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Shuffling Lattices

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C. Greene (*J. Combin. Theory Ser. A* **47** (1988), 126–131) studied a family of lattices denoted $\mathscr{W}_{m,n}$. In this paper, we generalize his work by defining a new operation on ordered pairs of lattices called *shuffling*. L_1 shuffle L_2 is denoted by $L_1 \bowtie L_2$. The resulting $L_1 \bowtie L_2$ are not necessarily lattices but are posets. The lattices $\mathscr{W}_{m,n}$ correspond to the special cases $\mathscr{B}_n \bowtie \mathscr{B}_m$. In this paper, we study the combinatorial structure of the shuffling operation and the resulting posets.

1. INTRODUCTORY DEFINITIONS

Let L be a finite lattice. $a \in L$ is called a *join-irreducible* element if $a = b \vee c$ implies that $a = b$ or $a = c$. Let $J(L)$ denote the set of join-irreducibles of L. Each element u of L can be written as the join of join-irreducible elements. In particular, $u = \bigvee \{a \in J(L) : u \geqslant a\}$. This allows us to label each element u of L by the join-irreducibles which are less than or equal to u. To do this, assign an integer $f(a)$ to each non-zero join-irreducible a so that $f : J(L) \backslash \hat{0} \to \{1, ..., n\}$ is a bijection. Then defined the *word* of u with respect to f and x to be $x_{i_1} x_{i_2} \cdots x_{i_t}$ where $\{i_1 < i_2 < \cdots < i_t\} = \{i : f^{-1}(i) \leqslant u\}$, and the *word* of $\hat{0}$ to be $\varnothing$. The *length* of u, denoted $|u|$, is the number of x's in the word corresponding to u, and $|L| = |J(L)| - 1$. $|L|$ is the length of the top element in L.

EXAMPLE 1. In Fig. 1, the value of the function f is written in bold to the left of each non-zero join-irreducible. The word corresponding to a particular element is written to the right of the element.

An important thing to note about the definition of words is that the order of the subscripts on the x's matters. That is to say, $x_4 x_2$ could not be the word corresponding to an element in any lattice. Also notice that the

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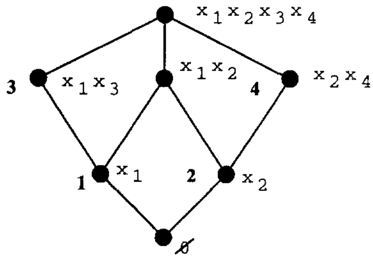


FIGURE 1

word corresponding to $\hat{0}$ is $\varnothing$ and to $\hat{1}$ is $x_1x_2\cdots x_n$. Since the map f can be recovered from the words of a lattice, we do not explicitly display it in future examples. For the remainder of the paper, we freely refer to elements of a lattice and their corresponding words as being one and the same.

In the next definition, we use the words of two given lattices, L_1 and L_2, to form a new poset. This new poset inherits much of its combinatorial structure from L_1 and L_2. The remainder of the paper studies the combinatorial properties of these posets.

DEFINITION. Let L_1 and L_2 be finite lattices. Let $\widehat{L_1}$ denote the dual lattice to L_1. Let $f: J(\widehat{L_1})\backslash\hat{0} \to \{1, ..., n\}$ and $g: J(L_2)\backslash\hat{0} \to \{1, ..., m\}$ be

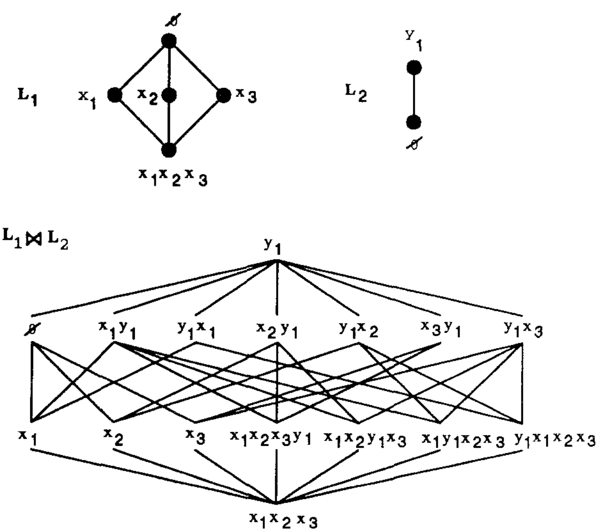


FIGURE 2

bijections. Take the words in $\widehat{L_1}$ with respect to f and x. Take the words in L_2 with respect to g and y. Define the poset L_1 *shuffle* L_2, denoted $L_1 \bowtie L_2$, as follows:

(i) The elements of $L_1 \bowtie L_2$ are all the possible words including the empty word $\varnothing$ using $\{x_1, ..., x_n, y_1, ..., y_m\}$ such that the x's by themselves form a word in $\widehat{L_1}$ and the y's form a word in L_2.

(ii) $\alpha \leqslant_{L_1 \bowtie L_2} \beta$ if the x-word in α (denoted α_x) is less than or equal to β_x (i.e., $\alpha_x \leqslant_{L_1} \beta_x$), $\alpha_y \leqslant_{L_2} \beta_y$, and whenever both x_i and y_j appear in both α and β, they appear in the same order.

Any poset of the form $L_1 \bowtie L_2$ is called a *shuffled* poset.

EXAMPLE 2. In Fig. 2, $\Pi_3 \bowtie \mathscr{B}_1$ is displayed.

The join-irreducible elements of the dual of a lattice $\widehat{L_1}$ are called *meet-irreducible* elements of L_1. Put another way, a is a meet-irreducible element if $a = b \wedge c$ implies $a = b$ or $a = c$. We abuse notation and use $|L_1|$ to mean $|L_1| = |J(L_1)| - 1$ whenever L_1 is the first lattice in a shuffled poset.

The ordering can be viewed as follows: to move up in $L_1 \bowtie L_2$, you may add y's subtract x's. In fact, note that the covering relation in $L_1 \bowtie L_2$ is given by

$$\alpha \succ_{L_1 \bowtie L_2} \beta \Leftrightarrow \left\{\begin{array}{ccc} \alpha_x \succ_{L_1} \beta_x & \text{and} & \alpha_y = \beta_y \\ & \text{or} & \\ \alpha_x = \beta_x & \text{and} & \alpha_y \succ_{L_2} \beta_y \end{array}\right\}.$$

$\alpha \succ \beta$ is called an *x-covering* if α is obtained by deleting some x's in β or a *y-covering* if α is obtained by adding some y's.

From the above example, we see that $L_1 \bowtie L_2$ need no be a lattice. The cases $\mathscr{B}_n \bowtie \mathscr{B}_m$, which have been studied in detail by C. Greene [G], are lattices. Although $L_1 \bowtie L_2$ is not in general a lattice, L_1, L_2, and $L_1 \times L_2$ are embedded in it. The first two come from $L_1 \simeq [x_1 \cdots x_n, \varnothing]$ and $L_2 \simeq [\varnothing, y_1 \cdots y_m]$. $L_1 \times L_2$ can be seen by taking all the words which have no y preceding an x. In some of the proofs that follow, we see that the meet and join are often defined. We write $\alpha \vee \beta$ for the sup(α, β) when defined, and $\alpha \wedge \beta$ for the inf(α, β) when defined.

