ADVANCES IN MATHEMATICS 105, 42–58 (1994)

The Branching Rules for Affine Lie Algebras

SHIRONG LU

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

1. INTRODUCTION

The representations of affine Lie algebras are one of the main tools in the construction of conformal field theories. They underlie some of the most important examples of rational CFT, namely the West–Zumino–Witten models [15]. In these models the Hilbert space is built from unitary representations of an affinie Lie algebra g at some positive integer level k. Another very useful tool is the coset construction [2]. Given an embedding $p \subset g$ of affine Lie algebras, one obtains a set of representations $U(\Lambda, \lambda)$ of the Virasoro algebra, which interwines with the action of p on $L(\Lambda)$. More precisely, we have

$$L(\Lambda) = \sum_{\lambda} U(\Lambda, \lambda) \otimes L(\lambda), \tag{1}$$

where the sum runs over all representations of p of level uk with k being the level of $L(\Lambda)$, u the index of $p \subset g$, and $U(\Lambda, \lambda)$ the subspace of p-highest weight vectors with weight λ. The central charge of the Virasoro algebra acting on $U(\Lambda, \lambda)$ is $c(g) - c(p)$ with

$$c(g) = \frac{k \dim \bar{g}}{k + h^{\vee}},$$

$\bar{g}$ being the finite part of g, $h^{\vee}$ the dual Coxeter number, and $c(g)$ the central charge of the Sugawara representation of the Virasoro algebra acting on $L(\Lambda)$. Let h and $\dot{h}$ be the Cartan subalgebras of g and p, respectively. One can choose them so that $\dot{h} \subset h$. Let $H = \{\tau \in C \mid \operatorname{Im} \tau > 0\}$ be the upper half plane. The normalized character χ_Λ of $L(\Lambda)$ is a holomorphic function on $H \times h$:

$$\chi_\Lambda(\tau, z) = q^{-c(g)/24} \operatorname{Tr}_{L(\Lambda)}(e^{2\pi i(\tau L_0 + z)}), \tag{2}$$

where as usual $q = e^{2\pi i \tau}$. Suppose that $z \in \dot{h}$, then from (1) we get

$$\chi_\Lambda(\tau, z) = \sum_{\lambda} b_\lambda^\Lambda(\tau)\, \chi_\lambda(\tau, z), \tag{3}$$

0001-8708/94 $9.00

where the branching function b_λ^Λ is

$$b_\lambda^\Lambda(\tau) = q^{(c(g)-c(p))/24}\,\mathrm{Tr}_{U(\Lambda,\lambda)}q^{L_0}. \tag{4}$$

Let Δ_+ be the set of positive roots of g, Q the root lattice, P the weight lattice, and W the Weyl group, and let $\bar{\Lambda}$ denote the "finite parts" of Λ, etc.; see [6]. The modular transformation properties of characters are given by [8]

$$\begin{cases} \chi_\Lambda(\tau+1, z) = e^{2\pi i s_\Lambda}\chi_\Lambda(\tau, z), \\ \chi_\Lambda\left(-\dfrac{1}{\tau}, \dfrac{z}{\tau}\right) = e^{\pi k i (z|z)/\tau} \displaystyle\sum_{M \in P_+^{(k)}} a(\Lambda, M)\,\chi_M(\tau, z), \end{cases} \tag{5}$$

where $s_\Lambda = (\Lambda + 2\rho \mid \Lambda)/2(k+h^\vee) - c(g)/24$, $\Lambda \in P_+^{(k)}$ is the set of dominant highest weight of level k, and

$$a(\Lambda, M) = i^{|\bar{\Delta}_+|}\,|\bar{P}/L|^{-1/2}(k+h^\vee)^{-1/2} \sum_{w \in \bar{W}} e^{-(\pi i/(k+h^\vee))(\bar{\Lambda}+\bar{\rho} \mid w(\bar{M}+\bar{\rho}))}, \tag{6}$$

with L being the subset of $\bar{Q}$ spanned by all long roots. Using the Weyl character formula, one can rewrite (6) as

$$a(\Lambda, M) = a(\Lambda, k\Lambda_0)\,\mathrm{Tr}_{\bar{M}} \exp \frac{2\pi i}{k+h^\vee}(\bar{\Lambda}+\bar{\rho}),$$

where $\mathrm{Tr}_{\bar{M}}$ denotes the trace over the finite dimensional $\bar{g}$-module with highest weight $\bar{M}$. One knows that $a(\Lambda, M)$ is a unitary symmetric matrix.

Due to (3), the transformation formula for the branching function b_λ^Λ was obtained from that for the affine characters and was given by [9]

$$\begin{cases} b_\lambda^\Lambda(\tau+1) = e^{2\pi i (s_\Lambda - \dot{s}_\lambda)} b_\lambda^\Lambda(\tau), \\ b_\lambda^\Lambda\left(-\dfrac{1}{\tau}\right) = \displaystyle\sum_{M \in P_+^{(k)}, \mu \in \dot{P}_+^{(uk)}} a(\Lambda, M)\,\dot{a}(\lambda, \mu')\,b_\mu^M(\tau), \end{cases} \tag{7}$$

where μ' denotes the highest weight of the contravariant representation of $L(\mu)$ and the dot denotes the objects attached to p. Note that some of these results were generalized to the so-called modular invariant representations, cf. [10, 11, 12].

Consider the restriction to the Heisenberg subalgebra; the function $\eta^{-l} b_\lambda^\Lambda$ turns out to be the string function, which counts the weight multiplicities. It is shown that the string functions for all unitary highest weight representation for $A_1^{(1)}$ are the indefinite Hecke modular form, see [7, 8]. However, it should be pointed out that the string functions for the modular invariant modules are not necessarily in modular form even for $A_1^{(1)}$, see [13].

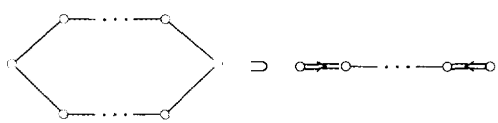
The problem of determining the string functions of all level 1 modules for $C_l^{(1)}$ was stated as an open problem in [6]. Only recently a formula was found in [9] and written in terms of Virasoro characters by considering the decomposition with respect to the subalgebra $C_{l-1}^{(1)} \oplus A_1^{(1)}$. In this paper we first give a new formula for the string functions of all level 1 modules for $C_l^{(1)}$. Our formula will be expressed in terms of quadratic theta functions of degree $l-1$. By comparing our formula with the one obtained by Kac and Wakimoto, we get some curious identities between theta functions. Next, motivated by the recent paper [12], we find a general formula expressing general branching functions in terms of string functions, which generalizes one of the main results in [12] by Kac and Wakimoto on the branching function for the winding subalgebras. An an application we compute explicitly the branching functions for the complementary decompositions.

I thank Professor V. Kac for sending me the paper [12] which motivated us to write a part of this paper and Professor R. Griess for some helpful discussions.

2. On the Level One String Functions of $C_l^{(1)}$

In this section we use $\sim$ to refer to an object attached to the affine Lie algebra $A_{2l-1}^{(1)}$. We first prove a result on the branching function for the embedding $C_l^{(1)} \subset A_{2l-1}^{(1)}$ by using only the modular invariance and transformation formula. This result was obtained before in [3] by a complicated calculation. We then show how to use this formula to derive the string functions for all level 1 modulles for $C_l^{(1)}$.

We have a canonical embedding $C_l^{(1)} \subset A_{2l-1}^{(1)}$ induced by the involution of a Dynkin diagram



$A_{2l-1}^{(1)}$ $\qquad$ $C_l^{(1)}$

Namely let $\tilde{e}_i, \tilde{f}_i, \tilde{h}_i$ $(0 \leqslant i \leqslant 2l-1)$ be the Chevalley generators of $A_{2l-1}^{(1)}$. Then the elements e_i, f_i, h_i $(0 \leqslant i \leqslant l)$ given by $(x = e, f, h)$

$$x_0 = \tilde{x}_0, \qquad x_i = \tilde{x}_i + \tilde{x}_{2l-i} \, (1 \leqslant i \leqslant l-1), \qquad x_l = \tilde{x}_l,$$

together with the derivation d, generate a subalgebra g which can be identified with $C_l^{(1)}$.

