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A Representation of SFP

GUO-QIANG ZHANG*

Artificial Intelligence Laboratory,
Department of Electrical Engineering and Computer Science,
The University of Michigan, Ann Arbor, Michigan 48109-2110

A sequent structure of Gentzen style entailment with identity, weakening, and cut rules is given and shown to be basic to information systems. A category of strongly finite sequent structures is proved to be equivalent to **SFP**. Constructions such as the Plotkin powerdomain and the function space are provided, as well as a cpo (comple partial order) of such systems to give meanings to recursively defined domains.

1. INTRODUCTION

Information systems were introduced by Scott (Scott, 1982) initially with the intention to make domain theory accessible to a wider audience. In this representation the idea of *information* is made explicit—every element is built out of *infons*. It gives a logical approach to domain theory, so that properties of domains can be derived from assumptions about the entailment between propositions expressing properties of computation.

Information systems are a representation of Scott domains, which form a foundational framework for denotational semantics of programming languages. As shown by Plotkin (Plotkin, 1976), however, they cannot treat parallelism and concurrency adequately in some aspects. A more general framework is the category **SFP** (we call the objects of this category SFP domains) introduced by Plotkin (Plotkin, 1976). It has been an open question how best SFP domains can be represented as information systems since the work of Scott. A couple of attempts were made to find an extension of Scott's representation to the **SFP** case. However, none of these seems to provide a fully satisfactory treatment of SFP domains.

This paper gives an information system representation of **SFP**. A more basic structure called a *sequent structure* is shown to determine a major part of the axioms for information systems. Except for the axiom which requires the consistency of each single token, axioms for information systems are derivable properties of the sequent structures. A category of

* Current address: Department of Computer Science, The University of Georgia, Athens, GA 30602.

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special kind of sequent structures called strongly finite ones is shown equivalent to **SFP**. Constructions such as the Plotkin powerdomain and the function space are given, as well as a cpo (comple partial order) of such systems to give meanings to recursively defined domains. Finally, related works and similar representations for other categories of domains are discussed.

The representation is given with the intention of keeping as close as possible to the spirit of Scott's original construction. In this way, a more concrete, logical account of **SFP** is provided, as well as a framework in which domain equations can be solved up to equality within the category.

2. Sequent Structures

One of the proof systems for propositional logic is the sequent calculus due to Gentzen (see Gallier, 1986). For each logical operator there are two logical rules: one is the left rule and the other the right rule. The structural rules of Gentzen sequent calculus include the identity axiom, the weakening rule, and the cut rule. However, the cut rule has been shown redundant by a theorem called cut-elimination.

Stripping off the logical operators and the corresponding rules, we get from Gentzen's sequent calculus *a sequent structure*, abbreviated as SS.

Definition 2.1. A sequent structure, SS in short, is a pair

$$\underline{A} = (A, \vdash),$$

where A is a set of propositions and $\vdash$ is the entailment relation, satisfying the following rules:

$$(\textit{Identity}) \qquad a \vdash a,$$

$$(\textit{Weakening}) \qquad \frac{X' \supseteq X \quad X \vdash Y \quad Y \subseteq Y'}{X' \vdash Y'},$$

$$(\textit{Cut}) \qquad \frac{X \vdash Y, a \quad a, X' \vdash Y'}{X, X' \vdash Y, Y'}.$$

These notations follow the standard convention. In particular, by $X \vdash Y$ we mean that conjunction of Xs implies disjunction of Ys. Moreover, by $X, X' \vdash Y, Y'$ we mean $X \cup X' \vdash Y \cup Y'$.

It is important to note that when we write $X \vdash Y$, X and Y are always finite sets of propositions, with X *non-empty*. The non-emptiness of X plays an essential role in ensuring a complete partial order to be constructed from an SS. Note also that in the absence of logical operators and their related rules, *Cut* should not be eliminated. In fact, *Cut* is the key to the following theorem.

THEOREM 2.1. *Let* $(A, \vdash)$ *be a sequent structure. Then*

1. $a \in X \Rightarrow X \vdash a$,

   $a \in X \Rightarrow a \vdash X$,
2. $((\forall b \in Y.X \vdash b) \;\&\; Y \vdash Z) \Rightarrow X \vdash Z$,
3. $(X \vdash Y \;\&\; Y' \subseteq Y \;\&\; \forall b \in Y'.b \vdash Z) \Rightarrow X \vdash (Y \backslash Y'), Z$,
4. $(\forall b \in Y.X, b \vdash \emptyset) \;\&\; X \vdash Y \Rightarrow X \vdash \emptyset$,
5. $(\forall b \in Y.X \vdash b) \;\&\; X, Y \vdash \emptyset \Rightarrow X \vdash \emptyset$.

Proof. As an example, we prove the second conclusion in detail. This can be done by repeated applications of the cut rule. When Y is a singleton the conclusion follows by an application of *Cut*. In general, let $b \in Y$. We have $X \vdash b$ and $b, (Y \backslash \{b\}) \vdash Z$. Applying the cut rule we get $X, (Y \backslash \{b\}) \vdash Z$. Now choose a $b' \in (Y \backslash \{b\})$, if possible. We have $X \vdash b'$ and $b', (Y \backslash \{b, b'\}), X \vdash Z$. Applying the cut rule again, we get $X, (Y \backslash \{b, b'\}) \vdash Z$. Repeating this process a number of times until $Y \backslash \{b, b', b'', ...\}$ is empty, we get $X \vdash Z$. ∎

A sequent structure determines a family of subsets of propositions called its elements.

DEFINITION 2.2. The elements $|\underline{A}|$, of a sequent structure $\underline{A} = (A, \vdash)$, consists of subsets x of propositions which are closed under entailment:

$$(X \subseteq x \;\&\; X \vdash Y) \Rightarrow x \cap Y \neq \emptyset.$$

THEOREM 2.2. *For a sequent structure* $\underline{A}$, $|\underline{A}|$ *ordered by inclusion is a cpo.*

Proof. Since X must be non-empty for any entailment $X \vdash Y$, $\emptyset$ is an element. It is certainly the bottom of the partial order $(|\underline{A}|, \subseteq)$.

Let S be a directed subset of $|\underline{A}|$. It is easy to check that $\bigcup S \in |\underline{A}|$; this is because for any finite subset X of $\bigcup S$ there is an $s \in S$ such that $X \subseteq s$, by the directedness of S and finiteness of X. It is then trivial to show that $\bigcup S$ is closed under entailment. ∎

It is interesting to see how sequent structures relate to information systems. An information system is a structure describing the logical relations among propositions that can be made about computation. It consists of a set of propositions, a consistency predicate, and an entailment relation specified in Definition 2.3. We actually use a definition slightly different from the original one given in (Scott, 1982), without a distinguished Δ standing for the proposition that is always true. The definition here is the same as the one given in (Larsen and Winskel, 1984).