There are a few troubling aspects to the definition of $L_1 \bowtie L_2$ which should be discussed. First, $L_1 \bowtie L_2 \not\simeq L_2 \bowtie L_1$. A tedious calculation shows that $L_2 \bowtie L_1 \simeq \widehat{\widehat{L_1} \bowtie \widehat{L_2}}$. Second, as the next two examples show, the definition of $L_1 \bowtie L_2$ is depends on the functions f and g. Surprisingly, this dependence does not pose a problem as most of the results in this paper do not depend on f or g. It is understood in the statements of theorems that $L_1 \bowtie L_2$ means all possible $L_1 \bowtie L_2$'s arising from shufling L_1 and L_2.

EXAMPLES 3 AND 4. In Figs. 3 and 4, L_1 and L_2 are the same, but the functions f are different. Note that the resulting $L_1 \bowtie L_2$ are not isomorphic.

Take $\alpha < \beta$ in $L_1 \bowtie L_2$. There are a two sets of elements in $[\alpha, \beta]$ which arise several times in what follows. First, let $\mathscr{A}(\alpha, \beta)$ be the set $\{\gamma \in [\alpha, \beta] : \gamma_x = \alpha_x$ and $\gamma_y = \beta_y\}$. So, the elements in $\mathscr{A}(\alpha, \beta)$ are all the elements in $[\alpha, \beta]$ whose word has maximal length. Second, let $\varphi(\alpha, \beta)$ be the unique element in $[\alpha, \beta]$ whose word has minimal length. This element can be described by $\varphi(\alpha, \beta)_x = \beta_x$, $\varphi(\alpha, \beta)_y = \alpha_y$, and the ordering of the x's and y's in $\varphi(\alpha, \beta)$ is the same as in α. The real value of $\varphi(\alpha, \beta)$ is shown below in Proposition 1.1.

EXAMPLE 5. In $\mathscr{B}_3 \bowtie \mathscr{B}_2$, let $\alpha = x_1 x_2 x_3$ and $\beta = y_1 x_2 y_2$. Then $\mathscr{A}(\alpha, \beta) = \{x_1 y_1 x_2 x_3 y_2,\ x_1 y_1 x_2 y_2 x_3,\ y_1 x_1 x_2 x_3 y_2,\ y_1 x_1 x_2 y_2 x_3\}$ and $\varphi(\alpha, \beta) = x_2$.

PROPOSITION 1.1. *If $\gamma \in [\alpha, \beta]$, then $\varphi(\alpha, \beta) \vee \gamma$ and $\varphi(\alpha, \beta) \wedge \gamma$ are defined.*

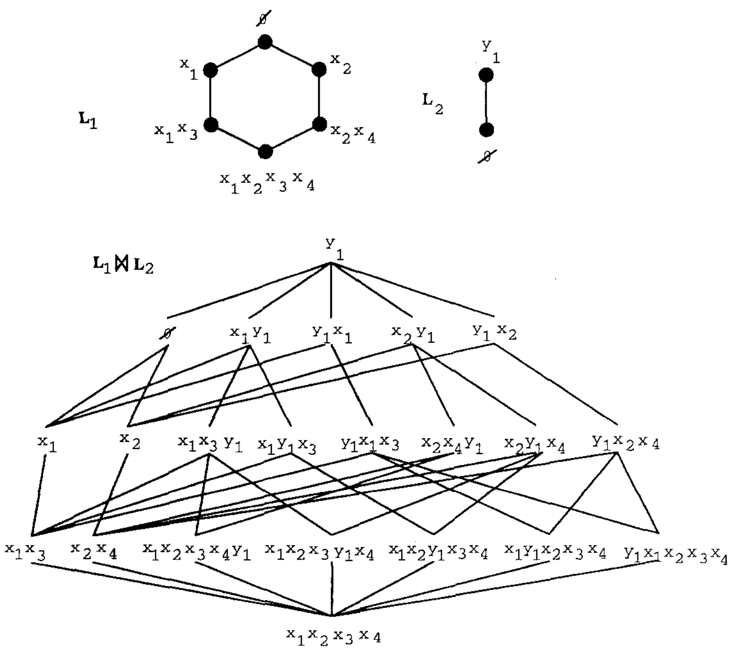


FIGURE 3

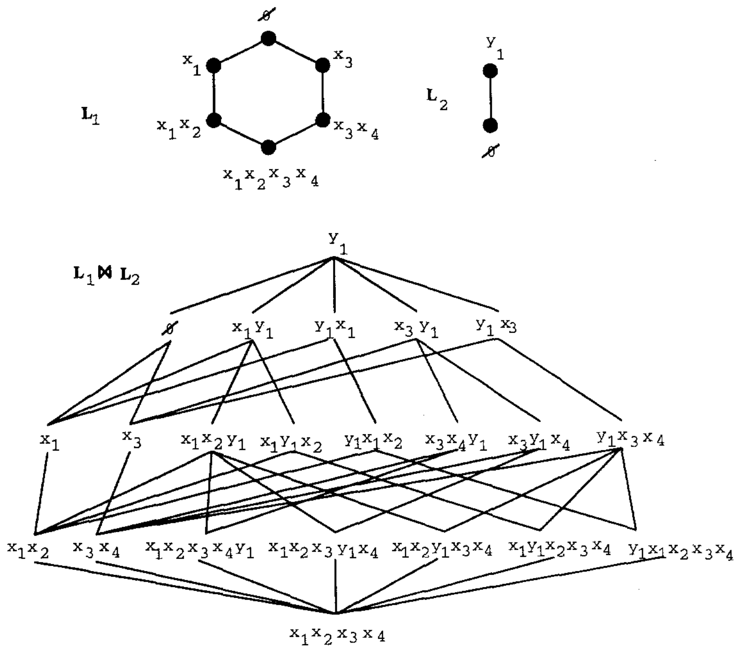


FIGURE 4

Proof. The join is defined by $(\varphi(\alpha, \beta) \vee \gamma)_x = \beta_x$, $(\varphi(\alpha, \beta) \vee \gamma)_y = \gamma_y$, and the ordering of the x's and y's is the same as in γ. An alternative description is that the join is obtained by removing from γ all the x's which do not appear in β. It is a straightforward calculation to check that this is indeed the join.

The meet is defined by $(\varphi(\alpha, \beta) \vee \gamma)_x = \gamma_x$, $(\varphi(\alpha, \beta) \vee \gamma)_y = \alpha_y$, and the ordering of the x's and y's is the same as in γ. More clearly put, remove all the y's from γ which do not appear in α. Another straightforward calculation shows that this is the meet. ∎

2. Rank Properties

In this section, we discuss the rank properties of $L_1 \bowtie L_2$. The following proposition gives the rank function of $L_1 \bowtie L_2$. After that we describe a large class of rank-symmetric shuffled posets.

PROPOSITION 2.1. *$L_1 \bowtie L_2$ is a ranked poset if and only if both L_1 and L_2 are ranked lattices.*

Proof. ($\Rightarrow$) Since L_1 and L_2 are embedded as intervals in $L_1 \bowtie L_2$, this direction is clear.

