OPTIMAL ASSIGNMENT OF DUE DATES AND STARTING TIMES TO IDENTICAL JOBS ON A SINGLE MACHINE WITH RANDOM PROCESSING TIME

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Abstract

We consider the problem of assigning optimal due dates and optimal starting times to a set of identical jobs on a single machine when processing time on the machine is random. There are N identical jobs ready to be scheduled in the facility. Processing time at the machine is random with known distribution and raw material is available at no additional cost. There is an earliness cost for holding a finished job an extra unit of time and a tardiness cost for being short an extra unit of time past the due date. There is also a cost for quoting an uncompetitive due date for each job in the set, this cost being zero if the quoted due date does not exceed a certain "acceptable" value A. The objective is to minimize the expected total cost of quoting the due dates and scheduling the jobs in the set. The optimal due dates

and the optimal starting times are determined analytically. They are the unique solutions to a set of first order conditions. We show that there exists an optimal solution where the due date of each job is at least equal to A, with the exception of the first job to be processed. The optimal starting time for a particular job in the set is described by a simple *wait-until* policy. This optimal policy is completely determined by a single critical number, which represents the optimal planned lead time for that job. We show that the optimal planned lead times are non-increasing with the position of the job in the sequence, with the exception of the planned lead time of the first job to be processed being the smallest. Finally we show that adding another job results in quoting earlier (or the same) due dates to the jobs in the preexisting set.

1 The Problem

1.1 Introduction

A set of N identical jobs are ready to be scheduled for processing on a machine. The optimal due dates for these identical jobs need to be quoted before any processing occurs on the machine. There are no other jobs in the facility, raw material is available at no additional cost and the machine cannot process more than one job at a time. The jobs consist of projects that must be completed once started in order to be delivered to different customers, hence preemption is not allowed. Job N is the job with the earliest due date, hence the job to be started first since all jobs are identical. The processing time τ at the machine is random with known distribution F. Once the due dates d_i^*, $i = 1, .., N$ of the jobs have been quoted, it is required to determine the optimal starting policy $y_i^* (l_i, d_{i-1}, ..., d_1)$, i.e. the optimal waiting time before starting the processing of job i, $i = 1, ..., N$, given that d_i is l_i units of time away and given the previously quoted due dates $d_{i-1}, ..., d_1$. Obviously, $y_N^* (l_N, d_{N-1}, ..., d_1) = 0$. This observation stems from the fact that having assumed job N is ready to be processed, we would like to quote its due date as early as possible, hence $l_N \equiv d_N$. The objective is to minimize the cost of quoting the due dates and scheduling jobs N through 1. A holding cost h per unit time is incurred if a job is completed before its quoted due date and a shortage cost p per unit time is incurred otherwise. The cost of quoting an uncompetitive due date is $C(.)$, assumed to be a strictly increasing function of the due date, convex, continuous, twice differentiable, and zero for a due date no greater the acceptable limit A (Jones [10]). A is a value determined by the market and by the customer conception of how long is she willing to wait before her order is delivered.

1.2 Background

Considerable research has been done on assigning optimal due-dates for the single machine scheduling problem with earliness/tardiness penalties. In their surveys, Baker [1] and Cheng [3] report of no analytical work done with the machine having random processing time. Further work with deterministic processing time have been done by De, assuming a given common due date in [7] and assigning distinct due dates in [8], and by Cheng [4] assigning the same time window (flow allowance) to all jobs. Random machine processing time has been considered in conjunction with random due dates as in De [6] and Emmons [9] with the objective of minimizing the weighted number of tardy jobs. We are not aware of any past research that considers random processing time and assigns distinct optimal due dates with earliness and tardiness penalties. Cheng [5] describes a model that assigns optimal due dates in the presence of tardiness/earliness penalties, and in which the due dates are random. In his model, $d_i = p_i + k_i$ for each job i, where d_i is the due date, p_i is the random processing time and k_i is a job waiting allowance, a decision variable. Cheng assumes in his model that the distribution of w_i, the time elapsed until the start of job i processing time, is given. This model suffers from two serious deficiencies which prevents it from addressing and analyzing the problem that the author has set to.

The first deficiency is that one cannot quote a random due date. The second deficiency is that if job i is processed before job j, then the distribution of w_j depends on p_i and k_i, hence a) the search for k_j^* must be carried using sequential decision making i.e. using the information given by the realization of p_i and b) $f_j(w_j)$ is not data but rather a function of k_i.

This paper is organized as follows. In section 2 we analyze the case when $N = 2$, i.e. determine the optimal due dates d_2^* and d_1^* and the optimal starting policy $y_1^*(l_1)$ for job 1. We also analyze the case when $N = 3$ to illustrate the derivation of the optimal starting policy when there is more than one remaining job to be processed, a situation that is not present when $N = 2$. Hence for $N = 3$ we determine $y_2^*(l_2, d_1)$, $y_1^*(l_1)$, d_3^*, d_2^* and d_1^*. We show in this section that for $N = 2$ there exists an optimal solution where the due date of the second job to be processed is at least equal to A and that for $N = 3$, there exists an optimal solution where the due date of the second and the third job to be processed is at least equal to A. We also show for $N = 3$ that, a) the optimal planned lead time (that completely determines the optimal starting policy) of the second job to be processed is at least equal to the optimal planned lead time of the third job and that b) adding the third job results in quoting an earlier (or the same) due date for the second job than the one quoted in the case when $N = 2$, i.e. when the second job was the last job to be processed. We discuss in section 3 the economic interpretation of the first order conditions that give rise to the optimal due dates and to the optimal starting policy in sections 2. We also discuss in section 3 the managerial insights provided by the practical results obtained in section 2. We generalize in section 4 for $N > 3$. Section 4 may be skipped if the reader is not interested in the mathematics. We conclude in section 5 by suggesting some further directions in research.

2 Dynamic Programming Formulation

2.1 Two-Jobs Model

Suppose that $N = 2$. We will use backward stochastic dynamic programming to determine d_2^*, d_1^* and $y_1^*(l_1)$. The first stage is triggered when job 2 is done processing. Figure 1 depicts the time advances in a two-job model. The first stage problem is defined as following:

$$J_1^*(l_1) = \text{Min}_{y_1 \geq 0} h \int_0^{l_1 - y_1} [(l_1 - y_1) - t] f_1(t)\, dt + p \int_{l_1 - y_1}^{\infty} [t - (l_1 - y_1)] f_1(t)\, dt \qquad (1)$$

where the first term is the first term is the expected holding cost and the second term is the expected shortage cost. It can be easily checked that $J_1(l_1)$ is convex in y_1 by differentiating it twice. Therefore, the optimal solution $y_1^*(l_1)$ to the first stage problem is obtained by differentiating equation (1) with respect to y_1 and setting to zero. Doing this we get the following *wait-until* policy, where we wait $l_1 - X_1^*$ units of time before processing the job if $l_1 - X_1^* \geq 0$, and process immediately otherwise.

$$y_1^*(l_1) = \begin{cases} l_1 - X_1^* & \text{if } l_1 \geq X_1^* \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

where $X_1^* = F_1^{-1}\left[\frac{p}{p+h}\right]$ is called the optimal planned lead time for job 1. Figure 1 shows that $l_1 = l_2 + r_1 - \tau_2$. Hence the second stage problem is defined as following:

$$\begin{aligned} \text{Min } J_2(d_2, r_1) \;=\; & C(d_2) + C(d_2 + r_1) + h \int_0^{d_2} (d_2 - u) f_2(u)\, du + \\ & p \int_{d_2}^{\infty} (u - d_2) f_2(u)\, du + E[J_1^*(l_2 + r_1 - \tau_2)] \qquad (3) \\ & \text{s.t. } d_2, r_1 \geq 0 \end{aligned}$$

Our goal is to show that the Hessian of $J_2(d_2, r_1)$ is non-negative. The Hessian of the first four terms is non-negative by assumption and from the first stage analysis. Suppose that $J_1^*(l_1)$ is convex in l_1, then we are done. Our goal is to show that $J_1^*(l_1)$ is convex in l_1. Substituting (2) in (1), we get

$$J_1^*(l_1) = \begin{cases} h\int_0^{X_1^*}(X_1^* - t) f_1(t)\,dt + p\int_{X_1^*}^{\infty}(t - X_1^*) f_1(t)\,dt & l_1 \geq X_1^* \\ h\int_0^{l_1}(l_1 - t) f_1(t)\,dt + p\int_{l_1}^{\infty}(t - l_1) f_1(t)\,dt & X_1^* \geq l_1 \end{cases} \tag{4}$$