DEFINITION 2.3. An information system is a structure $\underline{A} = (A, Con, \vdash)$, where

- A is a set of propositions,
- $Con \subseteq \mathrm{Fin}(A)$, the consistent sets,
- $\vdash \subseteq Con \times A$, the entailment relation,

which satisfy

1. $X \subseteq Y \;\&\; Y \in Con \Rightarrow X \in Con$,
2. $a \in A \Rightarrow \{a\} \in Con$,
3. $X \vdash a \;\&\; X \in Con \Rightarrow X \cup \{a\} \in Con$,
4. $a \in X \;\&\; X \in Con \Rightarrow X \vdash a$,
5. $(\forall b \in Y. X \vdash b \;\&\; Y \vdash c) \Rightarrow X \vdash c$.

Notation. We write $\mathrm{Fin}(A)$ for the set of finite subsets of A. Write $X \subseteq^{\mathrm{fin}} y$ to mean that X is a finite subset of y.

DEFINITION 2.4. A sequent structure $\underline{A}$ is called deterministic if it satisfies the following axioms:

- $X \vdash Y \Rightarrow |Y| \leqslant 1$,
- $a \not\vdash \varnothing$.

The first axiom allows the possibility of $X \vdash \varnothing$ or $X \vdash \{a\}$ for some $a \in A$ only. The first kind of entailment specifies the consistency predicate for an information system, and the second kind corresponds to entailments of the form $X \vdash a$. These make it possible to get an information system from a deterministic sequent structure.

THEOREM 2.3. *A deterministic sequent structure determines an information system.*

Proof. Let $(A, \vdash)$ be a deterministic sequent structure. Define Con to be the set $\{X \mid X \not\vdash \varnothing\}$. It follows from Theorem 2.1 that the five axioms for an information system are derivable from the sequent structure. ∎

It is easy to see that for a deterministic sequent structure, its elements are exactly the same as those of the information system it determines. Therefore, for a deterministic sequent structure $\underline{A}$, $(|\underline{A}|, \subseteq)$ is a Scott domain.

3. SFP Domains

This section gives a brief review of some results on SFP domains. More importantly, it introduces a new characterization of finite elements in the function space of SFP domains. This characterization is indispensable not only because it is relatively simple, but also because it provides the definition of approximable mappings on strongly finite sequent structures, as well as the construction of function space.

SFP domains are directed limits of Sequence of Finite inductive Partial orders introduced by Plotkin (Plotkin, 1976). Rather than presenting the original definition given by Plotkin, we mention a practical characterization of SFP domains in terms of minimal upper bounds.

Let D be a cpo. A minimal upper bound of a subset $X \subseteq D$ is an upper bound of X which is not strictly greater than any other upper bound of X. We write $\bowtie(X)$ for the set of minimal upper bounds of X. $\bowtie(X)$ is said to be complete if whenever u is an upper bound of X, $v \sqsubseteq u$ for some $v \in \bowtie(X)$. Note that $\bowtie(\emptyset) = \{\bot\}$.

For a subset X of D, let

$$U^0(X) = X,$$
$$U^{i+1}(X) = \{u \mid u \in \bowtie(S) \;\&\; S \subseteq^{\text{fin}} U^i(X)\}, \qquad \text{for} \quad i \geqslant 0.$$

It is convenient to write $U^*(X)$ for $\bigcup_{i \geqslant 0} U^i(X)$.

THEOREM 3.1 (Plotkin). *D is an SFP domain if and only if the following three statements hold:*

1. *D is ω-algebraic,*
2. *$\forall X \subseteq^{\text{fin}} D^0 . \bowtie(X)$ is complete,* and
3. *$U^*(X)$ is finite.*

The category **SFP** consists of the SFP domains as objects and the continuous functions as morphisms. Important constructions in this category include the function space and the powerdomains, of which we give a brief account now.

Many questions reduce to the issue of characterizing finite elements in the function space. Gunter (Gunter, 1987) obtained a characterization of functions in terms of maxima in $D^0 \times E^0$, where D^0 and E^0 stand for the sets of finite elements of D and E, respectively. An equivalent, but more effective form is derived by Abramsky (Abramsky, 1991). Our characterization is different from these, and suits our purpose better (see the discussion section). We actually present a stronger result, so that it may also be applied elsewhere.

Recall that an algebraic cpo D is said to have property m if every subset of D has a complete set of minimal upper bounds.

DEFINITION 3.1. Let D, E be algebraic cpos with property m. A set $T \subseteq D^0 \times E^0$ is called joinable if

$$\forall T' \subseteq^{\text{fin}} T. [\forall a \in \bowtie (\pi_1 T') \exists b \in \bowtie (\pi_2 T'). (a, b) \in T].$$

A function $f: D \to E$ is a step function if there is a finite joinable set

$$\{(a_i, b_i) \mid i \in I\}$$

such that $f = \lambda x \in D. \bigsqcup \{b_i \mid i \in I \,\&\, a_i \sqsubseteq x\}$.

Here π_1 and π_2 are projections to the first and the second component, respectively. Note that $T \subseteq D^0 \times E^0$ is joinable if and only if

$$\begin{aligned} \forall T' \subseteq^{\text{fin}} T \exists X \subseteq \pi_1 T \exists Y. \quad & \bowtie (\pi_1 T') = X \,\& \\ & \bowtie (\pi_2 T') = Y \,\& \\ & \forall a \in X \exists b \in Y. (a, b) \in T. \end{aligned}$$

Putting joinability in yet another slightly different form, $T \subseteq D^0 \times E^0$ is joinable if and only if

$$\forall T' \subseteq^{\text{fin}} T \, \forall X \, \forall Y.$$
$$[\bowtie (\pi_1 T') = X \,\&\, \bowtie (\pi_2 T') = Y] \Rightarrow \forall a \in X \exists b \in Y. (a, b) \in T.$$

These equivalent descriptions of joinable sets may be adopted in different contexts for different purposes.

THEOREM 3.2. *Let D, E be algebraic cpos with property m, and $f: D \to E$ a continuous function. We have*

1. $\mu f =_{\text{def}} \{(a, b) \mid a \in D^0, b \in E^0 \,\&\, b \sqsubseteq f(a)\}$ *is a joinable set,*
2. *For any joinable set T, $\lambda x. \bigsqcup \{b \mid (a, b) \in T \,\&\, a \sqsubseteq x\}$ is a continuous function,*
3. $f = \lambda x \in D. \bigsqcup \{b \mid (a, b) \in \mu f \,\&\, a \sqsubseteq x\}$.

Proof. Let D, E be algebraic cpos with property m, and f a continuous function from D to E. For any finite subset $T' = \{(a_i, b_i) \mid i \in I\}$ of μf and any minimal upper bound a of $\pi_1 T'$, we have

$$f(a) \sqsupseteq f(a_i) \sqsupseteq b_i$$

for each i in I. This implies that $f(a) \sqsupseteq b$ for some b in $\bowtie \{b_i \mid i \in I\}$, by virtue of property m. Therefore, $(a, b) \in \mu f$, noting that a minimal upper bound of a finite set of finite elements is finite. This proves the first conclusion; i.e., μf is a joinable set.