($\Leftarrow$) We claim that

$$\mathrm{rk}_{L_1 \bowtie L_2}(\alpha) = \mathrm{rk}_{L_1}(\alpha_x) + \mathrm{rk}_{L_2}(\alpha_y)$$

is the rank function of $L_1 \bowtie L_2$. This is proven by induction on the order of $L_1 \bowtie L_2$. First note that the above formula correctly gives 0 as the rank for $x_1 \cdots x_n$. Take $\alpha \in L_1 \bowtie L_2$. Suppose $\alpha \succ \beta$ is an x-covering. Using the induction hypothesis, we may assume that $\mathrm{rk}_{L_1 \bowtie L_2}(\beta) = \mathrm{rk}_{L_1}(\beta_x) + \mathrm{rk}_{L_2}(\beta_y)$. Then

$$\begin{aligned}\mathrm{rk}_{L_1 \bowtie L_2}(\alpha) &= \mathrm{rk}_{L_1}(\alpha_x) + \mathrm{rk}_{L_2}(\alpha_y)\\ &= (\mathrm{rk}_{L_1}(\beta_x) + 1) + \mathrm{rk}_{L_2}(\beta_y)\\ &= \mathrm{rk}_{L_1}(\beta_x) + \mathrm{rk}_{L_2}(\beta_y) + 1\\ &= \mathrm{rk}_{L_1 \bowtie L_2}(\beta) + 1.\end{aligned}$$

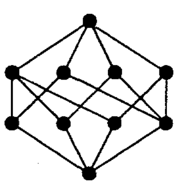
Likewise if $\alpha \succ \beta$ is a y-covering, $\mathrm{rk}_{L_1 \bowtie L_2}(\alpha) = \mathrm{rk}_{L_1 \bowtie L_2}(\beta) + 1$. This demonstrates the claim. It also shows that $L_1 \bowtie L_2$ is ranked. ▌

DEFINITION. The *extended Whitney numbers* of a lattice, L, are given by

$$W_i(k) = \sum_{\substack{x \in L \\ \mathrm{rk}(x) = i}} \binom{|x|}{k}.$$

A lattice is called *k-rank-symmetric* if for a fixed k, $W_i(k) = W_{\mathrm{rk}(L) - i + k}(k)$ for all i where $0 \leqslant i \leqslant \mathrm{rk}(L)$ and $0 \leqslant \mathrm{rk}(L) - i + k \leqslant \mathrm{rk}(L)$ ($\Leftrightarrow 0 \leqslant i - k \leqslant \mathrm{rk}(L)$). Finally, L is called *full k-rank-symmetric* if it is k-rank-symmetric for all k.

EXAMPLE 6. Let L be the lattice in Fig. 5. The values $W_i(k)$ for L are given in the adjacent table. By comparing the appropriate values in the table, we see that L is full k-rank-symmetric.



$W_i(k)$	$k=0$	1	2	3	4
$i=3$	1	4	6	4	1
2	4	9	6	1	
1	4	4			
0	1				

FIGURE 5

In the case $k=0$, the extended Whitney numbers are the usual Whitney numbers of the second kind, and 0-rank-symmetry is the same as rank-symmetry.

PROPOSITION 2.2. *$\mathscr{B}_n$ are full k-rank-symmetric.*

Proof. Fix n, k, and i. In $\mathscr{B}_n$, the elements of rank i are the i-subsets of $\{1,...,n\}$, and the length of the word associated with such an element is i. The following calculation gives the desired result:

$$\begin{aligned}
W_i(k) &= \binom{n}{i}\binom{i}{k} \\
&= \left(\frac{n!}{(n-i)!\,i!}\right)\left(\frac{i!}{(i-k)!\,k!}\right) \\
&= \left(\frac{n!}{(n-i)!\,(i-k)!\,k!}\right)\left(\frac{(n-i+k)!}{(n-i+k)!}\right) \\
&= \left(\frac{n!}{(n-i+k)!\,(i-k)!}\right)\left(\frac{(n-i+k)!}{(n-i)!\,k!}\right) \\
&= \binom{n}{n-i+k}\binom{n-i+k}{k} \\
&= W_{n-i+k}(k). \quad \blacksquare
\end{aligned}$$

THEOREM 2.3. *If L_1 and L_2 are lattices with full k-rank-symmetry, then $\widehat{L_1} \bowtie L_2$ is rank-symmetric.*

Proof. Note that that rank of $\widehat{L_1} \bowtie L_2$ is given by $\mathrm{rk}_{\widehat{L_1} \bowtie L_2}(\alpha) = \mathrm{rk}(L_1) - \mathrm{rk}_{L_1}(\alpha_x) + \mathrm{rk}_{L_2}(\alpha_y)$. Let $d(i,j,k)$ be the number $\alpha \in \widehat{L_1} \bowtie L_2$ such that $\mathrm{rk}_{L_1}(\alpha_x) = i$, $\mathrm{rk}_{L_2}(\alpha_y) = j$ and k of the y's in α appear in the first $|\alpha_x|$ positions of α. Then

$$\begin{aligned}
d(i,j,k) &= \Bigg(\sum_{\substack{a \in L_1 \\ \mathrm{rk}_{L_1}(a)=i}} \binom{|a|}{k}\Bigg)\Bigg(\sum_{\substack{b \in L_1 \\ \mathrm{rk}_{L_2}(b)=j}} \binom{|b|}{k}\Bigg) \\
&= W_i(k)\, W_j(k) \\
&= W_{\mathrm{rk}(L_1)+k-i}(k)\, W_{\mathrm{rk}(L_2)+k-j}(k) \\
&= d(\mathrm{rk}(L_1)+k-i,\, \mathrm{rk}(L_2)+k-j,\, k).
\end{aligned}$$

We use this relationship to show the rank-symmetry of $L_1 \bowtie L_2$.

$$
\begin{aligned}
&|\{\alpha \in L_1 \bowtie L_2 : \mathrm{rk}_{\widehat{L_1 \bowtie L_2}}(\alpha) = t\}| \\
&\quad = \sum_{\substack{i,\, j \\ \mathrm{rk}(L_1) - i + j = t}} \sum_k d(i, j, k) \\
&\quad = \sum_{\substack{i,\, j \\ \mathrm{rk}(L_1) - i + j = t}} \sum_k d(\mathrm{rk}(L_1) + k - i, \mathrm{rk}(L_2) + k - j, k) \\
&\quad = \sum_{\substack{i', j' \\ \mathrm{rk}(L_1) - (\mathrm{rk}(L_1) + k - i') \\ + (\mathrm{rk}(L_2) + k - j') = t}} \sum_k d(i', j', k) \\
&\quad = \sum_{\substack{i', j' \\ \mathrm{rk}(L_1) - i' + j' \\ = \mathrm{rk}(L_1) + \mathrm{rk}(L_2) - n}} \sum_k \sum_k d(i', j', k) \\
&\quad = |\{\alpha \in L_1 \bowtie L_2 : \mathrm{rk}_{\widehat{L_1 \bowtie L_2}}(\alpha) = \mathrm{rk}(\widehat{L_1} \bowtie L_2) - t\}|. \quad \blacksquare
\end{aligned}
$$

This theorem combined with the previous proposition gives that $\mathscr{W}_{m,n} = \mathscr{B}_n \bowtie \mathscr{B}_m$ are rank-symmetric.

3. Labelings

Labelings of the Hasse diagram of a poset are of great interest in algebraic combinatorics. Two most important types of labelings are R-labelings and EL-labelings. Definitions of these can be found in [S]. In this section, we show that the poset $L_1 \bowtie L_2$ inherits these labelings from L_1 and L_2.

Proposition 3.1. *If L_1 and L_2 are R-labelable, then $L_1 \bowtie L_2$ is R-labelable.*

Proof. ($\Leftarrow$) Since L_1 and L_2 are embedded in $L_1 \bowtie L_2$ as intervals, this follows directly from the definition of R-labelable.