It is easy to see that (4) is continuous and differentiable at $l_1 = X_1^*$. Finally, differentiating $J_1^*(l_1)$ twice shows that it is convex in l_1 and hence the Hessian of $J_2(d_2, r_1)$ is non-negative. To determine d_2^* and r_1^*, we substitute l_1 by $(d_2 + r_1 - \tau_2)$ in (4), apply the expectation operator, differentiate (3) with respect to d_2 and r_1 and set to zero. Doing this we get

$$\begin{aligned} E[J_1^*(d_2 + r_1 - \tau_2)] &= h\int_{d_2+r_1-X_1^*}^{d_2+r_1}\int_0^{d_2+r_1-u}(d_1 - u - t) f_1(t) f_2(u)\,dtdu + \\ & p\int_{d_2+r_1-X_1^*}^{d_2+r_1}\int_{d_2+r_1-u}^{\infty}(t + u - d_2 - r_1) f_1(t) f_2(u)\,dtdu + \\ & p\int_{d_2+r_1}^{\infty}(\mu_1 + u - d_2 - r_1) f_2(u)\,du + \\ & \left[h\int_0^{X_1^*}(X_1^* - t) f_1(t)\,dt + \right. \\ & \left. p\int_{X_1^*}^{\infty}(t - X_1^*) f_1(t)\,dt\right]\int_0^{d_2+r_1-X_1^*} f_2(u)\,du \end{aligned} \tag{5}$$

and hence

$$\begin{aligned} \frac{\delta J_2(d_2, r_1)}{\delta r_1} &= C'(d_2 + r_1) + \left[h\int_0^{X_1^*}(X_1^* - t) f_1(t)\,dt + \right. \\ & \left. p\int_{X_1^*}^{\infty}(t - X_1^*) f_1(t)\,dt\right] f_2(d_2 + r_1 - X_1^*) + \\ & h\int_{d_2+r_1-X_1^*}^{d_2+r_1}\int_0^{d_2+r_1-u} f_1(t) f_2(u)\,dtdu - \\ & h f_2(d_2 + r_1 - X_1^*)\int_0^{X_1^*}(X_1^* - t) f_1(t)\,dt - \\ & p\int_{d_2+r_1-X_1^*}^{d_2+r_1}\int_{d_2+r_1-u}^{\infty} f_1(t) f_2(u)\,dtdu + p f_2(d_2 + r_1)\int_0^{\infty} t f_1(t)\,dt - \\ & p f_2(d_2 + r_1 - X_1^*)\int_{X_1^*}^{\infty}(t - X_1^*) f_1(t)\,dt - \end{aligned}$$

$$p\int_{d_2+r_1}^{\infty} f_2(u)\,du - p\mu_1 f_2(d_2+r_1) = 0 \tag{6}$$

$$\frac{\delta J_2(d_2,r_1)}{\delta d_2} = C'(d_2) + h\int_0^{d_2} f_2(u)\,du - p\int_{d_2}^{\infty} f_2(u)\,du = 0 \tag{7}$$

which reduce to

$$\begin{aligned}\frac{\delta J_2(d_2,r_1)}{\delta r_1} &= C'(d_2+r_1) + h\int_{d_2+r_1-X_1^*}^{d_2+r_1}\int_0^{d_2+r_1-u} f_1(t)\,f_2(u)\,dtdu - \\ &\quad p\int_{d_2+r_1-X_1^*}^{d_2+r_1}\int_{d_2+r_1-u}^{\infty} f_1(t)\,f_2(u)\,dtdu - p\int_{d_2+r_1}^{\infty} f_2(u)\,du = 0 \qquad (8)\\ \frac{\delta J_2(d_2,r_1)}{\delta d_2} &= C'(d_2) + h\int_0^{d_2} f_2(u)\,du - p\int_{d_2}^{\infty} f_2(u)\,du + \frac{\delta J_2(d_2,r_1)}{\delta r_1} = 0 \qquad (9)\end{aligned}$$

$d_2^* \geq 0$ can be determined easily from (9). d_2^* satisfies

$$d_2^* = F^{-1}\left[\frac{p - C'(d_2^*)}{p+h}\right] \leq X_1^* \tag{10}$$

It can be seen that $r_1^* \geq 0$ since substituting r_1 by 0 in (8) gives

$$\begin{aligned}\frac{\delta J_2(d_2,r_1)}{\delta r_1}|_{r_1=0} &= C'(d_2^*) + h\int_0^{d_2^*}\int_0^{d_2^*-u} f_1(t)\,f_2(u)\,dtdu - \\ &\quad p\int_0^{d_2^*}\int_{d_2^*-u}^{\infty} f_1(t)\,f_2(u)\,dtdu - p\int_{d_2^*}^{\infty} f_2(u)\,du \\ &= C'(d_2^*) + (h+p)\int_0^{d_2^*}\int_0^{d_2^*-u} f_1(t)\,f_2(u)\,dtdu - p \\ &\leq C'(d_2^*) + (h+p)\int_0^{d_2^*} f_2(u)\,du - p = \frac{\delta J_2(d_2,r_1)}{\delta d_2}|_{d_2=d_2^*} = 0\end{aligned}$$

Denote the integral terms in (8) by $\Psi_2^1(d_2^*+r_1)$. $\Psi_2^1(d_2^*+r_1)$ is non-decreasing in r_1 and vanishes at $r_1 \geq \overline{\tau} + X_1^* - d_2^*$ where $\overline{\tau}$ is the largest realization of the machine processing time. This can be shown by differentiating it with respect to r_1. Doing this we get

$$\begin{aligned}\Psi_2^{'1}(d_2^*+r_1) &= h\int_{d_2^*+r_1-X_1^*}^{d_2^*+r_1} f_1(d_2^*+r_1-u)\,f_2(u)\,du - h\int_0^{X_1^*} f_2(d_2^*+r_1-X_1^*)\,f_1(t)\,dt \\ &\quad +p\int_{d_2^*+r_1-X_1^*}^{d_2^*+r_1} f_1(d_2^*+r_1-u)\,f_2(u)\,du - p\int_0^{\infty} f_2(d_2^*+r_1-X_1^*)\,f_1(t)\,dt\end{aligned}$$

$$+p\int_{X_1^*}^{\infty} f_2\left(d_2^*+r_1-X_1^*\right) f_1(t)\, dt + pf_2\left(d_2^*+r_1-X_1^*\right)$$

$$= (h+p)\int_{d_2^*+r_1-X_1^*}^{d_2^*+r_1} f_1\left(d_2^*+r_1-u\right) f_2(u)\, du \geq 0$$

Hence $\Psi_2^1\left(d_2^*+r_1\right) \leq 0\ \forall r_1 \geq 0$. Note also that $C'\left(d_2^*+r_1\right)=0$ for $r_1 \leq A-d_2^*$. Therefore letting $d_1 = d_2 + r_1$ we get

$$d_1^* = \begin{cases} \{x, x \in [\overline{\tau}+X_1^*, A]\} & \text{if } \overline{\tau}+X_1^* \leq A \\ \geq A & \text{otherwise} \end{cases} \tag{11}$$

2.2 Three-Jobs Model

Before extending the problem to N jobs, it is necessary to analyze the case when there are three jobs in order to illustrate the computation of $y_2^*(l_2, d_1) \equiv y_2^*(l_2, r_1)$ where $r_1 = d_1 - d_2$. In a three jobs problem, job 2 is not started immediately as in a two jobs problem. Suppose its due date has already been set and cannot be changed. Hence its starting time must depend on the remaining time till its due date and the due date of job 1 at the time job 3 is done processing, since this is when stage 2 is triggered. As a result, it also depends on the optimal planned lead time of job 1. Figure 2 depicts the time advances in a three jobs problem. In a three jobs problem, the decision variables are d_3, r_2 and r_1 at stage 3 (where $r_2 = d_2 - d_3$), y_2 at stage 2 and y_1 at stage 1. y_1^* is given by the optimal starting policy defined in (2). To determine y_2^*, we solve

$$\begin{aligned} J_2^*(l_2, r_1) &= \text{Min}_{y_2 \geq 0}\ h\int_0^{l_2-y_2} \left[(l_2-y_2)-t\right] f_2(t)\, dt + \\ &\quad p\int_{l_2-y_2}^{\infty} \left[t-(l_2-y_2)\right] f_2(t)\, dt + E\left[J_1^*(l_1)\right] \end{aligned} \tag{12}$$

To solve (12), we substitute l_1 by $(l_2 - y_2 + r_1 - \tau_2)$ in (4), substitute $l_2 - y_2$ by X_2 in (12), differentiate (12) with respect to X_2 and set it to zero. Note that convexity in X_2 is conserved since the Hessian of $J_2(d_2, r_1)$ is non-negative in a two jobs problem and

$C(.)$ is convex (equation (3)). Doing this, we obtain a first order condition similar to equation (7), but with X_2 instead of d_2 and without the due date cost terms.

$$\frac{dJ_2(l_2,r_1)}{dX_2} = h\int_0^{X_2} f_2(u)\,du - p\int_{X_2}^{\infty} f_2(u)\,du + \Psi_2^1(X_2+r_1) \tag{13}$$

and hence y_2^* is given by the following *wait-until* starting policy:

$$y_2^*(l_2,r_1) = \begin{cases} l_2 - X_2^* & \text{if } l_2 \geq X_2^* \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where X_2^*, the optimal planned lead time of job 2, satisfies (13). Note that $X_2^* \geq X_1^*$ since substituting X_2 by X_1^* in (13) gives

$$\frac{dJ_2(l_2,r_1)}{dX_2}|_{X_2=X_1^*} = h\int_0^{X_1^*} f_2(u)\,du - p\int_{X_1^*}^{\infty} f_2(u)\,du + \Psi_2^1(X_1^*+r_1)$$

But we have shown previously that $\Psi_2^1(d_2^*+r_1) \leq 0 \; \forall r_1 \geq 0$. Using the same arguments, we also have that $\Psi_2^1(X_1^*+r_1) \leq 0$. As a result $X_2^* \geq X_1^*$. To determine d_3^*, r_2^* and r_1^*, we solve

$$\begin{aligned} \text{Min } J_3(d_3,r_2,r_1) \;=\;& C(d_3) + C(d_3+r_2) + C(d_3+r_2+r_1) + \\ & h\int_0^{d_3}(d_3-v)f_3(v)\,dv + p\int_{d_3}^{\infty}(v-d_3)f_3(v)\,dv + \\ & E\left[J_2^*(d_3+r_2-\tau_3,r_1)\right] \\ & \text{s.t. } d_3,r_2,r_1 \geq 0 \end{aligned} \tag{15}$$

