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On Soliton Equations of Exceptional Type

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The main purpose of this paper is to present an explicit formula for the general hierarchy of soliton equations constructed by Kac–Wakimoto from the basic representation of an arbitrary affine Kac–Moody algebra. The results turn out that the differential operators of the corresponding Hirota bilinear equations can be written explicitly in terms of skew Schur functions for both principal and homogeneous hierarchies. The principal hierarchy includes the classical KP and KdV equations. The homogeneous hierarchy turns out to be related to the classical non-linear Schrödinger equation for type $A_1^{(1)}$ and to the classical 2-dimensional Toda lattice equation for type $A_2^{(1)}$.

INTRODUCTION

The connection between the soliton theory and the representation theory of classical affine Kac–Moody algebras was developed by Date, Jimbo, Kashiwara, and Miwa [DJKM, DKM], using the boson–fermion correspondence in 2-dimensional quantum field theory and certain vertex operator realizations of the algebras.

The basic representations of the classical affine Kac–Moody Lie algebra were constructed on a polynomial ring by using vertex operators. The group orbit of the highest weight vector is an infinite dimensional Grassmann manifold. Its defining equations on the space of polynomial functions, expressed in the form of differential equations, turn out to be the soliton equations (cf. [DJKM]). For instance, the Kadomtsev–Petviashvili (KP) hierarchy can be constructed by using the basic representations of the Lie algebra $gl(\infty)$, while the Korteweg–de Vries (KdV) hierarchy can be done by using the basic representations of the simplest affine Kac–Moody Lie algebra $\hat{sl}(2)$.

Recently, Kac–Wakimoto constructed the hierarchy associated to an arbitrary affine Kac–Moody algebras in a unified way [KW], including the exceptional ones. Their construction is based on the explicit realization of

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the basic representation $L(\Lambda_0)$ (cf. [LW, KKJW, FK, S]). By calculating the action of generalized Casimir operator on the tensor product $L(\Lambda_0)\otimes L(\Lambda_0)$, they obtained a generating series of Hirota bilinear equations which characterize the τ-functions. The soliton solutions to these equations can be obtained from the action of vertex operators on the vacuum vector.

The main purpose of this paper is to find an explicit formula for the general hierarchy of soliton equations constructed by Kac–Wakimoto [KW]. The result turns out to be that the corresponding Hirota bilinear equation can be explicitly written in terms of skew Schur functions for both principal and homogeneous realizations. Note that the similar result for KP and modified KP hierarchy can be found in [L], where the problem was posed by V. Kac.

The paper is organized as follows. We begin with the two constructions (principal and homogeneous) of the basic representation of Kac–Moody algebra in Section 1. Some results on Schur function needed are reviewed in Section 2. In Section 3 we give an explicit formula of Hirota bilinear differential equation for the principal hierarchy, while a formula for the homogeneous hierarchy is presented in Section 4. In the last section the formula for the BKP hierarchy is found.

Througout the paper, our Lie algebras and their representations are all defined over the field $\mathbf{C}$ of complex numbers. Symbols $\mathbf{Z}$ and $\mathbf{N}$ stand for the set of all integers, all positive integers, respectively. We use notations and basic definitions of Ref. [K] unless otherwise specified.

1. Two Constructions of the Basic Representation

Let us recall first the construction of the basic representations in [KKLW] which is called the principal realization of the basic module.

Let $\hat{g}$ be an affine Kac–Moody Lie algebra of type $X_N^{(k)}$ (with $X=A, D$, or E) and rank l over the complex field $\mathbf{C}$. Let $c\in\hat{g}$ be the canonical central element and h the Coxeter number of $\hat{g}$.

Let $\hat{g}=\bigoplus \hat{g}_j$ be the principal gradition of $\hat{g}$, E (resp. E_+) be the set with multiplicities of all (resp. all positive) exponents of $\hat{g}$. For each $j\in E$, one can choose $H_j\in\hat{g}_j$ such that

$$[H_i, H_j]=i\delta_{i,-j}c.$$

The subalgebra $\hat{s}=\mathbf{C}c+\sum_{j\in E}\mathbf{C}H_j$ is called a principal Heisenberg subalgebra of $\hat{g}$.

For each $i\in\mathbf{Z}$ and $r=1, ..., l$, there exist elements $X_i^{(r)}\in\hat{g}_i$ such that

$$[H_j, X_i^{(r)}]=\beta_{r,j}X_{i+j}^{(r)} \qquad \text{for some} \quad \beta_{r,j}\in\mathbf{C},$$
$$[d, X_i^{(r)}]=iX_i^{(r)}.$$

The elements H_j, $X_i^{(r)}$, c, d form a basis of $\hat{g}$. Since the Lie algebra $\hat{g}$ carries a non-degenerate invariant bilinear form $(\cdot|\cdot)$ (cf. [KW]), one can choose $Y_i^{(r)} \in \hat{g}_i$ ($i \in \mathbf{Z}$, $r = 1, ..., l$) such that

$$(Y_i^{(r)} \mid X_j^{(s)}) = \delta_{r,s}\delta_{i,-j}, \qquad (Y_i^{(r)} \mid \hat{s}) = (X_i^{(s)} \mid \hat{s}) = 0.$$

Then it follows from the invariance of $(\cdot|\cdot)$ that

$$[H_j, Y_i^{(r)}] = -\beta_{r,j} Y_{i+j}^{(r)},$$
$$[d, Y_i^{(r)}] = i Y_i^{(r)}.$$

Put

$$X^{(r)}(z) = \sum_{i \in Z} X_i^{(r)} z^{-i},$$
$$Y^{(r)}(z) = \sum_{i \in Z} Y_i^{(r)} z^{-i},$$

then the basic representation $L(\Lambda_0)$ is constructed on the space $L(\Lambda_0) = \mathbf{C}[x_j; j \in E_+]$ by the vertex operators

$$H_i \mapsto \alpha_i \qquad \text{for} \quad j \in E,$$

$$X^{(r)}(z) \mapsto \tilde{X}^{(r)}(z) = C_r \exp\left(\sum_{j \in E_+} \frac{\beta_{r,j}\alpha_{-j}}{j} z^j\right) \exp\left(-\sum_{j \in E_+} \frac{\beta_{r,-j}\alpha_j}{j} z^{-j}\right),$$

$$Y^{(r)}(z) \mapsto \tilde{Y}^{(r)}(z) = D_r \exp\left(-\sum_{j \in E_+} \frac{\beta_{r,j}\alpha_{-j}}{j} z^j\right) \exp\left(\sum_{j \in E_+} \frac{\beta_{r,-j}\alpha_j}{j} z^{-j}\right),$$

$$c \mapsto 1,$$

$$d \mapsto -\sum_{j \in E_+} \alpha_{-j}\alpha_j,$$

where $\alpha_j = \partial/\partial x_j$, $\alpha_{-j} = jx_j$ for $j \in E_+$ and

$$C_r = -h^{-1}(\rho \mid X_0^{(r)}),$$
$$D_r = -h^{-1}(\rho \mid Y_0^{(r)}).$$

Note that this construction obtained in [LW, KKLW, FK, S] is called the principal realization for the simply laced or twisted affine Kac–Moody algebra of type $X_N^{(k)}$.

