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Solutions Analytic in Time Are Not Dense

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For the wave equation

$$(\partial_t + a\partial_x) u = 0, \tag{1}$$

with $a(t, x)$ real valued, we ask whether the solutions which are real analytic with respect to the time variable t are dense. The question is motivated by the study of low frequency control by Bardos, Lebeau, and Rauch; see [1] and [2]. The answer depends on the regularity of the coefficient a. For example, if a is real analytic in t, x, the Cauchy–Kovalevsky theorem implies that the solutions real analytic in t, x are dense since it suffices to approximate the initial data by polynomials and then to solve the resulting approximate problems using the Cauchy–Kovalevsky theorem [3].

Somewhat surprisingly, the same positive result is valid when $a = a(x)$ is only C^1 in x [4]. The proof is by regularization in t with a Gaussian kernel

$$j_\varepsilon = ce^{-t^2/\varepsilon^2}.$$

We show that the hypothesis that a is independent of t cannot be replaced by the weaker assumption that a is real analytic in t, even when it is C^∞ in x.

First we present a preliminary result whose proof is particularly simple. If the coefficient a is required only to be C^∞ in t, x, then there are smooth a for which the only C^1 solutions to (1) which are real analytic in t are the constant solutions. Take

$$a := \begin{cases} 1 + e^{-1/(t+x)}, & t + x > 0, \\ 1, & t + x \leqslant 0. \end{cases}$$

Then a is C^∞ in t and x.

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THEOREM. *For $T>0$, if $u\in C^1([0, T]\times[-T, T])$ satisfies* (1) *and is real analytic in t, then u is constant.*

Proof. For $x_0\in\,]-T, 0[$, consider $u(t, x_0)$. By the uniqueness of analytic continuation, the value of u at points (t, x_0) for $t>-x_0$ is uniquely determined by $u(t, x_0)$ for $t\in\,]0, -x_0[$.

Note that u satisfies the equation

$$(\partial_t+\partial_x)u=0 \tag{2}$$

in the region below the line $t+x=0$ since here (2) is identical to (1). By considering the characteristics of (2) with speed $dx/dt=1$, we see that the solution below $t+x=0$ extends to a solution u' above $t+x=0$, and that this solution is analytic in time. Then by the uniqueness of analytic continuation, $u'=u$ below $t+x=0$ implies that $u'=u$ above the line $t+x=0$. Thus u solves both (1) and (2) in $[0, T]\times[-T, 0]$.

Subtracting (2) from (1) in the region above $t+x=0$, we obtain

$$(a-1)\,\partial_x u=0.$$

Since $a>1$ in this region, $\partial_x u=0$, and then $\partial_t u=0$, so u is constant in the portion of $]0, T[\times]-T, 0[$ above $t+x=0$.

Then by the uniqueness of analytic continuation, u is constant along lines $x=x_0<0$, and therefore in $[0, T]\times[-T, 0]$. Let C denote the value of u in this region.

By tracing the characteristics of (1) into the region where x is positive, we find for each $x\in[0, T]$ a nonempty open interval in $[0, T]\times\{x\}$ on which $u=C$.

Again using the uniqueness of analytic continuation, we see that $u=C$ on $[0, T]\times\{x\}$. Therefore $u=C$ in $[0, T]\times[-T, T]$. ∎

In the above example, the coefficient a is smooth but not real analytic in t. Next we give an example of an a which is real analytic in t, and for which the solutions real analytic in t are not dense. Set

$$a_\varepsilon(t, x):=\sin t+x_+e^{-\varepsilon/x}, \qquad \text{where} \quad x_+=\begin{cases} x, & x>0,\\ 0, & x\leqslant 0.\end{cases}$$

Then a_ε is analytic in t and smooth in t, x.

THEOREM. *There is an $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$, the $C^1(\mathbf{R}^2)$ solutions which are analytic in time are 2π-periodic in time. In particular these solutions are not dense in the sense that their $\mathscr{D}'(\mathbf{R}^2)$ closure contains no nontrivial solutions with Cauchy data supported in $x>e$.*

Proof. The characteristic curves for (1) have speed $dx/dt = a_\varepsilon = \sin t + x_+ e^{-\varepsilon/x}$, and the solution u is constant along characteristics. Note that since the vector field is $(1, a_\varepsilon)$, all characteristics can be extended for all real t, and can be expressed as $\gamma(t) = (t, x(t))$. In the entire plane the vector field is 2π-periodic in t, so translation by any multiple of 2π in t gives exactly the same characteristics. In the region $x < 0$, the characteristics are given by $x(t) = -\cos t + c$ with constant c.