Our goal is to show that the Hessian of $J_3(d_3,r_2,r_1)$ is non-negative. The Hessian of the first five terms is non-negative by assumption and from the first stage analysis. Suppose that the Hessian of $J_2^*(l_2,r_1)$ is non-negative, then we are done. Our goal is to show that the Hessian of $J_2^*(l_2, r_1)$ is non-negative. Substituting (14) in (12), we get

$$J_2^*(l_2, r_1) = \begin{cases} h\int_0^{X_2^*}(X_2^* - u) f_2(u)\, du + p\int_{X_2^*}^{\infty}(u - X_2^*) f_2(u)\, du + \\ E\left[J_1^*(X_2^* + r_1 - \tau_2)\right] & l_2 \geq X_2^* \\ h\int_0^{l_2}(l_2 - u) f_2(u)\, du + p\int_{l_2}^{\infty}(u - l_2) f_2(u)\, du + \\ E\left[J_1^*(l_2 + r_1 - \tau_2)\right] & X_2^* \geq l_2 \end{cases} \tag{16}$$

Using (13), it is easy to see that (16) is continuous and differentiable at $l_2 = X_2^*$. Since $J_1^*(l_1)$ is convex, then the Hessian of $J_2^*(l_2, r_1)$ is non-negative and therefore the Hessian of $J_3(d_3, r_2, r_1)$ is non-negative. To determine d_3^*, r_2^* and r_1^*, we substitute l_2 by $(d_3 + r_2 - \tau_3)$, apply the expectation operator, differentiate (15) with respect to d_3, r_2 and r_1 and set to zero. Doing this we get

$$\begin{aligned} E\left[J_2^*(d_3 + r_2 - \tau_3, r_1)\right] = \; & h\int_0^{d_3+r_2-X_2}\int_0^{X_2}(X_2 - u) f_2(u) f_3(v)\, du dv + \\ & p\int_0^{d_3+r_2-X_2}\int_{X_2}^{\infty}(u - X_2) f_2(u) f_3(v)\, du f dv + \\ & h\int_{d_3+r_2-X_2}^{d_2}\int_0^{d_3+r_2-v}(d_3 + r_2 - v - u) f_2(u) f_3(v)\, du dv + \\ & p\int_{d_3+r_2-X_2}^{d_3+r_2}\int_{d_3+r_2-v}^{\infty}(u + v - d_3 - r_2) f_2(u) f_3(v)\, du dv + \\ & p\int_{d_3+r_2}^{\infty}(\mu_2 + v - d_3 - r_2) f_3(v)\, dv + \\ & E\left[J_1^*(X_2 + r_1 - \tau_2)\right]\int_0^{d_3+r_2-X_2} f_3(v)\, dv + \\ & \int_{d_3+r_2-X_2}^{\infty} E\left[J_1^*(d_3 + r_2 + r_1 - v - \tau_2)\right] f_3(v)\, dv \end{aligned} \tag{17}$$

and hence

$$\begin{aligned} \frac{\delta J_3(d_3, r_2, r_1)}{\delta r_1} = \; & C'(d_3 + r_2 + r_1) + E'\left[J_1^*(X_2 + r_1 - \tau_2)\right]\int_0^{d_3+r_2-X_2} f_3(v)\, dv + \\ & \int_{d_3+r_2-X_2}^{\infty} E'\left[J_1^*(d_3 + r_2 + r_1 - \tau_2 - v)\right] f_3(v)\, dv \end{aligned}$$

But $E'\left[J_1^*(d_2^* + r_1 - \tau_2)\right] = \Psi_2^1(d_2^* + r_1)$ as defined in a two jobs problem. Substituting we get

$$\begin{aligned}\frac{\delta J_3(d_3,r_2,r_1)}{\delta d_1} &= C'(d_3+r_2+r_1)+\Psi_2^1(X_2+r_1)\int_0^{d_3+r_2-X_2} f_3(v)\,dv + \qquad (18)\\ & h\int_{d_3+r_2-X_2}^{d_3+r_2+r_1-X_1^*}\int_{d_3+r_2+r_1-X_1^*-v}^{d_3+r_2+r_1-v}\int_0^{d_3+r_2+r_1-u-v} f_1(t)\,f_2(u)\,f3(v)\,dtdudv + \\ & h\int_{d_3+r_2+r_1-X_1^*}^{d_3+r_2+r_1}\int_0^{d_3+r_2+r_1-v}\int_0^{d_3+r_2+r_1-u-v} f_1(t)\,f_2(u)\,f3(v)\,dtdudv - \\ & p\int_{d_3+r_2-X_2}^{d_3+r_2+r_1-X_1^*}\int_{d_3+r_2+r_1-X_1^*-v}^{d_3+r_2+r_1-v}\int_{d_3+r_2+r_1-u-v}^{\infty} f_1(t)\,f_2(u)\,f3(v)\,dtdudv - \\ & p\int_{d_3+r_2-X_2}^{d_3+r_2+r_1-X_1^*}\int_{d_3+r_2+r_1-v}^{\infty} f_2(u)\,f3(v)\,dudv - \\ & p\int_{d_3+r_2+r_1-X_1^*}^{d_3+r_2+r_1}\int_0^{d_3+r_2+r_1-v}\int_{d_3+r_2+r_1-u-v}^{\infty} f_1(t)\,f_2(u)\,f3(v)\,dtdudv - \\ & p\int_{d_3+r_2+r_1-X_1^*}^{d_3+r_2+r_1}\int_{d_3+r_2+r_1-v}^{\infty} f_2(u)\,f3(v)\,dudv - \int_{d_3+r_2+r_1}^{\infty} f_3(v)\,dv = 0\end{aligned}$$

Also

$$\begin{aligned}\frac{\delta J_3(d_3,r_2,r_1)}{\delta r_2} &= C'(d_3+r_2)+C'(d_3+r_2+r_1)+\\ & h\int_{d_3+r_2-X_2}^{d_3+r_2}\int_0^{d_3+r_2-v} f_2(u)\,f3(v)\,dudv - \\ & p\int_{d_3+r_2-X_2}^{d_3+r_2}\int_{d_3+r_2-v}^{\infty} f_2(u)\,f3(v)\,dudv - p\int_{d_3+r_2}^{\infty} f_3(v)\,dv + \\ & \int_{d_3+r_2-X_2}^{\infty} E'\left[J_1^*(d_3+r_2+r_1-\tau_2-v)\right] f_3(v)\,dv = 0 \qquad (19)\end{aligned}$$

and finally

$$\frac{\delta J_3(d_3,r_2,r_1)}{\delta d_3} = C'(d_3)+h\int_0^{d_3} f_3(v)\,dv - p\int_{d_3}^{\infty} f_3(v)\,dv + \frac{\delta J_3(d_3,r_2,r_1)}{\delta r_2} = 0 \qquad (20)$$

d_3^* is obtained from (20). r_2^*, r_1^* and X_2^* are determined simultaneously using (13) and the following set of first-order conditions:

$$\begin{aligned}\frac{\delta J_3(d_3,r_2,r_1)}{\delta r_1} &= C'(d_3^*+r_2+r_1)+\Psi_2^1(X_2+r_1)\int_0^{d_3^*+r_2-X_2} f_3(v)\,dv + \\ & \Psi_3^1(d_3^*+r_2,d_3^*+r_2+r_1) = 0 \qquad (21)\\ \frac{\delta J_3(d_3,r_2,r_1)}{\delta r_2} &= C'(d_3^*+r_2)+C'(d_3^*+r_2+r_1)+\\ & \Psi_3^2(d_3^*+r_2)+\Psi_3^1(d_3^*+r_2,d_3^*+r_2+r_1) = 0 \qquad (22)\end{aligned}$$

where $\Psi_3^1(d_3^* + r_2, d_3^* + r_2 + r_1)$ and $\Psi_3^2(d_3^* + r_2)$ are as defined in (18) and (19). Clearly d_3^* is equal to the due date of the first job to be processed in a two jobs problem, hence $d_3^* \leq X_1^* \leq X_2^*$. We want to show that $r_2^* \geq 0$. To show this, equation (19) can be rewritten as

$$\begin{aligned}
\frac{\delta J_3(d_3, r_2, r_1)}{\delta r_2} &= C'(d_3^* + r_2) + C'(d_3^* + r_2 + r_1) + \\
&\quad h \int_{d_3^*+r_2-X_2}^{d_3+r_2} \int_0^{d_3^*+r_2-v} f_2(u) f3(v)\, du dv - \\
&\quad p \int_{d_3^*+r_2-X_2}^{d_3^*+r_2} \int_{d_3^*+r_2-v}^{\infty} f_2(u) f3(v)\, du dv - p \int_{d_3^*+r_2}^{\infty} f_3(v)\, dv + \\
&\quad \frac{\delta J_3(d_3, r_2, r_1)}{\delta r_1} - C'(d_3^* + r_2 + r_1) - \\
&\quad \Psi_2^1(X_2 + r_1) \int_0^{d_3^*+r_2-X_2} f_3(v)\, dv = 0 \qquad (23)
\end{aligned}$$

Substituting r_2 by 0 in (23) and using the fact that $d_3^* \leq X_2^*$ we get

$$\begin{aligned}
\frac{\delta J_3(d_3, r_2, r_1)}{\delta r_2}\Big|_{r_2=0} &= C'(d_3^*) + h \int_0^{d_3^*} \int_0^{d_3^*-v} f_2(u) f_3(v)\, du dv - \\
&\quad p \int_0^{d_3^*} \int_{d_3^*-v}^{\infty} f_2(u) f_3(v)\, du dv - p \int_{d_3^*}^{\infty} f_3(v)\, dv \\
&= C'(d_3^*) + (h+p) \int_0^{d_3^*} \int_0^{d_3^*-v} f_2(u) f_3(v)\, du dv - p \\
&\leq C'(d_3^*) + (h+p) \int_0^{d_3^*} f_3(v)\, dv - p = \frac{\delta J_3(d_3, r_2, r_1)}{\delta d_3}\Big|_{d_3=d_3^*} = 0
\end{aligned}$$