For convenience in the proof of the second conclusion, we abbreviate the function constructed out of the joinable set T as g. For any $x \in D$,

$$T(x) =_{\text{def}} \{b \mid (a, b) \in T \;\&\; a \sqsubseteq x\}$$

is directed. In fact, let b_1 and b_2 be two elements in $T(x)$. By definition there are $a_1 \sqsubseteq x$, $a_2 \sqsubseteq x$ such that (a_1, b_1), (a_2, b_2) are both in T. But T is joinable. Therefore,

$$\forall a \in \bowtie \{a_1, a_2\}\ \exists b \in \bowtie \{b_1, b_2\}.(a, b) \in T.$$

Because x is above both a_1 and a_2, $\bowtie \{a_1, a_2\}$ is non-empty. Hence, $\bowtie \{b_1, b_2\}$ must also be non-empty, and $T(x) \cap \bowtie \{b_1, b_2\}$ is non-empty. This shows the directedness of $T(x)$. Once the directedness of $T(x)$ is established, we know that g is well-defined. The monotonicity of g is trivial. Its continuity follows from the fact that the pairs in T are made of finite elements.

The key to the proof of the third conclusion is to apply the following property of a continuous function f: for any finite element b,

$$b \sqsubseteq f(x) \Rightarrow b \sqsubseteq f(a)$$

for some finite element a below x. We leave the details to the reader. ∎

Thus any continuous function from D to E can be represented as a joinable set. We are more interested in finite joinable sets, however, especially the representability of finite elements of the function space in this form. Although every finite joinable set represents a finite element in the function space, the reverse is not true. It is the case, however, for SFP domains.

THEOREM 3.3. *Let D, E be SFP domains. A function is a finite element in $[D \to E]$ if and only if it equals some step function.*

Proof. Let T be a finite joinable set, and let

$$[\lambda x \in D. \bigsqcup \{b \mid (a, b) \in T \;\&\; a \sqsubseteq x\}] \sqsubseteq \bigsqcup \{f_j \mid j \in J\},$$

where $\{f_j \mid j \in J\}$ is a directed set of continuous functions. For any $(a', b') \in T$,

$$b' \sqsubseteq \bigsqcup \{b \mid (a, b) \in T \,\&\, a \sqsubseteq a'\} \sqsubseteq \bigsqcup_{j \in J} f_j(a').$$

It follows that $b' \sqsubseteq f_j(a')$ for some $j \in J$. Therefore, $(a', b') \in \mu f_j$. However, $\mu f \subseteq \mu g$ if $f \sqsubseteq g$, and T is finite. Hence $T \subseteq \mu f_k$ for some $k \in J$, which shows that the step function determined by T is a finite element.

For the second half of the conclusion, it is enough to prove that every continuous function can be approximated by step functions. Let $f: D \to E$ be a continuous function. Because D and E are SFP domains, we can enumerate the elements of μf and write

$$\mu f = \{(a_i, b_i) \mid i \in \omega\}.$$

Take T_0 to be the least joinable set $\{(\perp_D, \perp_E)\}$, and T_1 be $T_0 \cup \{(a_0, b_0)\}$. In general, from T_i we can construct a finite joinable set $T_{i+1} \subseteq \mu f$ which contains both T_i and $\{(a_i, b_i)\}$: All we have to do is to find a $\bowtie$-closed finite set containing

$$\pi_1 T_i \cup \{a_i\}$$

and pair every element in it with some appropriate element in E^0 to make a joinable set. The limit of the step functions determined by this sequence of finite joinable sets can be shown equal to the original f. ∎

This theorem indicates that finite joinable sets correspond to exactly the finite elements for SFP domains. With respect to the order on continuous functions, we have the following theorem.

THEOREM 3.4. *Let D, E be SFP domains and $f, g: D \to E$ continuous functions determined (in light of Theorem 3.2) by joinable sets $S, T \subseteq D^0 \times E^0$, respectively. We have*

$$f \sqsubseteq g \Leftrightarrow \forall (a, b) \in S\, \exists (a', b') \in T.(b \sqsubseteq b' \,\&\, a' \sqsubseteq a).$$

Proof. The proof of '$\Leftarrow$' is straightforward. To show the other way round, let $f \sqsubseteq g$ and $(a, b) \in S$. This implies $b \sqsubseteq f(a) \sqsubseteq g(a)$. Therefore,

$$b \sqsubseteq \bigsqcup \{b'' \mid (a'', b'') \in T \,\&\, a'' \sqsubseteq a\}.$$

However, from the proof of Theorem 3.2 we know that

$$\{b'' \mid (a'', b'') \in T \,\&\, a'' \sqsubseteq a\}$$

is a directed set. Therefore $b \sqsubseteq b'$ for some (a', b') in T, with $a' \sqsubseteq a$. ∎

We now turn to powerdomain constructions. Let D be an ω-algebraic cpo. There are three preorders on the powerset $\mathscr{P}(D)$: $\lesssim_0$, $\lesssim_1$, and $\lesssim_2$. They are defined as

$$
\begin{array}{lll}
A \lesssim_0 B & \text{if} \ \ \forall b \in B\, \exists a \in A . a \sqsubseteq b, & \\
A \lesssim_1 B & \text{if} \ \ \forall a \in A\, \exists b \in B . a \sqsubseteq b, & \text{and} \\
A \lesssim_2 B & \text{if} \ \ A \lesssim_0 B \;\&\; A \lesssim_1 B. &
\end{array}
$$

Powerdomains can be constructed from these preorders by a technique called *ideal completion* (Winskel, 1985). Let $(P, \lesssim)$ be a preorder with a least element. An ideal of $(P, \lesssim)$ is a subset $X \subseteq P$ which is downwards-closed and directed (hence non-empty). The set of ideals of $(P, \lesssim)$ is written as $I(P)$. $(I(P), \subseteq)$ is an algebraic domain with isolated elements $\{q \in P \mid q \lesssim p\}$ for $p \in P$. $(I(P), \subseteq)$ is called the ideal completion of $(P, \lesssim)$.

The Smyth powerdomain, the Hoare powerdomain, and the Plotkin powerdomain of an ω-algebraic cpo D are the ideal completions of $(M[D], \lesssim_0)$, $(M[D], \lesssim_1)$, and $(M[D], \lesssim_2)$, respectively, where $M[D]$ consists of the finite, non-empty, subsets of D^0.

Although conceptually simple, the powerdomains constructed from ideal completions have the disadvantage that they do not really live in the powerset $\mathscr{P}(D)$. The elements of $M[D]$ are already subsets of D, so the ideals are some sets of subsets of D, and are elements of $\mathscr{P}(\mathscr{P}(D))$ instead of $\mathscr{P}(D)$.

For the purpose of this paper a more concrete presentation of powerdomains is needed, so that the elements of the powerdomains are subsets of D. The idea is to pick up the biggest element (with respect to set inclusion) in each equivalent class induced by the three preorders (See Hrbacek, 1989 also).