The second realization called homogeneous realization of the basic representation was constructed for all simply-laced affine Kac–Moody Lie algebras (cf. [FK]), which can be described as follows:

Let g be a finite dimensional Lie algebra of rank l and type A-D-E. Let Δ be the root system and Q be the root lattice of g. Choose on Q a $\mathbf{C}$-valued symmetric, invariant bilinear form $(\cdot|\cdot)$ normalized by the condition $(\alpha \mid \alpha)=2$ for all roots $\alpha \in \Delta$. There is a (non-symmetric) bilinear form $R: Q \times Q \mapsto Z$ such that

$$(\alpha \mid \beta)=R(\alpha, \beta)+R(\beta, \alpha).$$

(one way to construct R is to choose the (directed) Dynkin diagram labelled by simple roots α_i and put $R(\alpha_i, \alpha_i)=1$, $R(\alpha_i, \alpha_j)=0$ for $i \neq j$ except when there is an arrow $\alpha_i \rightarrow \alpha_j$ for which we put $R(\alpha_i, \alpha_j)=-1$).

Define $\varepsilon(\alpha, \beta)=(-1)^{R(\alpha, \beta)}$, then ε is a 2-cocycle of the group Q with values in $\{\pm 1\}$ satisfying two additional properties,

$$\text{(a)} \quad \varepsilon(\alpha, \beta)\, \varepsilon(\beta, \alpha)=(-1)^{(\alpha \mid \beta)},$$

$$\text{(b)} \quad \varepsilon(\alpha, -\alpha)=\varepsilon(\alpha, 0)=1.$$

Let $\underline{h}$ be the Cartan subalgebra spanned by the set of α_i $(i=1, 2, \ldots, l)$. One can choose a root vector E_α for each root $\alpha \in \Delta$ such that

$$g=\underline{h} \oplus \sum_{\alpha \in \Delta} \mathbf{C} E_\alpha$$

and they satisfy the following commutation relations:

$$\begin{aligned}
&[\underline{h}, \underline{h}]=0, && [h, E_\alpha]=(\alpha \mid h)\, E_\alpha \quad \text{for} \quad h \in \underline{h},\\
&[E_\alpha, E_\beta]=0, && \text{if} \quad \alpha+\beta \notin \Delta \cup \{0\},\\
&[E_\alpha, E_{-\alpha}]=-\alpha, && [E_\alpha, E_\beta]=\varepsilon(\alpha, \beta)\, E_{\alpha+\beta}, \quad \text{if} \quad \alpha+\beta \in \Delta.
\end{aligned}$$

The bilinear form $(\cdot|\cdot)$ can be extended to g by putting

$$(\underline{h} \mid E_\alpha)=0 \quad \text{and} \quad (E_\alpha \mid E_\beta)=-\delta_{\alpha,-\beta}.$$

Then the affine Kac–Moody algebra $\hat{g}$ associated to g is

$$\hat{g}=\mathbf{C}[t, t^{-1}] \otimes g+\mathbf{C} c+\mathbf{C} d$$

with commutation relation given by

$$\begin{aligned}
[x(m), y(n)] &= [x, y](m+n)+m(x \mid y)\, \delta_{m,-n} c,\\
[d, x(m)] &= m x(m),\\
[c, \hat{g}] &= 0,
\end{aligned}$$

where $x(m)=t^m \otimes x$ for $m \in \mathbf{Z}$, $x \in g$.

Choose a basis u_j of $\underline{h}$ and a dual basis u^j of $\underline{h}$ with respect to $(\cdot|\cdot)$. The homogeneous realization of the basic representation $L(\Lambda_0)$ of $\hat{g}$ can be constructed in the space (cf. [FK])

$$L(\Lambda_0)=\mathbf{C}[x]\otimes\mathbf{C}[Q]=\mathbf{C}[x_k^{(i)};1\leqslant i\leqslant l,k\in\mathbf{N}]\otimes\sum_{\alpha\in Q}\mathbf{C}e^{\alpha}$$

with $\hat{g}$ acting as follows:

$$u_j(-k)=kx_k^{(j)},\qquad u^j(k)=\partial/\partial x_k^{(j)},\qquad 1\leqslant j\leqslant l,\quad k\in\mathbf{N};$$

$$H(0)(f\otimes e^{\beta})=\beta(H)\,f\otimes e^{\beta}\qquad\text{for}\quad H\in\underline{h},$$

$$d(f\otimes e^{\beta})=\left(-\sum_{k\geqslant 1}\sum_{i=1}^{l}kx_k^{(i)}\frac{\partial f}{\partial x_k^{(i)}}+\frac{1}{2}|\beta|^2f\right)\otimes e^{\beta},$$

$$E_{\gamma}(-k)(f\otimes e^{\beta})=\varepsilon(\gamma,\beta)\,X_k(\gamma)(f\otimes e^{\beta})\qquad\text{for}\quad\gamma\in\Delta,$$

where

$$X(\gamma,z)=\sum_{k\in Z}X_k(\gamma)\,z^k$$

$$:=z^{|\gamma|^2/2}\left(\exp\sum_{j\in E_+}\frac{\gamma(-j)}{j}z^j\right)\left(\exp-\sum_{j\in E_+}\frac{\gamma(j)}{j}z^{-j}\right)\otimes e^{\gamma}z^{\partial_{\gamma}}$$

is the vertex operator defined generally for any element γ in Q.

Here the operator $z^{\partial_{\gamma}}$ acts as

$$z^{\partial_{\gamma}}(f\otimes e^{\beta})=z^{(\beta\,|\,\gamma)}f\otimes e^{\beta}.$$

There is a natural Z_+-gradation on the space $L(\Lambda_0)$ defined by $\deg e^{\beta}:=\frac{1}{2}(\beta\mid\beta)$ and $\deg x_k^{(j)}:=k$. We let, for brevity, $x=(x_k^{(j)})_{1\leqslant j\leqslant l,\,k\in N}$.