We want to show that $r_1^* \geq 0$. Substituting r_1 by 0 in (21) and $C'(d_3^* + r_2^*)$ from (22) we get

$$\begin{aligned}
\frac{\delta J_3(d_3, r_2, r_1)}{\delta r_1}\Big|_{r_1=0} &= C'(d_3^* + r_2^*) + \Psi_2^1(X_2) \int_0^{d_3^*+r_2^*-X_2} f_3(v)\, dv + \Psi_3^1(d_3^* + r_2^*, d_3^* + r_2^*) \\
&= 2\Psi_2^1(X_2) \int_0^{d_3^*+r_2^*-X_2} f_3(v)\, dv + \Psi_3^1(d_3^* + r_2^*, d_3^* + r_2^*) - \Psi_3^2(d_3^* + r_2^*)
\end{aligned}$$

But we have shown previously that $\Psi_2^1(d_2^* + r_1) \leq 0, \forall r_1 \geq 0$. Using the same arguments we also have $\Psi_2^1(X_2) \leq 0$. Therefore it is sufficient to show that $\Psi_3^1(d_3^* + r_2^*, d_3^* + r_2^*) \leq \Psi_3^2(d_3^* + r_2^*)$ and we are done. In fact comparing the holding cost coefficients in both expressions we get

$$\int_{d_3^*+r_2^*-X_2}^{d_3^*+r_2^*-X_1^*} \int_{d_3^*+r_2^*-X_1^*-v}^{d_3^*+r_2^*-v} \int_0^{d_3^*+r_2^*-u-v} f_1(t) f_2(u) f3(v)\, dtdudv +$$
$$\int_{d_3^*+r_2^*-X_1^*}^{d_3^*+r_2^*} \int_0^{d_3^*+r_2^*-v} \int_0^{d_3^*+r_2^*-u-v} f_1(t) f_2(u) f3(v)\, dtdudv \leq$$
$$\int_{d_3^*+r_2^*-X_2}^{d_3^*+r_2^*-X_1^*} \int_0^{d_3^*+r_2^*-v} \int_0^{d_3^*+r_2^*-u-v} f_1(t) f_2(u) f3(v)\, dtdudv +$$
$$\int_{d_3^*+r_2^*-X_1^*}^{d_3^*+r_2^*} \int_0^{d_3^*+r_2^*-v} \int_0^{d_3^*+r_2^*-u-v} f_1(t) f_2(u) f3(v)\, dtdudv =$$
$$\int_{d_3^*+r_2^*-X_2^*}^{d_3^*+r_2^*} \int_0^{d_3^*+r_2^*-v} \int_0^{d_3^*+\hat{r}_2^*-u-v} f_1(t) f_2(u) f3(v)\, dtdudv \leq$$
$$\int_{d_3^*+r_2^*-X_2}^{d_3^*+r_2^*} \int_0^{d_3^*+r_2^*-v} f_2(u) f3(v)\, dudv$$

Comparing the shortage cost coefficients, the terms with single and double integrals cancel and only the terms with triple integrals in $\Psi_3^1(d_3^* + r_2^*, d_3^* + r_2^*)$ remain. Therefore $\Psi_3^1(d_3^* + r_2^*, d_3^* + r_2^*) \leq \Psi_3^2(d_3^* + r_2^*)$ and $r_1^* \geq 0$. Equations (21) and (22) can be rewritten as

$$C'(d_3^* + r_2 + r_1) = -\Psi_2^1(X_2 + r_1) \int_0^{d_3^*+r_2-X_2} f_3(v)\, dv - \Psi_3^1(d_3^* + r_2, d_3^* + r_2 + r_1) \quad (24)$$

and

$$C'(d_3^* + r_2) + C'(d_3^* + r_2 + r_1) = -\Psi_3^2(d_3^* + r_2) - \Psi_3^1(d_3^* + r_2, d_3^* + r_2 + r_1) \quad (25)$$

Denote by R_3^1 and R_3^2 the right-hand side of (24) and (25) respectively. Differentiating R_3^1 with respect to r_1 and using the fact that $X_1^* = F^{-1}[p/(p+h)]$ we get

$$\frac{\delta R_3^1}{\delta r_1} = -(h+p)\left[\int_{d_3^*+r_2-X_2}^{d_3^*+r_2+r_1-X_1^*} \int_{d_3^*+r_2+r_1-X_1^*-v}^{d_3^*+r_2+r_1-v} f_1(d_3^* + r_2 + r_1 - u - v) f_2(u) f3(v)\, dudv + \right.$$
$$\left. \int_{d_3^*+r_2+r_1-X_1^*}^{d_3^*+r_2+r_1} \int_0^{d_3^*+r_2+r_1-v} f_1(d_3^* + r_2 + r_1 - u - v) f_2(u) f3(v)\, dudv \right]$$
$$-\Psi_2'^1(X_2 + r_1) \int_0^{d_3^*+r_2-X_2} f_3(v)\, dv \leq 0$$

Differentiating R_3^1 with respect to r_2 we get

$$\frac{\delta R_3^1}{\delta r_1} = \frac{\delta R_3^1}{\delta r_2} + \Psi_2^1 (X_2 + r_1) f_3 (d_3^* + r_2 - X_2) - \Psi_2^1 (X_2 + r_1) f_3 (d_3^* + r_2 - X_2) \leq 0$$

Differentiating R_3^2 with respect to r_2 and using the facts that $X_1^* = F^{-1}[p/(p+h)]$ and X_2^* satisfies (13) we get

$$\begin{aligned}\frac{\delta R_3^2}{\delta r_2} &= -(h+p) \int_{d_3^*+r_2-X_2}^{d_3^*+r_2} f_2 (d_3^* + r_2 - v) f_3 (v)\, dv \\ &- (h+p) \left[\int_{d_3^*+r_2-X_2}^{d_3^*+r_2+r_1-X_1^*} \int_{d_3^*+r_2+r_1-X_1^*-v}^{d_3^*+r_2+r_1-v} f_1 (d_3^* + r_2 + r_1 - u - v) f_2 (u) f3 (v)\, dudv + \right. \\ &\left. \int_{d_3^*+r_2+r_1-X_1^*}^{d_3^*+r_2+r_1} \int_0^{d_3^*+r_2+r_1-v} f_1 (d_3^* + r_2 + r_1 - u - v) f_2 (u) f3 (v)\, dudv \right] \leq 0\end{aligned}$$

Note that R_3^1 and R_3^2 vanish at $d_3^*+r_2+r_1 \geq 2\overline{\tau}+X_1^*$, $d_3^*+r_2 \geq \overline{\tau}+X_2^*$ and $r_1 \geq \overline{\tau}+X_2^*-X_1^*$. Note also that $X_2^* \leq \overline{\tau}+X_1^*$ since substituting X_2 in (13) by $\overline{\tau}+X_1^*$ gives $h \geq 0$. Consider figure 3 and the fact that equation (25) can be rewritten using equation (24) as

$$C' (d_3^* + r_2^*) = -\Psi_3^2 (d_3^* + r_2^*) + \Psi_2^1 (X_2 + r_1) \int_0^{d_3^*+r_2^*-X_2} f_3 (v)\, dv \qquad (26)$$

where $\Psi_2^1 (X_2 + r_1) \leq 0$ and

$$\Psi_3^{\prime 2} (d_3^* + r_2) = (h+p) \int_{d_3^*+r_2-X_2}^{d_3^*+r_2} f_2 (d_3^* + r_2 - v) f_3 (v)\, dv \geq 0$$

From figure 3, if $A \geq 2\overline{\tau} + X_1^*$ then there exists $r_2^* = A - d_3^*$ and $r_1^* = A - d_3^* - r_2^*$. If $\overline{\tau} + X_2^* \leq A \leq 2\overline{\tau} + X_1^*$ then $r_1^* \geq A - d_3^* - r_2^*$ and there exists $r_2^* = A - d_3^*$. Otherwise, $r_2^* \geq A - d_3^*$ and $r_1^* \geq A - d_3^* - r_2^*$. Letting $d_1 = d_3 + r_2 + r_1$ and $d_2 = d_3 + r_2$, we have that if $A \geq 2\overline{\tau} + X_1^*$, then there exists $d_2^* = A$ and $d_1^* = A$. If $\overline{\tau} + X_2^* \leq A \leq 2\overline{\tau} + X_1^*$ then $d_1^* \geq A$ and there exists $d_2^* = A$. Otherwise, $d_1^* \geq A$ and $d_2^* = A$. Before leaving the three jobs problem we want to compare the optimal due date of the second job to be processed in a two jobs problem to the optimal due date of the second job to be processed in a three jobs problem to study the effect of adding a third job on the optimal due date of the second job to be processed. In a two jobs problems, r_1^* is given by

$$C'\left(d_2^* + r_1^*\right) = -\Psi_2^1\left(d_2^* + r_1^*\right) \tag{27}$$

where

$$\Psi_2'^1\left(d_2^* + r_1\right) = (h+p)\int_{d_2^*+r_1-X_1^*}^{d_2^*+r_1} f_1\left(d_2^* + r_1 - u\right) f_2(u)\, du \geq 0 \tag{28}$$

as was shown previously. In a three jobs problem, r_2^* satisfies equation (26). As can be seen in figure 4, the right-hand sides of (27) and (26) are equal at $r_1 = 0$ and $r_2 = 0$ respectively since $d_2^* = d_3^* \leq X_1^* \leq X_2^*$. Furthermore, the derivative of the right-hand side of equation (26) is steeper than the derivative of the right-hand side of equation (27), that is

$$-(h+p)\int_{d_3^*+r_2-X_2}^{d_3^*+r_2} f_2\left(d_3^* + r_2 - v\right) f_3(v)\, dv + \Psi_2^1\left(X_2 + r_1\right) f_3\left(d_3^* + r_2^* - X_2\right) \leq$$
$$-(h+p)\int_{d_2^*+r_1-X_1^*}^{d_2^*+r_1} f_1\left(d_2^* + r_1 - u\right) f_2(u)\, du$$

since $d_2^* = d_3^*$ and $X_1^* \leq X_2^*$. As a result, the right-hand side of (26) intersects $C'\left(d_3^* + r_2\right)$ at a smaller value than the one at which the right-hand side of (27) intersects $C'\left(d_2^* + r_1\right)$, i.e. $r_2^* \leq r_1^*$ and hence the optimal due date of the second job to be processed in a two jobs problem is at least equal to the optimal due date of the second job to be processed in a three jobs problem.