Let D be an ω-algebraic cpo. Define, on the powerset $\mathscr{P}(D)$, the three operations

$$
\begin{array}{ll}
Cl_{\mathrm{S}}(A) = \uparrow A, & \\
Cl_{\mathrm{H}}(A) = \downarrow A, & \text{and} \\
Cl_{\mathrm{P}}(A) = Cl_{\mathrm{S}}(A) \cap Cl_{\mathrm{H}}(A), &
\end{array}
$$

where $\uparrow A = \{x \in D \mid \exists a \in A . a \sqsubseteq x\}$, and $\downarrow A = \{x \in D \mid \exists a \in A . x \sqsubseteq a\}$. Clearly these operations are idempotent; i.e., $Cl(Cl(A)) = Cl(A)$ for every $A \in \mathscr{P}(D)$.

Note that $Cl_{\mathrm{S}}(A) \subseteq Cl_{\mathrm{S}}(B)$ if and only if $B \lesssim_0 A$. The Smyth powerdomain of an ω-algebraic domain D consists of, as the finite elements, all subsets $Cl_{\mathrm{S}}(A)$ with A a non-null finite set of D^0. Such a collection of elements form a partial order under the superset ordering, and we have to add to it the limits (least upper bounds) to get a cpo. The limit point of an ω-increasing chain

$$Cl_S(A_0) \sqsubseteq_0 Cl_S(A_1) \cdots \sqsubseteq_0 Cl_S(A_i) \sqsubseteq_0 \cdots$$

is the intersection $\bigcap_{i \in \omega} Cl_S(A_i)$.

Similarly, $Cl_H(A) \subseteq Cl_H(B)$ if and only if $A \sqsubseteq_1 B$, and the Hoare powerdomain of an ω-algebraic domain D consists of, as finite elements, all subsets $Cl_H(A)$ with A a non-null finite set of D^0. The limit point of an ω-increasing chain

$$Cl_H(A_0) \subseteq Cl_H(A_1) \cdots \subseteq Cl_H(A_i) \subseteq \cdots$$

is the union

$$\bigcup_{i \in \omega} Cl_H(A_i).$$

However, the Plotkin powerdomain of an ω-algebraic cpo D is slightly complicated. It consists of, as finite elements, all subsets $Cl_P(A)$ with A a non-empty finite set of D^0. The order is $\sqsubseteq_2$. The limit point of an ω-increasing chain

$$Cl_P(A_0) \sqsubseteq_2 Cl_P(A_1) \cdots \sqsubseteq_2 Cl_P(A_i) \sqsubseteq_2 \cdots$$

is defined to be the closure $Cl_P(A)$, where A consists of the least upper bounds of increasing chains

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \cdots \sqsubseteq a_n \sqsubseteq \cdots$$

such that $\forall i \in \omega . a_i \in A_i$.

Write $\mathscr{P}_P(D)$ for the set consisting of closures $Cl_P(B)$ with $B \in M[D]$ together with all the limits of ω-increasing chains given above. It has to be shown that the limit $Cl_P(A)$ is indeed the least upper bound with respect to $\sqsubseteq_2$ in $\mathscr{P}_P(D)$ (note that it is not possible to prove that the limit is the least upper bound in the powerset $\mathscr{P}(D)$). The following theorem justifies the construction of the Plotkin powerdomain. In particular, the limit construction gives least upper bounds.

THEOREM 3.5. *Given an ω-algebraic cpo D, $(\mathscr{P}_P(D), \sqsubseteq_2)$ is the Plotkin powerdomain. It is an ω-algebraic cpo isomorphic to the ideal completion of the preorder $(M[D], \sqsubseteq_2)$.*

Proof. See Zhang, 1991. ∎

The rest of the section is about a useful characterization of minimal upper bounds. Recall that a subset O of a cpo D is Scott open if for any directed set Q, $\bigsqcup Q \in O$ implies $Q \cap O \neq \emptyset$. Note that $\uparrow x$ is open if and only if $x \in D^0$, where

$$\uparrow x = \{ y \in D \mid x \sqsubseteq y \}.$$

Open sets of this form are called *prime* since they are the complete primes in the lattice of Scott open sets of D, in the sense that if they are dominated (with respect to set-inclusion) by a join they are dominated by an element of it.

PROPOSITION 3.1. *Let A, B be finite sets of isolated elements of an ω-algebraic domain D. $\bowtie A = B$ and B is complete if and only if*

$$\bigcap_{a \in A} \uparrow a = \bigcup_{b \in B} \uparrow b \qquad \text{and} \qquad \forall b, b' \in B. (b \sqsubseteq b' \Rightarrow b = b').$$

Proof. See (Gunter, 1985, p. 61). One should be careful that the requirement on B in the proposition cannot be dropped. ▮

This proposition suggests a useful *quasi-conjunctive* property. A set P of prime open sets of domain D is quasi-conjunctive if for all non-null finite subset S of P there is a subset R of P such that

$$\bigcap S = \bigcup R.$$

4. STRONGLY FINITE SEQUENT STRUCTURES

This section introduces a category of strongly finite sequent structures, denoted as SFSS. It then shows that this category is equivalent to SFP.

For technical convenience we assume the proposition set of a sequent structure to be countable, and each proposition is consistent ($a \not\vdash \emptyset$). To be more precise sometimes we write a SS with subscripts, such as $\underline{A} = (A, \vdash_{\underline{A}})$. We abbreviate

$$Y \in \mathrm{Fin}(A) \,\&\, \forall a \in Y. X \vdash b \qquad \text{as} \qquad X \vdash^* Y,$$

and

$$X \vdash Y \,\&\, \forall b \in Y. b \vdash^* X \qquad \text{as} \qquad X \rightleftharpoons Y.$$

Note that as suggested by the notation, $\rightleftharpoons$ is not a symmetric relation.

DEFINITION 4.1. A sequent structure $\underline{A}$ is called strongly finite (SFSS) if it satisfies the axiom of finite closure: for all $X \subseteq^{\text{fin}} A$ there is a finite super set $Y \supseteq X$ such that

$$(Y \supseteq Y' \neq \varnothing) \Rightarrow \exists Y'' \subseteq Y.\, Y' \rightleftharpoons Y'',$$

as well as the atomicity axiom: $a \vdash X \Rightarrow \exists b \in X. a \vdash b$.

We call such Y's $\rightleftharpoons$-closed super sets of X.

It follows from $Y' \rightleftharpoons Y''$ that $Y' \vdash Y''$ and $\forall b \in Y''.\{b\} \vdash^* Y'$. Therefore, if we make the logical relations explicit and write $Y' = \{a_i \mid i \in I\}$ and $Y'' = \{b_j \mid j \in J\}$, then $Y' \rightleftharpoons Y''$ means $\bigwedge a_i \Leftrightarrow \bigvee b_j$. The axiom of finite closure says that for any finite set of propositions there is a super set which has the property that any conjunction of propositions of a subset of the super set is equivalent to a disjunction of propositions of some subset of the super set. This is more or less a restatement of quasi-conjunctive closedness.

It can be easily checked that for any SFSS $\underline{A}$,

$$\bar{a} =_{\text{def}} \{b \mid b \in A \;\&\; a \vdash b\}$$

is a finite element of $(|\underline{A}|, \subseteq)$, where $a \in A$.

