Basic equations*

The solution of the problem, as modelled in the last section, is therefore reduced to the solution of the equations governing the unsteady, one-dimensional isentropic flow with a time-dependent cross-sectional area. The governing differential equation can therefore be derived in the same manner as in any standard gas dynamics text, such as in reference [12], by adding the feature of a time-dependent area. The differential equations are as follows: continuity equation,

$$\frac{1}{F}\left\{\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}\right\}+\frac{1}{\rho}\left\{\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right\}+\frac{\partial u}{\partial x}=0; \tag{2}$$

momentum equation,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0; \tag{3}$$

equation of state,

$$p=\rho RT, \qquad a^2=\partial p/\partial \rho; \tag{4, 5}$$

Second Law of Thermodynamics,

$$p/\rho^k=C. \tag{6}$$

Here, ρ, u, p, T and a are, respectively, the density, velocity, pressure, temperature and sonic velocity of the fluid. The area of the channel is represented by F which, for the model adapted in this work, is a function of both x and t. As in the case of steady isentropic flow, the energy equation is not an independent equation and therefore not included. For a given $F(x, t)$, equations (2)–(6) are the five equations for the solution of five dependent variables (ρ, u, p, T and a). By eliminating p, ρ and T from equations (2)–(6) and introducing the dimensionless variables,

$$X=x/L, \qquad Z=t@/L, \qquad U=u/@, \qquad A=a/@, \qquad @=\sqrt{kRT_t}, \tag{7}$$

equations (2) and (3) become

$$\frac{k-1}{2}\frac{A}{F}\left\{\frac{\partial F}{\partial Z}+U\frac{\partial F}{\partial X}\right\}+U\frac{\partial A}{\partial X}+\frac{k-1}{2}A\frac{\partial U}{\partial X}+\frac{\partial A}{\partial Z}=0, \tag{8}$$

$$\frac{k-1}{2}\left\{\frac{\partial U}{\partial Z}+U\frac{\partial U}{\partial X}\right\}+A\frac{\partial A}{\partial X}=0, \tag{9}$$

which can be used to solve for U and A. Once these two variables are solved, the remaining three variables, namely, p, ρ, and T, can then be solved from equations (4)–(6).

Following established practice in gas dynamics, the boundary conditions at $x=0$ and $x=L$ for unsteady isentropic processes can be obtained by using the energy equation

$$\frac{u^2}{2}+C_pT=C_pT_t,$$

which, in terms of dimensionless variables, gives

$$\text{for } X=0,\qquad \frac{U^2(Z,0)}{2}+\frac{A^2(Z,0)}{k-1}=\frac{1}{k-1}, \tag{10}$$

$$\text{for } X=1,\qquad \frac{U^2(Z,1)}{2}+\frac{A^2(Z,1)}{k-1}=\frac{1}{k-1}. \tag{11}$$

2.2. *The vibrating surface*

As discussed in the introduction, the vibrating surface of the duct is modelled using a localized change of area, $F(X,Z)$. Let F_0 represent the cross-sectional area of the duct. The vibration of the duct wall is modelled using a local change in the cross-sectional area with frequency ω and a decreasing amplitude from the point of vibration, as shown in Figure 1. In terms of dimensionless quantities, the change of area is modelled by

$$\Delta F=\epsilon F_0\,\mathrm{e}^{-\alpha(X-X_0)^2}\sin\bar{\omega}Z, \tag{12}$$

where $\bar{\omega}=\omega(2\pi L)/@$. The parameter ϵF_0 is the amplitude of the change of area at the point of vibration, and α is a parameter which indicates the extent of *localization* of the vibration. In this paper, data are generated using two values of α, namely 1000 and 100, respectively. For these values of α, the amplitude of vibration drops to 1 per cent of its maximum value at 7 per cent and 21 per cent, respectively, of the axial distance on either side of the location of vibration. The localized nature of the vibration is therefore insured. In terms of the dimensionless variables, the area of the duct, F, can be written as