3 Economic Interpretation

In this problem, the due dates must be quoted before any processing occurs on the machine. However, due to the randomness in the processing times, once the due dates have been quoted and processing has started, then the starting time of the next job in the sequence must be determined given the set of predetermined due dates of the jobs remaining to be processed. In this section we shall provide an economic interpretation to the first-order conditions that give rise to the optimal due dates (equations (22), (21) and (8)) and the optimal starting times (equation (13)), in the two and three jobs problems analyzed in the previous section.

3.1 Optimal Starting Times

Consider the three jobs problem. Suppose that there remains one job that has not been processed yet and whose due date have been already set. Then its starting time is determined by (2), determined completely by the solution of the classical Newscat problem which balances the tardiness cost p and the earliness cost h to find the optimal starting time X_1^*. The problem is more complicated when there remains two unprocessed jobs whose due dates have already been set. As in the previous case, the starting time for the next job is determined by (14), determined completely by the solution to (13). Equation (13) has a very appealing economic interpretation. It can be rewritten as

$$\begin{aligned} \frac{dJ_2(l_2, r_1)}{dX_2} &= h\left[Pr\{\tau_2 \leq X_2\} + Pr\{\tau_2 \geq X_2 - X_1^* + r_1, \tau_1 + \tau_2 \leq X_2 + r_1\}\right] - \\ & \quad p\left[Pr\{\tau_2 \geq X_2\} + Pr\{\tau_2 \geq X_2 - X_1^* + r_1, \tau_1 + \tau_2 \geq X_2 + r_1\}\right] = 0 \quad (29) \end{aligned}$$

It illustrates the combined impact of the marginal costs associated with each of the two jobs, on the decision to determine the optimal planned lead time X_2^*, i.e. the time window inside which the next job (job 2 in our case) must be processed. Again, the effect of job 2 is the one of the Newsalien problem, indicated by the first probability term inside the marginal holding and shortages cost brackets in the middle side of (29). The effect of the second job (job 1) on the current decision is less myopic in nature. Marginal savings in holding cost due to waiting an extra unit of time before starting job 2 are acheived only if the processing time of job 2 continues past the predetermined starting time of the next job *and* job 1 processing time did not end past its due date. While the second condition is a reminder of the savings achieved in the newsgirl problem, the first condition complicates the non-myopicity of the decision process, in the sense that no marginal savings in holding cost of job 1 due to waiting an extra unit of time before starting job 2 are acheived if some slack time is realized between the completion of job 2 and the start of job 1. Similarly, the marginal increases in shortage cost of job 1 due to waiting an extra unit of time before starting job 2 occur only if the processing time of job 2 continues past the predetermined starting time of the next job *and* job 1 processing time does end past its due date. Equivalently, no marginal increases in shortage cost of job 1 due to waiting an extra unit of time before starting job 2 are incurred if some slack time is realized between the completion of job 2 and the start of job 1. This information agrees with the intuition that job 1 has no impact on the starting time of job 2 if it is certain that some slack time will be realized after the completion of job 2. If $X_2^* \leq X_1^* - r_1$, then it is predetermined *a priori* that no slack is allowed between the two jobs and job 1 is rushed immediately after the completion of job 2. In that case

$$Pr\{y_1 \leq 0\} = Pr\{\tau_2 \geq X_2^* - (X_1^* - r_1)\} = 1$$

For a three jobs problem, we have shown that $X_2^* \geq X_1^* \geq X_1^* - r_1$ hence we never decide *a priori* to rush the next job and X_2^* is indeed determined by (29). This property can be generalized for larger number of jobs. We prove it for any number of jobs in the next section.

3.2 Optimal Due Dates

Equations (22), (21) and (8) also have an appealing economic interpretation. They can be rewritten respectively as

$$C'(d_3^* + r_2) \;\; = \;\; -hPr\{\tau_3 \geq d_3^* - X_2 + r_2, \tau_{32} \leq d_3^* + r_2\} +$$

$$pPr\{\tau_3 \geq d_3^* - X_2 + r_2, \tau_{32} \geq d_3^* + r_2\} \quad (30)$$

$$C'(d_3^* + r_2 + r_1) = -hPr\{\tau_3 \geq d_3^* - X_2 + r_2, \tau_{32} \leq d_3^* + r_2 + r_1 - X_1^*, \tau_{31} \leq d_3^* + r_1 + r_2\} + pPr\{\tau_3 \geq d_3^* - X_2 + r_2, \tau_{32} \geq d_3^* + r_2 + r_1 - X_1^*, \tau_{31} \geq d_3^* + r_1 + r_2\} \quad (31)$$

$$C'(d_2^* + r_1) = -hPr\{\tau_2 \geq d_2^* - X_1^* + r_1, \tau_{21} \leq d_2^* + r_1\} + pPr\{\tau_2 \geq d_2^* - X_1^* + r_1, \tau_{21} \geq d_2^* + r_1\} \quad (32)$$

The marginal costs associated with the first job to be processed are obvious as illustrated in (9) and (??) for a two and three jobs problem respectively. Determining the optimal due date for the next job in the sequence is slightly more complicated. Consider (30). For a three jobs problem, the marginal increases in holding cost associated with job 2 due to quoting a due date one unit of time longer are incurred only if job 3 is completed past the predetermined starting time of job 2 *and* job 3 is completed before its quoted due date. In other words, no marginal costs in holding cost are incurred due to delaying delivery one unit of time if some slack is realized after the completion of job 3. On the other hand, marginal savings in shortage cost associated with job 2 due to quoting a due date one unit of time longer are acheived only if job 3 is completed past the predetermined starting time of job 2 *and* job 3 is completed after its due date. For each job, the combined marginal effects of increases in holding cost and savings in shortage cost is negatively decreasing with increasing values for the quoted due date of that job, as the positively decreasing right-hand side of equations (30) and (31) indicate. In other words, the tardiness argument is stronger than the earliness argument for job 2 and 1. Consider job 2 and equation (30). This is due to the fact that marginal savings and marginal increases occur jointly, only when there is no slack after the completion of job 3. Moreover, savings occur only if the processing time of job 2 exceeds its due date, while increases occur only if it does not. As a result, the rate of the marginal savings is positive and the rate of marginal increases is negative because the higher the due date of job 2, the more likely the processing time of job 2 will exceeds it if no slack is going to be realized after the completion of job 3.

Equation (30) and (31) illustrate the intuitive fact that if there was no cost for quoting an uncompetitive due date, then one would quote due date values at least equal to $\overline{\tau}+X_2^*$ for job 2 and $2\overline{\tau}+X_1^*$ for job 1. However if that cost exists and $\text{Max}\{A, 2\overline{\tau}+X_1^*, \overline{\tau}+X_2^*\} = A$, then $\overline{\tau}+X_2^* \leq d_2^* \leq A$ and $2\overline{\tau}+X_1^* \leq d_1^* \leq A$. These ranges of multiple optimal due date values represents the guaranteed slack that the manager will have after the completion of job 3 and after the completion of job 2 respectively under the optimal starting policy. We assumed the marginal effect of uncompetitive due date cost to be positively increasing with increasing values for the quoted due date of that job. In instances when A is sufficiently small so that the latter does not apply, the quoted due date must be larger than A if the combined marginal effects of the three costs is negative at A, and exactly A otherwise. As a result, if the cost of quoting an uncompetitive the due date is linear to the right of A with slope c, it is more likely to quote A when c is high. Several additional observations can be made from equations (30) and (31). The higher is p and the smaller is h, the slower is the rate of negatively decreasing combined marginal effects of savings in holding cost and costs in shortage cost, hence the more likely that it is higher in abslolute value than $C'(A)$ and the further is the due date. Furthermore, the larger is the processing time variance, the higher is the term containing p and the less likely is that the due date is A. Therefore, the tradeoffs are that high p/low h and high variance increase the quoted due date, force us to produce early and keep a high chance of introducing slack time between the processing of consecutive jobs, while the cost of quoting an uncompetitive due date have the opposite effect and ensures that jobs are rushed without any slack in between. Finally, the analysis presented shows that $r_i^* \geq 0$, $i = 1, ..., N-1$ and hence quoting a common due date for all the jobs is suboptimal in single machine problems with random processing time and earliness/tardiness costs.

4 Extension to N-Jobs

Extending the problem to $N > 3$ jobs, the problem becomes for $1 \leq i \leq NN - 1$:

$$J_i^*(l_i, r_{i-1}, ..., r_1) = \text{Min}_{y_i \geq 0} \; h\int_0^{l_i - y_i} [(l_i - y_i) - t] f_i(t)\, dt + p\int_{l_i - y_i}^{\infty} [t - (l_i - y_i)] f_i(t)\, dt + E\left[J_{i-1}^*(l_i - y_i + r_{i-1} - \tau_i, r_{i-2}, ..., r_1)\right] \quad (33)$$

and for $i = N$:

$$\begin{aligned} \text{Min } J_N(d_N, r_{N-1}, ..., r_1) &= C(d_N) + C(d_N + r_{N-1}) + ... + C(d_N + r_{N-1} + ... + r_1) + \\ & h\int_0^{d_N} (d_N - t) f_N(t)\, dt + p\int_{d_N}^{\infty} (t - d_N) f_N(t)\, dt + \\ & E\left[J_{N-1}^*(d_N + r_{N-1} - \tau_N, r_{N-2}, ..., r_1)\right] \quad (34) \\ & \text{s.t. } d_N, r_{N-1}, ..., r_1 \geq 0 \end{aligned}$$

where $r_i = d_{i+1} - d_i$, $i = 1, ..., N-1$.