THEOREM 4.1. *If $\underline{A}$ is a strongly finite sequent structure, then $|\underline{A}|$ is an ω-algebraic cpo with all its non-bottom finite elements being of the form $\bar{a}$, where $a \in A$.*

Proof. We already know that all $\bar{a}$'s are finite elements. If x is a finite element of $|\underline{A}|$, then clearly

$$x = \bigcup \{\bar{a} \mid a \in x\}.$$

Now we show that $\{\bar{a} \mid a \in x\}$ is directed. Let $a, b \in x$. By the axiom of finite closure there is some finite Y such that $\{a, b\} \vdash Y$, $\forall c \in Y$. $\{c\} \vdash \{a\}$ and $\{c\} \vdash \{b\}$. However, x is closed under entailment, so $\{a, b\} \vdash Y$ implies that there is some $c_0 \in Y$ such that $c_0 \in x$. For this particular c_0 we also have $\{c_0\} \vdash \{a\}$ and $\{c_0\} \vdash \{b\}$. Hence $\bar{a} \subseteq \overline{c_0}$ and $\bar{b} \subseteq \overline{c_0}$, which means $\{\bar{a} \mid a \in x\}$ is directed. As x is a finite element there is some $\overline{a_0} \in \{\bar{a} \mid a \in x\}$ with $x \subseteq \overline{a_0}$, which is only possible when $x = \overline{a_0}$. We have shown that every finite element of $|\underline{A}|$ is of the form $\bar{a}$. The ω-algebracity of $|\underline{A}|$ then follows easily. ∎

Moreover, SFSSs represent SFP domains.

THEOREM 4.2. *If $\underline{A}$ is a strongly finite sequent structure then $|\underline{A}|$ is an SFP domain.*

Proof. Let S be a finite, consistent subset of finite elements of $|\underline{A}|$. We know from the previous theorem that S can be written as

$$S = \{\overline{a_1}, \overline{a_2}, ..., \overline{a_n}\}$$

with a_i's in A. By the axiom of finite closure there is some finite Y such that

$$\{a_i \mid 1 \leqslant i \leqslant n\} \rightleftharpoons Y.$$

Since S is consistent, i.e.,

$$\exists b \in A . \forall i . \overline{a_i} \subseteq \bar{b},$$

the above Y must be non-empty because $\overline{a_i} \subseteq \bar{b}$ implies $\{b\} \vdash \{a_i\}$ for each i, and $\{b\} \nvdash \varnothing$.

Let Y' be a subset of Y such that $\forall b, b' \in Y' . \{b\} \vdash \{b'\} \Rightarrow b = b'$. It is not difficult to check that

$$\bowtie(S) = \{\bar{b} \mid b \in Y'\},$$

which implies that $\bowtie(S)$ is finite and non-empty.

$\bowtie(S)$ is complete because assuming $x \supseteq s$ for all $s \in S$ we have

$$x \supseteq \{a_i \mid 1 \leqslant i \leqslant n\}.$$

By closedness of x under entailment there is some $b \in Y$ with $b \in x$. Therefore $x \supseteq \bar{b}$.

Similarly, by the axiom of finite closure we know that S has a $\bowtie$-closed finite super set. ∎

On the other hand, an SFP domain determines a strongly finite sequent structure in the way described below.

Definition 4.2. Let D be an SFP domain. Define

$$IS(D) = (\mathbf{P\Omega}(D), \vdash),$$

where $\mathbf{P\Omega}(D)$ is the set $\{\uparrow d \mid d \in D^0 \backslash \{\bot_D\}\}$ and

$$X \vdash Y \quad \text{iff} \quad \bigcap X \subseteq \bigcup Y.$$

It is routine to show that $IS(D)$ is a SS. When $\bigcap X = \bigcup Y$ we have both $\bigcap X \subseteq \bigcup Y$ and $b \subseteq \bigcup Y \subseteq a$ for any $a \in X$, $b \in Y$, i.e., $X \rightleftharpoons Y$. Hence the axiom of finite closure follows from the quasi-conjunctive closedness of SFP domains.

We now introduce morphisms on SFSSs called approximable mappings. This makes SFSSs a category. Approximable mappings show how sequent structures are related to one another and they correspond to continuous functions between the associated SFP domains. The way approximable mappings defined is slightly different from the traditional one on information systems. The canonical character of strongly finite sequent structures makes it possible to specify approximable mappings as relations on propositions rather than on consistent sets.

DEFINITION 4.3. Let $\underset{\sim}{A} = (A, \vdash_A)$, $\underset{\sim}{B} = (B, \vdash_B)$ be SFSSs. An approximable mapping from $\underset{\sim}{A}$ to $\underset{\sim}{B}$ is a relation $R \subseteq A \times B$ which satisfies aRb & $b \vdash_B b' \Rightarrow aRb'$, and

$$\forall S \subseteq^{\text{fin}} R\, \forall X\, \forall Y. (\pi_1 S \rightleftharpoons_A X \mathbin{\&} \pi_2 S \rightleftharpoons_B Y) \Rightarrow \forall a \in X\, \exists b \in Y. aRb.$$

One can immediately recognize that this is an exact translation of one of the equivalent forms of joinable sets given in Definition 3.1. Note that for domains the set of minimal upper bounds of any set exists, and is unique. Reflected in Definition 3.1, this fact implies that the quantifications for X and Y there do not really matter. For SFSSs, however, a similar kind of uniqueness does not hold. Therefore universal quantifications are used here for X and Y, to make the approximable mappings "saturated."

PROPOSITION 4.1. *For an approximable mapping R we have*

$$a \vdash a' \mathbin{\&} a'Rb' \Rightarrow aRb'.$$

Proof. This is because when $a \vdash a'$ we have $a' \rightleftharpoons \{a, a'\}$. The desired property now follows by taking $\{(a', b')\} \subseteq R$, $X = \{a, a'\}$, $Y = \{b'\}$, and applying Definition 4.3. ▮

PROPOSITION 4.2. *Strongly finite sequent structures with approximable mappings form a category, written as* **SFSS**.

Proof. Identities are given by $a\, Id_A b$ if $a \vdash_A b$.

We check that approximable mappings compose. Other axioms for a category can be checked similarly. Let $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ be SFSSs and $R: \underset{\sim}{A} \to \underset{\sim}{B}$ and $S: \underset{\sim}{B} \to \underset{\sim}{C}$ be approximable mappings. Let $R \circ S$ be the relational composition. We show that $R \circ S$ is an approximable mapping. Suppose, for a finite set I, $\forall i \in I. a_i (R \circ S)\, c_i$ and $\{a_i \mid i \in I\} \rightleftharpoons_A X$, $\{c_i \mid i \in I\} \rightleftharpoons_C Z$. There exists a $u_i \in B$ such that $a_i R u_i$, $u_i S c_i$ for any $i \in I$. Let $\{u_i \mid i \in I\} \rightleftharpoons_B Y$. The existence of such Y follows from the strong finiteness. Since R is an approximable mapping, $\forall p \in X\, \exists q \in Y. pRq$. But for each $q \in Y$ we have qSc_i for all $i \in I$. Therefore there exists some $r \in Z$ such that qSr, since $q \rightleftharpoons q$. Hence $\forall p \in X\, \exists r \in Z. p(R \circ S)\, r$. ▮

Note that the universal quantifications over X and Y in Definition 4.3 is essential for approximable mappings to compose.