2. Schur Functions

We now review some basic results about symmetric functions. One can find basic definitions and notations in [M]. Let

$$\Lambda=\mathbf{C}[x_1,x_2,...]$$

be the polynomial ring in infinitely many variables $x_1, x_2, \ldots$.

Define $p_j(x_1,x_2,...)$ by the generating series

$$\sum_{j\geqslant 0}p_j(x)\,t^j=\exp\left(\sum_{k\geqslant 1}x_kt^k\right).$$

For a partition $\lambda=(\lambda_1\ \lambda_2 \cdots \lambda_l)$, we put

$$p_\lambda(x)=\det(p_{\lambda_i-i+j}(x)).$$

Then it is well known that the set of all functions p_λ ($\lambda \in \mathrm{Par}$), called Schur functions, forms a basis of the ring Λ.

Note that if we let

$$x_n=\frac{1}{n}\sum_{j=1}^{\infty}\xi_j^n,$$

then $p_\lambda(x)=s_\lambda(\xi)$ is symmetric in $(\xi_1, \xi_2, ...)$ and is the classic Schur function studied in [M]. One sees that if we let $\deg x_j=j$, then p_λ is homogeneous of degree $|\lambda|$.

The multiplication of two Schur functions p_λ and p_μ can be written as a linear combination of Schur functions

$$p_\mu p_\nu=\sum_\lambda c_{\mu\nu}^\lambda p_\lambda \tag{1}$$

with the coefficients $c_{\mu\nu}^\lambda \in Z_+$ which can be computed by the celebrated Littlewood–Richardson rule.

Suppose $\lambda=(\lambda_1, \lambda_2, ...)$ and $\mu=(\mu_1, \mu_2, ...)$ are partitions with $\mu_i \leqslant \lambda_i$ for all i. We then write $\mu \leqslant \lambda$. The skew Schur function $p_{\lambda/\mu}$ is defined by

$$p_{\lambda/\mu}(x)=\det(p_{\lambda_i-\mu_j-i+j}(x)).$$

We then have the following well-known property.

THEOREM 2.1 [M]. *If $\lambda \geqslant \mu$. Then*

$$p_{\lambda/\mu}=\sum_\nu c_{\mu\nu}^\lambda p_\nu.$$

PROPOSITION 2.1. *The following identity holds*

$$\exp\left(\sum_{n\geqslant 1} x_n y_n\right)=\sum_\lambda p_\lambda(\tilde{x})\, p_\lambda(y),$$

where $\tilde{x}=(x_1, x_2/2, x_3/3, ...)$.

Proof. Introduce new variables $\xi=(\xi_1, \xi_2, ...)$, $\eta=(\eta_1, \eta_2, ...)$, by putting for $k \geqslant 1$,

$$x_k=(\xi_1^k+\xi_2^k+\cdots),$$

$$y_k=\frac{1}{k}(\eta_1^k+\eta_2^k+\cdots).$$

Then we have

$$p_\lambda(\tilde{x}) = s_\lambda(\xi), \qquad p_\lambda(y) = s_\lambda(\eta).$$

$$\begin{aligned}\exp \sum_{k \geqslant 1} x_k y_k &= \exp \sum_{k \geqslant 1} \frac{1}{k}\left(\sum_{i,j} (\xi_i \eta_j)^k\right) \\ &= \exp \sum_{i,j} \sum_{k \geqslant 1} \frac{1}{k} (\xi_i \eta_j)^k \\ &= \exp \sum_{i,j} (-\log(1 - \xi_i \eta_j)) \\ &= \prod_{i,j} (1 - \xi_i \eta_j)^{-1}.\end{aligned}$$

But it is known that (cf. [M])

$$\prod_{i,j} (1 - \xi_i \eta_j)^{-1} = \sum_\lambda s_\lambda(\xi)\, s_\lambda(\eta).$$

Therefore

$$\exp \sum_{k \geqslant 1} x_k y_k = \sum_\lambda s_\lambda(\xi)\, s_\lambda(\eta) = \sum_\lambda s_\lambda(\tilde{x})\, s_\lambda(y)$$

which finishes the proof.

Now suppose λ is a partition of n. Let χ^λ be the character of symmetric group of S_n corresponding to λ. Define

$$\gamma^\lambda_{\mu\nu} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, \chi^\mu(\sigma)\, \chi^\nu(\sigma).$$

In the same way as we prove Proposition 2.1, we obtain the following result from [M, p. 63].

PROPOSITION 2.2. *We have*

$$p_\lambda(x_1 y_1, x_2 y_2, ...) = \sum_{\mu, \nu} \gamma^\lambda_{\mu\nu} p_\mu(\tilde{x})\, p_\nu(y).$$

Since $\gamma^{(n)}_{\mu\nu} = \delta_{\mu,\nu}$, it then follows

COROLLARY 2.1. $p_n(x_1 y_1, x_2 y_2, ...) = \sum_{\mu \vdash n} p_\mu(\tilde{x})\, p_\mu(y)$.

PROPOSITION 2.3 [M]. *The following identity holds*

$$n x_n = \sum_{s=0}^{n-1} (-1)^s p_{(n-s,\, 1^s)}(x).$$

3. Principal Hierarchy of A-D-E and Exceptional Type

Consider the basic representation of simply laced or twisted affine Kac–Moody algebra $\hat{g}$ of rank l and type $X_N^{(k)}$ (with $X=A$, D, or E) on the space $L(\Lambda_0)=\mathbf{C}[x_j : j\in E_+]$ (see Section 1). Let G be the simply connected algebraic group over $\mathbf{C}$ with the Lie algebra g. The corresponding affine Kac–Moody group $\hat{G}$ is then the central extention of $G(\mathbf{C}[t, t^{-1}])$ by $\mathbf{C}^\times$. The representation $L(\Lambda_0)$ of $\hat{g}$ can be extended to give a projective representation of the group $\hat{G}$.

Then we have the following due to Kac–Wakimoto.