($\Rightarrow$) Let λ_1 be an R-labeling of L_1 and λ_2 be an R-labeling of L_2. Let l be the maximal value taken by λ_1. Then construct a labeling λ of $L_1 \bowtie L_2$ as follows:

$$
\lambda(\alpha \prec \beta) = \begin{cases} \lambda_1(\alpha_x \prec \beta_x), & \text{if } \alpha \prec \beta \text{ is an } x\text{-covering;} \\ \lambda_2(\alpha_y \prec \beta_y) + l, & \text{if } \alpha \prec \beta \text{ is a } y\text{-covering.} \end{cases}
$$

We claim that λ is an R-labeling of $L_1 \bowtie L_2$. Take $\gamma < \delta$ in $L_1 \bowtie L_2$. Suppose $\gamma = \gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_t = \delta$ is a saturated chain satisfying

$$
\lambda(\gamma_0 \prec \gamma_1) \leqslant \lambda(\gamma_1 \prec \gamma_2) \leqslant \cdots \leqslant \lambda(\gamma_{t-1} \prec \gamma_t).
$$

By the construction of λ, all the coverings which correspond to x-coverings get a smaller label than any of the coverings which correspond to a y-covering. Thus for some i, $\gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_i$ are all x-coverings, and $\gamma_i \prec \gamma_{i+1} \prec \cdots \prec \gamma_t$ are all y-coverings. In particular, γ_i must be $\varphi(\gamma, \delta)$. Thus the interval $[\gamma, \gamma_i]_{L_1 \bowtie L_2}$ labeled with λ is the same the interval $[\gamma_x, \delta_x]_{L_1}$ labeled with λ_1. Since λ_1 is an R-labeling of L_1, there is only one way to go from γ to γ_i so that λ is weakly increasing. Likewise since λ_2 is an R-labeling of L_2, there is only one way to go from γ_i to δ. Shifting the labels by l does not alter the fact that the labeling is an R-labeling. Thus, λ is an R-labeling of $L_1 \bowtie L_2$. ∎

COROLLARY 3.2. *If L_1 and L_2 are both R-labelable and Cohen–Macaulay, then*

$$\mu_{L_1 \bowtie L_2}(\hat{0}, \hat{1}) = \binom{|L_1| + |L_2|}{|L_1|} \mu_{L_1}(\hat{0}, \hat{1})\, \mu_{L_2}(\hat{0}, \hat{1}).$$

Proof. In an R-labelable Cohen–Macaulay poset, $\mu(\hat{0}, \hat{1})$ equals up to sign the number of saturated chains from $\hat{0}$ to $\hat{1}$ with strictly decreasing labels (for a proof of this, see [S]). In Section 5, we show that the shuffle of two Cohen–Macaulay lattices is Cohen–Macaulay. By this result and the previous proposition, $L_1 \bowtie L_2$ is again an R-labelable Cohen–Macaulay poset.

Using the canonical labeling defined in the previous proof, any chain with strictly increasing labels must consist first of a sequence of y-coverings and then a sequence of x-coverings. Thus such a chain must pass through an element of $\mathscr{A}(\hat{0}, \hat{1})$. Note that if $\alpha \in \mathscr{A})(\hat{0}, \hat{1})$, then $[\hat{0}, \alpha] \simeq L_2$ and $[\alpha, \hat{1}] \simeq L_1$. The following calculation gives the result:

$$\begin{aligned}
&\mu_{L_1 \bowtie L_2}(\hat{0}, \hat{1}) \\
&\quad = \sum_{\alpha \in \mathscr{A}(\hat{0}, \hat{1})} (\#\{\text{chains from } \hat{0} \text{ to } \alpha \text{ with strictly decreasing labels}\}) \\
&\qquad\qquad (\#\{\text{chains from } \alpha \text{ to } \hat{1} \text{ with strictly decreasing labels}\}) \\
&\quad = \sum_{\alpha \in \mathscr{A}(\hat{0}, \hat{1})} (\#\{\text{chains in } L_2 \text{ with strictly decreasing labels}\}) \\
&\qquad\qquad (\#\{\text{chains in } L_1 \text{ with strictly decreasing labels}\}) \\
&\quad = |\mathscr{A}(\hat{0}, \hat{1})|\, \mu_{L_1}(\hat{0}, \hat{1}) \mu_{L_2}(\hat{0}, \hat{1}) \\
&\quad = \binom{|L_1| + |L_2|}{|L_1|} \mu_{L_1}(\hat{0}, \hat{1})\, \mu_{L_2}(\hat{0}, \hat{1})
\end{aligned}$$

It is easy to check that the the signs work out. ∎

This result is generalized in the next section.

PROPOSITION 3.3. *If L_1 and L_2 are EL-labelable, then $L_1 \bowtie L_2$ is EL-labelable.*

Proof. The same labeling which gave an R-labeling of $L_1 \bowtie L_2$, gives an EL-labeling. ∎

4. MÖBIUS FUNCTION

In this section, we describe the Möbius function and characteristic polynomial of $L_1 \bowtie L_2$. We need some preliminary results. The first is a theorem due to H. Crapo [C].

THEOREM 4.1 (Crapo). *For any locally finite poset P, any order-preserving map $f: P \to Q$, and elements $a, b \in P$ such that $a < b$ and $f(a) < f(b)$*

$$\mu_P(a, b) = \sum_{z: f(z) = f(b)} \mu_f(a, z)\, \mu_P(z, b),$$

where $\mu_f(a, z) = \mu_S(\hat{0}, \hat{1})$ with $S = [a, z] \backslash \{c : f(c) = f(z) \text{ and } c \neq z\}$.

LEMMA 4.2. *If L is a lattice which is embedded (as a poset) in a finite poset P with $\hat{0}$ and $\hat{1}$ such that*

(i) $\hat{1}_P = \hat{1}_L$,

(ii) $\hat{0}_P < \hat{0}_L$, *and*

(iii) *every element in P save $\hat{1}$ is less than or equal to a coatom of L (the only elements in P which are covered by $\hat{1}$ are coatoms of L),*

then $\mu_P(\hat{0}, \hat{1}) = 0$.

Proof. For $a \in P$, let $w(a) = \min\{l \in L : l > \mathrm{a}\}$. Since L is a lattice, $w(a)$ is well-defined. By condition (iii), $w(a) < \hat{1}$ for all $a < \hat{1}$. Then

$$\begin{aligned}
0 &= \sum_{x \in L \backslash \hat{1}} \mu_L(x, \hat{1})[\mu(\hat{0}, x) - \mu_P(\hat{0}, x)] \\
&= \sum_{x \in L \backslash \hat{1}} \mu_L(x, \hat{1})\, \mu_P(\hat{0}, x) + \sum_{x \in L \backslash \hat{1}} \sum_{\{a \in P : \hat{0} \leqslant a < x\}} \mu_L(x, \hat{1})\, \mu_P(\hat{0}, a) \\
&= \sum_{x \in L \backslash \hat{1}} \left\{ \sum_{\{y \in L : x \leqslant y < \hat{1}\}} \mu_L(y, \hat{1}) \right\} \mu_P(\hat{0}, x) \\
&\quad + \sum_{a \in P \backslash L} \left\{ \sum_{\{y \in L : w(a) \leqslant y < \hat{1}\}} \mu_L(y, \hat{1}) \right\} \mu_P(\hat{0}, a) \\
&= \sum_{x \in L \backslash \hat{1}} \{-\mu_L(\hat{1}, \hat{1})\}\, \mu_P(\hat{0}, x) + \sum_{a \in P \backslash L} \{-\mu_L(\hat{1}, \hat{1})\}\, \mu_P(\hat{0}, a) \\
&= - \sum_{\hat{0} \leqslant a \leqslant \hat{1}} \mu_P(\hat{0}, a) \\
&= \mu_P(\hat{0}, \hat{1}). \quad ∎
\end{aligned}$$

We can now give a complete description of the Möbius function for any shuffled poset.

