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# Space–Time Wavelet Basis for the Continuity Equation[1]

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We construct $L^2$-orthonormal bases of vector-valued wavelets for divergence-free vector fields in four dimensions. They have exponential decay and any degree of smoothness one chooses for the construction. Although a momentum vortex field construction is possible in the special case of four dimensions, our method can be applied to an arbitrary number of dimensions. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

Phase cell decompositions have applications to many areas of the mathematical sciences, from signal analysis to the analysis of long-distance correlations in statistical mechanics. The phase cells index a set of expansion functions for analyzing a configuration. They also label locations in phase space about which the functions are more or less concentrated in both position coordinates and Fourier transform coordinates. Since the Fourier transform imposes severe limits on the amount of localization possible in phase space for a complete set of phase cells, the idea of wavelet expansions is to break up this obstacle into an indefinite hierarchy of scales, where the wavelets are well-localized in the scale-commensurate sense. Since many of the mathematical sciences involve differential equations of one kind or another, we are currently interested in the construction of complete sets of wavelets satisfying differential constraints.

In a previous paper [1] we constructed an orthonormal basis of wavelets for the space of square-integrable divergence-free vector fields in three dimensions. These wavelets are exponentially localized with an arbitrary degree of smoothness and number of vanishing moments. Such a basis will be useful in the analysis of the mechanics of incompressible fluid flow [2,3].

Although our construction in [1] may appear peculiar to three dimensions, we shall establish in this paper that it actually generalizes to an arbitrary number of dimensions. Our construction here will be confined to four dimensions only. The divergence-free condition in four dimensions is the down-to-earth continuity equation for conserved currents, so we expect such a basis of space–time wavelets to be useful in that context, whether the physics is relativistic or not. On the other hand, the multidimensional generalization of our construction will become clear in principle as we describe the four-dimensional construction. In any case, the complexity of the algebra grows with the dimension. We suppress most of our calculations, but they are already much lengthier in four dimensions than in three.

We have learned that Lemarié [4,5] has constructed a nonorthogonal basis of wavelets for divergence-free vector fields in an arbitrary number of dimensions. His wavelets have compact support as well as an arbitrary degree of smoothness. Although they are not orthogonal, the dual basis is as easy to construct and also consists of compactly supported functions.

Recall that in three dimensions our basic construction was to minimize $\int(d\vec{A})^2$ for a vector field $\vec{A}$ with respect to different sets of loop integral averaging constraints on $\vec{A}$ and a gauge-fixing condition. We then took the exterior derivative $d\vec{A}_\gamma$ of our solutions $\vec{A}_\gamma$ as our $L^2$-orthogonal divergence-free wavelets. This approach works because the exterior derivative of a three-dimensional vector field is just the curl, whose divergence is identically zero. Obviously the exterior derivative of a vector field in four dimensions is

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an antisymmetric tensor with six independent components, so we cannot play quite the same game here.

The idea in four dimensions is to minimize $\int(d\mathbb{F})^2$ for an antisymmetric tensor field $\mathbb{F}$ with respect to sets of averaging constraints and a gauge-fixing condition. We then apply the exterior derivative to our tensor solutions to obtain $L^2$-orthogonal *vector* wavelets that are automatically divergence-free. The point is that in *4 dimensions* the exterior derivative of $\mathbb{F}$ is a *pseudovector field*. The exterior derivative of such a pseudovector is identically zero and can be written as the divergence of the corresponding real vector field.

What is the geometric nature of the averaging constraints? In the case of minimizing $\int(d\vec{A})^2$ for a vector field $\vec{A}$, the geometric objects whose translates were averaged were canonical loop integrals, where the canonical loops were oriented boundaries of plaquettes. In our case they are the oriented boundary integrals that the usual generalization of Stoke's Theorem relates our particular exterior derivative to. Since $d\mathbb{F}$ is the exterior derivative of an antisymmetric tensor, the boundary integrals are normal surface integrals instead of the path integrals arising in the vector case. The surfaces consist of the plaquette faces of the three-dimensional cubes or "hyperplaquettes," where the normal directions on the surface boundaries are antisymmetric tensors. If $\vec{e}_\mu$ denotes the unit $\mu$th-coordinate vector and if we consider, say, a three-dimensional cube in the $\mu\nu\lambda$-coordinate hyperplane with vertices on $\mathbb{Z}^4$, then the normal directions are given by

$$\mathbb{N} = \vec{e}_\mu \wedge \vec{e}_\nu, \quad \mathbb{N} = \vec{e}_\lambda \wedge \vec{e}_\mu, \quad \mathbb{N} = \vec{e}_\nu \wedge \vec{e}_\lambda,$$

for the max $-x_\lambda-$, max $-x_\nu-$, and max $-x_\mu-$faces, respectively—or by the same rule corresponding to a permutation of $\mu, \nu, \lambda$. The cyclic relation is necessary for the orientation of the surface boundary to be consistent with the exterior derivative of $\mathbb{F}$.

We minimize the quadratic form

$$\sum_{\mu,\nu,\lambda} \int \left( \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} \right)^2 dx \qquad (1.1)$$

on antisymmetric tensors $\mathbb{F}$ with respect to the constraints

$$\sum_\mu \frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0, \qquad (1.2)$$

$$\int \eta(x-\vec{m}) \iint_{S_{\mu\nu\lambda}+x} \mathbb{F}\cdot\mathbb{N}\, d\sigma dx = s_{\mu\nu\lambda}(\vec{m}), \quad \vec{m}\in\mathbb{Z}^4, \quad (1.3)$$

where $\eta$ is an averaging function satisfying the Meyer functional equation

$$\eta(x) = \sum_{\vec{m}} c_{\vec{m}}\, \eta(2x-\vec{m}) \qquad (1.4)$$

for some sequence $\{c_{\vec{m}}\}$, $S_{\mu\nu\lambda}$ is the surface of the unit cube in the $\mu\nu\lambda$-coordinate hyperplane with minimum-coordinate vertex at the origin, the dot product here signifies contraction product of tensors, and the numbers $s_{\mu\nu\lambda}(\vec{m})$ associated with these constraints must satisfy certain conditions. First, there are compatibility conditions: $s_{\mu\nu\lambda}(\vec{m})$ must be antisymmetric in the indices and it must satisfy the lattice exterior derivative condition

$$\begin{aligned}[s_{\mu\nu\lambda}(\vec{m}+\vec{e}_\sigma) - s_{\mu\nu\lambda}(\vec{m})] - [s_{\sigma\mu\nu}(\vec{m}+\vec{e}_\lambda) - s_{\sigma\mu\nu}(\vec{m})] \\ + [s_{\lambda\sigma\mu}(\vec{m}+\vec{e}_\nu) - s_{\lambda\sigma\mu}(\vec{m})] \\ - [s_{\nu\lambda\sigma}(\vec{m}+\vec{e}_\mu) - s_{\nu\lambda\sigma}(\vec{m})] = 0. \quad (1.5)\end{aligned}$$

This equation must hold if our "hyperplaquette" constraints are to be compatible, and it will play a crucial role in our calculations. Second, $s_{\mu\nu\lambda}(\vec{m})$ must satisfy an averaging constraint itself it our solutions on different scales are to be orthogonal. The constraint is

$$\sum_{\vec{m}} c_{\vec{m}} \sum_{\iota,\iota',\iota''=0}^{1} s_{\mu\nu\lambda}(\vec{m}+2\vec{n}+\iota\,\vec{e}_\mu+\iota'\,\vec{e}_\nu+\iota''\,\vec{e}_\lambda) = 0, \quad (1.6)$$

and we omit the proof that it guarantees orthogonality. It is entirely similar to the proof of the parallel claim in three dimensions [1].