Proposition 1 $y_i^*(l_i, d_{i-1}, ..., d_1) \equiv y_i^*(l_i, r_{i-1}, ..., r_1)$, *the optimal waiting time before processing of job i is started, given that d_i is l_i units of time away and given the quoted due dates $d_{i-1}, ..., d_1$, is expressed by*

$$y_i^*(l_i, r_{i-1}, ..., r_1) = \begin{cases} l_i - X_i^* & \text{if } l_i \geq X_i^* \\ 0 & \text{otherwise} \end{cases} \quad (35)$$

where X_i^*, the optimal planned lead time of job i, solves $dJ_i(l_i, r_{i-1}, ..., r_1)/dX_i = 0$ (after subtituting $l_i - y_i$ by X_i).

It is true for $i = 1$ and 2. To prove this for $3 \leq i \leq N$, we assume that $J_{i-1}^*(l_{i-1}, r_{i-2}, ..., r_1)$ is convex in l_{i-1}, hence (35) is true for job i, and show that this implies $J_i^*(l_i, r_{i-1}, ..., r_1)$ is convex in l_i, hence (35) is true for job $i+1$. In fact, substituting $y_i^*(l_i, r_{i-1}, ..., r_1)$ in

$J_i(l_i, r_{i-1}, ..., r_1)$, we get

$$J_i^*(l_i, r_{i-1}, ..., r_1) = \begin{cases} h\int_0^{X_i^*}(X_i^* - t) f_i(t)\,dt + p\int_{X_i^*}^{\infty}(t - X_i^*) f_i(t)\,dt + \\ E\left[J_{i-1}^*(X_i^* + r_{i-1} - \tau_i, r_{i-2}, ..., r_1)\right] & l_i \geq X_i^* \\ J_i(X_i + y_i, r_{i-1}, ..., r_1)\,|_{\{y_i=0, X_i=l_i\}} = \\ h\int_0^{l_i}(l_i - t) f_i(t)\,dt + p\int_{l_i}^{\infty}(t - l_i) f_i(t)\,dt + \\ E\left[J_{i-1}^*(l_i + r_{i-1} - \tau_i, r_{i-2}, ..., r_1)\right] & l_i \leq X_i^* \end{cases} \tag{36}$$

It is clearly convex in l_i for $l_i \leq X_i^*$ since the first 2 terms are convex in l_i and we assumed that $J_{i-1}^*(l_{i-1}, r_{i-2}, ..., r_1)$ is convex in l_{i-1}, hence convex in l_i. Furthermore $dJ_i^*(l_i, r_{i-1}, ..., r_1)/dl_i = 0$ at $l_i = X_i^*$ since we assumed that X_i^* solves $dJ_i(l_i, r_{i-1}, ..., r_1)/dX_i = 0$ (after subtituting $l_i - y_i$ by X_i), hence solves $dJ_i(X_i + y_i, r_{i-1}, ..., r_1)\,|_{\{y_i=0, X_i=l_i\}}/dl_i = 0$.

Proposition 2 *X_i^*, $1 \leq i \leq N-1$ solves the following equation (after subtituting $l_i - y_i$ by X_i):*

$$\begin{aligned} \frac{dJ_i(l_i, r_{i-1}, ..., r_1)}{dX_i} &= h\sum_{j=1}^{i} Pr\left\{\sum_{k=j}^{i}\tau_k \leq \sum_{k=j}^{i-1} r_k + X_i, \sum_{k=j+1}^{i}\tau_k \geq \sum_{k=j}^{i-1} r_k + X_i - X_j^*,\right. \\ &\left.\sum_{k=j+2}^{i}\tau_k \geq \sum_{k=j+1}^{i-1} r_k + X_i - X_{j+1}^*, ...\right\} - \\ &p\sum_{j=1}^{i} Pr\left\{\sum_{k=j}^{i}\tau_k \geq \sum_{k=j}^{i-1} r_k + X_i, \sum_{k=j+1}^{i}\tau_k \geq \sum_{k=j}^{i-1} r_k + X_i - X_j^*,\right. \\ &\left.\sum_{k=j+2}^{i}\tau_k \geq \sum_{k=j+1}^{i-1} r_k + X_i - X_{j+1}^*, ...\right\} = 0 \end{aligned} \tag{37}$$

It is true for $i = 1$ and 2. To prove this for $3 \leq i \leq N-1$, assume that it is true for i. $J_i^*(l_i, r_{i-1}, ..., r_1)$ is given by (36) and $J_{i+1}(l_{i+1}, r_i, ..., r_1)$ is given by (33). Substituting l_i by $l_{i+1} - y_{i+1} + r_i - \tau_{i+1}$ in (33), letting $l_{i+1} - y_{i+1} = X_{i+1}$, differentiating (33) with respect to X_{i+1}^*, setting it to zero, and after doing further manipulations we get

$$\begin{aligned} \frac{dJ_{i+1}(l_{i+1}, r_i, ..., r_1)}{dX_{i+1}} &= hPr\{\tau_{i+1} \leq X_{i+1}\} - pPr\{\tau_{i+1} \geq X_{i+1}\} + \\ &E\left[\frac{dJ_i^*(X_{i+1} + r_i - \tau_{i+1}, r_{i-1}, ..., r_1)}{dX_{i+1}}\right] = 0 \end{aligned} \tag{38}$$

but from (36), we have that

$$\frac{dJ_i^*\left(X_{i+1}+r_i-\tau_{i+1},r_{i-1},...,r_1\right)}{dX_{i+1}}=\begin{cases}0 & \text{if } \tau_{i+1}\leq X_{i+1}+r_i-X_i^* \\ \frac{dJ_i(l_i,r_{i-1},...,r_1)}{dl_i}\Big|_{l_i=X_{i+1}+r_i-\tau_{i+1}} & \text{otherwise}\end{cases} \tag{39}$$

For $\tau_{i+1}\geq X_{i+1}+r_i-X_i^*$, it is given by the middle side of (37) evaluated at $X_i = X_{i+1}+r_i-\tau_{i+1}$. Substituting this latter in (38) gives the following first order condition:

$$h\left(Pr\left\{\tau_{i+1}\leq X_{i+1}\right\}+\sum_{j=1}^{i}\left[\int_{X_{i+1}+r_i-X_i^*}^{\infty}Pr\left\{\sum_{k=j}^{i+1}\tau_k\leq\sum_{k=j}^{i}r_k+X_{i+1},\right.\right.\right.$$
$$\left.\left.\left.\sum_{k=j+1}^{i+1}\tau_k\geq\sum_{k=j}^{i}r_k+X_{i+1}-X_j^*,\ \sum_{k=j+2}^{i+1}\tau_k\geq\sum_{k=j+1}^{i}r_k+X_{i+1}-X_{j+1}^*,....\right\}f_{i+1}(u)\,du\right]\right)-$$
$$p\left(Pr\left\{\tau_{i+1}\geq X_{i+1}\right\}+\sum_{j=1}^{i}\left[\int_{X_{i+1}+r_i-X_i^*}^{\infty}Pr\left\{\sum_{k=j}^{i+1}\tau_k\geq\sum_{k=j}^{i}r_k+X_{i+1},\right.\right.\right.$$
$$\left.\left.\left.\sum_{k=j+1}^{i+1}\tau_k\geq\sum_{k=j}^{i}r_k+X_{i+1}-X_j^*,\ \sum_{k=j+2}^{i+1}\tau_k\geq\sum_{k=j+1}^{i}r_k+X_{i+1}-X_{j+1}^*,....\right\}f_{i+1}(u)\,du\right]\right)=0$$

which reduces to

$$\begin{aligned}\frac{dJ_{i+1}\left(l_{i+1},r_i,...,r_1\right)}{dX_{i+1}} &= hPr\left\{\tau_{i+1}\leq X_{i+1}\right\}-pPr\left\{\tau_{i+1}\geq X_{i+1}\right\}+\\ &h\sum_{j=1}^{i}Pr\left\{\sum_{k=j}^{i+1}\tau_k\leq\sum_{k=j}^{i}r_k+X_{i+1},\ \sum_{k=j+1}^{i+1}\tau_k\geq\sum_{k=j}^{i}r_k+X_{i+1}-X_j^*,\right.\\ &\left.\sum_{k=j+2}^{i+1}\tau_k\geq\sum_{k=j+1}^{i}r_k+X_{i+1}-X_{j+1}^*,....\right\}-\\ &h\sum_{j=1}^{i}Pr\left\{\sum_{k=j}^{i+1}\tau_k\geq\sum_{k=j}^{i}r_k+X_{i+1},\ \sum_{k=j+1}^{i+1}\tau_k\geq\sum_{k=j}^{i}r_k+X_{i+1}-X_j^*,\right.\\ &\left.\sum_{k=j+2}^{i+1}\tau_k\geq\sum_{k=j+1}^{i}r_k+X_{i+1}-X_{j+1}^*,....\right\}=0\end{aligned}$$

and finally to

$$\begin{aligned}\frac{dJ_{i+1}\left(l_{i+1},r_i,...,r_1\right)}{dX_{i+1}} &= h\sum_{j=1}^{i+1} Pr\left\{\sum_{k=j}^{i+1}\tau_k \leq \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1} - X_j^*,\right.\\ &\left.\sum_{k=j+2}^{i+1}\tau_k \geq \sum_{k=j+1}^{i} r_k + X_{i+1} - X_{j+1}^*, ...\right\} -\\ &h\sum_{j=1}^{i+1} Pr\left\{\sum_{k=j}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1} - X_j^*,\right.\\ &\left.\sum_{k=j+2}^{i+1}\tau_k \geq \sum_{k=j+1}^{i} r_k + X_{i+1} - X_{j+1}^*, ...\right\} = 0 \end{aligned} \tag{40}$$

and we are done.