Denote by $\tilde{\Lambda}_j$ $(0 \leqslant j \leqslant 2l-1)$, Λ_k $(0 \leqslant k \leqslant l)$ the fundamental weights of $A_{2l-1}^{(1)}$ and $C_l^{(1)}$, respectively. Let

$$\tilde{s}_j := s_{\tilde{\Lambda}_j} = \frac{j(2l-j)}{4l} - \frac{2l-1}{24},$$

$$s_k := s_{\Lambda_k} = \frac{k(2l+2-k)}{4(l+2)} - \frac{l(2l+1)}{24(l+2)}.$$

Then by definition we have

$$b_{j,k}^{(l)} := b_{\Lambda_k}^{\tilde{\Lambda}_j}(A_{2l-1}^{(1)}, C_l^{(1)}) = q^{\tilde{s}_j - s_k} \sum_{n \neq Z} \mathrm{mult}_{\tilde{\Lambda}_j}(\Lambda_k - n\delta)\, q^n. \tag{8}$$

Given two integers j, k so that $j \equiv k+1 \bmod 2$, the indefinite Hecke modular form is defined by

$$\theta_{j,k}^{(l)}(\tau) = \sum_{\substack{(x,y) \in R^2 \\ -|x| < y \leqslant |x| \\ (x,y) \text{ or } (1/2-x,\, 1/2+y) \in \mu(j,k) + Z^2}} \mathrm{sign}(x)\, q^{(l+2)x^2 - ly^2}, \tag{9}$$

where $\mu(j,k) = (j/2(l+2), k/2l)$. We have the following

$$\theta_{j,k}^{(l)} = \theta_{j,2l-k}^{(l)}, \qquad \theta_{0,k}^{(l)} = \theta_{l+2,k}^{(l)} = 0.$$

PROPOSITION 2.1. *$\eta(\tau)^2 b_{j,k}^{(l)} = \theta_{k+1,j}^{(l)}$ if $j \equiv k \pmod 2$ and 0 otherwise.*

Proof. According to the transformation formula for the Hecke from (see [9]), we have

$$\eta\left(-\frac{1}{\tau}\right)^{-2} \theta_{k+1,j}^{(l)}\left(-\frac{1}{\tau}\right)$$

$$= \frac{1}{\sqrt{l(l+2)}} \sum_{\substack{1 \leqslant r \leqslant l+1 \\ 0 \leqslant s \leqslant 2l-1 \\ r+s \equiv 1 \bmod 2}} \sin\frac{\pi(k+1)r}{l+2} \cos\frac{\pi js}{l}\, \eta(\tau)^{-2} \theta_{r,s}^{(l)}(\tau)$$

$$= \frac{1}{\sqrt{l(l+2)}} \sum_{\substack{0 \leqslant r \leqslant l \\ 0 \leqslant s \leqslant l \\ r \equiv s \bmod 2}} \varepsilon_s \sin\frac{\pi(k+1)(r+1)}{l+2} \cos\frac{\pi js}{l}\, \eta(\tau)^{-2} \theta_{r+1,s}^{(l)}(\tau).$$

Here we use $\theta_{r,2l-s}^{(l)}(\tau) = \theta_{r,s}^{(l)}(\tau)$ and

$$\varepsilon_s = \begin{cases} 1 & \text{if } 1 \leqslant s \leqslant l-1, \\ 2 & \text{if } s = 0, l. \end{cases}$$

From the calculation of the number $a(\Lambda, M)$ (cf. [9]), we have

$$\tilde{a}(\tilde{\Lambda}_j, \tilde{\Lambda}_s) = \frac{1}{\sqrt{2l}} e^{\pi i j s / l}, \qquad a(\Lambda_r, \Lambda'_k) = \sqrt{\frac{2}{l+2}} \sin \frac{\pi(r+1)(k+1)}{l+2}.$$

It follows from Eq. (7) that

$$b_{j,k}^{(l)}\left(-\frac{1}{\tau}\right) = \sum_{\substack{0 \leqslant s \leqslant 2l-1 \\ 0 \leqslant r \leqslant l \\ r \equiv s \bmod 2}} \tilde{a}(\tilde{\Lambda}_j, \tilde{\Lambda}_s)\, a(\Lambda_r, \Lambda'_k)\, b_{s,r}^{(l)}(\tau)$$

$$= \frac{1}{\sqrt{l(l+1)}} \sum_{\substack{0 \leqslant s \leqslant l \\ 0 \leqslant r \leqslant l \\ r \equiv s \bmod 2}} \varepsilon_s \sin \frac{\pi(k+1)(r+1)}{l+2} \cos \frac{\pi j s}{l}\, b_{s,r}^{(l)}(\tau).$$

Hence $\eta^{-2}\theta_{k+1,j}^{(l)}$ and $b_{j,k}^{(l)}$ have the same transformation formula. Comparing the polar parts, we see that

$$\eta^{-2}\theta_{k+1,j}^{(l)} = q^{\theta}(1 + a_1 q + a_2 q_2 + \cdots)$$

with

$$\theta = \tfrac{1}{4}(k+1)^2(l+2)^{-1} - \tfrac{1}{4} j^2 l^{-1} - \max(0, \tfrac{1}{2}(j-k)) - \tfrac{1}{12}.$$

Since $\tilde{\Lambda}_j|_{C_l^{(1)}} = \Lambda_j$ for $0 \leqslant j \leqslant l$, $\Lambda_k \in P(\Lambda_j)$ iff $j \equiv k \bmod 2$ and

$$\Lambda_k - \max(0, \tfrac{1}{2}(k-j))\delta \in \max(\Lambda_j),$$

we have

$$b_{j,k}^{(l)} = q^{b}(1 + b_1 q + b_2 q^2 + \cdots),$$

with

$$b = \tilde{s}_j - s_k + \max(0, \tfrac{1}{2}(k-j)).$$

It is now easy to check that

$$\tilde{s}_j - s_k = \tfrac{1}{2}(j-k) + \tfrac{1}{4}(k+1)^2\,(l+2)^{-1} - \tfrac{1}{4} j^2 l^{-1} - \tfrac{1}{12}$$

and $\theta = b$. Therefore have

$$\eta(\tau)^{-2}\theta_{k+1,j}^{(l)} = b_{j,k}^{(l)}. \quad \blacksquare$$

Now we show how to use this to find a new formula for the string functions of all level one modules for $C_l^{(1)}$. Based on the embedding $C_l^{(1)} \subset A_{2l-1}^{(1)}$, the formula will be obtained by using the explicit character formula for the basic modules of $A_{2l-1}^{(1)}$.

Let $\{\tilde{\alpha}_i \mid i=0, 1,..., 2l-1\}$ be the set of simple roots for the affine Lie algebra of type $A_{2l-1}^{(1)}$. Define

$$\alpha_i = \begin{cases} \tilde{\alpha}_i & \text{if} \quad i=0, l, \\ \frac{1}{2}(\tilde{\alpha}_i + \tilde{\alpha}_{2l-i}) & \text{if} \quad 1 \leqslant i \leqslant l-1. \end{cases}$$

Then $\{\alpha_i \mid i=0, 1, ..., l\}$ forms a set of simple roots for $C_l^{(1)}$. Let

$$\tilde{M} = \sum_{i=1}^{2l-1} Z\tilde{\alpha}_i,$$

$$M = \sum_{i=1}^{l-1} Z(2\alpha_i) + Z\alpha_l \subset \tilde{M}.$$

We have from [6] that

$$\chi_{\tilde{\Lambda}_j} = c_{\tilde{\Lambda}_j}^{\tilde{\Lambda}_j} \Theta_{\tilde{\Lambda}_j}, \tag{10}$$

where the theta function is defined by

$$\Theta_{\tilde{\Lambda}_j} = q^{|\tilde{\Lambda}_j|^2/2} \sum_{\gamma \in \tilde{M}} e^{t_\gamma(\tilde{\Lambda}_j)} \tag{11}$$

and $t_\gamma(\tilde{\Lambda}_j) = \tilde{\Lambda}_j + \gamma - ((\tilde{\Lambda}_j \mid \gamma) + \frac{1}{2}|\gamma|^2)\tilde{\delta}$.