We emphasize that the constrained minimization problem solved here in four dimensions is quite different from the problem solved in [6,7], which involves the exterior derivative of a *vector* field in four dimensions. The purpose of the authors was to construct gauge field modes, and at first glance it may seem that they accomplish our purpose as well, and in a more straightforward way. After all, their gauge-fixing constraint is precisely the divergence-free condition on $\vec{A}$. Unfortunately, their solution is not exponentially localized. They can gauge away this problem, and this suits their own purposes, but this option also destroys the divergence-free condition.

In our context the gauge-fixing constraint is (1.2) and is needed for obtaining a unique solution for $\mathbb{F}$. Although this constraint is a divergence-free condition on the antisymmetric tensor field, it has nothing to do with the divergence-free condition on the vector field that we ultimately obtain.

The format of this paper is similar to that of [1] in some respects. In the next section we derive the momentum expression for the solution of the unconstrained minimization problem obtained by introducing a constraint parameter $\alpha < \infty$. In the $\alpha = \infty$ limit we have to invert a singular matrix on the orthogonal complement of its kernel. In Section 3 we do just that and then take the exterior derivative of the antisymmetric-tensor-valued solution which this inversion computes for us in momentum space. As in [1,6,7], the solution does not have exponential decay, but the exterior derivative does.

Following [1], we choose our averaging function $\eta$ to be a compactly supported spline of arbitrary degree. Specifically,

$$\hat{\eta}(p) = \prod_{\mu} \hat{\chi}(p_\mu)^N, \tag{1.7}$$

where $\chi$ is the characteristic function of [0,1]. We may choose $N$ as large as we like, and the resulting divergence-free wavelets are of class $C^{N-\varepsilon}$ with vanishing moments of order $\leqslant N$. As usual, the basis constructed by our constrained minimization technique is inter-scale orthogonal but not intrascale orthogonal as it stands. Translation-invariant orthogonalization on each scale is a canonical procedure if the overlap matrix is positive definite with positive lower bound. The proof of that property is nontrivial but differs in no interesting way from the proof in the three-dimensional case [1], so we omit it.

In Section 4 we prove that the exterior derivative of our solution does indeed have exponential decay, but our proof is unappealing. Instead of deriving the property from some characteristic of our construction, we simply prove that the momentum expression is real analytic. This means that one has to *compute* the restricted inverse matrix mentioned above. This had to be done in the three-dimensional case [1] as well, but in the four-dimensional case the calculation is tedious.

In Section 5 we take time out to discuss completeness—an issue that is often lost in the complexity of our constructions. We do not say much about it in [1], but here we explain why one so rarely worries about it. We give the proof for scalar one-dimensional wavelets in the context of the constrained minimization approach, and then we describe how one extends the argument to our geometric sophistication of that approach.

Finally, it is clear how to generalize the construction to arbitrary dimension $d$. By regarding the divergence of a vector field as just the exterior derivative of a pseudovector field, one needs only to construct antisymmetric $(d-2)$-order tensor wavelets orthogonal with respect to the Sobolev pseudonorm based on the exterior derivative of such tensors. The averaging constraints are based on the unit-normal-tensor $(d-2)$-dimensional surface integrals over the boundaries of the $(d-1)$-dimensional hyperplaquettes.

## 2. PRELIMINARY MOMENTUM EXPRESSION

We deal with the constrained minimization in the same way as it was dealt with in [6,7]. In this case the quadratic form

$$\frac{1}{6}\sum_{\mu,\nu,\lambda}\int\left(\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}\right)^2 dx$$
$$+\alpha^2\sum_{\nu}\int\left(\sum_{\mu}\frac{\partial F_{\mu\nu}}{\partial x_\mu}\right)^2 dx$$
$$+\alpha^2\sum_{\mu,\nu,\lambda}\sum_{\vec{m}}\left[s_{\mu\nu\lambda}(\vec{m})-\int\eta(x-\vec{m})\iint_{S_{\mu\nu\lambda}+x}\mathbb{F}\cdot\mathbb{N}\,d\sigma dx\right]^2 \tag{2.1}$$

is minimized on the space of antisymmetric tensors. We then take the $\alpha=\infty$ limit of the $\alpha$-dependent solution.

Our first step in obtaining that solution is to write the averaging functional as an inner product—as we did in three dimensions [1]. In this case we have

$$\int\eta(x-\vec{m})\iint_{S_{\mu\nu\lambda}+x}\mathbb{F}\cdot\mathbb{N}\,d\sigma dx = \int\mathbb{F}(x)\cdot\mathbb{R}^{\mu\nu\lambda}(x-\vec{m})dx, \tag{2.2}$$

$$\mathbb{R}^{\mu\nu\lambda}(x)=\iint_{S_{\mu\nu\lambda}+x-\vec{e}_\mu-\vec{e}_\nu-\vec{e}_\lambda}\eta\mathbb{N}\,d\sigma. \tag{2.3}$$

Now for the orientation of the surface to be consistent with the exterior derivative, we must have

$$\mathbb{N}=\vec{e}_\mu\wedge\vec{e}_\nu,\qquad \mathbb{N}=\vec{e}_\lambda\wedge\vec{e}_\mu,\qquad \mathbb{N}=\vec{e}_\nu\wedge\vec{e}_\lambda,$$

for the max $-x_\lambda-$, max $-x_\nu-$, and max $-x_\mu-$faces of $S_{\mu\nu\lambda}$, respectively. Since this orientation is changed by permutations of $\mu,\nu,\lambda$, we are obviously summing over all possible surface orientations in four dimensions of a three-dimensional cube as well as over such cubes that live on the unit-scale lattice. In momentum space we get

$$\begin{aligned}\hat{R}^{\mu\nu\lambda}_{\mu'\nu'}(p) &= \hat{\eta}(p)\hat{\chi}(p_{\mu'})\hat{\chi}(p_{\nu'})\\ &\times[(\delta_{\mu'\mu}\delta_{\nu'\nu}-\delta_{\mu'\nu}\delta_{\nu'\mu})(1-e^{ip_\lambda})\\ &+(\delta_{\mu'\lambda}\delta_{\nu'\mu}-\delta_{\mu'\mu}\delta_{\nu'\lambda})(1-e^{ip_\nu})\\ &+(\delta_{\mu'\nu}\delta_{\nu'\lambda}-\delta_{\mu'\lambda}\delta_{\nu'\nu})(1-e^{ip_\mu})],\end{aligned} \tag{2.4}$$

and it is useful to observe that matrices of the form

$$A_{\mu''\nu'',\mu'\nu'}=\delta_{\mu''\mu'}p_{\nu''}p_{\nu'}, \tag{2.5}$$
$$B_{\mu''\nu'',\mu'\nu'}=\delta_{\nu''\nu'}p_{\mu''}p_{\mu'} \tag{2.6}$$

annihilate $\hat{\mathbb{R}}^{\mu\nu\lambda}(p)$ because

$$p_{\mu'}\hat{\chi}(p_{\mu'})=i(e^{ip_{\mu'}}-1). \tag{2.7}$$

This is relevant to the propagator, which we now calculate.