$$F=F_0\{1+\epsilon\,\mathrm{e}^{-\alpha(X-X_0)^2}\sin\bar{\omega}Z\}=F_0(1+\epsilon G_0), \tag{13}$$

which can be substituted into equation (8). Equations (8) and (9) then become

$$\epsilon\frac{k-1}{2}A\,\mathrm{e}^{-\alpha(X-X_0)^2}\{\bar{\omega}\cos\bar{\omega}Z-2\alpha U_0(X-X_0)\sin\bar{\omega}Z\}+U\frac{\partial A}{\partial X}+\frac{k-1}{2}A\frac{\partial U}{\partial X}+\frac{\partial A}{\partial Z}=0, \tag{14}$$

$$\frac{k-1}{2}\left\{\frac{\partial U}{\partial Z}+U\frac{\partial U}{\partial X}\right\}+A\frac{\partial A}{\partial X}=0. \tag{15}$$

2.3. *Perturbation expansions*

We will now expand U and A in series form as

$$U=U_0+\epsilon U_1+\cdots,\qquad A=A_0+\epsilon A_1+\cdots, \tag{16, 17}$$

where ϵ is the perturbation parameter. Substituting the series from equations (16) and (17) into equations (14) and (15) and separating terms of power of ϵ, we obtain, for ϵ^0,

$$A=\text{constant},\qquad U_0=\text{constant}, \tag{18}$$

and for ϵ^1,

$$\frac{\partial A_1}{\partial Z}+\frac{k-1}{2}A_0\frac{\partial U_1}{\partial X}+U_0\frac{\partial A_1}{\partial X}+\frac{k-1}{2}A_0\,\mathrm{e}^{-\alpha(X-X_0)^2}$$

$$\times\{\bar{\omega}\cos\bar{\omega}Z-2\alpha U_0(X-X_0)\sin\bar{\omega}Z\}=0, \quad (19)$$

$$\frac{k-1}{2}\frac{\partial U_1}{\partial Z}+\frac{k-1}{2}U_0\frac{\partial U_1}{\partial X}+A_0\frac{\partial A_1}{\partial X}=0. \quad (20)$$

The zeroth order solutions, U_0 and A_0, represent the undisturbed flow. For isentropic flows in a constant area duct, both U_0 and A_0 are constants. For a given Mach number in the duct, the pressure and the temperature are given by

$$p_I=\frac{p_t}{\{1+(k-1)M_I^2/2\}^{k/(k-1)}}, \qquad T_I=\frac{T_t}{\{1+(k-1)M_I^2/2\}},$$

from which

$$A_0=\{a/@\}=\sqrt{T_I/T_t}, \qquad U_0=M_I\sqrt{kRT_I}/@. \quad (21)$$

The first order solutions, U_1 and A_1, are the responses of the fluid flow to the vibrating wall of the duct. In view of the fact that equation (19) involves two terms, with one containing a cosine function and the other a sine function, this suggests the application of the superposition principle to separate the dependent variables into two terms, as follows:

$$U_1=U_{11}+U_{12}, \qquad A_1=A_{11}+A_{12}. \quad (22)$$

Equations (19) and (20) are now separated into two systems of boundary value problems. For U_{11} and A_{11}, we have

$$\frac{\partial A_{11}}{\partial Z}+\frac{k-1}{2}A_0\frac{\partial U_{11}}{\partial X}+U_0\frac{\partial A_{11}}{\partial X}+\frac{k-1}{2}A_0\bar{\omega}\,\mathrm{e}^{-\alpha(X-X_0)^2}\cos\bar{\omega}Z=0, \quad (23)$$

$$\frac{k-1}{2}\frac{\partial U_{11}}{\partial Z}+\frac{k-1}{2}U_0\frac{\partial U_{11}}{\partial X}+A_0\frac{\partial A_{11}}{\partial X}=0, \quad (24)$$

subject to the boundary conditions

$$U_0U_{11}(0)+\frac{2}{k-1}A_0A_{11}(0)=0, \qquad U_0U_{11}(1)+\frac{2}{k-1}A_0A_{11}(1)=0. \quad (25)$$