PROPOSITION 4.3. *Let R be an approximable mapping from $\underline{A}$ to $\underline{B}$. Define $f_R: |\underline{A}| \to |\underline{B}|$ by*

$$f_R(x) = \{b \in B \mid \exists a \in x . aRb\}.$$

Then f_R is a continuous function from $|\underline{A}|$ to $|\underline{B}|$.

Proof. Let $x \in |\underline{A}|$ and let $R: \underline{A} \to \underline{B}$ be an approximable mapping. To show that $f_R(x) \in |\underline{B}|$ let $Y \subseteq f_R(x)$ and $Y \vdash_{\underline{B}} Z$. For each $b \in Y$ there is some $a \in x$ such that aRb. Write X for such a collection a a's. Because $\underline{A}$ and $\underline{B}$ are strongly finite there are X', Y' such that $X \rightleftharpoons_{\underline{A}} X'$ and $Y \rightleftharpoons_{\underline{B}} Y'$. This means we have $X \subseteq x$ and $X \vdash_{\underline{A}} X'$, which implies $X' \cap x \neq \emptyset$. Now let $u_0 \in X' \cap x$. By Definition 4.3 there is a $v_0 \in Y'$ such that $u_0 R v_0$. Thus $v_0 \in f_R(x)$. Also we have $v_0 \vdash_{\underline{B}} Z$, which implies $v_0 \vdash_{\underline{B}} c$ for some $c \in Z$. Therefore $c \in f_R(x) \cap Z$.

The monotonicity of f_R is obvious. It also preserves least upper bounds of directed sets of $|\underline{A}|$; for assuming $b \in f_R(\bigsqcup P)$ with P directed, there is some $a \in \bigcup P$ such that aRb. There is, therefore, some $y \in P$ for which $a \in y$. Hence $b \in f_R(y)$ and

$$f_R\left(\bigcup P\right) \subseteq \bigcup \{f_R(y) \mid y \in P\},$$

enough for the equality to hold. ▮

We now show that the category of strongly finite sequent structures and the category of SFP domains are equivalent. Informally this result implies that strongly finite information systems and SFP domains are essentially the same. We will use one of MacLane's results (MacLane, 1971, p. 91) in the proof. By this result, a functor F determines an equivalence of the categories if it is full and faithful, and each SFP domain D is isomorphic to $F(\underline{A})$ for some SFSS $\underline{A}$.

THEOREM 4.3. **SFSS** *is equivalent to* **SFP**.

Proof. Let F: **SFSS** → **SFP** be the functor given by

$$F(\underline{A}) = |\underline{A}|$$

$$F(R) = f_R.$$

That each SFP domain D is isomorphic to $F(\underline{A})$ for some SFSS $\underline{A}$ is easy to check. It remains to show that F is full and faithful. First we show that

F is full. Let $\underline{A}$ and $\underline{B}$ be SFSSs and f: $F(\underline{A}) \to F(\underline{B})$ a continuous function. Define a relation $R \subseteq A \times B$ by letting aRb if $b \in f(\bar{a})$. We check that this relation is an approximable mapping from $\underline{A}$ to $\underline{B}$. Let $\{(a_i, b_i) \mid i \in I\}$ be a finite subset of R. Assume that

$$\{a_i \mid i \in I\} \rightleftharpoons_{\underline{A}} X$$

and

$$\{b_i \mid i \in I\} \rightleftharpoons_{\underline{B}} Y.$$

For any $a \in X$, $a \vdash^* \{a_i \mid i \in I\}$. Thus we have $b_i \in f(\overline{a_i}) \subseteq f(\bar{a})$ for any $i \in I$. Now $\{b_i \mid i \in I\} \vdash_{\underline{B}} Y$. Therefore $f(\bar{a}) \cap Y \neq \varnothing$. This means for some $b \in Y$, $b \in f(\bar{a})$, or aRb.

We now show that the continuous function f_R determined by the above R is actually equal to f. Let $x \in |\underline{A}|$. Suppose $b \in f_R(x)$. By definition there is some $a \in x$, aRb. That is, $b \in f(\bar{a})$. Therefore $b \in f(x)$, by the monotonicity of f. Thus $f_R(x) \subseteq f(x)$. On the other hand, let $b \in f(x)$. By the continuity of f there is some $a \in x$ such that $b \in f(\bar{a})$. Hence aRb and $b \in f_R(x)$. This means $f(x) \subseteq f_R(x)$. Hence $f = f_R$.

Second, we show that F is faithful. Suppose R, S: $\underline{A} \to \underline{B}$ are approximable mappings such that $f_R = f_S$. Let aRb. Then $b \in f_S(\bar{a})$. This means for some $a' \in \bar{a}$, $a'Sb$. By Proposition 4.1, aSb. Therefore, $R \subseteq S$. By symmetry, $S \subseteq R$ and hence $R = S$. ▌

5. Constructions

This section introduces constructions on SFSS. We focus on the more complicated function space and powerdomain constructions while leaving the simpler ones such as sum, product, and lifting to the reader.

To give a construction of function space, we need an auxiliary definition first.

Definition 5.1. Let $\underline{A} = (A, \vdash_{\underline{A}})$ and $\underline{B} = (B, \vdash_{\underline{B}})$ be SFSSs. A finite subset m of $A \times B$ is said to be a molecule if

$$\begin{aligned} \forall m' \subseteq m\, \exists X \subseteq \pi_1 m\, \exists Y.\quad & \pi_1 m' \rightleftharpoons_{\underline{A}} X, \\ & \pi_2 m' \rightleftharpoons_{\underline{B}} Y, \qquad \text{and} \\ & \forall a \in X\, \exists b \in Y.(a, b) \in m. \end{aligned}$$

We can observe that molecules are an exact translation of one of the equivalent descriptions of joinable sets given after Definition 3.1.

DEFINITION 5.2 (Function Space). Let $\underset{\sim}{A} = (A, \vdash_A)$ and $\underset{\sim}{B} = (B, \vdash_B)$ be SFSSs. Their function space, $\underset{\sim}{A} \to \underset{\sim}{B}$, is the sequent structure $\underset{\sim}{C} = (C, \vdash)$ given by

- $C = \{m \mid m \text{ is a molecule in } A \times B\}$.
- $X \vdash Y$ iff $(\forall m' \in X. \{m\} \vdash \{m'\}) \Rightarrow \exists m'' \in Y. \{m\} \vdash \{m''\}$, where $\{m\} \vdash \{m'\} \Leftrightarrow \forall \alpha' \in m' \, \exists \alpha \in m. \{\pi_1 \alpha'\} \vdash_A \{\pi_1 \alpha\} \;\&\; \{\pi_2 \alpha\} \vdash_B \{\pi_2 \alpha'\}$.

Unlike Scott's information system where one can use (X, Y) as propositions for the function space, we have to use a more complicated form of proposition. The reason for not being able to use a simpler form of proposition can be illustrated by the following example.