Now according to the Eq. (3) we have

$$\chi_{\tilde{\Lambda}_j} = \sum_{k=0}^{l} b_{\Lambda_k}^{\tilde{\Lambda}_j} \chi_{\Lambda_k}. \tag{12}$$

Since $(b_{\Lambda_k}^{\tilde{\Lambda}_j})_{0 \leqslant j,k \leqslant l}$ is non-degenerate, denote by $(d_{\tilde{\Lambda}_j}^{\Lambda_k})_{0 \leqslant j,k \leqslant l}$ the inverse of $(b_{\Lambda_k}^{\tilde{\Lambda}_j})$, we get from (12) by multiplying $(d_{\tilde{\Lambda}_j}^{\Lambda_k})$

$$\chi_{\Lambda_k} = \sum_{j=0}^{l} d_{\tilde{\Lambda}_j}^{\Lambda_k} \chi_{\tilde{\Lambda}_j}. \tag{13}$$

Now define

$$\bar{M} = \sum_{i=1}^{l-1} Z\tilde{\alpha}_i, \qquad Q = \sum_{i=1}^{l-1} Z\alpha_i,$$

then $\tilde{M} = M \oplus \bar{M}$ and there is an isomorphism $\sim: \bar{Q} \cong \bar{M}$ defined by putting $\sim(\alpha_i) = \tilde{\alpha}_i$. Hence for $0 \leqslant j \leqslant l$ we have from (11)

$$\begin{aligned} \Theta_{\tilde{\Lambda}_j} d^{|\tilde{\Lambda}_j|^2/2} & \sum_{\gamma \in \tilde{M}} e^{t_\gamma(\tilde{\Lambda}_j)} \\ &= q^{|\tilde{\Lambda}_j|^2/2} \sum_{\gamma \in M} \sum_{\alpha \in \bar{M}} e^{t_\gamma t_\alpha(\tilde{\Lambda}_j)} \\ &= q^{|\tilde{\Lambda}_j|^2/2} \sum_{\gamma \in M} \sum_{\alpha \in \bar{M}} e^{t_\gamma(\tilde{\Lambda}_j + \alpha - ((\tilde{\Lambda}_j \mid \alpha) + (1/2)|\alpha|^2)\delta)} \end{aligned}$$

$$= \sum_{\alpha \in \bar{M}} q^{|\bar{\Lambda}_j + \alpha|^2/2} \sum_{\gamma \in M} e^{t_\gamma(\bar{\Lambda}_j + \alpha)}$$

$$= \sum_{\alpha \in \bar{Q}} q^{(1/2)(|\bar{\Lambda}_j + \bar{\alpha}|^2 - |\Lambda_j + \alpha|^2)} \Theta_{\Lambda_j + \alpha}.$$

Plugging this into (10) and using the property

$$\Theta_{\Lambda_j + \alpha + \beta} = \Theta_{\Lambda_j + \alpha} \text{ for any } \alpha \in \bar{Q}, \beta \in M,$$

Eq. (13) becomes

$$\begin{aligned} \chi_{\Lambda_k} &= \sum_{j=0}^{l} d_{\Lambda_j}^{\Lambda_k} \chi_{\bar{\Lambda}_j} \\ &= \sum_{j=0}^{l} d_{\Lambda_j}^{\Lambda_k} c_{\bar{\Lambda}_j}^{\bar{\Lambda}_j} \sum_{\alpha \in \bar{Q}} q^{(1/2)(|\bar{\Lambda}_j + \bar{\alpha}|^2 - |\Lambda_j + \alpha|^2)} \Theta_{\Lambda_j + \alpha} \\ &= \sum_{\alpha \in \bar{Q}/2\bar{Q}} \left(\sum_{j=0}^{l} d_{\Lambda_j}^{\Lambda_k} c_{\bar{\Lambda}_j}^{\bar{\Lambda}_j} \sum_{\beta \in 2\bar{Q}} q^{(1/2)(|\bar{\Lambda}_j + \bar{\alpha} + \bar{\beta}|^2 - |\Lambda_j + \alpha + \beta|^2)} \right) \Theta_{\Lambda_j + \alpha}. \end{aligned} \tag{14}$$

Note that from [6] we have

$$\chi_{\Lambda_k} = \sum_{\substack{\mu \in \Lambda_k + Q + C\delta \\ \mathrm{mod}(M + C\delta)}} c_\mu^{\Lambda_k} \Theta_\mu = \sum_{\alpha \in \bar{Q}/2\bar{Q}} c_{\Lambda_k + \alpha}^{\Lambda_k} \Theta_{\Lambda_k + \alpha}. \tag{15}$$

Now if $\alpha \in \bar{Q}/2\bar{Q}$ is fixed, then there is $\alpha' \in \bar{Q}/2\bar{Q}$ so that

$$\Lambda_j + \alpha' \equiv \Lambda_k + \alpha (\mathrm{mod}\ M) \text{ iff } j \equiv k \text{ mod } 2),$$

where if this is the case, then $\alpha' = \alpha + \alpha_{j,k}$ and $\alpha_{j,k}$ is given by

$$\alpha_{j,k} = \begin{cases} \alpha_{k+1} + \alpha_{k+3} + \cdots + \alpha_{j-1} & \text{if } \ k < j, \\ \alpha_{j+1} + \alpha_{j+3} + \cdots + \alpha_{k-1} & \text{otherwise.} \end{cases}$$

Since the theta functions $\Theta_{\Lambda_k + \alpha}$ ($\alpha \in \bar{Q}/2\bar{Q}$) are linearly independent over holomorphic functions on the upper half plane (cf. [6, Chap. 13]), by comparing formulas (14) and (15) we get

$$c_{\Lambda_k + \alpha}^{\Lambda_k} = \sum_{\substack{0 \leq j \leq l \\ j \equiv k \text{ mod } 2}} d_{\Lambda_j}^{\Lambda_k} c_{\bar{\Lambda}_j}^{\bar{\Lambda}_j} \sum_{\beta \in 2\bar{Q}} q^{(1/2)(|\bar{\Lambda}_j + \bar{\alpha}_{j,k} + \bar{\alpha} + \bar{\beta}|^2 - |\Lambda_j + \alpha_{j,k} + \alpha + \beta|^2)}.$$

Choosing $\alpha = \alpha_{k,i}$ and putting

$$\lambda_{i,j} = \Lambda_i - \frac{j}{l} \Lambda_l + \frac{j-i}{2} \alpha_l = \Lambda_i + (i-j) \Lambda_{l-1} + \left(\frac{l-1}{l} j - i \right) \Lambda_l,$$

we then obtain by some direct calculations

THEOREM 2.2. *The string functions of all level* 1 *modules for* $C_l^{(1)}$ *are given by the following*: *if* $k \equiv i$ (mod 2), *then*

$$c_{\Lambda_i}^{\Lambda_k} = \eta(\tau)^{-(2l-1)} \sum_{\substack{0 \leq j \leq l \\ j \equiv i \bmod 2}} d_{\bar{\Lambda}_j}^{\Lambda_k} \sum_{\beta \in 2\bar{Q}} q^{|\lambda_{i,j} + \beta|^2/2}, \tag{16}$$

and 0 *otherwise.*

The following formula can be derived from [8].