THEOREM 4.3. *For* $\alpha \leqslant \beta$ *in* $L_1 \bowtie L_2$,

$$\mu_{L_1 \bowtie L_2}(\alpha, \beta) = |\mathscr{A}(\alpha, \beta)| \, \mu_{L_1}(\alpha_x, \beta_x) \, \mu_{L_2}(\alpha_y, \beta_y)$$

Proof. Take $\alpha \leqslant \beta$ in $L_1 \bowtie L_2$. Let $f\colon [\alpha, \beta] \to [\alpha, \beta]$ be defined by $y \mapsto \gamma \vee \varphi(\alpha, \beta)$. From Proposition 1.1, this is well-defined. From the proof of that proposition, we get that $\{\gamma : f(\gamma) = \beta\} = \{\gamma : \gamma_y = \beta_y\}$. The elements $\mathscr{A}(\alpha, \beta)$ are the minimal elements of this set. (See Fig. 6.)

Case 1. $f(\alpha) < f(\beta)$. Apply Crapo's Theorem (Theorem 4.1) where P and Q are $[\alpha, \beta]$ and f is as above,

$$\begin{aligned}
\mu_{L_1 \bowtie L_2}(\alpha, \beta) &= \sum_{\gamma : f(\gamma) = \beta} \mu_f(\alpha, \gamma) \, \mu_{[\alpha, \beta]}(\gamma, \beta) \\
&= \sum_{\gamma \in \mathscr{A}(\alpha, \beta)} \mu_f(\alpha, \gamma) \, \mu_{[\alpha, \beta]}(\gamma, \beta) \\
&\quad + \sum_{\substack{\gamma \in [\alpha, \beta] : \gamma > \delta \\ \text{for some } \delta \in \mathscr{A}(\alpha, \beta)}} \mu_f(\alpha, \gamma) \, \mu_{[\alpha, \beta]}(\gamma, \beta) \\
&= |\mathscr{A}(\alpha, \beta)| \, \mu_{L_2}(\alpha_x, \beta_x) \, \mu_{L_1}(\alpha_y, \beta_y) \\
&\quad + \sum_{\substack{\gamma \in [\alpha, \beta] : \gamma > \delta \\ \text{for some } \delta \in \mathscr{A}(\alpha, \beta)}} \mu_f(\alpha, \gamma) \, \mu_{[\alpha, \beta]}(\gamma, \beta). \qquad (1)
\end{aligned}$$

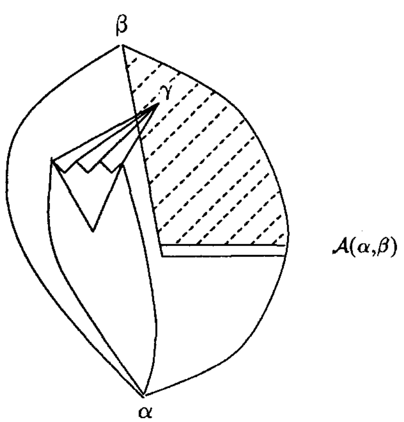


FIGURE 6

We claim that $\mu_f(\alpha, \gamma) = 0$ for all γ strictly than an element of $\mathscr{A}(\alpha, \beta)$. Fix γ. Then $\gamma_x > \alpha_x$ and $\gamma_y = \beta_y$. More simply put, γ contains all of the y's in β but does not contain some of x's in α.

We want to apply Lemma 4.2 with $P = [\alpha, \gamma] \backslash \{\delta : f(\delta) = f(\gamma) \text{ and } \delta \neq \gamma\}$. So we need to define a lattice in P which meets the properties set forth in Lemma 4.2. Let $L = \{\delta \in P : \delta_y = \gamma_y\}$. With the induced ordering from $L_1 \bowtie L_2$, L forms a lattice which isomorphic to $[\alpha_x, \gamma_x]_{L_1}$.

Clearly $\hat{1}_P = \gamma = \hat{1}_L$. Since $f(\alpha) < f(\beta)$, $\alpha_y < \gamma_y$. Thus $\alpha \notin L$ and $\hat{0}_P < \hat{0}_L$. To show the third condition, suppose $\delta \neq \gamma \in P$. Then $\delta <_{L_1 \bowtie L_2} \gamma$ and $\delta_y <_{L_2} \gamma_y$. Take $a \in L_2$ such that γ_y covers an in L_2 and $\alpha >_{L_2} \delta_y$. Construct ζ in P by taking γ and removing all the y's that do not appear in word corresponding to a. From the construction of ζ, $\zeta \in P$ and $\zeta > \delta$. Also $\zeta_y \prec_{L_2} \gamma_y$ and $\zeta_x = \gamma_x$. So γ covers ζ in $L_1 \bowtie L_2$ and thus also in P.

The conclusion of Lemma 4.2 gives $\mu_P(\hat{0}, \hat{1}) = \mu_f(\alpha, \gamma) = 0$. Applying this to (1) gives the desired result.

Case 2. $f(\alpha) = \beta$. In this case, $\alpha_x = \beta_x$. So $[\alpha, \beta] \simeq [\alpha_y, \beta_y]$. Also note that $\mathscr{A}(\alpha, \beta)$ contains but one element, β. Then

$$\begin{aligned} \mu_{L_1 \bowtie L_2}(\alpha, \beta) &= \mu_{L_2}(\alpha_y, \beta_y) \\ &= \underbrace{|\mathscr{A}(\alpha, \beta)|}_{=1} \underbrace{\mu_{L_1}(\alpha_x, \beta_x)}_{=1} \mu_{L_2}(\alpha_y, \beta_y). \quad \blacksquare \end{aligned}$$

One special case of this result is that $\mu_{L_1 \bowtie L_2}(\hat{0}, \hat{1}) = \binom{|L_1| + |L_2|}{|L_1|}\ {}_{L_1}(\hat{0}, \hat{1})$ $\mu_{L_2}(\hat{0}, \hat{1})$.

In order to describe the characteristic polynomial of $L_1 \bowtie L_2$, we need the following polynomial defined for any finite ranked lattice L:

$$\chi_L^*(\lambda, k) = \sum_{a \in L} \mu_L(\hat{0}, a) \binom{|a| + k}{|a|} \lambda^{\mathrm{rk}(L) - \mathrm{rk}(a)}.$$

As $\chi_L(\lambda) = \chi_L^*(\lambda, 0)$, we have that χ_L^* is a generalization of the characteristic polynomial. Since $\binom{|a|+k}{|a|}$ is a polynomial in k of degree $|a|$ and $|a|$ is bounded, χ_L^* is indeed a polynomial in λ and k.

The χ_L^* do not, in general, have nice expressions. One exception is the case $\mathscr{B}_n$. In [G], C. Greene essentially proved the following.