For U_{12} and A_{12}, we have

$$\frac{\partial A_{12}}{\partial Z}+\frac{k-1}{2}A_0\frac{\partial U_{12}}{\partial X}+U_{12}\frac{\partial A_{12}}{\partial X}-2\alpha\frac{k-1}{2}A_0U_0(X-X_0)\,\mathrm{e}^{-\alpha(X-X_0)^2}\sin\bar{\omega}Z=0 \quad (26)$$

$$\frac{k-1}{2}\frac{\partial U_{12}}{\partial Z}+\frac{k-1}{2}\frac{\partial U_{12}}{\partial X}+A_0\frac{\partial A_{12}}{\partial X}=0, \quad (27)$$

subject to the boundary conditions

$$U_0U_{12}(0)+\frac{2}{k-1}A_0A_{12}(0)=0, \qquad U_0U_{12}(1)+\frac{2}{k-1}A_0A_{12}(1)=0. \quad (28)$$

Solutions of $U_{11}(Z,X)$, $A_{11}(Z,X)$, $U_{12}(Z,X)$ and $A_{12}(Z,X)$ can be obtained by the method of complex superposition by writing

$$U_{11}(Z,X)=\mathrm{Re}\,\{\{U_{11R}+\mathrm{i}U_{11I}\}\,\mathrm{e}^{\mathrm{i}\bar{\omega}Z}\}, \qquad A_{11}(Z,X)=\mathrm{Re}\,\{\{U_{11R}+\mathrm{i}U_{11I}\}\,\mathrm{e}^{\mathrm{i}\bar{\omega}Z}\}, \quad (29, 30)$$

$$U_{12}(Z, X) = \mathrm{Im}\,\{\{U_{12R} + \mathrm{i}U_{12I}\}\,\mathrm{e}^{\mathrm{i}\bar{\omega}Z}\}, \qquad A_{12}(Z, X) = \mathrm{Im}\,\{\{U_{12R} + \mathrm{i}U_{12I}\}\,\mathrm{e}^{\mathrm{i}\bar{\omega}Z}\}, \tag{31, 32}$$

where Re and Im are, respectively, the real and imaginary parts of the complex functions. Substituting equations (29)–(32) into equations (23)–(28) and separating the various terms, a system of boundary value problems consisting of eight simultaneous ordinary differential equations will be obtained for the solution of U_{11R}, U_{11I}, A_{11R}, A_{11I}, U_{12R}, U_{12I}, A_{12R} and A_{12I}, respectively. Since the solution procedure is routine [13], details will be omitted here.

2.4. *The sound pressure level*

The pressure at any section of the channel is given by

$$p/p_t = A^{2k/(k-1)} = (A_0 + \epsilon A_1)^{2k/(k-1)}$$

or, expanded by the binomial theorem,

$$\frac{p}{p_t} = A_0^{2k/(k-1)}\left\{1 + \epsilon\frac{A_1}{A_0}\right\}^{2k/(k-1)} \approx A_0^{2k/(k-1)}\left\{1 + \epsilon\frac{2k}{k-1}\frac{A_1}{A_0}\right\} = \frac{p_l}{p_t}\left\{1 + \epsilon\frac{2k}{k-1}\frac{A_1}{A_0}\right\}.$$

The pressure pulse can, therefore, be written as

$$p' = p - p_l = \epsilon p_l\frac{2k}{k-1}\frac{A_1}{A_0}, \tag{33}$$

from which the root mean square pressure can be found by

$$p_{rms} = \sqrt{(1/N)\sum_{n=1}^{N}(p')_n^2}. \tag{34}$$

The sound pressure level is

$$L_p = 20\log_{10}\{p_{rms}/p_{ref}\} \quad \text{(dB)}, \tag{35}$$

where $p_{ref} = 0{\cdot}00002$ N/m^2.