PROPOSITION 2.3. *If* $k \equiv j$ (mod 2), *then we have*

$$d_{\bar{\Lambda}_j}^{\Lambda_k} = \eta^{-1} \sum_{\substack{n \in Z \\ n \equiv \pm j (\bmod 2l)}} (-1)^{(1/2)(n+k)} q^{(l(l+2)/8)(n/l + (k+1)/(l+2))^2}. \tag{17}$$

Proof. Let $e_{k,j}^{(l)}$ denote the right-hand side of (17). Then by Proposition 2.1, we have

$$(d_{\bar{\Lambda}_j}^{\Lambda_k}) = (b_{\Lambda_k}^{\bar{\Lambda}_j})^{-1} = \eta^2 (\theta_{k+1,j}^{(l)})^{-1}.$$

But it follows from [8] that

$$(\theta_{k+1,j}^{(l)})^{-1} = \eta^{-3} (e_{k,j}^{(l)}).$$

Hence the proposition follows. ∎

EXAMPLE 2.4. Let $l = 2$. Given $a, b \in \frac{1}{2}Z$, we define

$$\vartheta_{a,b}(\tau) = \sum_{n \in Z} q^{a(n + b/2a)^2}. \tag{18}$$

Then $\vartheta_{a,b} = \vartheta_{a,-b} = \vartheta_{a,2a-b}$. Formulas (16) and (17) give us

$$c_{\Lambda_0}^{\Lambda_k} = \eta^{-4}(-1)^{k/2} (\vartheta_{2,0} \vartheta_{4,k+1} - \vartheta_{2,2} \vartheta_{4,3-k}),$$
$$c_{\Lambda_2}^{\Lambda_k} = \eta^{-4}(-1)^{k/2} (\vartheta_{2,2} \vartheta_{4,k+1} - \vartheta_{2,0} \vartheta_{4,3-k}),$$

where $k = 0, 2$ and

$$c_{\Lambda_1}^{\Lambda_1} = \eta^{-4} (\vartheta_{4,0} - \vartheta_{4,4}) \vartheta_{2,1}.$$

Comparing our formula with the Kac–Wakimoto formula given in [9], we get

$$\vartheta_{2,0} \vartheta_{4,1} - \vartheta_{2,2} \vartheta_{4,3} = \eta (\vartheta_{12,1} - \vartheta_{12,7}),$$
$$\vartheta_{2,2} \vartheta_{4,1} - \vartheta_{2,0} \vartheta_{4,3} = \eta (\vartheta_{12,5} - \vartheta_{12,11}),$$
$$(\vartheta_{4,0} - \vartheta_{4,4}) \vartheta_{2,1} = \eta (\vartheta_{12,2} - \vartheta_{12,10}).$$

Note that only the last identity can be derived by using the Euler and Jacobi triple product identity (cf. [14]). Since in this case we have by either formula

$$c^{\Lambda_1}_{\Lambda_1} = \eta^{-1} \prod_{n \in N} \left(\frac{1+q^n}{1-q^n} \right).$$

However, the first two given non-trivial identities.

3. A Formula on Branching Functions

Let g be a affine Lie algebra of type $X_N^{(k)}$ and rank l, and let h be its Cartan subalgebra generated by $\{d, \alpha_0^\vee, \alpha_1^\vee, ..., \alpha_l^\vee\}$. The dual h^* is then generated by $\{\Lambda_0, \alpha_0, \alpha_1, ..., \alpha_l\}$. We define a symmetric non-degenerate invariant bilinear form on h^* by (cf. [6])

$$(\alpha_i \mid \alpha_j) = a_i^{-1} a_i^\vee a_{i,j},$$

$$(\Lambda_0 \mid \alpha_j) = a_0^{-1} \delta_{0,j},$$

$$(\Lambda_0 \mid \Lambda_0) = 0,$$

where $(a_{i,j})$ denotes the Cartan matrix of g, and a_i and $a_i^\vee$ $(i = 0, 1, ..., l)$ are the numerical invariants defined in [6].

We identify h with its dual h^* by using the above bilinear form. By this identification we have $a_0 \Lambda_0 = a_0^\vee d$, $a_i \alpha_i = a_i^\vee \alpha_i^\vee$.

Given $\Lambda \in h^*$, let $L(\Lambda)$ be the irreducible highest weight module with highest weight Λ. Let

$$\mathrm{ch}_\Lambda(h) = \sum_\lambda \mathrm{mult}_\Lambda(\lambda) e^{(\lambda \mid h)}, \quad h \in h^*, \tag{19}$$

be the character of $L(\Lambda)$. Define the modular anomaly of $\Lambda \in h^*$ by

$$s_\Lambda = \frac{|\Lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee} \quad \text{with } k = \Lambda(c) \tag{20}$$

and the normalized character by

$$\chi_\Lambda = e^{-s_\Lambda \delta} \, \mathrm{ch}_\Lambda. \tag{21}$$

Note that χ_Λ depends only on Λ (mod $C\delta$).

For $\lambda \in P_+^{(k)}$, $k \in N$, let $(q = e^{-\delta})$

$$A_\lambda = q^{|\lambda|^2/2k} \sum_{w \in W} \varepsilon(w)\, e^{w(\lambda)}. \tag{22}$$

Then we have by the Weyl–Kac character formula $(\Lambda \in P_+)$

$$\chi_\Lambda = \frac{A_{\Lambda+\rho}}{A_\rho}. \tag{23}$$

Recall that, given $\lambda \in h^*$, the string function c_λ^Λ of the g-module $L(\Lambda)$ is defined by

$$c_\lambda^\Lambda(\tau) = q^{s_\Lambda - |\lambda|^2/2k} \sum_{n \in Z} \operatorname{mult}_\Lambda(\lambda - n\delta)\, q^n, \tag{24}$$

where as usual $q = e^{-\delta}$, Note that c_λ^Λ depends only on $\lambda \bmod C\delta$, that $c_\lambda^\Lambda = 0$ implies $\Lambda - \lambda \in Q$, and that $c_{w\lambda}^\Lambda = c_\lambda^\Lambda$ for any $w \in W$.

Let g and $\tilde{g}$ be affine Lie algebras associated to simple finite-dimensional Lie algebras. Suppose there is a embedding $\iota : g \hookrightarrow \tilde{g}$ which preserves the triangular decompositions and satisfies

(a) $\iota(\Lambda_0) = u^{-1}\tilde{\Lambda}_0, \qquad \iota(\delta) = u\tilde{\delta}$,

(b) $\iota(Q^\vee) \subset \tilde{Q}^\vee$,

(c) $(\iota(\lambda) \mid \iota(\mu)) = (\lambda \mid \mu) \qquad \text{for} \quad \lambda, \mu \in h^*$.

The natural injective homorphism $W \hookrightarrow \tilde{W}$ is also denoted by ι. We then have

$$\iota(w) \cdot \iota(\lambda) = \iota(w \cdot \lambda) \qquad \text{for} \quad w \in W, \quad \lambda \in h^*.$$

It follows from the complete reducibility theorem [6, Chap. 10] that any integrable $\tilde{g}$-module can be decomposed into a sum of (usually infinite many) irreducible highest weight modules of level uk as a g-module.