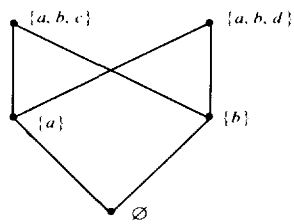
EXAMPLE. Consider a SFSS $\underset{\sim}{A} = (A, \vdash)$, where

$$A = \{a, b, c, d\},$$

$\vdash$ is given by

$$\{a, b\} \vdash \{c, d\},$$
$$\{c\} \vdash \{a\}, \{c\} \vdash \{b\},$$
$$\{d\} \vdash \{a\}, \{d\} \vdash \{b\},$$
$$\{a, b, c, d\} \vdash \varnothing.$$

This SFSS represents the SFP domain pictured as follows:



Now consider the function space from $\underset{\sim}{A}$ to itself. If one were to use propositions of the form (u, v) with $u, v \in A$ and $\{(u, v)\} \vdash \{(u', v')\}$ iff $\{u'\} \vdash \{u\}$ and $\{v\} \vdash \{v'\}$ (which is reasonable), one would only have an entailment

$$\{(a, a), (b, b)\} \vdash \{(c, c), (c, d), (d, c), (d, d)\}$$

but without any Y such that

$$\{(a, a), (b, b)\} \rightleftharpoons Y;$$

i.e., the resulting information system would not be strongly finite.

There is a general guideline according to which we can test the definition of entailment. $X \vdash Y$ is a sequent structure should mean that

$$\bigcap \{\uparrow \bar{a} \mid a \in X\} \subseteq \bigcup \{\uparrow \bar{b} \mid b \in Y\}$$

in the topology of the domain. In other words, for any point x above all $\bar{a}$ with $a \in X$, this x should be above some point $\bar{b}$ for $b \in Y$. Since the domains concerned are algebraic, this x can be chosen from the set of finite elements, which are of the form $\bar{a}$.

According to Theorem 3.4, it is reasonable to have, for molecules m, m',

$$\{m\} \vdash \{m'\} \Leftrightarrow \forall \alpha' \in m' \; \exists \alpha \in m . \{\pi_1 \alpha'\} \vdash_{\underline{A}} \{\pi_1 \alpha\} \; \& \; \{\pi_2 \alpha\} \vdash_{\underline{B}} \{\pi_2 \alpha'\}.$$

Therefore,

$$(\forall m' \in X . \{m\} \vdash \{m'\}) \Rightarrow \exists m'' \in Y . \{m\} \vdash \{m''\}$$

is just a direct translation of the corresponding topological relation. It has, however, the unwelcome feature that it contains a universal quantifier over propositions in C. Nevertheless, this unwelcome feature is removable. We come back to this point after the following proposition, which shows that the construction of function spaces preserves strongly finite sequent structures.

PROPOSITION 5.1. *If $\underline{A}$ and $\underline{B}$ are strongly finite then $\underline{A} \to \underline{B}$ is strongly finite.*

It is routine to check that $\underline{A} \to \underline{B}$ is a SS. To show that $\underline{A} \to \underline{B}$ satisfies the finite closure axiom, let $X \subseteq^{\text{fin}} C$. Since $\underline{A}$ and $\underline{B}$ are strongly finite we know that there are P, Q, $\rightleftharpoons_{\underline{A}}$ and $\rightleftharpoons_{\underline{B}}$-closed, respectively, with $P \supseteq \bigcup \{\pi_1 m \mid m \in X\}$ and $Q \supseteq \bigcup \{\pi_2 m \mid m \in X\}$. It is enough to show that

$$\{m \in C \mid \pi_0 m \subseteq P \; \& \; \pi_1 m \subseteq Q\}$$

is finite, is $\rightleftharpoons$-closed, and contains X. However, this is a corollary of Theorem 4.3 and the following lemma.

LEMMA 5.1. *Let D, E be SFP domains, and Z a finite set of finite joinable sets over $D^0 \times E^0$. In the function space $D \to E$,*

$$\Pi \underset{\text{def}}{=} \{T \mid T \textit{ is joinable}, \; \pi_1 T \subseteq G \; \& \; \pi_2 T \subseteq Q\}$$

specifies a collection of functions which forms a finite, $\bowtie$-closed super set of the functions specified by Z. Here G is a finite, $\bowtie$-closed super set of $\pi_1 \bigcup Z$, and Q is a finite, $\bowtie$-closed super set of $\pi_2 \bigcup Z$.

Proof. Let Z' be a subset of Π. It is enough to show that given any minimal upper bound f of the functions determined by the elements in Z', there is already some member T in Π which defines the same function f.

Assume f *is* such a function. We can construct the T in the following way. Write Z' as $\{(a_i, b_i) \mid i \in I\}$, where I is finite, and consider $\{(a_j, b_j) \mid j \in J\}$, with $J \subseteq I$. For any $a \in \bowtie\{a_j \mid j \in J\}$, choose a b in $\bowtie\{b_j \mid j \in J\}$ such that $f(a) \sqsupseteq b$. This is possible because f, being a minimal upper bound f of the functions determined by the elements in Z', has the property that $f(a_i) \sqsupseteq b_i$ for all i in I. Now update Z' by adding the pair (a, b) to it, and repeat this process until a joinable set is formed. This process terminates because $G \times Q$ is a finite set. However, when this process terminates, we get a joinable set T such that

$$\bigcup Z' \subseteq T \subseteq \mu f.$$

Moreover, it can be seen that the step function determined by T equals f. ∎

Therefore, we have the following rule, telling us how to get $\rightleftharpoons$-closed sets in the function space of strongly finite sequent structures:

$$P, Q \rightleftharpoons\text{-closed} \;\&\; \bigcup \{\pi_1 m \mid m \in F\} \subseteq P \;\&\; \bigcup \{\pi_2 m \mid m \in F\} \subseteq Q$$

$$\Rightarrow \{m \in C \mid \pi_0 m \subseteq P \;\&\; \pi_1 m \subseteq Q\} \text{ is a } \rightleftharpoons\text{-closed super set of } F.$$

For convenience we write $\Pi(F)$ for the set

$$\{m \in C \mid \pi_0 m \subseteq P \;\&\; \pi_1 m \subseteq Q\}$$

given above. We now have an effective (but equivalent to that provided in Definition 5.2) definition for entailment in the function space:

$$X \vdash Y \qquad \text{iff} \quad \forall m \in \Pi(X).(\forall m' \in X.\{m\} \vdash \{m'\}) \Rightarrow \exists m'' \in Y.\{m\} \vdash \{m''\}.$$

The following isomorphism is expected.

PROPOSITION 5.2. *If* $\underline{A}$ *and* $\underline{B}$ *are SFSSs then*

$$|\underline{A} \to \underline{B}| \cong |\underline{A}| \to |\underline{B}|.$$

Powerdomain constructions can also be given on strongly finite sequent structures.