Combining equations (33)–(35), we obtain

$$L_p = 20\log_{10}\left\{\frac{\epsilon}{p_{ref}}\sqrt{\frac{1}{N}\sum_{n=1}^{N}\frac{4k^2}{(k-1)^2}p_l^2\frac{A_l^2}{A_0^2}}\right\}, \tag{36}$$

where the function $A_1(X)$ is obtained from the solution of the system of boundary value problem as described in the analysis in the previous section. Since $A_1(X)$ is a function of the axial distance, equation (36) therefore gives an analytical expression of the sound pressure level as a function of the axial distance, X. It is seen therefore that the present analysis provides a method for the derivation of an analytical expression of the sound pressure level as a function of the axial distance and the physical variables involved.

3. NUMERICAL SOLUTIONS

With the analytical expression of the sound pressure level derived in the last section as a function of the axial distance, it is possible to explore the effects of any physical parameter involved in the problem as needed. In the brief discussion here, only a few simple solutions will be generated. These results are for the case in which the location of excitation is arbitrarily selected at the mid-point of the duct. From equation (36), it is seen that the pattern of the sound pressure level distribution along the duct will be dependent on the amplitude of the vibration, ϵ, the degree of localization, α, the frequency of vibration, f, and the Mach number at the entrance section of the duct, M, respectively.

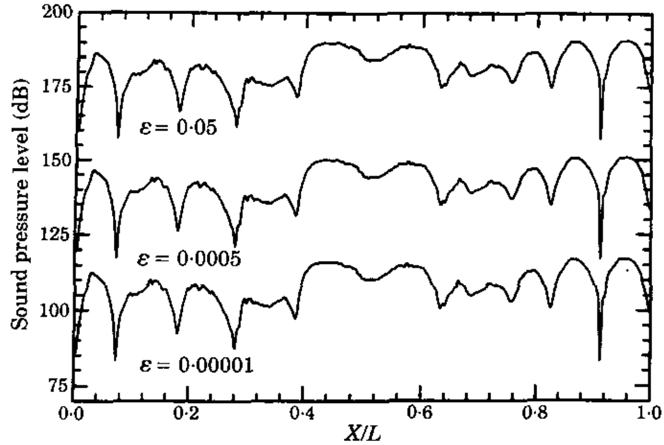

Figure 2. The sound pressure level distribution for $\alpha = 100$, $f = 1000$ Hz, $M_i = 0{\cdot}2$.

The effect of ϵ can be seen easily from equation (36), which shows that the difference in L_p for two values of ϵ's is

$$\Delta L_p = L_{p2} - L_{p1} = 20 \log_{10}(\epsilon_2/\epsilon_1). \tag{37}$$

For $\epsilon_1 = 0{\cdot}00001$ and $\epsilon_2 = 0{\cdot}0005$, ΔL_p is equal to 34 and for $\epsilon_1 = 0{\cdot}0005$ and $\epsilon_2 = 0{\cdot}05$, ΔL_p is equal to 38. In the figures to be presented, three values of ϵ will be given. Equation (37) gives the distances between curves of different values of ϵ. One physical significance of these results is that a localized vibration of even extremely small amplitude will generate sound of considerable sound pressure level.

The effect of the frequency, f, is more profound, as shown in Figures 2–5. Figures 2 and 3 are for $\alpha = 100$ and an entrance Mach number of 0·2. The difference between the figures lies in the frequency, f. Figure 2 is for a frequency of 1000 Hz, while Figure 3 is for a frequency of 2000 Hz. Increasing the frequency is seen to create a region at the location of excitation where the sound pressure level increases relatively smoothly to a maximum value and then decreases to a regular pattern. The creation of such a region in a way *pushes* the peaks and valleys of the sound pressure level to both sides of this band. This implies that energy is evenly distributed along the duct at low frequency excitation, but the energy