If $\tilde{\Lambda} \in \tilde{P}_+^{(k)}$ let $L(\tilde{\Lambda})$ be the irreducible highest weight module of highest weight $\tilde{\Lambda}$. Then the character of $L(\tilde{\Lambda})$ can be rewritten as (cf. [6])

$$\chi_{\tilde{\Lambda}} = \sum_{\tilde{\xi} \in \tilde{h}^* \pmod{C\delta}} q^{|\tilde{\xi}|^2/2k}\, e^{\tilde{\xi}}\, c_{\tilde{\xi}}^{\tilde{\Lambda}}. \tag{25}$$

Now let $\hat{h}^*$ be the orthogonal complement of h^* in $\tilde{h}^*$ with respect to the bilinear form $(\cdot \mid \cdot)$. Since $(\cdot \mid \cdot)$ is non-degenerate, we have $\tilde{h}^* = \iota(h^*) \oplus \hat{h}^*$. Note that $\iota(W)$ leaves $\hat{h}^*$ invariant. If $\tilde{\xi} \in \tilde{h}^*$ write

$$\tilde{\xi} = \iota(\xi) + \eta, \qquad \text{with} \quad \xi \in h^*, \quad \eta \in \hat{h}^*.$$

Then as a function on h^* we have $(q=e^{-\tilde{\delta}})$

$$\begin{aligned}
\chi_{\tilde{\Lambda}} A_\rho &= q^{u|\rho|^2/2h^\vee} \sum_{w\in W} \sum_{\tilde{\xi}\in\tilde{h}^*(\mathrm{mod}\, C\tilde{\delta})} q^{|\tilde{\xi}|^2/2k} \varepsilon(w)\, e^{\tilde{\xi}+w(\rho)} c^{\tilde{\Lambda}}_{\tilde{\xi}} \\
&= q^{u|\rho|^2/2h^\vee} \sum_{w\in W} \sum_{\tilde{\xi}\in\tilde{h}^*(\mathrm{mod}\, C\tilde{\delta})} q^{|\tilde{\xi}|^2/2k} \varepsilon(w)\, e^{\iota(w)\tilde{\xi}+w(\rho)} c^{\tilde{\Lambda}}_{\iota(w)\tilde{\xi}} \\
&= q^{u|\rho|^2/2h^\vee} \sum_{\tilde{\xi}\in\tilde{h}^*(\mathrm{mod}\, C\tilde{\delta})} q^{|\tilde{\xi}|^2/2k} \sum_{w\in W} \varepsilon(w)\, e^{\iota(w)\tilde{\xi}+w(\rho)} c^{\tilde{\Lambda}}_{\tilde{\xi}} \\
&= q^{u|\rho|^2/2h^\vee} \sum_{\eta\in\hat{h}^*} \sum_{\xi\in h^*(\mathrm{mod}\, C\delta)} q^{|\iota(\xi)+\eta|^2/2k} \sum_{w\in W} \varepsilon(w)\, e^{\iota(w)\iota(\xi)+w(\rho)} c^{\tilde{\Lambda}}_{\eta+\iota(\xi)} \\
&= q^{u|\rho|^2/2h^\vee} \sum_{\eta\in\hat{h}^*} \sum_{\xi\in h^*(\mathrm{mod}\, C\delta)} q^{|\iota(\xi)+\eta|^2/2k} \sum_{w\in W} \varepsilon(w)\, e^{\iota(w(\xi))+w(\rho)} c^{\tilde{\Lambda}}_{\eta+\iota(\xi)} \\
&= q^{u|\rho|^2/2h^\vee} \sum_{\eta\in\hat{h}^*} \sum_{\xi\in h^*(\mathrm{mod}\, C\delta)} q^{|\iota(\xi)+\eta|^2/2k} \sum_{w\in W} \varepsilon(w)\, e^{w(\xi+\rho)} c^{\tilde{\Lambda}}_{\eta+\iota(\xi)}
\end{aligned}$$

Note that $c^{\tilde{\Lambda}}_{\eta+\iota(\xi)}=0$ unless $\eta+\iota(\xi)\in\tilde{\Lambda}+\tilde{Q}+C\tilde{\delta}$. Since $\iota(Q^\vee)\subset\tilde{Q}^\vee$, we may assume that if $\xi+\rho$ is integral and also regular with respect to W, then there is a unique $\lambda\in P_+^{(uk)}$ and $\sigma\in W$ such that

$$\xi+\rho\equiv\sigma(\lambda+\rho)\ (\mathrm{mod}\ C\delta).$$

Therefore we have $\xi+\rho=\sigma(\lambda+\rho)+a\delta$ for some $a\in C$ and

$$\begin{aligned}
\chi_{\tilde{\Lambda}} A_\rho &= q^{u|\rho|^2/2h^\vee} \sum_{\eta\in\hat{h}^*} \sum_{\substack{\lambda\in P_+^{(uk)}\\ \sigma\in W}} q^{|\eta+\iota(\sigma(\lambda+\rho)-\rho)|^2/2k+au} \\
&\quad\times \sum_{w\in W} \varepsilon(w)\, e^{w\sigma(\lambda+\rho)+a\delta} c^{\tilde{\Lambda}}_{\eta+\iota(\sigma(\lambda+\rho)-\rho)} \\
&= q^{u|\rho|^2/2h^\vee} \sum_{\eta\in\hat{h}^*} \sum_{\substack{\lambda\in P_+^{(uk)}\\ \sigma\in W}} q^{|\eta+\iota(\sigma(\lambda+\rho)-\rho)|^2/2k} \\
&\quad\times \sum_{w\in W} \varepsilon(w)\, e^{w\sigma(\lambda+\rho)} c^{\tilde{\Lambda}}_{\eta+\iota(\sigma(\lambda+\rho)-\rho)} \\
&= \sum_{\lambda\in P_+^{(uk)}} \sum_{\eta\in\hat{h}^*} \sum_{\sigma\in W} \varepsilon(\sigma)\, q^{|\eta+\iota(\sigma(\lambda+\rho)-\rho)|^2/2k+u|\rho|^2/2h^\vee-u|\lambda+\rho|^2/2(uk+h^\vee)} \\
&\quad\times A_{\lambda+\rho} c^{\tilde{\Lambda}}_{\eta+\iota(\sigma(\lambda+\rho)-\rho)}.
\end{aligned}$$

Using the identity

$$\frac{|\sigma(\lambda+\rho)-\rho|^2}{2k}+\frac{u|\rho|^2}{2h^\vee}-\frac{u|\lambda+\rho|^2}{2(uk+h^6)}=\frac{h^\vee(uk+h^\vee)}{2k}\left|\frac{\sigma(\lambda+\rho)}{uk+h^\vee}-\frac{\rho}{h^\vee}\right|^2,$$

we have obtained.

THEOREM 3.1. *If* $\tilde{\Lambda} \in \tilde{P}_{+}^{(k)}$, $\lambda \in P_{+}^{(uk)}$, *then the branching function for the embedding* $\imath: g \subsetneq \tilde{g}$ *is given by*

$$b_{\lambda}^{\tilde{\Lambda}} = \sum_{\eta \in \tilde{h}^*} q^{|\eta|^2/2k} \sum_{\sigma \in W} \varepsilon(\sigma)\, q^{h^\vee(uk+h^\vee)/2k\, |\sigma(\lambda+\rho)/uk+h^\vee - \rho/h^\vee|^2}\, c_{\eta + \imath(\sigma(\lambda+\rho)-\rho)}^{\tilde{\Lambda}}, \tag{26}$$

where $q = e^{-\tilde{\delta}}$.

One of the special case is when $\imath(h^*) = \tilde{h}^*$. Then we call g a regular subalgebra of $\tilde{g}$ due to the definition in the finite-dimensional case. We have the following simple formula.

COROLLARY 3.2. *If* g *is a regular subalgebra of* $\tilde{g}$, *then*

$$b_{\lambda}^{\tilde{\Lambda}} = \sum_{\sigma \in W} \varepsilon(\sigma)\, q^{(h^\vee(uk+h^\vee)/2k)|\sigma(\lambda+\rho)/(uk+h^\vee) - \rho/h^\vee|^2}\, c_{\imath(\sigma(\lambda+\rho)-\rho)}^{\tilde{\Lambda}}. \tag{27}$$

Remark 3.3. Given an affine Lie algebra $\tilde{g}$, there may exist many regular subalgebras, for example, $E_8^{(1)} \supset A_8^{(1)}$, $E_8^{(1)} \supset D_8^{(1)}$, etc.