Theorem 3.1 [KM]. *A non-zero element τ of $L(\Lambda_0)$ lies in the orbit $\hat{G}.1$ if and only if τ satisfies the hierarchy of Hirota bilinear differential equations,*

$$\left[-2h\sum_{j\in E_+} jy_jD_j+\sum_{r=1}^{l} b_r\sum_{n\geqslant 1} p_n^{(E)}(2\beta_{r,j}y_j)\, p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j}D_j\right)\right]\\ \times e^{\sum_{j\in E_+} y_jD_j}\tau\cdot\tau=0, \tag{2}$$

where $b_r=(\rho \mid X_0^{(r)})(\rho \mid Y_0^{(r)})$ and $p_n^{(E)}(x)$ $(n\in\mathbf{Z}_+)$ are defined by

$$\sum_{n\geqslant 0} p_n^{(E)}(x)z^n=\exp\left(\sum_{j\in E_+} x_jz^j\right).$$

The hierarchy (2) of Hirota bilinear equation is called the principal hierarchy of type $X_N^{(k)}$. Now define

$$P(x,y)=\left[-2h\sum_{j\in E_+} jy_jx_j+\sum_{r=1}^{l} b_r\sum_{n\geqslant 1} p_n^{(E)}(2\beta_{r,j}y_j)\, p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j}x_j\right)\right]\\ \times e^{\sum_{j\in E_+} y_jx_j}. \tag{3}$$

Since the Schur function $p_\lambda(y)$ $(\lambda\in\mathrm{Par})$ forms a basis of the polynomial ring $\mathbf{C}[y_1, y_2, ...]$. The function $P(x, y)$ can be written as a linear combination of $p_\lambda(y)$. Now define $Q_\lambda(x)$ by the following

$$P(x,y)=\sum_{\lambda\in\mathrm{Par}} Q_\lambda(x)\, p_\lambda(y). \tag{4}$$

Then the principal hierarchy (2) is equivalent to the following Hirota bilinear equation

$$Q_\lambda(\partial/\partial u)(\tau(x+u)\,\tau(x-u))|_{u=0}=0. \tag{5}$$

The following theorem gives an explicit formula for the differential operators Q_λ.

THEOREM 3.2. *The differential operator Q_λ is given by*

$$Q_\lambda(x)=\Bigg[-2h\sum_{\substack{r\geq 1,\, s\geq 0\\ r+s\in E_+}}(-1)^s x_{r+s}p^{(E)}_{\lambda/(r,1^s)}(\tilde{x})$$

$$+\sum_{r=1}^{l} b_r \sum_{\mu\neq 0} p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right)p^{(E)}_{|\mu|}\left(-\frac{\beta_{r,-j}}{j}x_j\right)p^{(E)}_{\lambda/\mu}(\tilde{x})\Bigg], \tag{6}$$

where

$$p^{(E)}_{\lambda/\mu}(x)=\det(p^{(E)}_{\lambda_i-\mu_j-i+j}(x)).$$

Proof. Since

$$p^{(E)}_n(x)=p_n(x)|_{x_j=0,\, j\in E^+}.$$

We then have from Corollary 2.1

$$p^{(E)}_n(2\beta_{r,j}y_j)=\sum_{\mu\vdash n}p_\mu(y)\,p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right). \tag{7}$$

By Proposition 2.1, we see

$$e^{\sum_{j\in E_+}y_jx_j}=\sum_{\nu\in \mathrm{Par}}p^{(E)}_\nu(\tilde{x})\,p_\nu(y). \tag{8}$$

Therefore we obtain from Theorem 2.1 and identities (1), (7), (8)

$$\begin{aligned}
p^{(E)}_n(2\beta_{r,j}y_j)\,e^{\sum_{j\in E_+}y_jx_j}&=\sum_{\substack{\mu\vdash n\\ \nu\in\mathrm{Par}}}p_\mu(y)\,p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right)p^{(E)}_\nu(\tilde{x})\,p_\nu(y)\\
&=\sum_{\substack{\mu\vdash n\\ \nu\in\mathrm{Par}}}p_\mu(y)\,p_\nu(y)\,p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right)p^{(E)}_\nu(\tilde{x})\\
&=\sum_{\substack{\mu\vdash n\\ \nu\in\mathrm{Par}}}\left(\sum_\lambda c^\lambda_{\mu\nu}p_\lambda(y)\right)p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right)p^{(E)}_\nu(\tilde{x})\\
&=\sum_\lambda p_\lambda(y)\sum_{\substack{\mu\vdash n\\ \nu\in\mathrm{Par}}}c^\lambda_{\mu\nu}p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right)p^{(E)}_\nu(\tilde{x})\\
&=\sum_\lambda p_\lambda(y)\sum_{\mu\vdash n}\left(\sum_{\nu\in\mathrm{Par}}c^\lambda_{\mu\nu}p^{(E)}_\nu(\tilde{x})\right)p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right)\\
&=\sum_\lambda p_\lambda(y)\sum_{\mu\vdash n}p^{(E)}_{\lambda/\mu}(\tilde{x})\,p^{(E)}_\mu\left(\frac{2\beta_{r,j}}{j}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n\geqslant 1} p_n^{(E)}(2\beta_{r,j}y_j)\, p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j}x_i\right) e^{\sum_{i\in E_+} y_j x_j}\\
&\quad=\sum_{n\geqslant 1}\sum_{\lambda} p_\lambda(y) \sum_{\mu\vdash n} p_{\lambda/\mu}^{(E)}(\tilde{x})\, p_\mu^{(E)}\left(\frac{2\beta_{r,-j}}{j}\right) p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j}x_j\right)\\
&\quad=\sum_{\lambda} p_\lambda(y) \sum_{\mu\neq\varnothing} p_\mu^{(E)}\left(\frac{2\beta_{r,-j}}{j}\right) p_{|\mu|}^{(E)}\left(-\frac{\beta_{r,-j}}{j}x_j\right) p_{\lambda/\mu}^{(E)}(\tilde{x}).
\end{aligned}\tag{9}$$

On the other hand, we have from Proposition 2.3

$$jy_j=\sum_{s=0}^{j-1}(-1)^s p_{(j-s,\,1^s)}(y).\tag{10}$$

Using again Theorem 2.1 and identities (1), (8), (10), it then follows that

$$\begin{aligned}
\sum_{j\in E_+} jy_jx_je^{\sum_{j\in E_+} y_jx_j}&=\left(\sum_{j\in E_+}\sum_{s=0}^{j-1}(-1)^s p_{(j-s,\,1^s)}(y)\,x_j\right)\left(\sum_{\mu\in \mathrm{Par}} p_\mu(y)\, p_\mu^{(E)}(\tilde{x})\right)\\
&=\sum_{\mu}\sum_{j\in E_+}\sum_{s=0}^{j-1}(-1)^s p_{(j-s,\,1^s)}(y)\, p_\mu(y)\, x_j p_\mu^{(E)}(\tilde{x})\\
&=\sum_{\mu}\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\in E_+}}(-1)^s p_{(r,\,1^s)}(y)\, p_\mu(y)\, x_{r+s} p_\mu^{(E)}(\tilde{x})\\
&=\sum_{\mu}\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\in E_+}}(-1)^s\left(\sum_{\lambda} c_{\mu(r,\,1^s)}^{\lambda} p_\lambda(y)\right) x_{r+s} p_\mu^{(E)}(\tilde{x})\\
&=\sum_{\lambda} p_\lambda(y)\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\in E_+}}(-1)^s\left(\sum_{\mu} c_{\mu(r,\,1^s)}^{\lambda} p_\mu^{(E)}(\tilde{x})\right) x_{r+s}\\
&=\sum_{\lambda} p_\lambda(y)\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\in E_+}}(-1)^s x_{r,s} p_{\lambda/(r,\,1^s)}^{(E)}(\tilde{x}).
\end{aligned}\tag{11}$$