If we write the differential part of the quadratic form in terms of momentum integrals, that contribution is

$$\frac{1}{6}\sum_{\mu,\nu,\lambda}\int |p_\mu \widehat{F}_{\nu\lambda}(p) + p_\lambda \widehat{F}_{\mu\nu}(p) + p_\nu \widehat{F}_{\lambda\mu}(p)|^2 dp$$

$$+ \alpha^2 \sum_{\mu}\int \left|\sum_{\nu} p_\nu \widehat{F}_{\mu\nu}(p)\right|^2 dp. \quad (2.8)$$

Straightforward algebra, in which the antisymmetry of $F_{\mu\nu}$ is exploited, reduces this quadratic form to

$$\frac{1}{2}\sum_{\mu,\nu}\sum_{\mu',\nu'}\int \widehat{F}_{\mu\nu}(p)^* \Delta_\alpha(p)_{\mu\nu,\mu'\nu'} \widehat{F_{\mu'\nu'}}(p)dp, \quad (2.9)$$

$$\Delta_\alpha(p)_{\mu\nu,\mu'\nu'} = p^2\delta_{\mu\mu'}\delta_{\nu\nu'} + (\alpha^2 - 1)\delta_{\mu\mu'}p_\nu p_{\nu'} + (\alpha^2 - 1)\delta_{\nu\nu'}p_\mu p_{\mu'}. \quad (2.10)$$

where * denotes complex conjugation. This matrix operator on tensors preserves the subspace of antisymmetric tensors.

*Note.* The special operator $\Delta_0(p)$ is positive definite on this subspace but not on the whole space of tensors.

For $\alpha \neq 0$, $\Delta_\alpha(p)$ is invertible on this subspace, and that inverse $C_\alpha(p)$ is given by

$$C_\alpha(p)_{\mu\nu,\mu'\nu'} = \frac{1}{p^2}\delta_{\mu\mu'}\delta_{\nu\nu'} + \left(\frac{1}{\alpha^2} - 1\right)\frac{1}{p^4}\delta_{\mu\mu'}p_\nu p_{\nu'} + \left(\frac{1}{\alpha^2} - 1\right)\frac{1}{p^4}\delta_{\nu\nu'}p_\mu p_{\mu'}. \quad (2.11)$$

Obviously $C_\alpha(p)\Delta_\alpha(p) = \Delta_\alpha(p)C_\alpha(p) = 1$ on the subspace—not on the whole space—because the matrix contribution

$$P_{\mu\nu,\mu'\nu'} = p_\mu p_{\mu'} p_\nu p_{\nu'} \quad (2.12)$$

annihilates antisymmetric tensors.

Now if we apply this propagator to $\widehat{\mathbb{R}}^{\mu\nu\lambda}(p)$, it follows from our observation about the matrices $A$ and $B$ that

$$C_\alpha(p)\widehat{\mathbb{R}}^{\mu\nu\lambda}(p) = \frac{1}{p^2}\widehat{\mathbb{R}}^{\mu\nu\lambda}(p). \quad (2.13)$$

We now turn to the actual solution of the minimization problem.

The integral–differential equation determining the minimization of (2.1) is seen to be

$$-\Delta F_{\mu'\nu'} + (1 - \alpha^2)\sum_{\mu,\nu}[\delta_{\mu\mu'}\partial_\nu\partial_{\nu'} + \delta_{\nu\nu'}\partial_\mu\partial_{\mu'}]F_{\mu\nu}$$

$$+ \alpha^2 \sum_{\mu,\nu,\lambda}\sum_{\vec{m}} R^{\mu\nu\lambda}_{\mu'\nu'}(\cdot - \vec{m})\int \mathbb{F}(x)\cdot \mathbb{R}^{\mu\nu\lambda}(x - \vec{m})dx$$

$$= \alpha^2 \sum_{\mu,\nu,\lambda}\sum_{\vec{m}} R^{\mu\nu\lambda}_{\mu'\nu'}(\cdot - \vec{m}) s_{\mu\nu\lambda}(\vec{m}). \quad (2.14)$$

Solving this equation by the same routine method that was applied to the three-dimensional analogue [1], we obtain the momentum expression for the solution:

$$\widehat{\mathbb{F}}(p) = \alpha^2 G(p)^* \cdot [1 + \alpha^2 M(p)]^{-1} C_\alpha(p)\widehat{\mathbb{R}}(p), \quad (2.15)$$

$$M(p)_{\mu\nu\lambda,\mu'\nu'\lambda'} = \sum_{\vec{\ell}} \widehat{\mathbb{R}}^{\mu\nu\lambda}(p + 2\pi\vec{\ell})^* \times C_\alpha(p + 2\pi\vec{\ell})\widehat{\mathbb{R}}^{\mu'\nu'\lambda'}(p + 2\pi\vec{\ell}), \quad (2.16)$$

$$G_{\mu\nu\lambda}(p) = \sum_{\vec{m}} s_{\mu\nu\lambda}(\vec{m})e^{i\vec{m}\cdot p}, \quad (2.17)$$

where $[1 + \alpha^2 M(p)]^{-1}$ acts on the tensor-valued oriented-cube vector $\widehat{\mathbb{R}}(p)$ as an oriented-cube vector. The dot product in (2.15) is the contraction over oriented-cube indices $(\mu, \nu, \lambda)$, while the dot product in (2.16) is the tensor inner product; but (2.13) reduces (2.15) and (2.16) to

$$\widehat{\mathbb{F}}(p) = \frac{\alpha^2}{p^2}G(p)^* \cdot [1 + \alpha^2 M(p)]^{-1}\widehat{\mathbb{R}}(p), \quad (2.18)$$

$$M(p)_{\mu\nu\lambda,\mu'\nu'\lambda'} = \sum_{\vec{\ell}} \frac{1}{(p + 2\pi\vec{\ell})^2} \times \widehat{\mathbb{R}}^{\mu\nu\lambda}(p + 2\pi\vec{\ell})^* \cdot \widehat{\mathbb{R}}^{\mu'\nu'\lambda'}(p + 2\pi\vec{\ell}). \quad (2.19)$$

Now it follows from a tedious but straightforward calculation that

$$M(p)_{\mu\nu\lambda,\mu'\nu'\lambda'} = f_\mu(p)^* f_\nu(p)^* f_\lambda(p)^* \times \langle M_0(p)_{\mu\nu\lambda,\mu'\nu'\lambda'}\rangle f_{\mu'}(p) f_{\nu'}(p) f_{\lambda'}(p), \quad (2.20)$$

$$f_\mu(p) \equiv e^{ip_\mu} - 1, \quad (2.21)$$

$$\langle h(p)\rangle \equiv \sum_{\vec{\ell}} \frac{1}{(p + 2\pi\vec{\ell})^2}|\hat{\eta}(p+2\pi\vec{\ell})|^2 h(p+2\pi\vec{\ell}), \quad (2.22)$$

$$M_0(p)_{\mu\nu\lambda,\mu''\nu''\lambda''} = \frac{1}{p_\mu^2 p_\nu^2}(\delta_{\mu\mu''}\delta_{\nu\nu''} - \delta_{\nu\lambda''}\delta_{\mu\mu''} + \delta_{\mu\nu''}\delta_{\nu\lambda''})$$
$$+ \frac{1}{p_\nu^2 p_\lambda^2}(\delta_{\nu\mu''}\delta_{\lambda\nu''} - \delta_{\nu\mu''}\delta_{\lambda\lambda''} + \delta_{\nu\nu''}\delta_{\lambda\lambda''})$$
$$+ \frac{1}{p_\lambda^2 p_\mu^2}(-\delta_{\mu\mu''}\delta_{\lambda\nu''} + \delta_{\mu\mu''}\delta_{\lambda\lambda''} - \delta_{\mu\nu''}\delta_{\lambda\lambda''}). \quad (2.23)$$