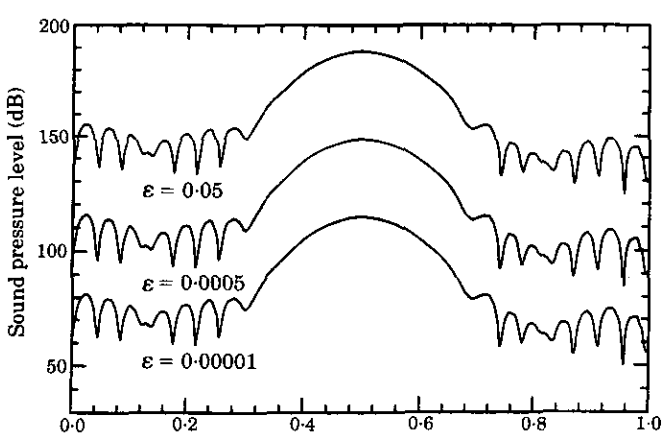

Figure 3. The sound pressure level distribution for $\alpha = 100$, $f = 2000$ Hz, $M_i = 0{\cdot}2$.

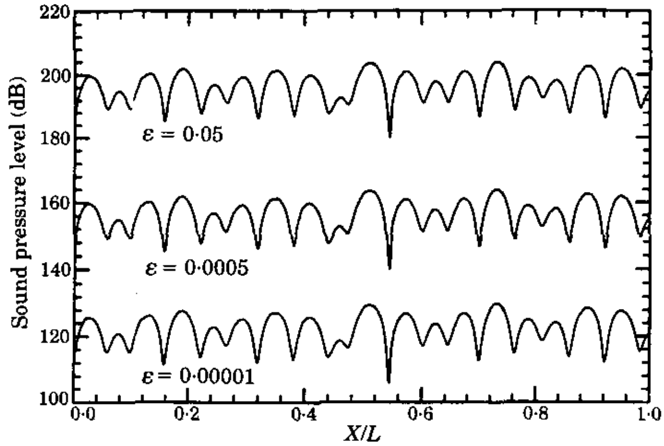

Figure 4. The sound pressure level distribution for $\alpha = 100$, $f = 1000$ Hz, $M_i = 0{\cdot}4$.

is concentrated in the region of excitation at high frequency. Similar results are observed by comparing Figures 4 and 5.

The effect of M_i on the sound pressure level can be seen clearly by comparing Figure 2 with Figure 4, where the Mach numbers are equal to 0·2 and 0·4, respectively. For the case in which M_i is equal to 0·2 (Figure 2), the sound pressure level distribution along the duct is seen to have a range of approximately 50 dB between its minimum and maximum values. For the case in which M_i is equal to 0·4 (Figure 4), the range is decreased to approximately 35 dB. Similar conclusions can be drawn by comparing Figure 3 with Figure 5. It is therefore seen that Figures 2–5 show that, by increasing M_i, the average of the sound pressure level is increased with larger values of M_i.

As stated earlier, the parameter α represents the degree of localization of the vibration. The typical influence can be seen by comparing Figure 3 with Figure 6. The vibration is more localized for α equal to 150 in Figure 6 than for the results in Figure 3, where the value of α is smaller, meaning less localized. The width of the significant change in the vicinity of the disturbance is seen to be narrower as the degree of localization of the disturbance is increased. Another feature, as seen from these two figures, is that the average sound pressure level over the length of the duct is higher when the vibration is more localized. The higher sound pressure level at the more localized vibration can be due to

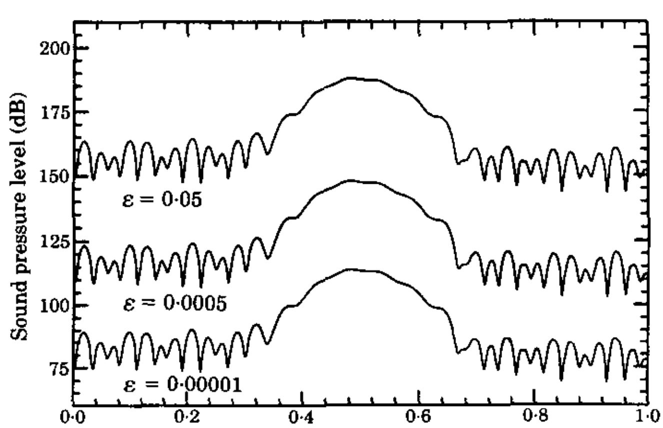

Figure 5. The sound pressure level distribution for $\alpha = 100$, $f = 2000$ Hz, $M_i = 0{\cdot}4$.