Combining the above identities (9), (11), and (3), we find

$$\begin{aligned}
P(x,y)=\sum_{\lambda} p_\lambda(y)\Bigg(&-2h\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\in E_+}}(-1)^s x_{r+s} p_{\lambda/(r+1^s)}^{(E)}(\tilde{x})\\
&+\sum_{r=1}^{l} b_r\sum_{\mu\neq\varnothing} p_\mu^{(E)}\left(\frac{2\beta_{r,-j}}{j}\right) p_{|\mu|}^{(E)}\left(-\frac{\beta_{r,-j}}{j}x_j\right) p_{\lambda/\mu}^{(E)}(\tilde{x})\Bigg).
\end{aligned}$$

Therefore it follows from the definition of Q_λ

$$Q_\lambda(x)=\Bigg(-2h\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\in E_+}}(-1)^s x_{r+s}p^{(E)}_{\lambda/(r+1^s)}(\tilde{x})$$
$$+\sum_{r=1}^{l} b_r \sum_{\mu\neq\varnothing} p^{(E)}_\mu\left(\frac{2\beta_{r,-j}}{j}\right)p^{(E)}_{|\mu|}\left(-\frac{\beta_{r,-j}}{j}x_j\right)p^{(E)}_{\lambda/\mu}(\tilde{x})\Bigg).$$

The proof is completed.

From Theorem 3.2, one can obtain the differential equations for the principal hierarchy up to any high order by computing the constants $\beta_{r,j}$ and b_r. The constants $\beta_{r,j}$ were computed in [D] for the affine Lie algebra of type $E^{(1)}_n$ $(n=6, 7, 8)$. The non-trivial equations of lowest degree in some cases were computed in [KW].

Now applied to $A^{(1)}_{l-1}$, we have from [KKLW]

$$\beta_{r,j}=(1-\varepsilon^{rj}),$$
$$b_r=(-l)\frac{\varepsilon^r}{(1-\varepsilon^r)^2},$$

where ε is a primitive lth root of unity.

Theorem 3.2 now gives the following

PROPOSITION 3.1. *The function τ lies in the orbit $GL_l(\mathbf{C}[t, t^{-1}])\cdot 1$ if and only if*

$$\Bigg[2\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\not\equiv 0\ (\mathrm{mod}\ l)}}(-1)^s x_{r+s}p^{(l)}_{\lambda/(r,1^s)}(\tilde{x})+\sum_{r=1}^{l-1}\frac{\varepsilon^r}{(1-\varepsilon^r)^2}$$
$$\times\sum_{\mu\neq\varnothing}p^{(l)}_\mu\left(\frac{2(1-\varepsilon^{rj})}{j}\right)p^{(l)}_{|\mu|}\left(-\frac{1-\varepsilon^{-rj}}{j}x_j\right)p^{(l)}_{\lambda/\mu}(\tilde{x})\Bigg]\tau\cdot\tau=0, \qquad (12)$$

where $p^{(l)}_n(x)$ is defined by the generating series,

$$\sum_{n\geqslant 0}p^{(l)}_n(x)\,t^n=\exp\Bigg(\sum_{\substack{k\geqslant 1\\ k\not\equiv 0\ (\mathrm{mod}\ l)}}x_k t^k\Bigg).$$

EXAMPLE 3.1. Consider the simplest case $A^{(1)}_1$. The corresponding principal hierarchy of (12) becomes ($\lambda\in\mathrm{Par}$)

$$\Bigg[\sum_{\substack{r\geqslant 1,\, s\geqslant 0\\ r+s\ \mathrm{odd}}}(-1)^s x_{r+s}\tilde{p}_{\lambda/(r,1^s)}(\tilde{x})$$
$$-\frac{1}{8}\sum_{\mu\neq\varnothing}\tilde{p}_\mu\left(\frac{4}{j}\right)\tilde{p}_{|\mu|}(-2\tilde{x})\,\tilde{p}_{\lambda/\mu}(\tilde{x})\Bigg]\tau\cdot\tau=0, \qquad (13)$$

where

$$\tilde{p}_\lambda(x) = p_\lambda(x_1, 0, x_3, 0, \ldots).$$

The unique non-trivial Hirota bilinear equation of lowest degree is obtained by choosing the partition $\lambda = (2, 2)$ in Eq. (13). One obtains the following equation by calculating directly the bracket in Eq. (13) (cf. [DJKM, KW]),

$$(D_1^4 - 4D_1 D_3)\, \tau \cdot \tau = 0. \tag{14}$$

Putting $x = x_1$, $t = x_3$, and all other $x_{2k-1} = \text{constant}$. Denote

$$u(x, t) = 2 \frac{\partial^2}{\partial x^2} (\log \tau(x, t, c_5, c_7, \ldots)).$$

Then Eq. (14) is equivalent to the classical KdV equation,

$$u_t = \tfrac{3}{2} u u_x + \tfrac{1}{4} u_{xxx}. \tag{15}$$

It is well known that the basic module $L(\Lambda_0)$ carries a contravariant Hermitian form defined by

$$\langle P(x), Q(x) \rangle = \left(P\left(\frac{\tilde{\partial}}{\partial x}\right) \bar{Q}(x) \right)\bigg|_{x=0},$$

where $\bar{Q}$ is obtained from Q by conjugating the coefficients.