If we list our oriented-cube directions in the order 432, 431, 421, 321, the matrix array for $M_0(p)$ is

$$\left[\begin{array}{cc}
\frac{1}{p_3^2 p_2^2} + \frac{1}{p_4^2 p_3^2} + \frac{1}{p_4^2 p_2^2} & \frac{1}{p_4^2 p_3^2} \\
\frac{1}{p_4^2 p_3^2} & \frac{1}{p_1^2 p_3^2} + \frac{1}{p_1^2 p_4^2} + \frac{1}{p_3^2 p_4^2} \\
-\frac{1}{p_4^2 p_2^2} & \frac{1}{p_4^2 p_1^2} \\
\frac{1}{p_3^2 p_2^2} & -\frac{1}{p_3^2 p_1^2}
\end{array}\right.$$

$$\times \begin{matrix} -\frac{1}{p_4^2p_2^2} & \frac{1}{p_3^2p_2^2} \\ \frac{1}{p_4^2p_1^2} & -\frac{1}{p_3^2p_1^2} \\ \frac{1}{p_4^2p_2^2}+\frac{1}{p_4^2p_1^2}+\frac{1}{p_2^2p_1^2} & \frac{1}{p_2^2p_1^2} \\ \frac{1}{p_2^2p_1^2} & \frac{1}{p_3^2p_2^2}+\frac{1}{p_3^2p_1^2}+\frac{1}{p_2^2p_1^2} \end{matrix} \Bigg]$$

It is easy to see that $M_0(p)$ is a rank 3 matrix which annihilates the vector

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

## 3. THE $\alpha = \infty$ LIMIT

Clearly $\langle M_0(p)\rangle$ is also a rank 3 matrix that annihilates

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix},$$

so we have the same problem that we had in three dimensions: the $\alpha = \infty$ limit of $\alpha^2(1 + \alpha^2 M(p))^{-1}$ cannot exist because

$$M(p) = D(p)\langle M_0(p)\rangle D(p)^*, \tag{3.1}$$

where in this case $D(p)$ is the $4 \times 4$ matrix

$$D(p)_{\mu\nu\lambda,\mu'\nu'\lambda'} = f_\mu(p) f_\nu(p) f_\lambda(p) \delta_{\mu\mu'} \delta_{\nu\nu'} \delta_{\lambda\lambda'} \tag{3.2}$$

with the four components labeled by 432, 431 421, and 321.

We have almost a routine attitude toward this special kind of problem, because it is solved by the lattice exterior derivative condition on the constraints. The singularity of the matrix arises from the nature of the averaging, which in turn imposes that lattice condition. This consistency is precisely what enables us to make sense of the $\alpha = \infty$ limit of the whole expression. Specifically, note that the kernel of $M(p)$ is generated by the vector

$$\begin{bmatrix} f_1(p)^* \\ -f_2(p)^* \\ f_3(p)^* \\ -f_4(p)^* \end{bmatrix},$$

while the dot product of this oriented-cube vector with $G(p)^*$ is zero as a result of writing the lattice exterior derivative condition in terms of Fourier series. Thus the $\alpha = \infty$ limit of (2.18) exists and is realized by the inversion of the matrix $M(p)$ as an operator on the orthogonal complement of its kernel.

Let $U$ be a unitary matrix which maps the vector

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

for example, pick

$$U = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ \sqrt{2} & \sqrt{2} & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}. \tag{3.3}$$

The point is that

$$U\langle M_0(p)\rangle U^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & J_{11} & J_{12} & J_{13} \\ 0 & J_{21} & J_{22} & J_{23} \\ 0 & J_{31} & J_{32} & J_{33} \end{bmatrix}, \tag{3.4}$$

where $J$ is an invertible $3 \times 3$ matrix. Let

$$I(p) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (J^{-1})_{11} & (J^{-1})_{12} & (J^{-1})_{13} \\ 0 & (J^{-1})_{21} & (J^{-1})_{22} & (J^{-1})_{23} \\ 0 & (J^{-1})_{31} & (J^{-1})_{32} & (J^{-1})_{33} \end{bmatrix} \tag{3.5}$$

be the matrix obtained from the admittedly tedious inversion of $J$. Then

$$L(p) = U^{-1} I(p) U \tag{3.6}$$

is the inversion of $\langle M_0(p)\rangle$ on the orthogonal complement of

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Now if we set

$$q_{\mu\nu} = \left\langle \frac{1}{p_\mu^2 p_\nu^2} \right\rangle \tag{3.7}$$

then obviously

$$\langle M_0(p)\rangle = \Bigg[ \begin{matrix} q_{23} + q_{34} + q_{24} & q_{34} \\ q_{34} & q_{13} + q_{14} + q_{34} \\ -q_{24} & q_{14} \\ q_{23} & -q_{13} \end{matrix}$$

$$\times \begin{bmatrix} -q_{24} & q_{23} \\ q_{14} & -q_{13} \\ q_{12}+q_{24}+q_{14} & q_{12} \\ q_{12} & q_{12}+q_{13}+q_{23} \end{bmatrix}. \tag{3.8}$$

After extracting $J$ from (3.4) we find that

$$\det J = \sum_{\substack{\{\mu,\nu\},\{\lambda,\sigma\},\{\kappa,\iota\} \text{ distinct as sets} \\ \text{no index value occurs more than twice}}} q_{\mu\nu} q_{\lambda\sigma} q_{\kappa\iota}. \tag{3.9}$$

We omit the matrix elements of $J$, as they depend on our choice of $U$ anyway.

*Remark.* By elementary combinatorics, our expression for $\det J$ consists of 16 terms. After the chore of inverting $J$, the remaining calculation in (3.6) is quite lengthy due to the number of terms in each matrix element of $J^{-1}$. However, one finally obtains

$$L(p)_{\kappa\iota} = \frac{c}{\det J} \sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} N^{\kappa\iota}_{\mu\nu,\lambda\sigma} q_{\mu\nu} q_{\lambda\sigma} \tag{3.10}$$

with $N^{\kappa\iota}_{\mu\nu,\lambda\sigma}$ given as follows.

*Case* A. $\kappa \neq \iota$ and $\kappa + \iota$ is even.

1. If $\{\mu,\nu\} \cap \{\lambda,\sigma\} = \phi$, then
   a. $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = 4$ if $\{\kappa,\iota\}$ is one of the sets.
   b. $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = -4$ if $\{\kappa,\iota\}$ meets both sets.
2. If $\{\mu,\nu\}$ and $\{\lambda,\sigma\}$ meet, then
   a. $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = 1$ if neither $\kappa$ nor $\iota$ lie in both sets.
   b. $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = -3$ if either $\kappa$ or $\iota$ lies in both sets.

*Case* B. $\kappa + \iota$ is odd. In this case we have the same rules as in Case A but with all signs reversed.

*Case* C. $\kappa = \iota$.

1. If $\{\mu,\nu\} \cap \{\lambda,\sigma\} = \phi$, then $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = 4$.
2. If $\{\mu,\nu\}$ and $\{\lambda,\sigma\}$ meet, then
   a. $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = 1$ if $\kappa$ does not lie in both sets.
   b. $N^{\kappa\iota}_{\mu\nu,\lambda\sigma} = 9$ if $\{\mu,\nu\} \cap \{\lambda,\sigma\} = \{\kappa\}$.

*Remark.* The sum in (3.10) consists of 15 terms.