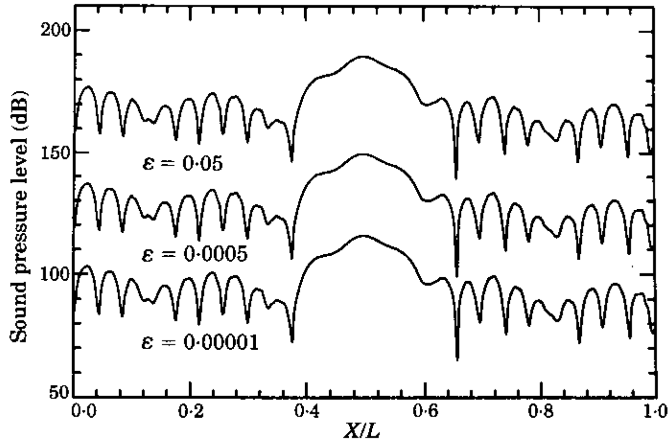

Figure 6. The sound pressure level distribution for $\alpha = 150$, $f = 2000$ Hz, $M_i = 0{\cdot}2$.

the gradient of the cross-sectional area, $\partial F/\partial X$, being greater than the gradient value at a less localized vibration.

4. CONCLUSIONS

A theoretical study of the sound generation by a localized vibration is made based on a transient gas dynamics model. This one-dimensional model is standard in the gas dynamics field where the transient flow through long ducts is analyzed. An analytical expression of the sound pressure level distribution along the axial length of the duct is obtained in terms of the four physical parameters in the formulation. The method used here is very general. By treating the disturbance as a perturbation to an otherwise steady flow in the solution procedure, the perturbation equations are, therefore, linear. This enables the extension of the analysis to the case in which the excitation is periodic but not harmonic. In such cases, the excitation can be represented by a Fourier series consisting of sine and/or cosine functions, where the principle of superposition can be applied. Each term can be solved by the present technique.

It should be emphasized that the problem under consideration is inherently a difficult one. If the localized vibration of the surface is considered to consist of an infinite number of sound sources simultaneously added to the duct with different magnitudes and phase angles, the resultant acoustic pressure or the sound pressure level will be the superposition of all these sources. Adding the complicated forward-and-reflected motion of pressure waves in a channel, it is understandable that a physical interpretation of the phenomena is difficult before further extensive research on the subject is conducted. The present letter, using a one-dimensional model which is standard in the unsteady gas dynamics field, does provide a way to obtain an analytical solution of the sound pressure level distribution and is, it is hoped, the basis for further theoretical and experimental studies in the future.

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APPENDIX: NOMENCLATURE

a	sonic velocity of the fluid
A	dimensionless sonic velocity
C_p	constant pressure specific heat of the fluid
D_h	hydraulic diameter
f	vibration frequency
F	duct cross-sectional area
I	sound intensity
k	adiabatic constant
L	length of the duct
L_I	sound intensity level (dB)
L_p	sound pressure level (dB)
M_I	inlet flow Mach number
N	number of time intervals per cycle
p	flow mean pressure
p'	pressure disturbance
P_I	pressure at Mach number M_I
p_{rms}	root-mean-square of pressure disturbance
p_t	pressure at temperature T_t
R	gas constant
T	temperature
T_I	temperature at Mach number M_I
T_t	reference temperature
u	flow velocity
U	dimensionless flow velocity
x	axial co-ordinate
x_0	location of vibration source
X	dimensionless co-ordinate
Z	dimensionless time
ϵ	area change factor
ρ	density
δ, α	parameters for the extent of localization
ω	angular frequency
$\bar{\omega}$	normalized angular frequency
@	reference sonic velocity at temperature T_t