Proposition 3 $X_i^* \geq X_{i-1}^*$, $i = 2, ..., N-1$

Proposition 4 r_i^*, $i = 1,..,N-1$ *satisfy the following set of first-order conditions:*

$$\begin{aligned} C'\left(d_N^* + r_{(N-1)} + ... + r_i\right) &= -hPr\left\{\sum_{k=i}^{N}\tau_k \leq \sum_{k=i}^{N-1} r_k + d_N, \sum_{k=i+1}^{n}\tau_k \geq \sum_{k=i}^{N-1} r_k + d_N - X_i,\right.\\ &\left.\sum_{k=i+2}^{n}\tau_k \geq \sum_{k=i+1}^{N-1} r_k + d_N - X_{i+1}, ...\right\}\\ &+pPr\left\{\sum_{k=i}^{N}\tau_k \geq \sum_{k=i}^{N-1} r_k + d_N, \sum_{k=i+1}^{N}\tau_k \geq \sum_{k=i}^{N-1} r_k + d_N - X_i,\right.\\ &\left.\sum_{k=i+2}^{N}\tau_k \geq \sum_{k=i+1}^{N-1} r_k + d_N - X_{i+1}, ...\right\} = 0 \end{aligned} \tag{41}$$

The proof is by differentiating (34) with respect to r_i and noting that $\delta J_N\left(d_N, r_{N-1}, ..., r_1\right)/\delta r_i$ is nothing but the due date cost terms plus the terms containing r_i in (37), for $i = 1, ..., N-1$.

Proposition 5 d_N^* *is given by:*

$$C'\left(d_N\right) = -hPr\left\{\tau_N \leq d_N\right\} + pPr\left\{\tau_N \geq d_N\right\} = 0 \tag{42}$$

The proof is by differentiating (34) with respect to d_N and noting that $\delta J_N(d_N, r_{N-1}, ..., r_1)/\delta d_N$ is nothing but the due date cost terms plus the terms containing X_N in (37).

Proposition 6 *Denoting by* $\Psi_N^i(r_i, ..., r_N, X_2, ..., X_{N-1})$ *the right-hand side of (41), we get for* $i = 1, ..., N-1$:

$$r_i^* = \begin{cases} \left\{x, x \in \left[(N-i)\overline{\tau} + X_i^* - d_N^* - \sum_{k=i+1}^{N-1} r_k^*, \; A - d_N^* - \sum_{k=i+1}^{N-1} r_k^*\right]\right\} & \textit{if } (N-i)\overline{\tau} + X_i^* \leq A \\ A - d_N^* - \sum_{k=i+1}^{N-1} r_k^* & \textit{if } (N-i)\overline{\tau} + X_i^* \geq A \textit{ and } \\ & \Psi_N^i\left(A - d_N^* - \sum_{k=i+1}^{N-1} r_k^*, ..., r_N, \right. \\ & \left. X_2, ..., X_{N-1}\right) \leq C'(A) \\ r_i^* & \textit{otherwise} \end{cases} \tag{43}$$

The proof is by showing that the derivative of $\Psi_N^i(r_i, ..., r_N, X_2, ..., X_{N-1})$ with respect to r_i is negative for $i = 1, ..., N-1$, and that it vanishes at $r_i = (N-i)\overline{\tau} + X_i^* - d_N^* - \sum_{k=i+1}^{N-1} r_k^*$.

Proposition 7 *Let* d_i^{N*} *be the quoted due date for job i in an N jobs problem.* $d_{(i+1)}^{(N+1)*} \leq d_i^{*N}$, $i = 1, ..., N$.

5 Conclusion

We have considered the problem of assigning optimal due dates and starting times to a set of identical jobs ready to be processed. We have shown that optimal quoted due dates can be obtained analytically by balancing the marginal effects of holding cost, shortage cost and cost of quoting an uncompetitive due date for each job, due to quoting a due date one unit of time longer. The optimal due date shave been shown to be at least equal to A, the "acceptable" value set by the market or by the cutomer conception, below which any quoted due date will be assigned without any extra cost. We have also shown that once the due dates are quoted, optimal starting times of subsequent jobs in the sequence can be also obtained analytically by balancing the marginal effects of holding and shortage costs, due to waiting an extra unit of time before starting the job with the earliest due date. The issue of sequencing the jobs was not raised because we assumed the jobs to be identical with same processing time distribution on the machine and same cost structure. However, a future direction on research could be one in which this assumption is relaxed. Hence it would be required to find the optimal sequence in which the jobs must be processed on the machine, their quoted due dates and the optimal starting time policy once processing has started. It would be interesting to determine necessary conditions on the cost structure and/or processing time distributions and parameters that will allow for some specific sequences to be optimal and to question whether these conditions are reasonable. Another direction in research may be the generalization of this model to serial production lines and flow shops.

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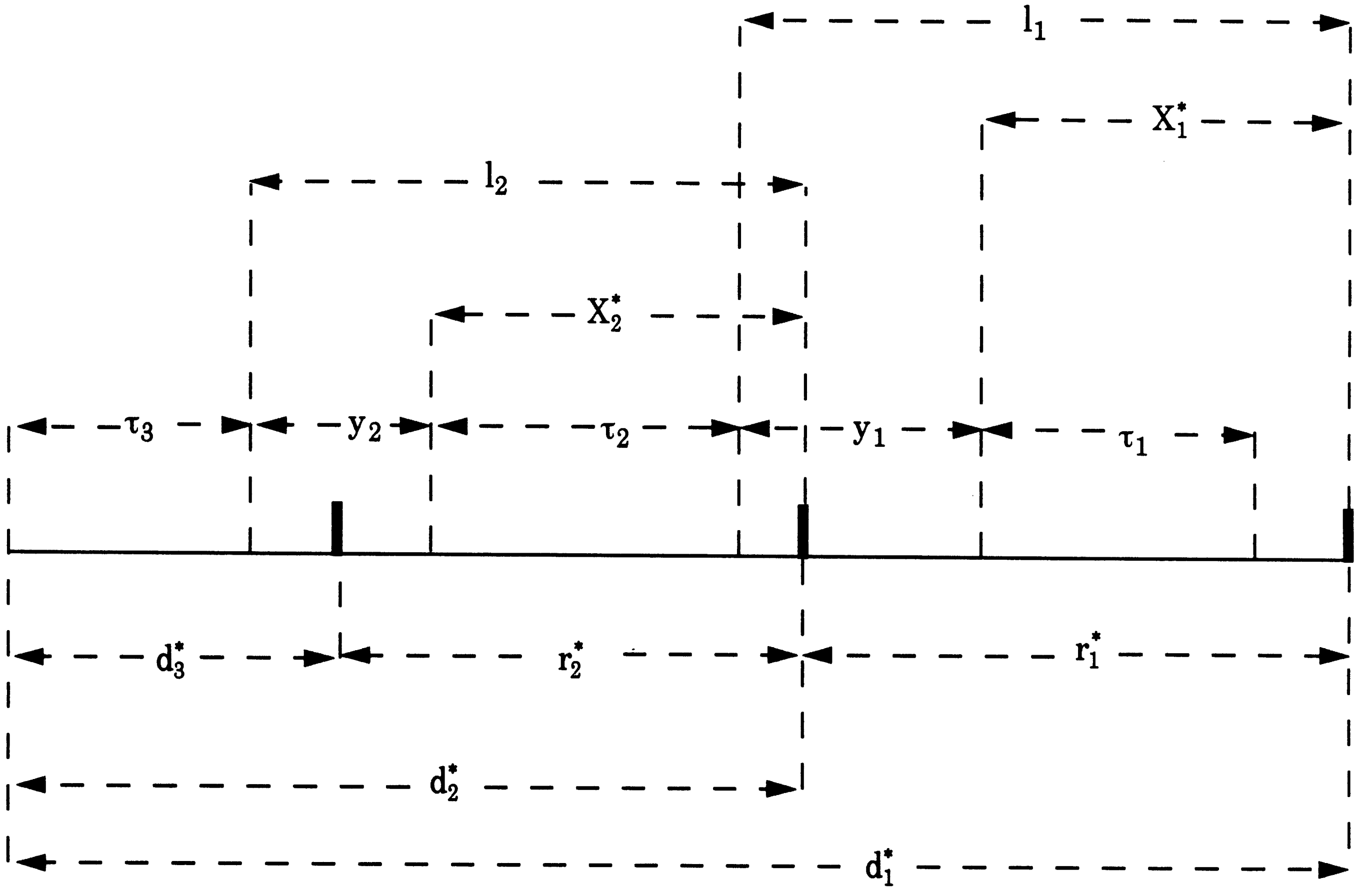