In summary, the solution $\mathbb{F}$ of the original constrained minimization problem is given by

$$\hat{\mathbb{F}}(p) = \frac{1}{p^2} G(p)^* \cdot D(p)^{-1*} L(p) D(p)^{-1} \hat{\mathbb{R}}(p), \tag{3.11}$$

$$\hat{R}_{\mu\nu}(p) = \frac{\hat{\eta}(p)}{p_\mu p_\nu} D(p) \omega_{\mu\nu}, \tag{3.12}$$

$$\omega_{\mu\nu} = \begin{bmatrix} \delta_{\mu4}\delta_{\nu2} - \delta_{\mu4}\delta_{\nu3} - \delta_{\mu3}\delta_{\nu2} \\ \delta_{\mu4}\delta_{\nu1} - \delta_{\mu4}\delta_{\nu3} - \delta_{\mu3}\delta_{\nu1} \\ \delta_{\mu4}\delta_{\nu1} - \delta_{\mu4}\delta_{\nu2} - \delta_{\mu2}\delta_{\nu1} \\ \delta_{\mu3}\delta_{\nu1} - \delta_{\mu3}\delta_{\nu2} - \delta_{\mu2}\delta_{\nu1} \end{bmatrix} \tag{3.13}$$

for $\mu > \nu$. Naturally, it is the exterior derivative of $\mathbb{F}(x)$ that we want:

$$B_{\mu\nu\lambda} = \frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{\partial}{\partial x_\nu} F_{\lambda\mu} + \frac{\partial}{\partial x_\mu} F_{\nu\lambda}. \tag{3.14}$$

Clearly,

$$\begin{aligned} \hat{B}_{\mu\nu\lambda}(p) &= ip_\lambda \hat{F}_{\mu\nu}(p) + ip_\nu \hat{F}_{\lambda\mu}(p) + ip_\mu \hat{F}_{\nu\lambda}(p) \\ &= \frac{1}{p^2} G(p)^* \cdot D(p)^{-1*} L(p) D(p)^{-1} \\ &\quad \times [ip_\lambda \hat{\zeta}_{\mu\nu}(p) + ip_\nu \hat{\zeta}_{\lambda\mu}(p) + ip_\mu \hat{\zeta}_{\nu\lambda}(p)] \end{aligned} \tag{3.15}$$

because the matrices act on $\hat{\mathbb{R}}(p)$ as an oriented-cube vector and not as an antisymmetric tensor. Hence

$$\begin{aligned} \hat{B}_{\mu\nu\lambda}(p) &= \frac{i}{p^2} G(p)^{\dagger *} D(p)^{-1*} L(p) \eta(p) \\ &\quad \times p_\mu p_\nu p_\lambda \left[ \frac{1}{p_\mu^2 p_\nu^2} \omega_{\mu\nu} + \frac{1}{p_\lambda^2 p_\mu^2} \omega_{\lambda\mu} + \frac{1}{p_\nu^2 p_\lambda^2} \omega_{\nu\lambda} \right], \end{aligned} \tag{3.16}$$

where we have written the vector dot product with the (oriented-cube) column vector $G(p)^*$ as the matrix product with the row vector $G(p)^{\dagger *}$.

## 4. EXPONENTIAL LOCALIZATION

As always, one establishes exponential decay of a function by showing that its Fourier transform is real-analytic and satisfies the decay bounds necessary for contour-shifting. We analyze only the component $\hat{B}_{432}(p)$, as the analysis of $\hat{B}_{431}(p)$, $\hat{B}_{421}(p)$, and $\hat{B}_{321}(p)$ is similar. By (3.13) and (3.16) we have

$$\hat{B}_{432}(p) = \frac{i}{p^2} G(p)^{\dagger *} D(p)^{-1*} L(p) \eta(p) \times p_4 p_3 p_2 \begin{bmatrix} -\frac{1}{p_4^2 p_3^2} - \frac{1}{p_2^2 p_4^2} - \frac{1}{p_3^2 p_2^2} \\ -\frac{1}{p_4^2 p_3^2} \\ \frac{1}{p_2^2 p_4^2} \\ -\frac{1}{p_3^2 p_2^2} \end{bmatrix}, \tag{4.1}$$

while multiplication of $L(p)$ with this column vector can be written as the column vector

$$-\frac{1}{\det J} \sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} K(p)_{\mu\nu,\lambda\sigma} q_{\mu\nu} q_{\lambda\sigma}, \tag{4.2}$$

where the column-vector-valued matrix $K(p)$ is given by

$$K^{\iota}(p) = \frac{1}{p_4^2 p_3^2}(N^{\iota 1}+N^{\iota 2}) + \frac{1}{p_2^2 p_4^2}(N^{\iota 1}-N^{\iota 3}) + \frac{1}{p_3^2 p_2^2}(N^{\iota 1}+N^{\iota 4}). \quad (4.3)$$

Hence

$$\hat{B}_{432}(p) = -i\frac{\hat{\eta}(p)}{p^2} p_4 p_3 p_2 \frac{1}{\prod_\mu f_\mu(p)^*} \frac{1}{\det J} \times \sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} q_{\mu\nu} q_{\lambda\sigma} \sum_\iota G^\iota(p)^* \times f_\iota(p)^* K^\iota(p)_{\mu\nu,\lambda\sigma}, \quad (4.4)$$

where we have used

$$D(p)\begin{bmatrix} f_1(p) & 0 & 0 & 0 \\ 0 & f_2(p) & 0 & 0 \\ 0 & 0 & f_3(p) & 0 \\ 0 & 0 & 0 & f_4(p) \end{bmatrix} = \prod_\mu f_\mu(p). \quad (4.5)$$

Our main task is to derive the expressions that arise when (4.3) is inserted in (4.2) for each $\iota$. We calculate (exploiting cancellations)