We next show that for sufficiently small ε all characteristics meet the half plane $x < 0$.

First we find ε_0 so that for $0 < \varepsilon < \varepsilon_0$, there is an a_ε-characteristic γ_ε which crosses both $x = 0$ and $x = e$.

We want γ_ε to cross into $x > e$, since for $xe^{-\varepsilon/x} > 1$, the speed a_ε of the characteristics is positive. If we consider only $\varepsilon < e$, then for $x > e$, we have

$$xe^{-\varepsilon/x} > e \cdot e^{-e/e} = e \cdot e^{-1} = 1,$$

so any characteristic through (t', x') for $x' > e$ has positive speed at all later times. Then since $xe^{-\varepsilon/x}$ is an increasing function of x, the speed a_ε will be bounded away from zero, so along the characteristic x increases to ∞ as t approaches ∞.

For any $x > 0$, as ε approaches 0, $e^{-\varepsilon/x}$ approaches 1. So as ε approaches 0, the speed a_ε of the characteristics approaches

$$\frac{dx}{dt} = a_0 := \sin t + x_+,$$

whose characteristics in the region $x \geqslant 0$ are given by

$$x(t) = be^t - \tfrac{1}{2}(\sin t + \cos t) \tag{3}$$

for constants b.

Now fix a point (t', x') with $x' > e$. For $0 \leqslant \varepsilon < e$, let γ_ε denote the a_ε characteristic of (1) through (t', x'). For $t > t'$ we have that $x(t) > x'$ on any γ_ε. By solving (3) for b in terms of x' and t', we find an explicit formula for the connected component of γ_0 intersected with $x \geqslant 0$ containing (t', x').

Suppose that γ_0 lies in the region $x \geqslant 0$ for all t. Then (3) holds for all t and satisfies $x(t) \geqslant 0$. But as t approaches $-\infty$, the first term goes to 0, and the second term oscillates between plus and minus $2^{-1/2}$. This is a contradiction, so for some t, $x(t) < 0$.

Choose a point (t'', x'') on γ_0 such that $x'' < 0$, and an open disk D about (t'', x'') lying in the region $x < 0$. Then as ε goes to 0, the a_ε-characteristics γ_ε through (t', x') will intersect the disk D. Pick ε_0 so that for $0 < \varepsilon < \varepsilon_0$, γ_ε intersects D. Thus these γ_ε cross the t axis and the line $x = e$.

Now we use these γ_ε to show that all characteristics passing through points in the half plane $x>0$ also pass through points in the half plane $x<0$, provided $0<\varepsilon<\varepsilon_0$.

Since characteristics cannot intersect each other, a characteristic which lies to the left of a characteristic γ at some time lies to the left of γ at all times $t \in \mathbf{R}$.

Suppose there is a characteristic γ' which is in the $x>0$ half plane for all times t. Then γ' intersects the x-axis at some $x_0>0$. Since γ_ε crosses the line $x=e$, beyond some time t_0 the x-coordinate of γ_ε is greater than x_0. Let γ_ε^k denote a translate of γ_ε by an amount $2k\pi$ in t, for integer values of k. Note that the periodicity in t of the vector field makes γ_ε^k an a_ε-characteristic as well. Then there is a γ_ε^k such that γ_ε^k intersects the x-axis at a point $(0, x_1)$ with $x_0<x_1$.

Since γ' lies to the left of γ_ε^k at time $t=0$, it does so at all times. However, since γ_ε crosses the t-axis, so does γ_ε^k; since γ' lies to the left of γ_ε^k, it must cross the t-axis also. Thus all a_ε-characteristics of (1) intersect the region $x<0$.

Since u is constant along characteristics and all characteristics in the region $x>0$ cross into the region $x<0$, if u vanishes for $x<0$ it vanishes everywhere.