Now $L(\Lambda_0) \otimes L(\Lambda_0)$ can be thought of as the space of polynomials on two sets of variables: $\mathbf{C}[x_j', x_j''; j \in E_+]$. We introduce new variables

$$x_j = \tfrac{1}{2}(x_j' + x_j''),$$
$$y_j = \tfrac{1}{2}(x_j' - x_j'').$$

The Hermitian form on the tensor product is induced from that on $L(\Lambda_0)$. We then have the orthogonal decomposition

$$L(\Lambda_0) \otimes L(\Lambda_0) = L(2\Lambda_0) \oplus L(2\Lambda_0)^\perp$$

with

$$L(2\Lambda_0)^\perp = \mathbf{C}[x] \otimes \mathrm{Hir}$$

and

$$\mathrm{Hir} = \mathbf{C}[y] \cap L(2\Lambda_0)^\perp$$

is a subspace of $\mathbf{C}[y_j; j \in E_+]$. Then it is easy to show by a standard argument

PROPOSITION 3.2. *The polynomials* $Q_\lambda(y)$ $(\lambda \in \mathrm{Par})$ *span the space* Hir.

Remark. $Q_\lambda(y)$ $(\lambda \in \mathrm{Par})$ are not linearly independent. Let Hir_n denote the space spanned by Q_λ $(\lambda \vdash n)$. Then by comparing the q-dimension we obtain

$$\sum_{n=0}^{\infty} \dim(\mathrm{Hir}_n)\, q^n = \dim_q L(\Lambda_0) - \frac{\dim_q L(2\Lambda_0)}{\dim_q L(\Lambda_0)},$$

where $\dim_q L(\Lambda)$ denotes the q-dimension of $L(\Lambda)$.

4. HOMOGENEOUS HIERARCHY OF TYPE A-D-E

Our goal in this section is to find an explicit expression of Hirota bilinear equations for the homogeneous hierarchy. It turns out that they can be written in terms of skew Schur functions depending on l partitions.

Now suppose $\hat{g}$ is a simply laced affine algebra of type $X_N^{(1)}$ and rank l $(X = A, D$ or $E)$ with the corresponding group $\hat{G}$. Choose a basis u_j of $\underline{h}$ and a dual basis u^j $(j = 1, 2, \ldots, l)$ as in Section 1. Define $P_n^\gamma(x)$, $Q_n^\gamma(x)$ for any $\gamma \in Q$ and $n \in \mathbf{Z}$ by

$$\sum_{n \geqslant 0} P_n^\gamma(x) = \exp\left(\sum_{k=1}^{\infty} \sum_{j=1}^{l} \langle \gamma, u_j \rangle\, x_k^{(j)} z^k\right)$$

$$\sum_{n \geqslant 0} Q_n^\gamma(x) = \exp\left(\sum_{k=1}^{\infty} \sum_{j=1}^{l} \langle \gamma, u^j \rangle\, x_k^{(j)} z^k\right).$$

Then we have the following result.

THEOREM 4.1 [KW]. *An element* $\tau = \sum_{\beta \in Q} \tau_\beta \otimes e^\beta$ *of* $L(\Lambda_0) = \mathbf{C}[x] \otimes \mathbf{C}[Q]$ *lies in the* $\hat{G}$ *orbit of the vacum vector* $1 \otimes e^0$ *if and only if the following differential equation is satisfied*

$$\left(2 \sum_{k \geqslant 1} \sum_{j=1}^{l} k y_k^{(j)} D_k^{(j)} + \tfrac{1}{2} |\alpha - \beta|^2\right) e^{\sum y_k^{(j)} D_k^{(j)}} \tau_\alpha \cdot \tau_\beta + \sum_{\gamma \in \Delta} \varepsilon(\gamma, \alpha - \beta)$$
$$\times \sum_{n \geqslant 0} Q_n^\gamma(2y)\, P_{n-2+(\gamma \mid \alpha - \beta)}^\gamma(-\tilde{D})\, e^{\sum y_k^{(j)} D_k^{(j)}} \tau_{\alpha - \gamma} \cdot \tau_{\beta + \gamma} = 0. \qquad (16)$$

The hierarchy (16) *is called the homogeneous hierarchy of type* $X_N^{(1)}$.

Now for any $j = 1, 2, ..., l$ and $\lambda \in \mathrm{Par}$, we let

$$x^{(j)} = (x_1^{(j)}, x_2^{(j)}, ...)$$

and denote $p_\lambda(x^{(j)}) = p_\lambda(x_1^{(j)}, x_2^{(j)}, ...)$.

If c is any constant we put

$$p_\lambda(c) = p_\lambda(c, c, ...).$$

Then we have the following

THEOREM 4.2. *Fix $\alpha, \beta \in Q$ and define for l partitions $\lambda_1, \lambda_2, ..., \lambda_l$,*

$$\begin{aligned} Q^{(0)}_{\lambda_1\lambda_2\cdots\lambda_l}(x) &= 2\sum_{j=1}^{l}\sum_{r\geqslant 1,\, s\geqslant 0}(-1)^s x^{(j)}_{r+s} p_{\lambda_1}(\tilde{x}^{(1)})\cdots p_{\lambda_j/(r,1^s)}(\tilde{x}^{(j)})\cdots p_{\lambda_l}(\tilde{x}^{(l)}) \\ &\quad + \tfrac{1}{2}|\alpha-\beta|^2 p_{\lambda_1}(\tilde{x}^{(1)})\cdots p_{\lambda_l}(\tilde{x}^{(l)}), \\ Q^{(\gamma)}_{\lambda_1\lambda_2\cdots\lambda_l}(x) &= \sum_{\mu_1, ..., \mu_l} p_{\mu_1}(2\langle\gamma, u^1\rangle)\cdots p_{\mu_l}(2\langle\gamma, u^l\rangle)\, P^{\gamma}_{\sum|\mu_i| - 2 + (\gamma\,|\,\alpha-\beta)}(-\tilde{x}) \\ &\quad \times p_{\lambda_1/\mu_1}(\tilde{x}^{(1)})\cdots p_{\lambda_l/\mu_l}(\tilde{x}^{(l)}). \end{aligned}$$

Then the homogeneous hierarchy is equivalent to the Hirota bilinear differential equation,

$$Q^{(0)}_{\lambda_1\lambda_2\cdots\lambda_l}\tau_\alpha\cdot\tau_\beta + \sum_{\gamma\in\Delta}\varepsilon(\gamma, \alpha-\beta)\, Q^{(\gamma)}_{\lambda_1\lambda_2\cdots\lambda_l}\tau_{\alpha-\gamma}\cdot\tau_{\beta+\gamma} = 0. \tag{17}$$

Proof. Denote for brevity

$$C^\gamma(\mu_1, ..., \mu_l) = p_{\mu_1}(2\langle\gamma, u^1\rangle)\cdots p_{\mu_l}(2\langle\gamma, u^l\rangle).$$