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{11}_{\mu\nu,\lambda\sigma} + N^{12}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 4q_{23}q_{24} + 12q_{13}q_{14} + 8q_{23}q_{14} + 8q_{13}q_{24} + 12q_{12}q_{13} + 4q_{12}q_{23} + 4q_{12}q_{24} + 12q_{12}q_{14}, \quad (4.6.1)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{21}_{\mu\nu,\lambda\sigma} + N^{22}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 12q_{23}q_{24} + 4q_{13}q_{14} + 8q_{23}q_{14} + 8q_{13}q_{24} + 4q_{12}q_{13} + 12q_{12}q_{23} + 4q_{12}q_{14} + 12q_{12}q_{24}, \quad (4.6.2)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{31}_{\mu\nu,\lambda\sigma} + N^{32}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 8q_{13}q_{24} + 4q_{23}q_{24} - 4q_{13}q_{14} - 8q_{23}q_{14} - 4q_{12}q_{13} + 4q_{12}q_{23} + 4q_{12}q_{24} - 4q_{12}q_{14}, \quad (4.6.3)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{41}_{\mu\nu,\lambda\sigma} + N^{42}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 8q_{24}q_{13} + 4q_{13}q_{14} - 4q_{23}q_{24} - 8q_{14}q_{23} + 4q_{12}q_{13} + 4q_{12}q_{14} - 4q_{12}q_{23} - 4q_{12}q_{24}, \quad (4.6.4)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{11}_{\mu\nu,\lambda\sigma} - N^{13}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 4q_{23}q_{13} + 8q_{12}q_{34} + 8q_{23}q_{14} + 12q_{13}q_{14} + 4q_{34}q_{13} + 4q_{34}q_{23} + 12q_{12}q_{13} + 12q_{12}q_{14}, \quad (4.7.1)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{21}_{\mu\nu,\lambda\sigma} - N^{23}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 4q_{13}q_{14} + 8q_{23}q_{14} - 4q_{13}q_{23} - 8q_{12}q_{34} + 4q_{12}q_{13} + 4q_{12}q_{14} - 4q_{13}q_{34} - 4q_{23}q_{34}, \quad (4.7.2)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{31}_{\mu\nu,\lambda\sigma} - N^{33}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = -(12q_{23}q_{13} + 8q_{23}q_{14} + 4q_{13}q_{14} + 8q_{12}q_{34} + 12q_{34}q_{13} + 12q_{34}q_{23} + 4q_{12}q_{13} + 4q_{12}q_{14}), \quad (4.7.3)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{41}_{\mu\nu,\lambda\sigma} - N^{43}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 4q_{13}q_{14} + 8q_{12}q_{34} - 4q_{23}q_{13} - 8q_{14}q_{23} + 4q_{12}q_{13} + 4q_{12}q_{14} - 4q_{34}q_{23} - 4q_{34}q_{13}, \quad (4.7.4)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{11}_{\mu\nu,\lambda\sigma} + N^{14}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 12q_{13}q_{14} + 8q_{13}q_{24} + 4q_{14}q_{24} + 8q_{12}q_{34} + 12q_{12}q_{13} + 12q_{12}q_{14} + 4q_{34}q_{14} + 4q_{34}q_{24}, \quad (4.8.1)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{21}_{\mu\nu,\lambda\sigma} + N^{24}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 4q_{13}q_{14} + 8q_{13}q_{24} - 4q_{14}q_{24} - 8q_{12}q_{34} + 4q_{12}q_{13} + 4q_{12}q_{14} - 4q_{34}q_{14} - 4q_{34}q_{24}, \quad (4.8.2)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{31}_{\mu\nu,\lambda\sigma} + N^{34}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 4q_{24}q_{14} + 8q_{13}q_{24} - 4q_{13}q_{14} - 8q_{12}q_{34} + 4q_{34}q_{14} + 4q_{34}q_{24} - 4q_{12}q_{13} - 4q_{12}q_{14}, \quad (4.8.3)$$

$$\sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} (N^{41}_{\mu\nu,\lambda\sigma} + N^{44}_{\mu\nu,\lambda\sigma}) q_{\mu\nu} q_{\lambda\sigma} = 12q_{24}q_{14} + 8q_{24}q_{13} + 4q_{14}q_{13} + 8q_{12}q_{34} + 12q_{34}q_{14} + 12q_{34}q_{24} + 4q_{12}q_{14} + 4q_{12}q_{13}. \quad (4.8.4)$$

Now to establish real analyticity of the momentum expression for $\hat{B}_{432}(p)$, we first make some basic observations analogous to those made in [1],

$$p^2 p_\mu^2 p_\nu^2 q_{\mu\nu} = |\hat{\eta}(p)|^2(1 + \Gamma_{\mu\nu}(p)), \quad (4.9)$$

$$p^2 \left(\prod_\mu p_\mu^4\right) \det J = |\hat{\eta}(p)|^6(1 + \Gamma(p)), \quad (4.10)$$

where $\Gamma(p)$ and $\Gamma_{\mu\nu}(p)$ are functions analytic and vanishing at $p = 0$. The latter relation follows from Formula (2.22) for $\det J$ and the identity

$$\left(\prod_\mu p_\mu^4\right) \sum_{\substack{\{\mu,\nu\},\{\lambda,\sigma\},\{\kappa,\iota\} \text{ distinct as sets} \\ \text{no index occurs more than twice}}} (p_\mu^2 p_\nu^2 p_\lambda^2 p_\sigma^2 p_\kappa^2 p_\iota^2)^{-1} = p^4. \quad (4.11)$$

This will settle our concerns about the analyticity of $p^2 \det J$ in (4.4), but this means that the numerator is multiplied by $\prod_\mu p_\mu^4$ as well. If we single out the $\iota = 1$ contribution in

(4.4), it is straightforward to see that there is a function $\Lambda_1(p)$ analytic and vanishing at $p = 0$ such that

$$\left(\prod_\mu p_\mu^4\right) \sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} K^1(p)_{\mu\nu,\lambda\sigma} q_{\mu\nu} q_{\lambda\sigma} = 12|\hat{\eta}(p)|^4(1 + \Lambda_1(p)). \quad (4.12)$$

One simply extracts the leading terms with the correspondence

$$q_{\mu\nu} \Leftrightarrow \frac{1}{p^2 p_\mu^2 p_\nu^2} |\hat{\eta}(p)|^2 \quad (4.13)$$

given by (4.9) and combines them algebraically. The "miracle" is that one obtains the perfect square

$$\frac{1}{p^4}\left(\sum_\mu p_\mu^2\right)^2 = 1 \quad (4.14)$$

for the leading contribution, while the remainder $\Lambda_1(p)$ is a quadratic combination of the functions $\Gamma_{\mu\nu}(p)$. There are similar functions $\Lambda_\iota(p)$, $\iota = 2, 3, 4$, such that

$$\left(\prod_\mu p_\mu^4\right) \sum_{\{\mu,\nu\},\{\lambda,\sigma\} \text{ distinct as sets}} K^\iota(p)_{\mu\nu,\lambda\sigma} q_{\mu\nu} q_{\lambda\sigma} = 4(-1)^\iota|\hat{\eta}(p)|^4(1 + \Lambda_\iota(p)). \quad (4.15)$$

Thus (4.4) becomes

$$\hat{B}_{432}(p) = -i\frac{\hat{\eta}(p)}{|\hat{\eta}(p)|^2} p_4 p_3 p_2 \frac{1}{\prod_\mu f_\mu(p)^*} \frac{1}{1+\Gamma(p)} \times \left[12(1 + \Lambda_1(p)) f_1(p)^* G_1(p)^* + 4\sum_{\iota\neq 1}(-1)^\iota(1 + \Lambda_\iota(p)) f_\iota(p)^* G^\iota(p)^*\right], \quad (4.16)$$

and at first glance, it may seem that we are in trouble, since the factor $f_1(p)^*$ in the denominator vanishes for $p_1 = 0$ and there is no factor $p_1$ in the numerator to cancel that singularity. Now the $G^1(p)^* f_1(p)^*$-terms in the brackets obviously solve this problem, but what about the other terms? The key observation is the lattice exterior derivative condition

$$G^1(p)f_1(p) - G^2(p)f_2(p) + G^3(p)f_3(p) - G^4(p)f_4(p) \equiv G_{432}(p)f_1(p) - G_{431}(p)f_2(p) + G_{421}(p)f_3(p) - G_{321}(p)f_4(p) = 0, \quad (4.17)$$

which enables us to write the bracketed expression as

$$16G^1(p)^* f_1(p)^* + 12\Lambda_1(p)G^1(p)^* f_1(p)^* + 4\sum_{\iota\neq 1}(-1)^\iota\Lambda_\iota(p)G^\iota(p)^* f_\iota(p)^*. \quad (4.18)$$