Now consider a solution u of (1) (with coefficient a_ε) which is real analytic in time and C^1 in t, x. In the left half plane, the characteristics of (1) are

$$\gamma(t)=(t, x(t))=(t, -\cos t+c)$$

with constants c, and u is constant along characteristics. So any γ which remains entirely in the left half plane, i.e., for which $c<-1$, gives us

$$\begin{aligned}u(t, -\cos t+c) &= u(\gamma(t))\\ &= u(\gamma(t+2\pi))\\ &= u(t+2\pi, -\cos(t+2\pi)+c)\\ &= u(t+2\pi, -\cos t+c),\end{aligned}$$

so that

$$u(t, x)=u(t+2\pi, x)$$

wherever (t, x) lies on a characteristic γ which is contained in the left half plane.

If $u(t, x)$ is a solution, then so is $w(t, x)=u(t, x)-u(t+2\pi, x)$, and u analytic in t implies that w is as well. We note that w vanishes on every characteristic which remains entirely in the $x \leqslant 0$ half plane.

The characteristic given by $x(t) = -\cos t - 1$ remains in $x \leqslant 0$, and every characteristic to its left (given by $x(t) = -\cos t + c$ for $c < -1$) remains in $x < 0$.

Thus for every $x_0 < 0$, there is a nonempty interval $]T, T'[$ such that for all $t \in]T, T'[$, (t, x_0) lies on a characteristic on which w vanishes (see Fig. 1). Then by the uniqueness of analytic continuation, $w(t, x_0)$ vanishes for all time t.

Thus w vanishes for $x < 0$ and therefore everywhere. So u is everywhere 2π-periodic in time.

To prove the last statement of the theorem, first note that any element of the $\mathscr{D}'(\mathbf{R}^2)$-closure of such solutions is a solution of (1) and is also 2π-periodic in time. It suffices to show that if u is a 2π-periodic solution with Cauchy data v supported in $x > e$, then $u \equiv 0$.

For such a v there is a $\delta > 0$ such that the support of v is contained in $x > e + \delta$. Let u be the solution of (1) with Cauchy data v. Then the support of u lies on the characteristics which intersect the support of v. So the support of u lies to the right of the characteristic $\gamma = (t, x(t))$ through $e + \delta$.

However, $x(t)$ increases to ∞ as t increases to ∞, and for any $x \in \mathbf{R}$, there is a $T(x)$ such that for $t > T(x)$, (t, x) does not lie in the support of u. Since u is 2π periodic in t, this implies that $u = 0$ in $\mathbf{R}^2$. ∎

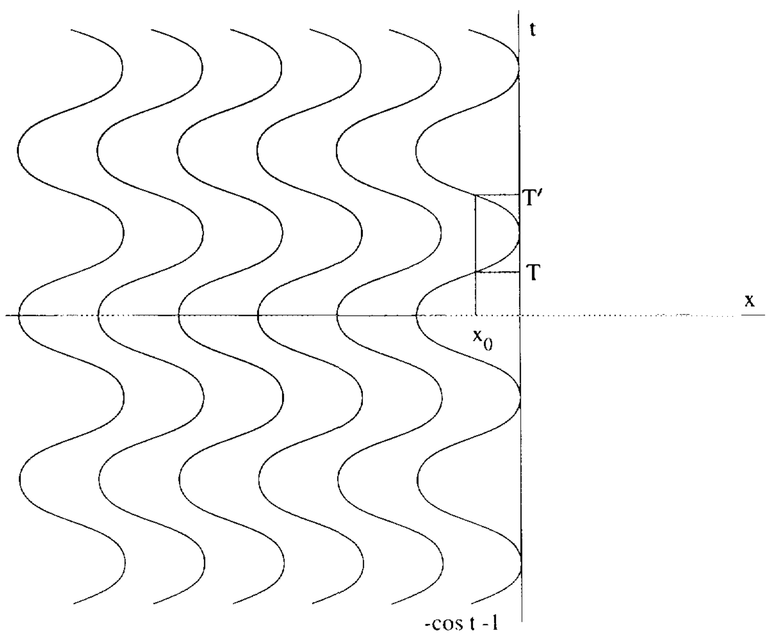

FIGURE 1

Remark. It is easy to construct second order wave equations, e.g., $(\partial_t + a\partial_x)\,\partial_t u = 0$, whose solutions real analytic in time are not dense.

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