EXAMPLE 3.4. Fix $u \in N$ relatively prime to $\tilde{a}_0$. Then

$$\alpha_0 = \tilde{a}_0^{-1}(u-1)\tilde{\delta} + \alpha_0 \in \tilde{\Delta}_{+}^{\mathrm{re}},$$

and $\Pi_{[u]} = \{\alpha_0, \alpha_i = \tilde{\alpha}_i\ (0 \leqslant i \leqslant l)\}$ forms a set of simple roots for the so-called winding subalgebra $g = \tilde{g}_{[u]}$ of $\tilde{g}$, which is isomorphic to $\tilde{g}$. For example, if

$$\tilde{g} = \bar{g} \otimes C[t, t^{-1}] + Cc + Cd,$$

which $\bar{g}$ being the finite parts of $\tilde{g}$, then

$$\tilde{g}_{[u]} = \bar{g} \otimes C[t^u, t^{-u}] + Cc + Cd.$$

Note that the canonical embedding $\imath: g \subsetneq \tilde{g}$ satisfies conditions (a)–(c) and that if $\lambda = \sum m_i \Lambda_i \in P_{+}^{(uk)}$, then $\imath(\lambda) = \sum m_i \tilde{\Lambda}_i + (1-u)\,k\Lambda_0$. Denote $\tilde{\lambda} = \sum m_i \tilde{\Lambda}_i$ and $\tilde{\sigma} = \tilde{s}_{i_1}\tilde{s}_{i_2}\cdots\tilde{s}_{i_r} \in \tilde{W}$ if $\lambda = \sum m_i \Lambda_i$ and $\sigma = s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ (s_i, $\tilde{s}_i$ are simple reflections). Then it is easy to see that

$$\imath(\sigma(\lambda+\rho)-\rho) = \tilde{\sigma}(\tilde{\lambda}+\tilde{\rho}) - \tilde{\rho}) + (1-u)\,k\tilde{\Lambda}_0.$$

Therefore we recover the following formula due to Kac and Wakimoto [12].

$$b_{\lambda}^{\tilde{\Lambda}} = \sum_{\sigma \in W} \varepsilon(\sigma)\, q^{h^\vee(uk+h^\vee)/2k\, |\sigma(\tilde{\lambda}+\tilde{\rho})/(uk+h^\vee) - \tilde{\rho}|h^\vee|^2}\, c_{\sigma(\tilde{\lambda}+\tilde{\rho}) - \tilde{\rho} + (1-u)k\tilde{\Lambda}_0}^{\tilde{\Lambda}}. \tag{28}$$

EXAMPLE 3.5. Let $\tilde{g} = X_N^{(k)}$, $X = A$–D–E and $\tilde{\Lambda} \in \tilde{P}_+^{(1)}$, $\lambda \in P_+^{(u)}$. Assume in addition that $\imath(Q) \subset \tilde{Q}$. Then $c^{\tilde{\Lambda}}_{\eta + \imath(\sigma(\lambda+\rho)-\rho)} \neq 0$ if and only if $\eta + \imath(\lambda) \in \tilde{\Lambda} + \tilde{Q} + C\tilde{\delta}$.

Note that we do not assume that g is of type A–D–E. Since for any $\sigma \in W$,

$$c^{\tilde{\Lambda}}_{\eta - \imath(\sigma(\lambda+\rho)-\rho)} = c^{\tilde{\Lambda}}_{\eta + \imath(\lambda)},$$

by Theorem 3.1 we get

$$b_\lambda^{\tilde{\Lambda}} = \left(\sum_{\sigma \in W} \varepsilon(\sigma)\, q^{h^\vee(u+h^\vee)/2\,|\sigma(\lambda+\rho)/(u+h^\vee)-\rho/h^\vee|^2} \right) \sum_{\eta \in \hat{h}^*} q^{|\eta|^2/2} c^{\tilde{\Lambda}}_{\eta + \imath(\lambda)}.$$

Let $P(\tilde{\Lambda}, \lambda) = (\tilde{\Lambda} - \imath(\lambda) + \tilde{Q} + C\tilde{\delta} \bmod C\tilde{\delta}) \cap \hat{h}^*$. Since we have for $\lambda \in P_+^{(u)}$ (cf. [12]),

$$\sum_{\sigma \in W} \varepsilon(\sigma) q^{uh^\vee/2\,|\sigma(\lambda)/u - \rho/h^\vee|^2} = q^{uh^\vee/2\,|\lambda/u - \rho/h^\vee|^2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda|\rho)})^{\text{mult}\,\alpha}, \quad (29)$$

$$c_{\tilde{\Lambda}}^{\tilde{\Lambda}} = q^{s_{\tilde{\Lambda}} - |\tilde{\Lambda}|^2/2} \prod_{n \in N} (1 - q^n)^{-\text{mult}\, n\tilde{\delta}}, \quad (30)$$

we get

$$\sum_{n \in Z} \text{mult}_{\tilde{\Lambda}}(\lambda - n\tilde{\delta})\, q_n = \frac{q^{(|\lambda|^2 - |\Lambda|)^2/2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda+\rho|\rho)})^{\text{mult}\,\alpha}}{\prod_{n \in N} (1 - q^n)^{\text{mult}\, n\tilde{\delta}}} \sum_{\in P(\tilde{\Lambda}, \lambda(} q^{|\eta|^2/2k}. \quad (31)$$

One notes that if g is a winding subalgebra of $\tilde{g}$, this result gives Theorem 2.2 in [12], which was originally conjectured in [5] for the simply laced A–D–E type. Note also that we may use this to obtain an explicit formula for the branching function for the decomposition with respect to the regular subalgebras. Many such examples were carried out for small levels in [9] by using the asymptotics of the branching functions.

EXAMPLE 3.6. Let $\tilde{g}$ be an affine Lie algebra of type $B_l^{(1)}$, $\hat{h}^* = 0$, and $\tilde{\Lambda} \in \tilde{P}_+^{(1)} = \{\Lambda_1, \Lambda_2, \Lambda_l\}$, $\lambda \in P_+^{(1)}$. Assume also that $\imath(Q) \subset \tilde{Q}$. Since we have from [8]

$$c_{\Lambda_0}^{\Lambda_0} + c_{\Lambda_0}^{\Lambda_1} = c_{\Lambda_1}^{\Lambda_0} + c_{\Lambda_1}^{\Lambda_1} = \prod_{n \in N} (1 - q^n)^{-l} (1 + q^{n-1/2}),$$

$$c_{\Lambda_l}^{\Lambda_l} = \prod_{n \in N} (1 - q^n)^{-l} (1 + q^n).$$

Then we have by the same argument as used in [12] that either $\text{mult}_{\tilde{\Lambda}}(\lambda - n\tilde{\delta}) = 0$, for all $n \in Z$ or $\iota(\lambda) \in \tilde{\Lambda}_0 + \tilde{Q}$ (resp., $\iota(\lambda) \in \tilde{\Lambda}_l + \tilde{Q}$)

$$\sum_{n \in Z} \text{mult}_{\tilde{\Lambda}_0}(\lambda - n\tilde{\delta}) q^n + \sum_{n \in Z} \text{mult}_{\tilde{\Lambda}_1}(\lambda - n\tilde{\delta})\, q^{n+1/2}$$
$$= q^{|\lambda|^2/2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda + \rho | \rho)})^{\text{mult}\,\alpha} \prod_{n \in N} (1 - q^n)^{-l} (1 + q^{n-1/2}), \quad (32)$$

and respectively,

$$\sum_{n \in Z} \text{mult}_{\tilde{\Lambda}_l}(\lambda - n\tilde{\delta})\, q^n$$
$$= q^{(|\lambda|^2 - |\Lambda_l|^2)/2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda + \rho | \rho)})^{\text{mult}\,\alpha} \prod_{n \in N} (1 - q^n)^{-l} (1 - q^n). \quad (33)$$

Note that one can generalize most of the results in this section to the general branching functions for the pair $g \subset \tilde{g}$ such that $\tilde{g}$ is a semisimple Lie algebra. As an example, we look at complementary decomposition in the next section.