However, this condition has to be applied again and more subtly. $\Lambda_\iota(p)$ consists of the contributions

$$c_{\mu\nu\lambda\sigma ij}\frac{1}{|\hat{\eta}(p)|^4}\left(\prod_\kappa p_\kappa^4\right)\left(q_{\mu\nu}q_{\lambda\sigma} - \frac{|\hat{\eta}(p)|^4}{p^4 p_\mu^2 p_\nu^2 p_\lambda^2 p_\sigma^2}\right)\frac{1}{p_i^2 p_j^2},$$

where $\{\mu, \nu\}$ and $\{\lambda, \sigma\}$ are distinct as sets and $\{i, j\}$ does *not* include 1. Let $\Lambda'_\iota(p)$ consist of those contributions for which $\{\mu, \nu\} \cap \{\lambda, \sigma\} = \{1\}$ and let $\Lambda''_\iota(p)$ consist of the remaining contributions. The second algebraic "miracle" is that

$$\Lambda'_2(p) = \Lambda'_3(p) = \Lambda'_4(p) \equiv \Lambda'(p), \quad (4.19)$$

which can be seen by collecting the products $q_{12}q_{13}$, $q_{12}q_{14}$, and $q_{13}q_{14}$ in our formulae (4.6–4.8) inserted in (4.15). Hence (4.17) reduces the bracketed expression still further to

$$16G^1(p)^* f_1(p)^* + 12\Lambda_1(p)G^1(p)^* f_1(p)^* + 4\Lambda'(p)G^1(p)^* f_1(p)^* + 4\sum_{\iota\neq 1}(-1)^\iota\Lambda''_\iota(p)G^\iota(p)^* f_\iota(p)^*,$$

and our problem is finally solved by the straightforward observation that $\Lambda''_\iota(p)$-contributions can all have a factor of $p_1^2$ extracted from them.

*Remark.* We have concentrated on a neighborhood of $p = 0$ because that is the hardest case. Analyzing a neighborhood of $p = 2\pi\,\vec{\ell}$ for $\vec{\ell} \in \mathbb{Z}^4\backslash\{0\}$ is a bit easier, but one would multiply and divide by $\prod_\mu(p_\mu + 2\pi\ell_\mu)^4$, etc.

Our analysis of the regularity properties is identical to that done in the three-dimensional case, so we omit it.

## 5. COMPLETENESS

We address here the issue of completeness for the set of wavelets we have constructed. Since our point of view differs from that of Meyer and his co-workers [8,9], it is instructive to first consider the scalar one-dimensional case in our own language and show that the substance of our completeness argument is not all that different.

Without loss we replace the real line $\mathbb{R}$ with the lattice $\delta\mathbb{Z}$ where the spacing $\delta > 0$ is arbitrary. The continuum limit poses no problem for the adaptation of our construction to this discrete setting. Let $\mu_\delta$ be the discrete measure with point mass $\delta$. There are two operators on the Hilbert space $L^2(\mu_\delta)$ – the averaging transformation $T$ and the minimizer

$M$. With $\{c_n\}$ the one-dimensional binomial sequence in the scaling relation (1.4) for the continuum $\eta$ given by

$$\hat{\eta} = \hat{\chi}^N, \tag{5.1}$$

the averaging transformation is defined by

$$(T\phi)(x) = \frac{1}{\sqrt{2}} \sum_n c_n \phi(2x + n\delta) \tag{5.2}$$

for an arbitrary lattice configuration $\phi \in L^2(\mu_\delta)$. The minimizer $M$ is defined such that the configuration $\psi = M\phi$ minimizes the norm with respect to the constraints

$$T\psi = \phi. \tag{5.3}$$

These operators lie at the heart of renormalization group analysis.

Obviously $M$ is a right inverse of $T$. On the other hand, $M\phi$ is automatically orthogonal to any configuration $T$ annihilates. Hence

$$P = MT \tag{5.4}$$

is an orthogonal projection because

$$\begin{aligned} P^2 &= M(TM)T = MT = P, \\ (P\psi, \psi) &= (P\psi, P\psi) + (P\psi, (1-P)\psi) \\ &= (P\psi, P\psi) \geqslant 0, \end{aligned} \tag{5.5}$$

where $P\psi = MT\psi$ is orthogonal to $(1-P)\psi$ as a consequence of the identity

$$\begin{aligned} T(1-P) &= T - TMT \\ &= 0. \end{aligned} \tag{5.6}$$

The same manipulation shows that the operators

$$P_\ell = M^\ell T^\ell \tag{5.7}$$

are orthogonal projections with

$$P_{\ell+1} P_\ell = P_\ell P_{\ell+1} = P_{\ell+1}. \tag{5.8}$$

Thus we have the orthogonal decomposition

$$L^2(\mu_\delta) = \sum_{\ell=0}^{\infty} \operatorname{ran}(P_\ell - P_{\ell+1}) \tag{5.9}$$

with $P_0 = 1$, and this is the basic part of the completeness argument for the wavelets defined below. It is easy to verify the property

$$\lim_{\ell\to\infty} P_\ell \psi = 0 \tag{5.10}$$

by computing the momentum expression for the minimizer—a routine calculation [10] which we omit here—but the key observation is that

$$\lim_{\ell\to\infty} T^\ell \psi = 0 \tag{5.11}$$

for square-summable $\psi$.

The decomposition (5.9) is the multiscale resolution of Meyer and Mallat in the lattice setting, and the lattice analog of a wavelet with length scale equal to $2^\ell \delta$ is

$$\psi_\ell^{(m)}(x) = \psi_\ell(x - 2^{\ell+1}\delta m), \quad x \in \delta\mathbb{Z}, \tag{5.12}$$

$$\psi_\ell = \psi_{\delta,\ell} = M^\ell \psi_{\delta,0}, \tag{5.13}$$

$$\psi_0(n\delta) = \psi_{\delta,0}(n\delta) = (-1)^n c_{1-n}. \tag{5.14}$$

In the continuum limit, the unit-scale wavelet located at the origin is given by

$$\lim_{\ell\to\infty} \psi_{2^{-\ell},\ell}(r2^{-\ell_0})$$

at a given fixed dyadic point $r2^{-\ell_0}$. We need to show that $\{\psi_\ell^{(m)}\}$ spans $(P_\ell - P_{\ell+1})$ for each $\ell$ if we are to prove completeness for this set of functions that we are really constructing by constrained minimization.

*Reminder.* Our wavelets are not intra-scale orthogonal in their initial construction, but, as usual, this additional orthogonality can be obtained, if desired, because there is no glitch here with boundedness for the inverse square root of the overlap matrix.

To see that $\psi_\ell^{(m)}$ actually lies in $\operatorname{ran}(P_\ell - P_{\ell+1})$, we note that

$$\begin{aligned} P_{\ell+1}\psi_\ell^{(m)} &= M^{\ell+1} T^{\ell+1} \psi_\ell^{(m)} \\ &= M^{\ell+1} T \psi_0^{(m)}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} (T\psi_0^{(m)})(r\delta) &= \frac{1}{\sqrt{2}} \sum_n c_n \psi_0(2r\delta - 2m\delta + n\delta) \\ &= \frac{1}{\sqrt{2}} \sum_n c_n (-1)^n c_{1-2r+2m-n} \\ &= 0, \end{aligned} \tag{5.16}$$

where the sum vanishes by antisymmetry under the index change $n \mapsto 1 - 2r + 2m - n$. Thus $P_{\ell+1}$ annihilates $\psi_\ell^{(m)}$, while $\psi_\ell^{(m)}$ lies in the range of $P_\ell$ because

$$\psi_\ell^{(m)} = M^\ell \psi_0^{(m)} \tag{5.17}$$

and $T^\ell$ is onto.