PROPOSITION 4.4.

$$\chi_{\mathscr{B}_n}^*(\lambda, k) = (\lambda - 1)^n \sum_{j \geq 0} \binom{n}{j} \binom{k}{j} \left(\frac{1}{1 - \lambda}\right)^j.$$

Proof. Compare the result in [G] for $\chi_{\mathscr{B}_m \bowtie \mathscr{B}_n}(\lambda)$ with the result obtained from the next theorem. $\blacksquare$

THEOREM 4.5. *If $L_1 \bowtie L_2$ is a ranked, then*

$$\chi_{L_1 \bowtie L_2}(\lambda) = \chi_{L_1}(\lambda)\, \chi^*_{L_2}(\lambda, |L_1|).$$

Proof. In the calculation below, we need that for a given $a \in L_1$ and $b \in L_2$,

$$\sum_{\substack{\alpha \in L_1 \bowtie L_2 \\ \alpha_x = a, \alpha_y = b}} |\mathscr{A}(\hat{0}, \alpha)| = \binom{|L_1| + |b|}{|L_1|}.$$

Let $\mathscr{F}$ be the set of all elements of $L_1 \bowtie L_2$ obtained by shuffling $x_1 \cdots x_n$ with the y-word corresponding to b. Clearly $\mathscr{F}$ has cardinality $\binom{|L_1| + |b|}{|L_1|}$. We want to show that $\mathscr{F} = \dot{\bigcup}_{\alpha \in L_1 \bowtie L_2, \alpha_x = a, \alpha_y = b} \mathscr{A}(\hat{0}, \alpha)$. If β is taken from one these $\mathscr{A}$'s, then β must contain all of the x's in $\hat{0}$, which is to say it must contain all of the x's, and it must contain all of the y's in α, which means it contains all of y's in the word corresponding to b. Hence $\beta \in \mathscr{F}$.

Suppose on the other hand that $\beta \in \mathscr{F}$. Then removing the x's from β which do not appear in the word for a, we get an element α of $L_1 \bowtie L_2$ such that $\alpha_x = a$ and $\beta_y = b$. Moreover, $\beta \in \mathscr{A}(\hat{0}, \alpha)$. Since, there was only one way to remove the x's from β to get α, the $\mathscr{A}$'s are disjoint. Hence, we have the desired equality

Using this, a tedious but routine calculation gives the theorem:

$$\begin{aligned}
\chi_{L_1 \bowtie L_2}(\lambda) &= \sum_{\alpha \in L_1 \bowtie L_2} \mu_{L_1 \bowtie L_2}(\hat{0}, \alpha)\, \lambda^{\mathrm{rk}(L_1 \bowtie L_2) - \mathrm{rk}(\alpha)} \\
&= \sum_{a \in L_1} \sum_{b \in L_2} \sum_{\substack{\alpha \in L_1 \bowtie L_2 \\ \alpha_x = a, \alpha_y = b}} \mu_{L_1 \bowtie L_2}(\hat{0}, \alpha)\, \lambda^{\mathrm{rk}(L_1 \bowtie L_2) - \mathrm{rk}(\alpha)} \\
&= \sum_{a \in L_1} \sum_{b \in L_2} \left\{ \sum_{\substack{\alpha \in L_1 \bowtie L_2 \\ \alpha_x = a, \alpha_y = b}} \mu_{L_1 \bowtie L_2}(\hat{0}, \alpha) \right\} \lambda^{(\mathrm{rk}(L_1) - \mathrm{rk}(a)) + (\mathrm{rk}(L_2) - \mathrm{rk}(b))} \\
&= \sum_{a \in L_1} \sum_{b \in L_2} \left\{ \sum_{\substack{\alpha \in L_1 \bowtie L_2 \\ \alpha_x = a, \alpha_y = b}} |\mathscr{A}(\hat{0}, \alpha)| \right\} \\
&\quad \times \mu_{L_1}(\hat{0}, a)\, \mu_{L_2}(\hat{0}, b)\, \lambda^{(\mathrm{rk}(L_1) - \mathrm{rk}(a)) + (\mathrm{rk}(L_2) - \mathrm{rk}(b))} \\
&= \sum_{a \in L_1} \sum_{b \in L_2} \left\{ \binom{|L_1| + |b|}{|L_1} \right\} \\
&\quad + \mu_{L_1}(\hat{0}, a)\, \mu_{L_2}(\hat{0}, b)\, \lambda^{(\mathrm{rk}(L_1) - \mathrm{rk}(a)) + (\mathrm{rk}(L_2) - \mathrm{rk}(b))} \\
&= \left\{ \sum_{a \in L_1} \mu_{L_1}(\hat{0}, a)\, \lambda^{\mathrm{rk}(L_1) - \mathrm{rk}(a)} \right\} \\
&\quad \times \left\{ \sum_{b \in L_2} \mu_{L_2}(\hat{0}, b) \binom{|L_1| + |b|}{|L_1|} \lambda^{\mathrm{rk}(L_2) - \mathrm{rk}(b)} \right\} \\
&= \chi_{L_1}(\lambda)\, \chi^*_{L_2}(\lambda, |L_1|). \quad \blacksquare
\end{aligned}$$

It should be noted that the labelings f and g of the join-irreducibles of L_1 and L_2 used to define $L_1 \bowtie L_2$ play a subtle role in the calculation of the Möbius function. Although f and g do not appear explicitly in Theorem 4.3, they affect the cardinalities of the $\mathscr{A}$'s. Thus they affect the values of the Möbius function. On the other hand, the choice of f and g has no effect on the characteristic polynomial.

5. Homology

The homology of $L_1 \bowtie L_2$ is obtained by applying a theorem due to Björner and Walker [BW]. We need one definition before stating the theorem. Let P be a poset with a $\hat{0}$ and $\hat{1}$. Two elements $a, b \in P$ are called *complements*, denoted $a \perp b$, if $a \vee b = \hat{1}$ and $a \wedge b = \hat{0}$. In the case that P is a lattice, this definition of complements corresponds to the usual notion of complements.

THEOREM 5.1 (Björner and Walker). *Let P be a poset, and suppose that $s \in P$ satisfies the conditions*:

(i) *$\{a \in P : a \perp s\}$ is an antichain in P.*

(ii) *For all $a \in P$, $a \vee s$ and $a \wedge s$ are defined.*

(iii) *Take a and b in P which are neither $\hat{0}$, $\hat{1}$, nor complements of s. If $a < b$, $s \wedge a = \hat{0}$, and $s \wedge b > \hat{0}$, then $(s \wedge b) \vee a$ is defined and is not $\hat{1}$. Then*

$$H_*(P) \simeq \bigoplus_{\{a \in P : a \perp s\}} [H_*([\hat{0}, a]) \otimes H_*([a, \hat{1}])].$$

This is actually a special case of their theorem, but suffices for our proposes.

THEOREM 5.2.

$$H_*(L_1 \bowtie L_2) \simeq \bigoplus_{i=1}^{\binom{|L_1| + |L_2|}{|L_1|}} H_*(L_1) \otimes H_*(L_2)$$

Proof. We apply Björner and Walker's theorem to $L_1 \bowtie L_2$ with s being the empty word $\varnothing$. $\{\alpha \in L_1 \bowtie L_2 : \alpha \perp \varnothing\}$ is the set of $\binom{|L_1| + |L_2|}{|L_1|}$ words which use all $|L_1|$ x's and all of the $|L_2|$ y's, otherwise known as $\mathscr{A}(\hat{0}, \hat{1})$. This set is clearly an antichain in $L_1 \bowtie L_2$. We showed in Proposition 3.4 that $\varnothing \vee \alpha$ and $\varnothing \wedge \alpha$ are defined for all $\alpha \in L_1 \bowtie L_2$. This takes care of conditions (i) and (ii).