Then we have by the definition of $Q^\gamma_n(x)$,

$$Q^\gamma_n(2y) = \sum_{\substack{\mu_1, ..., \mu_l \\ \sum|\mu_i| = n}} C^\gamma(\mu_1, ..., \mu_l)\, p_{\mu_1}(y^{(1)})\cdots p_{\mu_l}(y^{(l)}). \tag{18}$$

By using Theorem 2.1, Proposition 2.1, and (1), (18), we obtain

$$\begin{aligned} Q^\gamma_n(2y)\, e^{\sum y_k^{(j)} x_k^{(j)}} = \sum_{\substack{\mu_1, ..., \mu_l \\ \sum|\mu_i| = n \\ \nu_1, ..., \nu_l}} & C^\gamma(\mu_1, ..., \mu_l)\, p_{\mu_1}(y^{(1)})\cdots p_{\mu_l}(y^{(l)}) \\ & \times p_{\nu_1}(y^{(1)})\cdots p_{\nu_l}(y^{(l)})\, p_{\nu_1}(\tilde{x}^{(1)})\cdots p_{\nu_l}(\tilde{x}^{(l)}) \end{aligned}$$

$$= \sum_{\substack{\mu_1, \ldots, \mu_l \\ \sum |\mu_i| = n \\ \nu_1, \ldots, \nu_l}} C^{\gamma}(\mu_1, \ldots, \mu_l)$$

$$\times \left(\sum_{\lambda_1} c^{\lambda_1}_{\mu_1 \nu_1} p_{\lambda_1}(y^{(1)}) \right) \cdots \left(\sum_{\lambda_l} c^{\lambda_l}_{\mu_l \nu_l} p_{\lambda}(y^{(l)}) \right)$$

$$\times p_{\nu_1}(\tilde{x}^{(l)}) \cdots p_{\nu_l}(\tilde{x}^{(l)})$$

$$= \sum_{\substack{\lambda_1, \ldots, \lambda_l \\ \mu_1, \ldots, \mu_l \\ \sum |\mu_i| = n}} C^{\gamma}(\mu_1, \ldots, \mu_l)\, p_{\lambda_1}(y^{(1)}) \cdots p_{\lambda_l}(y^{(l)})$$

$$\times \left(\sum_{\nu_1} c^{\lambda_1}_{\mu_1 \nu_1} p_{\nu_1}(\tilde{x}^{(1)}) \right) \cdots \left(\sum_{\nu_l} c^{\lambda_l}_{\mu_l \nu_l} p_{\nu_l}(\tilde{x}^{(l)}) \right)$$

$$= \sum_{\substack{\lambda_1, \ldots, \lambda_l \\ \mu_1, \ldots, \mu_l \\ \sum |\mu_i| = n}} C^{\gamma}(\mu_1, \ldots, \mu_l)\, p_{\lambda_1}(y^{(1)}) \cdots p_{\lambda_l}(y^{(l)})$$

$$\times p_{\lambda_1/\mu_1}(\tilde{x}^{(1)}) \cdots p_{\lambda_l/\mu_l}(\tilde{x}^{(l)}).$$

Let us use the following abbreviation: for two sets of partitions

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_l), \qquad \vec{\mu} = (\mu_1, \ldots, \mu_l)$$

we put $|\vec{\mu}| = \sum |\mu_i|$ and

$$C^{\gamma}(\vec{\mu}) = C^{\gamma}(\mu_1, \ldots, \mu_l),$$

$$p_{\vec{\lambda}}(x) = p_{\lambda_1}(x^{(1)}) \cdots p_{\lambda_l}(x^{(l)}),$$

$$p_{\vec{\lambda}/\vec{\mu}}(x) = p_{\lambda_1/\mu_1}(x^{(1)}) \cdots p_{\lambda_l/\mu_l}(x^{(l)}).$$

Then we arrive at

$$\sum_{n \geq 0} Q^{\gamma}_n(2y)\, P^{\gamma}_{n-2+(\gamma \mid \alpha - \beta)}(-\tilde{x})\, e^{\sum y^{(j)}_k x^{(j)}_k}$$

$$= \sum_{n \geq 0} \sum_{\substack{\vec{\lambda}, \vec{\mu} \\ |\vec{\mu}| = n}} C^{\gamma}(\vec{\mu})\, p_{\vec{\lambda}}(y)\, p_{\vec{\lambda}/\vec{\mu}}(\tilde{x})\, P^{\gamma}_{n-2+(\gamma \mid \alpha - \beta)}(-\tilde{x})$$

$$= \sum_{\vec{\lambda}} p_{\vec{\lambda}}(y) \left(\sum_{\vec{\mu}} C^{\gamma}(\vec{\mu})\, P^{\gamma}_{|\vec{\mu}|-2+(\gamma \mid \alpha - \beta)}(-\tilde{x})\, p_{\vec{\lambda}/\vec{\mu}}(\tilde{x}) \right). \tag{19}$$

In the same way, using Theorem 2.1, Propositions 2.1 and 2.3, and (1) we obtain

$$\sum_{k=1}^{\infty} \sum_{j=1}^{l} (ky_k^{(j)} x_k^{(j)})\, e^{\sum y_k^{(j)} x_k^{(j)}}$$
$$= \sum_{\bar{\lambda}} p_{\bar{\lambda}}(y) \left(\sum_{j=1}^{l} \sum_{r \geqslant 1, s \geqslant 0} (-1)^s x_{r+s}^{(j)} p_{\lambda_1}(\tilde{x}^{(1)}) \cdots p_{\lambda_j/(r,1^s)}(\tilde{x}^{(j)}) \cdots p_{\lambda_l}(\tilde{x}^{(l)}) \right). \tag{20}$$

Combining Eqs. (19), (20), and Theorem 4.1 will finish the proof.

Let us consider the case for the simplest affine Kac–Moody algebra of type $A_1^{(1)}$. Choose $u_1 = \alpha_1$, $u^1 = \frac{1}{2}\alpha_1$ and let $\tau_n = \tau_{n\alpha_1}$. Then we have the following.