To conclude our completeness proof, consider an arbitrary configuration $\psi$ in the subspace and set

$$\phi = T^{\ell}\psi. \tag{5.18}$$

Clearly,

$$\psi = P_{\ell}\psi = M^{\ell}\phi, \tag{5.19}$$

while the annihilation of $\psi$ by $M^{\ell+1}T^{\ell+1} = P_{\ell+1}$ implies

$$T\phi = T^{\ell+1}\psi = 0 \tag{5.20}$$

because $M^{\ell+1}$ has zero kernel. Hence

$$\sum_n c_n\phi(2x + n\delta) = 0, \tag{5.21}$$

so if we let

$$h(p) = \sum_n c_n e^{inp}, \tag{5.22}$$

$$g(p) = \sum_n \phi(n\delta)e^{inp}, \tag{5.23}$$

we obtain the relation

$$g(p)h(p)^* + g(p+\pi)h(p+\pi)^* = 0 \tag{5.24}$$

whose general solution is

$$g(p) = e^{ip}h(p+\pi)^*\alpha(p) \tag{5.25}$$

with $\alpha(p)$ an arbitrary $\pi$-periodic function. Since

$$e^{ip}h(p+\pi)^* = \sum_n (-1)^n c_{1-n}e^{inp}, \tag{5.26}$$

the Fourier series

$$\alpha(p) = \sum_n \alpha_m e^{i2mp} \tag{5.27}$$

yields the expansion

$$\phi(n\delta) = \sum_m \psi_0(n\delta - 2m\delta)\alpha_m, \tag{5.28}$$

which is an expansion of $\phi$ in the $\psi_0^{(m)}$. Thus

$$\begin{aligned}\psi &= M^{\ell}\phi \\ &= \sum_m \alpha_m M^{\ell}\psi_0^{(m)} \\ &= \sum_m \alpha_m \psi_{\ell}^{(m)},\end{aligned} \tag{5.29}$$

which is the desired result.

Given the completeness argument for the scalar one-dimensional case, one can easily extend it to more complicated cases. Now, as explained in the Introduction, the construction of antisymmetric-tensor-valued wavelets orthogonal with respect to the exterior-derivative Sobolev norm is the key to obtaining the $L^2$-orthogonal divergence-free vector wavelets in four dimensions. The lattice version of this scheme is to replace the anti-symmetric-valued configurations with scalar-valued functions on the set of oriented plaquettes on $\delta\mathbb{Z}^4$. The Hilbert space norm is based on the lattice exterior derivative:

$$\|\mathbb{F}\|^2 = \delta^4 \sum_{\mu=0}^{3}\sum_{C\in\Pi_\delta^\mu} \mathbb{F}_{\partial C}^2, \tag{5.30}$$

where $\Pi_\delta^\mu$ is the set of oriented three-dimensional cubes on $\delta\mathbb{Z}^4$ perpendicular to the $\mu$-coordinate direction and

$$\begin{aligned}\mathbb{F}_{\partial C} = (&\mathbb{F}(P_{\mu\lambda}^+) - \mathbb{F}(P_{\mu\lambda}^-) + \mathbb{F}(P_{\lambda\sigma}^+) - \mathbb{F}(P_{\lambda\sigma}^-) \\ &+ \mathbb{F}(P_{\sigma\nu}^+) - \mathbb{F}(P_{\sigma\nu}^-))\delta^{-1}\end{aligned} \tag{5.31}$$

with $\mu, \nu, \lambda, \sigma$ distinct and $P_{\nu\lambda}^+$ (resp. $P_{\nu\lambda}^-$) as the $\nu\lambda$-coordinate face on the positive (resp. negative) $\sigma$-coordinate side of $C$, both plaquettes oriented in the same direction. The order $(\nu, \lambda, \sigma)$ is chosen such that $P_{\nu\lambda}^+$, $P_{\lambda\sigma}^+$, $P_{\sigma\nu}^+$ are oriented out of $C$ (resp. into $C$) if $\partial C$ has outward (resp. inward) orientation. The degeneracy of the Hilbert space norm is eliminated by defining the Hilbert space itself as the space of configurations $\mathbb{F}$ such that $\|\mathbb{F}\| < \infty$ and

$$\sum_{\nu=0}^{\infty}(\mathbb{F}(P_{\mu\nu} + \vec{x} + \delta\,\vec{e}_\nu) - \mathbb{F}(P_{\mu\nu} + \vec{x})) = 0, \quad \vec{x} \in \delta\mathbb{Z}^4, \tag{5.32}$$

where $P_{\mu\nu}$ is the plaquette on $\delta\mathbb{Z}^4$ in the $\mu\nu$-coordinate plane with vertex at the origin and orientation $\vec{e}_\mu \wedge \vec{e}_\nu$. This lattice divergence-free condition is not to be confused with the divergence-free property of the vector wavelets ultimately constructed. It is only a type of gauge-fixing condition.

Our geometric sophistication of the completeness argument has an additional twist. The averaging transformation $T_\ell$ for $\ell$ steps maps our Hilbert space onto the Hilbert space of functions on the set of oriented three-dimensional cubes on $\delta\mathbb{Z}^4$ annihilated by the fourth-order lattice exterior derivative and with the $L^2$-norm. Specifically,

$$T_\ell = T^{\ell}\partial, \tag{5.33}$$

where $\partial$ denotes the third-order lattice exterior derivative and

$$(T\mathbb{G})(C) = \frac{1}{4}\sum_n c_n \sum_{C'\in\mathcal{S}_c} \mathbb{G}(2C' + n\delta) \tag{5.34}$$

with $\mathscr{S}_c$ the set of subcubes of $C$ obtained from bisection in each direction. On the other hand the minimizer $M_\ell$ maps the latter Hilbert space back into the former one. $M_\ell$ is defined such that the configuration $\mathbb{F} = M_\ell \mathbb{G}$ minimizes the norm with respect to the constraints (5.32) and

$$\mathbb{G} = T_\ell \mathbb{F}. \tag{5.35}$$

The condition (5.32) makes it impossible to express $M_\ell$ as a power of one-step minimizers, but $M_\ell$ is still a right inverse of $T_\ell$, and $M_\ell T_\ell$ is still on orthogonal projection of the former Hilbert space.

*Note added in proof.* Shortly before the appearance of this article, these authors—together with Paul Uhlig—found an easy construction for the four-dimensional case [11]. It is a momentum vortex field construction that applies only to two, four, and eight dimensions.

## REFERENCES

1. G. Battle and P. Federbush, Divergence-free vector wavelets, *Michigan Math. J.* **40** (1993), 181.
2. P. Federbush, Local strong solution of the Navier–Stokes equations in terms of local estimates, preprint, Univ. of Michigan.
3. P. Federbush, Navier and Stokes meet the wavelet, *Comm. Math. Phys.* **155** (1993), 219.
4. P. Lemarié, Ondelettes vecteurs a divergence nulle, preprint, Université de Paris-Sud.
5. P. Lemarié, Analyses multi-resolutions non-orthogonales et ondelettes vecteurs a divergence nulle, preprint, Université de Paris-Sud.
6. P. Federbush, *Comm. Math. Phys.* **107** (1986), 319.
7. P. Federbush and C. Williamson, *J. Math. Phys.* **28**, No. 6 (1987), 1416.
8. P. Lemarié, *J. Math. Pures Appl.* **67** (1988), 227.
9. I. Daubechies, *Comm. Pure Appl. Math.* **41** (1988), 906.
10. K. Gawedzki and A. Kupiainen, *Comm. Math. Phys.* **77** (1980), 31.
11. G. Battle, P. Federbush, and P. Uhlig, Wavelets for quantum gravity and divergence-free wavelets, preprint.