Figure 2

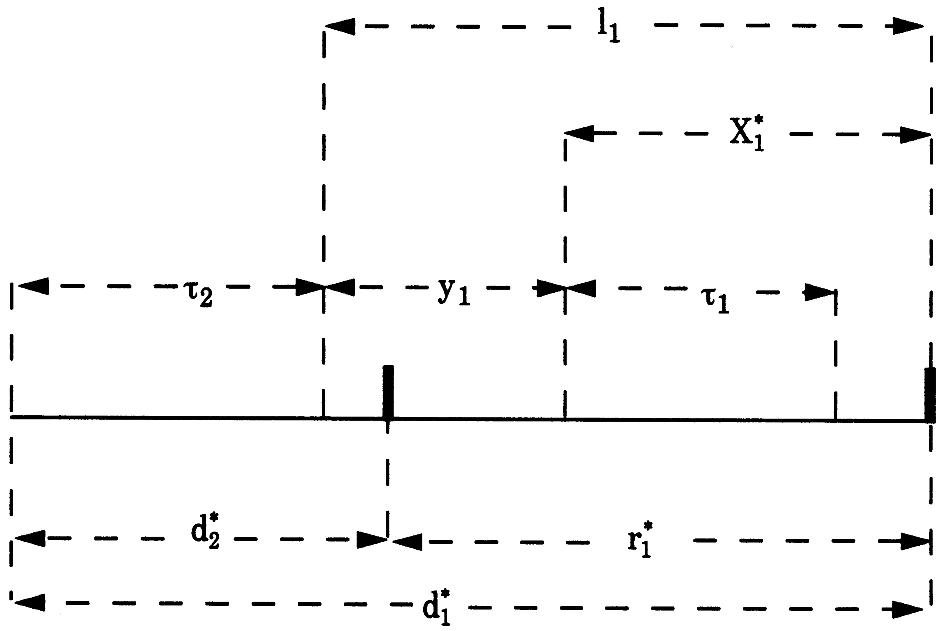
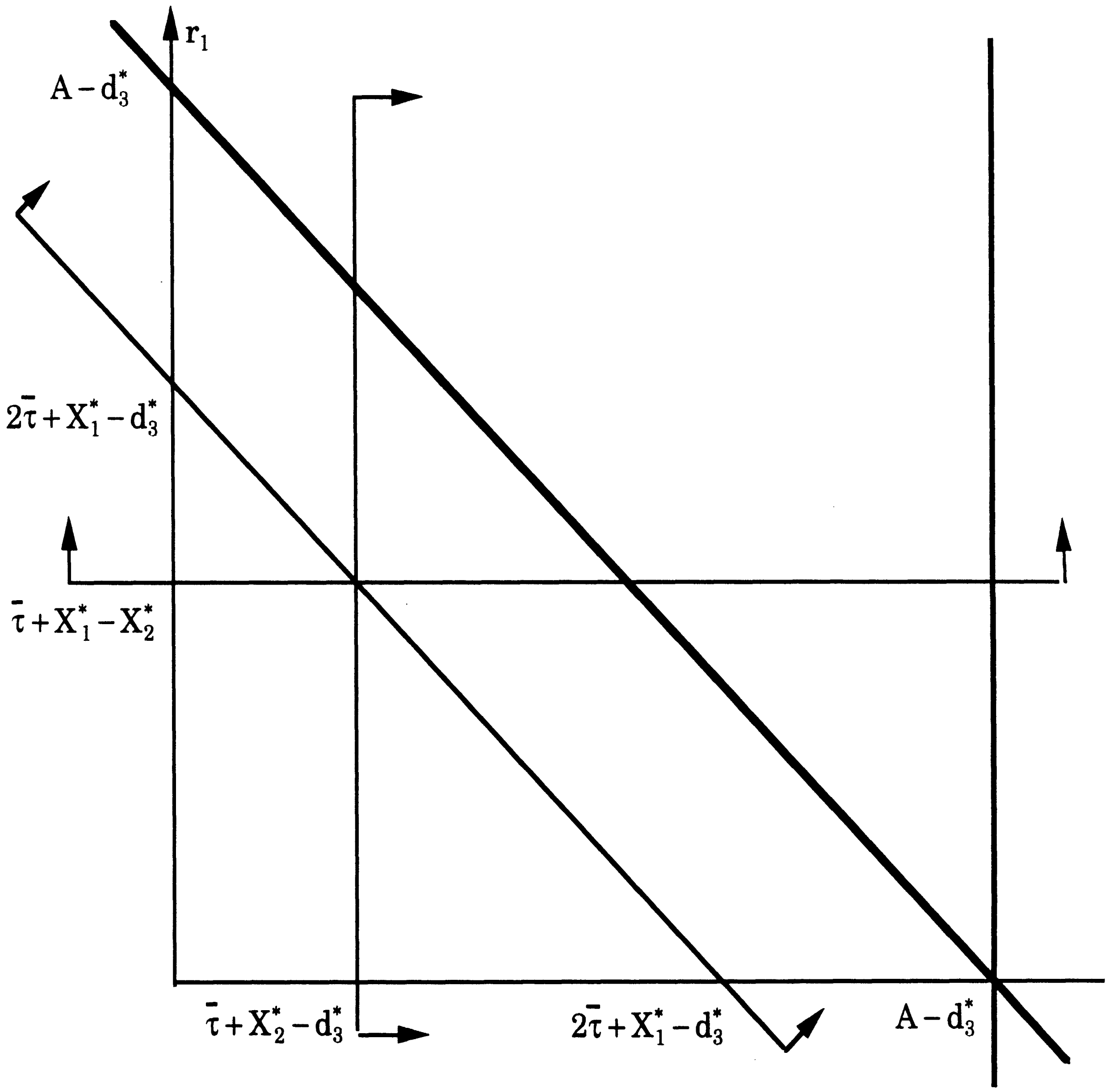
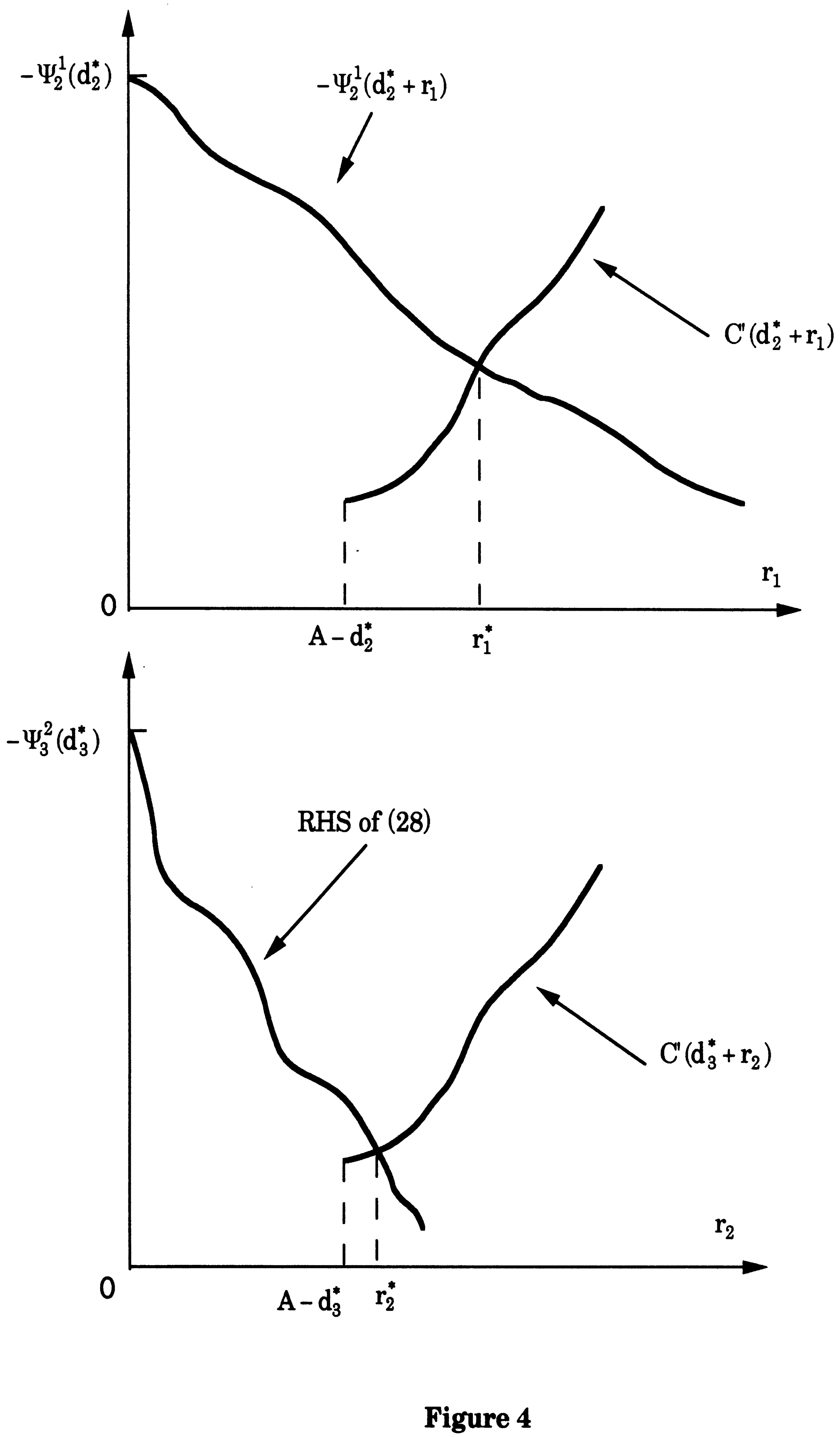


Figure 1



r_1
$A - d_3^*$
$2\bar{\tau} + X_1^* - d_3^*$
$\bar{\tau} + X_1^* - X_2^*$
$\bar{\tau} + X_2^* - d_3^*$
$2\bar{\tau} + X_1^* - d_3^*$
$A - d_3^*$

Figure 3



$-\Psi_2^1(d_2^*)$
$-\Psi_2^1(d_2^* + r_1)$
$C'(d_2^* + r_1)$
r_1
0
$A - d_2^*$
r_1^*
$-\Psi_3^2(d_3^*)$
RHS of (28)
$C'(d_3^* + r_2)$
r_2
0
$A - d_3^*$
r_2^*

Figure 4