4. On Complementary Decompositions

Recall that the affine Lie algebra g associated to a semisimple finite-dimensional Lie algebra $\bar{g}$ is constructed as follows. Let $\bar{g} = \bar{g}_1 \oplus g_2 \oplus \cdots \oplus \bar{g}_s$ be the decomposition into the simple Lie algeb ras, then

$$g = g'_1 \oplus g'_2 \oplus \cdots \oplus g'_s + Cd$$

where $g'_i = \bar{g}_i \otimes C[t, t^{-1}] - Cc_i$ with c_i being the central element of g'_i and d the common derivation which acts as $t(d/dt)$ on all g'_i. Let $\bar{h}_i$ be the Cartan subalgebra of $\bar{g}_i$, then the Abelian subalgebra

$$h = \sum (C\bar{h}_i + Cc_i) + Cd$$

is called the Cartan subalgebra of g. Put

$$g_i = g'_i + Cd \qquad \text{and} \qquad h_i = \bar{h}_i - Cc_i + Cd.$$

Now assumed that $s = 2$. In the following, we consider only the complementary decomposition even though the result can be stated under sopme general assumptions. Note that the complementary decompositions were studied in [4] for finding certain duality between branching functions. From now on we denote g_1 by g, g_2 by $\dot{g}$, and so on for the corresponding objects. Let us first consider the pair $(g, \tilde{g})$. Assume that there exists an embedding $\imath\colon h^* \subsetneq \tilde{h}^*$ satisfying

$$(1)\quad \imath(\Lambda_0) = \tilde{\Lambda}_0, \qquad \imath(\delta) = \tilde{\delta},$$
$$(2)\quad \imath(Q^\vee) \subset \tilde{Q}^\vee, \qquad \imath(\Delta) \in \tilde{\Delta},$$
$$(3)\quad (\imath(\lambda) \mid \imath(\mu)) = (\lambda \mid \mu) \qquad \text{for} \quad \lambda, \mu \in h^*.$$

The natural injective homorphisms of Weyl groups $W \subset \tilde{W}$ is also denoted by $\imath$. Then $\imath$ satisfies $\imath(w) \cdot \imath(\lambda) = \imath(\omega \cdot \lambda)$ for $w \in W$, $\lambda \in h^*$.

Now we may assume that there exists another embedding $\dot{\imath}\colon \dot{h}^* \subsetneq \tilde{h}^*$ satisfying conditions (1)–(3) with $\imath$, Λ_0, δ, etc., replaced by $\dot{\imath}$, $\dot{\Lambda}_0$, $\dot{\delta}$, etc. We may also assume that

$$(\imath(\lambda) \mid \dot{\imath}(\mu)) = 0, \qquad \text{for} \quad \lambda \in h^*, \ \mu \in \dot{h}^*. \tag{34}$$

Then it is easy to show that $\imath(W)$, $\dot{\imath}(\dot{W})$ commutes.

Let $\hat{h}^*$ be the orthogonal complement of $\imath(h^*) + \imath(\dot{h}^*)$ in $\tilde{h}^*$. Then $\hat{h}^*$ is a $\imath(W)\,\dot{\imath}(\dot{W})$ invariant subspace of $\tilde{h}^*$. According to the remark in the last section, we have

THEOREM 4.1. *If $\tilde{\Lambda} \in \tilde{P}_+^{(k)}$, $\Lambda \in P_+^{(k)}$, and $\dot{\Lambda} \in \dot{P}_+^{(k)}$, then the branching function for the complementary decomposition is given by*

$$b^{\tilde{\Lambda}}_{\Lambda, \dot{\Lambda}} = \sum_{\eta \in \hat{h}^*} q^{|\eta|^2/2k} \sum_{\sigma \in W} \sum_{\dot{\sigma} \in \dot{W}} \varepsilon(\sigma)\, \varepsilon(\dot{\sigma})$$
$$\times q^{h^\vee (k + h^\vee)/2k\, |\sigma(\Lambda + \rho)/(k + h^\vee) - \rho/h^\vee|^2 + \dot{h}^\vee (k + \dot{h}^\vee)/2k\, |\dot{\sigma}(\dot{\Lambda} + \dot{\rho})/(k + h^\vee) - \dot{\rho}/h^\vee|^2}$$
$$\times c^{\tilde{\Lambda}}_{\eta + \imath(\sigma(\Lambda + \rho) - \rho) + \dot{\imath}(\dot{\sigma}(\dot{\Lambda} + \dot{\rho}) - \dot{\rho}) - k\tilde{\Lambda}_0}. \tag{35}$$

Proof. From (34) we have

$$\imath(w)\, \dot{\imath}(\dot{w})\, \tilde{\Lambda}_0 = \imath(w)\, \tilde{\Lambda}_0 + \dot{\imath}(\dot{w})\, \tilde{\Lambda}_0 - k\tilde{\Lambda}_0.$$

Then we complete the proof by following the proof of Theorem 3.1.

The following pairs are examples of classical complementary pairs.

EXAMPLE 4.2.

$$(1)\quad A_l^{(1)} \oplus A_{m-1}^{(1)} \subset A_{l+m}^{(1)}$$
$$(2)\quad B_l^{(1)} \oplus B_m^{(1)} \subset B_{l+m}^{(1)}$$
$$(3)\quad C_l^{(1)} \oplus C_m^{(1)} \subset C_{l+m}^{(1)}$$
$$(4)\quad D_l^{(1)} \oplus D_m^{(1)} \subset D_{l+m}^{(1)}$$

(5) $A_{2l}^{(2)} \oplus A_{2m}^{(2)} \subset A_{2(l+m)}^{(2)}$

(6) $A_{2l-1}^{(2)} \oplus A_{2m-1}^{(2)} \subset A_{2(l+m)-1}^{(2)}$

(7) $D_{l+1}^{(2)} \oplus D_{m+1}^{(2)} \subset D_{l+m+1}^{(2)}$.

COROLLARY 4.3. *Assume that* $\hat{h}^* = 0$ *and that* $\tilde{g}$ *is of A–D–E type and* $\tilde{\Lambda} \in \tilde{P}_+^{(1)}$, $\Lambda \in P_+^{(1)}$, $\dot{\Lambda} \in \dot{P}_+^{(1)}$. *Then either all branching coefficients* $\operatorname{mult}_{\tilde{\Lambda}}(\Lambda \otimes \dot{\Lambda} - n\tilde{\delta}) = 0$ *or* $\iota(\Lambda) + i(\dot{\Lambda}) \in \tilde{\Lambda} + \tilde{\Lambda}_0 + \tilde{Q}$ *and we have a product expansion*

$$\sum_{n \in Z} \operatorname{mult}_{\tilde{\Lambda}}(\Lambda \otimes \dot{\Lambda} - n\tilde{\delta})\, q^n = q^{(|\Lambda|^2 + |\dot{\Lambda}|^2 - |\tilde{\Lambda}|^2)/2} \frac{\prod_{\alpha \in \Delta_+} (1 - q^{(\Lambda + \rho | \rho)})^{\operatorname{mult} \alpha} \sum_{\dot{\alpha} \in \dot{\Delta}_+} (1 - q^{(\dot{\Lambda} + \dot{\rho} | \dot{\rho})})^{\operatorname{mult} \dot{\alpha}}}{\prod_{\tilde{\alpha} \in \tilde{\Delta}_+} (1 - q^{(\tilde{\Lambda} + \tilde{\rho} | \tilde{\rho})})^{\operatorname{mult} \tilde{\alpha}}}. \tag{36}$$

Proof. Use formulas (29) and (30) and Theorem 4.1.