Take $\alpha < \beta$ neither of which are complements of $\varnothing$. Suppose $\alpha \wedge \varnothing = x_1 \cdots x_n$, which means that $\alpha_x = \hat{0}$. Suppose further that $\beta \wedge \varnothing > x_1 \cdots x_n$, which means $\beta_x > \hat{0}$. $\varnothing \wedge \beta$ is the element β_x as embedded in $[x_1 \cdots x_n, \phi]$. Then $(\varnothing \wedge \beta) \vee \alpha = \beta_x \vee \alpha$. This is a well-defined element. Just remove from α all of the x's which do not appear in β_x. By our supposition, α contains every x_i so this is possible. If $(\varnothing \wedge \beta) \vee \alpha = y_1 \cdots y_m$, then it would have to be the case that $\alpha_y = \hat{1}$. But then α would be a complement of $\varnothing$ which is a contradiction.

So the conditions of the theorem are met, and its application yields

$$H_*(L_1 \bowtie L_2) \simeq \bigoplus_{\{\alpha \in L_1 \bowtie L_2 : \alpha \perp \varnothing\}} H_*([x_1 \cdots x_n, \alpha]) \otimes H_*([\alpha, y_1 \cdots y_m]).$$

Noting that for $\alpha \in \mathscr{A}(\hat{0}, \hat{1})$, $[x_1 \cdots x_n, \alpha] \simeq L_2$ and $[\alpha, y_1 \cdots y_m] \simeq L_1$ gives the desired result. ▮

The previous result is, in fact, a particular case of the following theorem.

THEOREM 5.3. *Take $\alpha < \beta$ in $L_1 \bowtie L_2$. Then*

$$H_*([\alpha, \beta]) \simeq \bigoplus_{i=1}^{|\mathscr{A}(\alpha, \beta)|} H_*([\alpha_x, \beta_x]_{L_1}) \otimes H_*([\alpha_y, \beta_y]_{L_2}).$$

Proof. Apply Björner and Walker's theorem to $[\alpha, \beta]$ taking s to be $\varphi(\alpha, \beta)$. The set of complements to $\varphi(\alpha, \beta)$ is $\mathscr{A}(\alpha, \beta)$. The proof proceeds as in the above special case. ▮

COROLLARY 5.4. *$L_1 \bowtie L_2$ is Cohen–Macaulay if and only if L_1 and L_2 are both Cohen–Macaulay.*

Proof. $(\Rightarrow)$ Since L_1 and L_2 are embedded in $L_1 \bowtie L_2$ as intervals, L_1 and L_2 must be Cohen–Macaulay.

$(\Leftarrow)$ Take $\alpha < \beta \in L_1 \bowtie L_2$. Since L_1 and L_2 are Cohen–Macaulay, the only non-zero homologies in $[\alpha_y, \beta_y]_{L_1}$ and $[\alpha_y, \beta_y]_{L_2}$ are their top homologies. Thus, by the previous theorem, the only non-zero homology in $[\alpha, \beta]$ is the top homology. Therefore, $L_1 \bowtie L_2$ is Cohen–Macaulay. ▮

6. ZETA POLYNOMIAL

In this section, we discuss the zeta polynomial of $L_1 \bowtie L_2$. Unlike the rank function, Möbius function, homology, and even the characteristic polynomial, the zeta polynomial of $L_1 \bowtie L_2$ proves to be difficult to characterize.

For $\alpha \in \mathscr{A}(\hat{0}, \hat{1})$, let L_α the subposet of $L_1 \bowtie L_2$ whose elements are $\{\beta \in L_1 \bowtie L_2 :$ if both x_i and y_j appear in β, then they appear in same order as in $\alpha\}$. More simply put, L_α is set elements in $L_1 \bowtie L_2$ with the same ordering as in α. For all $\alpha \in \mathscr{A}(\hat{0}, \hat{1})$, $L_\alpha \simeq L_1 \times L_2$. For $\mathscr{S} \neq \varnothing \subseteq \mathscr{A}(\hat{0}, \hat{1})$, let $L_{\mathscr{S}} = \bigcap_{\alpha \in \mathscr{S}} L_\alpha$. There is no need to define $L_\varnothing$. It should be noted that these are not necessarily lattices.

EXAMPLE 7. Let $L_1 = \Pi_3$ and $L_2 = \mathscr{B}_2$. Let $\mathscr{S}$ be $\{x_1 x_2 x_3 y_1 y_2, x_1 y_1 x_2 y_2 x_3\}$. $L_{\mathscr{S}}$ is shown in Fig. 7.

The reason for introducing the $L_{\mathscr{S}}$'s is that any chain in $L_1 \bowtie L_2$ is a chain in at least one of the $L_{\mathscr{S}}$'s. A chain

$$\beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_s$$

in $L_1 \bowtie L_2$ can be described as a chain in L_1

$$\beta_{0_x} \leqslant \beta_{1_x} \leqslant \cdots \leqslant \beta_{s_x},$$

a chain in L_2

$$\beta_{0_y} \leqslant \beta_{1_y} \leqslant \cdots \leqslant \beta_{s_y},$$

and an ordering for mixing the x's and y's. This ordering of x's and y's can be described by an element $\alpha \in \mathscr{A}(\hat{0}, \hat{1})$. Thus the β_i's form a chain in L_α. The problem is that more than one element of $\mathscr{A}(\hat{0}, \hat{1})$ could describe the mixing for certain chains.

The first step in describing the zeta polynomial for $L_1 \bowtie L_2$ is a simple application of the principle of inclusion–exclusion:

$$Z_{L_1 \bowtie L_2}(s) = \sum_{\mathscr{S} \neq \varnothing \in \mathscr{A}(\hat{0}, \hat{1})} (-1)^{|\mathscr{S}|+1} Z_{L_{\mathscr{S}}}(s) \tag{2}$$

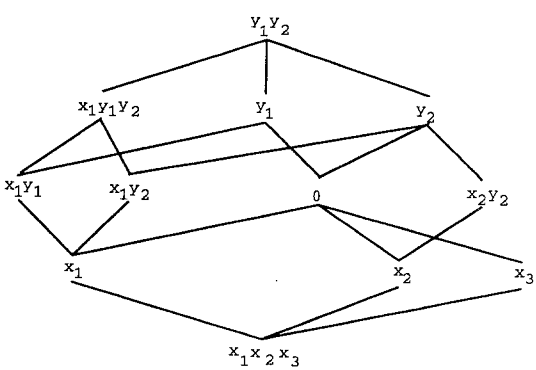


FIGURE 7

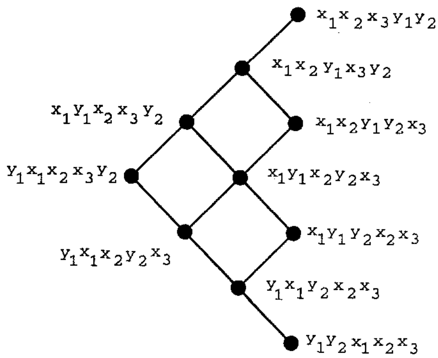


FIGURE 8

Define an ordering on $\mathscr{A}(\hat{0}, \hat{1})$ by associating with each $\alpha \in \mathscr{A}(\hat{0}, \hat{1})$ a partition $v = (v_m, ..., v_1)$ where v_i = the number of x's to left of y_i in α. Say that $\alpha \leqslant \beta$ if $v_\alpha \subseteq v_\beta$ as partitions. Call this poset $\mathscr{N}_{n,m}$.