COROLLARY 4.1. *The homogeneous hierarchy of type $A_1^{(1)}$ is given by*

$$\left[2 \sum_{r \geqslant 1, s \geqslant 0} (-1)^s x_{r+s} p_{\lambda/(r,1^s)}(\tilde{x}) + (m-n)^2 p_\lambda(\tilde{x}) \right] \tau_n \cdot \tau_m$$
$$+ (-1)^{m-n} \sum_{\mu \leqslant \lambda} p_\mu(2)\, p_{|\mu| - 2(m-n+1)}(-2\tilde{x})\, p_{\lambda/\mu}(\tilde{x})\, \tau_{n-1} \cdot \tau_{m+1}$$
$$+ (-1)^{m-n} \sum_{\mu \leqslant \lambda} p_\mu(-2)\, p_{|\mu| + 2(m-n-1)}(2\tilde{x})\, p_{\lambda/\mu}(\tilde{x})\, \tau_{n+1} \cdot \tau_{m-1} = 0. \tag{21}$$

Note that Eq. (21) is called the non-linear Schrödinger hierarchy since some of the equations of lower degree produce the classical non-linear Schrödinger equation (cf. [KW, P]).

EXAMPLE 4.1. Let $\lambda = (k)$, k is a positive integer. Then the above equation becomes

$$\left[2 \sum_{r=1}^{k} x_r p_{(k-r)}(\tilde{x}) + (m-n)^2 p_k(\tilde{x}) \right] \tau_n \cdot \tau_m$$
$$+ (-1)^{m-n} \sum_{r=0}^{k} p_r(2)\, p_{r-2(m-n+1)}(-2\tilde{x})\, p_{(k-r)}(\tilde{x})\, \tau_{n-1} \cdot \tau_{m+1}$$
$$+ (-1)^{m-n} \sum_{r=0}^{k} p_r(-2)\, p_{r+2(m-n-1)}(2\tilde{x})\, p_{(k-r)}(\tilde{x})\, \tau_{n+1} \cdot \tau_{m-1} = 0.$$

If $k = 2$, $m = n$, this gives ($D_2 \tau_n \cdot \tau_n = 0$ is trivial)

$$D_1^2 \tau_n \cdot \tau_n + 2\tau_{n-1} \cdot \tau_{n+1} = 0. \tag{22}$$

Note that if we put

$$u_n(x) = \log \frac{\tau_{n+1}}{\tau_n}(x, c_2, c_3, \ldots),$$

then one obtains (cf. [TB, KW]) the Hirota bilinear equations of 1-dimensional Toda lattice equation

$$(u_n)_{xx} = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}. \tag{23}$$

EXAMPLE 4.2. Consider the case of type $A_2^{(1)}$. For $i = 1, 2$, choose the basis $u_i = \alpha_i^\vee$ and its dual basis $u^i = \omega_i$ (the fundamental weights of $sl(3)$). Put $\theta = \alpha_1 + \alpha_2$, then we have

$$Q_{1,1}^{(0)}(x) = 4x_1^{(1)}x_1^{(2)},$$

$$Q_{1,1}^{(\gamma)}(x) = 4\langle \gamma, u^1 \rangle \langle \gamma, u^2 \rangle = \begin{cases} 4 & \text{if } \gamma = \pm\theta, \\ 0 & \text{otherwise.} \end{cases}$$

Take $\alpha = \beta = n\theta$, $\lambda_1 = \lambda_2 = (1)$ and let $\tau_n = \tau_{n\theta}$. Then the corresponding equation in Theorem 4.2 gives

$$D_1^{(1)}D_1^{(2)}\tau_n \cdot \tau_n + 2\tau_{n+1} \cdot \tau_{n-1} = 0. \tag{24}$$

Define $\tau_n(x, y) = \tau_n$ by putting $x_1^{(1)} = x$, $x_1^{(2)} = y$, and all other $x_k^{(i)} =$ constant. Denote

$$u_n(x, y) = \log \frac{\tau_{n+1}(x, y)}{\tau_n(x, y)}.$$

Then Eq. (24) is equivalent to the classical 2-dimensional Toda lattice equation (cf. [UT]),

$$(u_n)_{xy} = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}. \tag{25}$$

5. EXPLICIT FORMULA FOR THE BKP HIERARCHY

In this section we give an analogous result for BKP hierarchy which is associated to the spin representation of B_∞. The polynomial solutions to the BKP hierarchy turn out to be the so called Schur Q-functions (cf. [DJKM, Y]), which connect to the projective representation of symmetric group (cf. [St]).

Define $\tilde{p}_j(x)$ by the following

$$\sum_{j \geq 0} \tilde{p}_j(x)\, z^j = \exp\Bigg(\sum_{k \geq 1 \text{ odd}} x_k z^k \Bigg).$$

It is known [DJKM] that the BKP hierarchy is equivalent to the Hirota bilinear equation

$$\sum_{j \geqslant 1} \tilde{p}_j(-2y)\, \tilde{p}_j(2\tilde{D})\, e^{\sum_{j \text{ odd}} y_j D_j} \tau \cdot \tau = 0, \tag{26}$$

where $\tilde{D} = (\partial/\partial x_1, \frac{1}{2}(\partial/\partial x_2), \frac{1}{3}(\partial/\partial x_3), ...)$.

The following result can be deduced in the same way as we obtain the principal hierarchy. So the proof is omitted.

THEOREM 5.1. *A function $\tau(x)$ is a solution for the* BKP *hierarchy if and only if it satisfies the differential equations* ($\lambda \in \mathrm{Par}$)

$$\left[\sum_{\varnothing \neq \mu \leqslant \lambda} \tilde{p}_\mu\left(-\frac{2}{j}\right) \tilde{p}_{|\mu|}(2\tilde{x})\, \tilde{p}_{\lambda/\mu}(\tilde{x})\right] \tau \cdot \tau = 0, \tag{27}$$

where $\tilde{p}_{\lambda/\mu}(\tilde{x}) = \det(\tilde{p}_{\lambda_i - \mu_j + i - j}(\tilde{x}))$.

Note that some of these equations of form (27) of lower degrees can be found in [JM]. Define for any partition λ

$$\tilde{Q}_\lambda = \sum_{\varnothing \neq \mu \leqslant \lambda} \tilde{p}_\mu\left(-\frac{2}{j}\right) \tilde{p}_{|\mu|}(2\tilde{x})\, \tilde{p}_{\lambda/\mu}(\tilde{x}).$$

Let $\mathrm{BKP}(n)$ = the space spanned by $\{\tilde{Q}_\lambda; \lambda \vdash n\}$. Then the dimension of $\mathrm{BKP}(n)$ is given by

$$\dim \mathrm{BKP}(n) = p^{(o)}(n) - p^{(e)}(n),$$

where $p^{(o)}(n)$ (resp. $p^{(e)}(n)$) is the number of partitions into odd (resp. even) parts.

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