Remark 4.4. If the pair is $A_l^{(1)} \oplus A_{m-1}^{(1)} \subset A_{l+m}^{(1)}$, then $\dim \hat{h}^* = 1$. Put $d = (l+1, m)$, one has that $\iota(\Lambda_i) + i(\dot{\Lambda}_j) + \eta \in \tilde{\Lambda}_q + \tilde{\Lambda}_0 + \tilde{Q}$ for some $\eta \in \hat{h}^*$ if and only $s \equiv i + j (\operatorname{mod} d)$. By Theorem 4.1 and formula (30) we have either all $\operatorname{mult}_{\tilde{\Lambda}_s}(\Lambda_i \otimes \dot{\Lambda}_j - n\tilde{\delta}) = 0$ or $s \equiv i + j (\operatorname{mod} d)$ for which

$$\sum_{n \in Z} \operatorname{mult}_{\tilde{\Lambda}_s}(\Lambda_i \otimes \dot{\Lambda}_j - n\tilde{\delta})\, q^n = \frac{\prod_{\alpha \in \Delta_+} (1 - q^{(\Lambda_i + \rho | \rho)})^{\operatorname{mult} \alpha} \prod_{\dot{\alpha} \in \dot{\Delta}_+} (1 - q^{(\dot{\Lambda}_j + \dot{\rho} | \dot{\rho})})^{\operatorname{mult} \dot{\alpha}}}{\prod_{\tilde{\alpha} \in \tilde{\Delta}_+} (1 - q^{(\tilde{\Lambda}_s + \tilde{\rho} | \tilde{\rho})})^{\operatorname{mul} \tilde{\alpha}}} \times q^{(|\Lambda_i|^2 + |\dot{\Lambda}_j|^2 - |\tilde{\Lambda}_s|^2)/2} \left(\sum_{n \in Z} q^{m(l+1)(l+m+1)/d^2 (n + ((l+1)j - mi)d/m(l+1)(l+m+1))^2} \right). \tag{37}$$

Remark 4.5. If the pair is $B_l^{(1)} \oplus B_m^{(1)} \subset B_{l+m}^{(1)}$, $\tilde{\Lambda} \in \tilde{P}_+^{(1)}$, $\Lambda \in P_+^{(1)}$, $\dot{\Lambda} \in \dot{P}_+^{(1)}$, then either $\operatorname{mult}_{\tilde{\Lambda}}(\Lambda \otimes \dot{\Lambda} - n\tilde{\delta}) = 0$ for all $n \in Z$ or $\iota(\Lambda) + i(\dot{\Lambda}) \in \tilde{\Lambda} + \tilde{\Lambda}_0 + \tilde{Q}$ for which an explicit formula can be found from the formula on the string functions. We have for $\iota(\Lambda) + i(\dot{\Lambda}) \in 2\tilde{\Lambda}_0 + \tilde{Q}$,

$$\sum_{n \in Z} \operatorname{mult}_{\tilde{\Lambda}_0}(\Lambda \otimes \dot{\Lambda} - n\tilde{\delta})\, q^n + \sum_{n \in Z} \operatorname{mult}_{\tilde{\Lambda}_1}(\Lambda \otimes \dot{\Lambda} - n\tilde{\delta})\, q^{n + 1/2} = q^{(|\Lambda|^2 + |\dot{\Lambda}|^2)/2} \frac{\prod_{\alpha \in \Delta_+} (1 - q^{(\Lambda + \rho | \rho)})^{\operatorname{mult} \alpha} \prod_{\dot{\alpha} \in \dot{\Delta}_+} (1 - q^{(\dot{\Lambda} + \dot{\rho} | \dot{\rho})})^{\operatorname{mult} \dot{\alpha}}}{\prod_{\tilde{\alpha} \in \tilde{\Delta}_+} (1 - q^{(\tilde{\Lambda} + \tilde{\rho} | \tilde{\rho})})^{\operatorname{mult} \tilde{\alpha}}} \times \prod_{n \in N} (1 - q^n)^{-l} (1 + q^{n - 1/2}), \tag{38}$$

and for $\imath(\Lambda)+\dot{\imath}(\dot{\Lambda})\in\tilde{\Lambda}_l+\tilde{\Lambda}_0+\tilde{Q}$,

$$\sum_{n\in Z}\text{mult}_{\tilde{\Lambda}_l}(\Lambda\otimes\dot{\Lambda}-n\tilde{\delta})\,q^n$$
$$=q^{(|\Lambda|^2+|\dot{\Lambda}|^2-|\Lambda_l|^2)/2}\frac{\prod_{\alpha\in\Delta_+}(1-q^{(\Lambda+\rho|\rho)})^{\text{mult}\,\alpha}\prod_{\dot{\alpha}\in\dot{\Delta}_+}(1-q^{(\dot{\Lambda}+\dot{\rho}|\dot{\rho})})^{\text{mult}\,\dot{\alpha}}}{\prod_{\tilde{\alpha}\in\tilde{\Delta}_+}(1-q^{(\tilde{\Lambda}+\tilde{\rho}|\tilde{\rho})})^{\text{mult}\,\tilde{\alpha}}}$$
$$\times\prod_{n\in N}(1-q^n)^{-l}(1+q^n). \qquad (39)$$

Remark 4.6. If the pair is $C_l^{(1)}\oplus C_m^{(1)}\subset C_{l+m}^{(1)}$, we have the following duality by Kac–Wakimoto [9]: if $s\equiv i+j \bmod 2$, then

$$b^{\tilde{\Lambda}_s}_{\Lambda_i,\Lambda_j}(C^{(1)}_{l+m},C^{(l)}_l\oplus C^{(1)}_m)=b^{\Lambda_{l,i};\Lambda_{m,j}}_{\Lambda_{l+m,s}}(A_1^{(1)}\oplus A_1^{(1)},A_1^{(1)}), \qquad (40)$$

where $\Lambda_{n,r}=(n-r)\Lambda_0+r\Lambda_1$ denotes the weight for $A_1^{(1)}$. Note that the branching functions for the tensor product decomposition have been expressed in terms of string functions, see [1] and (12).

References

1. P. Bouwknegt, J. McCarthy, and K. Pilch, On the free field resolutions for the coset conformal field theories, preprint, 1990.
2. P. Goddard, A. Kent, and D. Olive, Virasoro algebras and coset models, *Phys. Lett. B* **152** (1985), 88–93.
3. M. Jimbo and T. Miwa, Irreducible decomposition of fundamental modules for $A_l^{(1)}$ and $C_l^{(1)}$, and Hecke modular forms, *Adv. Stud. Pure Math.* **4** (1984), 97–119.
4. M. Jimbo and T. Miwa, On a duality of branching rules for affine Lie algebras, *Adv. Stud. Pure Math.* **6** (1985), 17–65.
5. I. Frenkel, "Representations of Affine Lie Algebras, Hecke Modular Forms and KdV-type Equations," Lecture Notes in Math. Vol. 933, pp. 71–110, Springer-Verlag, Berlin, 1982.
6. V. Kac, "Infinite Dimensional Lie Algebras," 2nd ed., Cambridge Univ. Press, 1985.
7. V. Kac and D. Peterson, Affine Lie algebras and Hecke modular forms, *Bull. Amer. Math. Soc.* **3** (1980), 1057–1061.
8. V. Kac and D. Peterson, Infinite-dimensional Lie algebras, theta function, and modular forms, *Adv. in Math.* **53** (1984), 125–264.
9. V. Kac and M. Wakimoto, Modular and conformal invariance constraints in representation theory of affine algebras, *Adv. in Math.* **70** (1988), 156–236.
10. V. Kac and M. Wakimoto, Modular invariant representations of infinite-dimensional Lie algebras and superalgebras, *Proc. Natl. Acad. Sci. USA* **85** (1988), 4956–4960.
11. V. Kac and M. Wakimoto, "Classification of Modular Invariant Representations of Affine Algebras," Advanced Series in Mathematical Physics, Vol. 7, pp. 138–177, World Scientific, Singapore, 1989.
12. V. Kac and M. Wakimoto, Branching functions for winding subalgebras and tensor products, MIT preprint, 1991.
13. S. Lu, "Some Results on Modular Invariant Representations," Advanced Series in Mathematical Physics, Vol. 7, pp. 235–253, World Scientific, Singapore, 1989.
14. D. Mumford, "Tata Lectures on Theta," Birkhäuser, Boston, 1983.
15. E. Witten, Nonabelian bosonization in two dimensions, *Comm. Math. Phys.* **92** (1984), 455.