EXAMPLE 8. $\mathscr{N}_{3,2}$ is shown in Fig. 8. $\mathscr{N}_{m,n}$ is a lattice. In $\alpha \vee \beta$, x_i appears to the left of y_j if x_i appears to the left of y_j in either α or β. In $\alpha \wedge \beta$, x_i appears to the right y_j if x_i appears to the right of y_j in either α or β.

LEMMA 6.1. $L_{\mathscr{S}} = L_{\{\bigvee_{\mathscr{N}} \mathscr{S}, \bigwedge_{\mathscr{N}} \mathscr{S}\}}$.

Proof. Given a pair x_i and y_j, one of three things can occur based on the choice of $\mathscr{S}$. Either x_i is always to the left of y_j in elements of $\mathscr{S}$, x_i is always to the right of y_j, or x_i is to the left of y_j in some elements of $\mathscr{S}$ and y_j is to the right of x_i in some other elements. The first two cases give the ordering that x_i and y_j must maintain if they both appear in some element of $L_{\mathscr{S}}$. The last case stipulates that no element of $L_{\mathscr{S}}$ can contain both x_i and y_j. Note that to describe $L_{\mathscr{S}}$, all that we need to known is which one of these three cases applies for each pair x_i and y_j.

Thus, we want to show that for every pair x_i and y_j, the sets $\mathscr{S}$ and $\{\bigvee_{\mathscr{N}} \mathscr{S}, \bigwedge_{\mathscr{N}} \mathscr{S}\}$ restrict the ordering of x_i and y_i in the same manner. If x_i always appears to the right of y_j in elements of $\mathscr{S}$, then x_i must appear to the right of y_j in both $\bigvee_{\mathscr{N}} \mathscr{S}$ and $\bigwedge_{\mathscr{N}} \mathscr{S}$. Likewise if x_i is always to the left of y_j. If x_i appears to right of y_j in some element of $\mathscr{S}$ and x_i appears to the left of y_j in some other element of $\mathscr{S}$, then x_i appears to right of y_j in $\bigwedge_{\mathscr{N}} \mathscr{S}$ and x_i appears to left of y_j in $\bigvee_{\mathscr{N}} \mathscr{S}$. Hence the sets $\mathscr{S}$ and $\{\bigvee_{\mathscr{N}} \mathscr{S}, \bigwedge_{\mathscr{N}} \mathscr{S}\}$ place the same restrictions on the order in which the x's and y's can appear. This proves the lemma. ∎

LEMMA 6.2. *If L is lattice with at least two elements, then*

$$\mu_L(\hat{0}, \hat{1}) = \sum_{\substack{\mathscr{S} \subseteq L \\ \wedge \mathscr{S} = \hat{0} \vee \mathscr{S} = \hat{1}}} (-1)^{|\mathscr{S}|+1}.$$

Proof. Proof by induction on the cardinality of L. The lemma is easily checked to be valid in case when the cardinality of L is 2 (i.e., $L = \mathscr{B}_1$). Suppose the lemma is valid for all lattices of cardinality strictly less than that of L:

$$\begin{aligned}
0 &= \sum_{\mathscr{S} \subseteq L} (-1)^{|\mathscr{S}|+1} \\
&= \underset{\mathscr{S} = \varnothing}{-1} + \sum_{a > \hat{0}} \sum_{b \geqslant a} \Bigg\{ \sum_{\substack{\mathscr{S} \subseteq L \\ \wedge \mathscr{S} = a \vee \mathscr{S} = b}} (-1)^{|\mathscr{S}|+1} \Bigg\} \\
&\quad + \sum_{b \geqslant \hat{0}} \Bigg\{ \sum_{\substack{\mathscr{S} \subseteq L \\ \wedge \mathscr{S} = \hat{0} \vee \mathscr{S} = b}} (-1)^{|\mathscr{S}|+1} \Bigg\} \\
&= -1 + \underbrace{\sum_{a > 0} \sum_{b \geqslant a} \mu_L(a, b)}_{= 0 \text{ for } a \neq \hat{1}} + \underbrace{\sum_{\hat{1} > b \geqslant \hat{0}} \mu_L(\hat{0}, b)}_{= -\mu_L(\hat{0}, \hat{1})} + \sum_{\substack{\mathscr{S} \subseteq L \\ \wedge \mathscr{S} = \hat{0} \vee \mathscr{S} = \hat{1}}} (-1)^{|\mathscr{S}|+1} \\
&= -1 + \mu_L(\hat{1}, \hat{1}) - \mu_L(\hat{0}, \hat{1}) + \sum_{\substack{\mathscr{S} \subseteq L \\ \wedge \mathscr{S} = \hat{0} \vee \mathscr{S} = \hat{1}}} (-1)^{|\mathscr{S}|+1}.
\end{aligned}$$

Thus

$$\mu_L(\hat{0}, \hat{1}) = \sum_{\substack{\mathscr{S} \subseteq L \\ \wedge \mathscr{S} = \hat{0} \vee \mathscr{S} = \hat{1}}} (-1)^{|\mathscr{S}|+1}. \qquad \blacksquare$$

THEOREM 6.3. $Z_{L_1 \bowtie L_2}(s) = \sum_{\alpha \leqslant_{\mathscr{N}} \beta} \mu_{\mathscr{N}}(\alpha, \beta)\, Z_{L_{\{\alpha, \beta\}}}$

Proof. Apply Lemma 6.1 to (2), to get that

$$Z_{L_1 \bowtie L_2}(s) = \sum_{\mathscr{S} \neq \varnothing \subseteq \mathscr{A}(\hat{0}, \hat{1})} (-1)^{|\mathscr{S}|+1} Z_{L_{\{\vee \mathscr{S}, \wedge \mathscr{S}\}}}.$$

So for a given $\alpha < \beta$ in $\mathscr{A}(\hat{0}, \hat{1})$, the coefficient on $Z_{L_{\{\alpha, \beta\}}}(s)$ in $Z_{L_1 \bowtie L_2}(s)$ is

$$\sum_{\substack{\mathscr{S} \subseteq \mathscr{A}(\hat{0}, \hat{1}) \\ \wedge \mathscr{S} = \alpha \vee \mathscr{S} = \beta}} (-1)^{|\mathscr{S}|+1}$$

By Lemma 6.2, this coefficient is $\mu_{\mathscr{N}}(\alpha, \beta)$, which proves the theorem. $\blacksquare$

The advantage that this method provides in calculating the zeta polynomial is that for most choices of α and β, $\mu_{\mathcal{N}}(\alpha, \beta) = 0$. The difficulty in applying this theorem is that the $Z_{L_{\{\alpha,\beta\}}}$ are not easy to calculate.

References

[BW] A. Björner and J. Walker, A homotopy complementation formula for partially ordered sets, *European J. Combin.* **4** (1983), 11–19.

[C] H. Crapo, The Möbius function of a lattice, *J. Combin. Theory* **1** (1966), 126–131.

[G] C. Greene, Posets of Shuffles, *J. Combin. Theory Ser.* **47** (1988), 191–206.

[S] R. Stanley, "Enumerative Combinatorics, Vol I," Wadsworthe & Brooks/Cole, Monterey, CA, 1986.