DEFINITION 5.3. Let $\underline{A}$ be a strongly finite sequent structure. The Hoare powerdomain of $\underline{A}$ is the sequent structure $P_{\mathrm{H}}\underline{A} = (C, \vdash)$ where

- $C = \{B \mid B \subseteq^{\text{fin}} A \;\&\; B \neq \varnothing\}$,
- $X \vdash Y \Leftrightarrow \exists \beta \in Y. \left(\forall b \in \beta\, \exists a \in \bigcup X. a \vdash_{\underline{A}} b\right)$.

According to the definition, $\{\alpha\} \vdash \{\beta\}$ if and only if

$$\forall b \in \beta\, \exists a \in \alpha. a \vdash_{\underline{A}} b.$$

Therefore $\vdash$ on C is a reverse of the preorder $\precsim_{\mathrm{l}}$ introduced in Section 3. We want $X \rightleftharpoons Y$ to characterize the situation

$$\bigcap \{\uparrow\bar{\alpha} \mid \alpha \in X\} \subseteq \bigcup \{\uparrow\bar{\beta} \mid \beta \in Y\}.$$

This is equivalent to saying that whenever $\{\alpha_0\} \vdash \{\alpha\}$ for all $\alpha \in X$, $\{\alpha_0\} \vdash \{\beta_0\}$ for some $\beta_0 \in Y$. Clearly the entailment thus derived is equivalent to $X \vdash Y$ since we have $X \rightleftharpoons \{\bigcup X\}$.

We can think of $\alpha \in C$ as a logical formula $\bigwedge \{\Diamond a \mid a \in \alpha\}$, where intuitively a set S of processes satisfies $\Diamond a$ if there exists $p \in S$, p satisfies a. Then we have, for example,

$$\left(\bigwedge \{\Diamond a \mid a \in \alpha\}\right) \wedge \left(\bigwedge \{\Diamond b \mid b \in \beta\}\right) \Leftrightarrow \bigwedge \{\Diamond c \mid c \in \alpha \cup \beta\},$$

in the sense that a set of processes satisfies the proposition on the left hand side if and only if it satisfies the proposition on the right hand side (see Vickers, 1989).

PROPOSITION 5.3. *If $\underline{A}$ is a strongly finite sequent structure then so is $P_{\mathrm{H}}\underline{A}$.*

Proof. $P_{\mathrm{H}}\underline{A}$ is clearly a SS. Suppose $X \subseteq^{\text{fin}} C$. Let $\beta = \bigcup X$. Then $\beta \in C$, $X \vdash \{\beta\}$, and $\forall \alpha \in X \{\beta\} \vdash \{\alpha\}$ according to the definition of entailment. Therefore $X \rightleftharpoons \{\beta\}$, from which it is easy to see that the finite closure axiom holds. ∎

The definition of entailment for $P_{\mathrm{H}}\underline{A}$ implies that, as explained earlier, if $X \vdash Y$ then there is some $\beta \in Y$ such that $X \vdash \{\beta\}$. Thus $P_{\mathrm{H}}\underline{A}$ is actually deterministic. This indicates that $|P_{\mathrm{H}}\underline{A}|$ is always a Scott domain.

PROPOSITION 5.4. *$|P_{\mathrm{H}}\underline{A}|$ is isomorphic to $\mathscr{P}_{\mathrm{H}}\,|\underline{A}|$.*

Proof. We want to establish an order preserving one-one correspondence between the finite elements of $|P_H \underline{A}|$ and $\mathscr{P}_H |\underline{A}|$. Define

$$\theta: |P_H \underline{A}| \to \mathscr{P}_H |\underline{A}| \qquad \bar{\alpha} \mapsto Cl_H(\{\bar{a} \mid a \in \alpha\})$$

with $\alpha \in C$ and

$$\eta: \mathscr{P}_H |\underline{A}| \to |P_H \underline{A}| \qquad Cl_H(\{\bar{a} \mid a \in X\}) \mapsto \bar{X},$$

where $X \subseteq^{\text{fin}} A$. Suppose $\bar{\alpha} \subseteq \bar{\beta}$. Then $\alpha \in \bar{\beta}$, or $\{\beta\} \vdash \{\alpha\}$. But by definition we have $\forall a \in \alpha\, \exists b \in \beta . \{b\} \vdash_{\underline{A}} \{a\}$. Hence

$$\forall \bar{a} \in \{\overline{a'} \mid a' \in \alpha\}\, \exists \bar{b} \in \{\overline{b'} \mid b' \in \beta\} . \bar{a} \subseteq \bar{b}.$$

Therefore $\{\overline{a'} \mid a \in \alpha\} \lesssim_1 \{\overline{b'} \mid b' \in \beta\}$, which implies that θ is order preserving. It is easy to see that θ is one-one. That $\eta \circ \theta = \text{id}_{|P_H \underline{A}|}$ and $\theta \circ \eta = \text{id}_{\mathscr{P}_H |\underline{A}|}$ are also obvious. ∎

We now deal with the Smyth powerdomain.

DEFINITION 5.4. Let $\underline{A}$ be a strongly finite sequent structure. The Smyth powerdomain of $\underline{A}$ is the sequent structure $P_S \underline{A} = (C, \vdash)$ where[1]

- $C = \{B \mid B \subseteq^{\text{fin}} A \ \&\ B \neq \varnothing\}$,
- $X \vdash Y \Leftrightarrow \exists \beta \in Y . \forall \alpha_0 \in C . (\forall \alpha \in X . \alpha_0 \cap \alpha \neq \varnothing) \Rightarrow \alpha_0 \vdash_{\underline{A}} \beta$.

When $X = \{\alpha\}$, $Y = \{\beta\}$ where $\alpha, \beta \in C$, the entailment $\{\alpha\} \vdash \{\beta\}$ means that $\forall W \subseteq^{\text{fin}} A . (W \cap \alpha \neq \varnothing \Rightarrow \forall w \in W\, \exists b \in \beta . \{w\} \vdash_{\underline{A}} \{b\})$. In other words, $\{\alpha\} \vdash \{\beta\}$ if and only if $\forall a \in \alpha\, \exists b \in \beta . \{a\} \vdash \{b\}$. Therefore $\vdash_{P_S \underline{A}}$ on C is a reverse if the preorder $\lesssim_0$ introduced in Section 3 on the non-empty, finite sets of finite elements of $|\underline{A}|$.

Having agreed on what the entailment should be on singletons, we check that $X \vdash Y$ is equivalent to

$$\forall \alpha_0 \in C . (\forall \alpha \in X . \{\alpha_0\} \vdash \{\alpha\}) \Rightarrow \exists \beta \in Y . \{\alpha_0\} \vdash \{\beta\},$$

which we write as $X \vdash' Y$, a description of the situation

$$\bigcap \{\uparrow \bar{\alpha} \mid \alpha \in X\} \subseteq \bigcup \{\uparrow \bar{\beta} \mid \beta \in Y\}.$$

$X \vdash Y \Rightarrow X \vdash' Y$: Suppose $\alpha_0 \in C$ and $\forall \alpha \in X . \{\alpha_0\} \vdash \{\alpha\}$. Let $\beta_0 \in Y$ be such that

$$(\forall \alpha \in X . \alpha' \cap \alpha \neq \varnothing) \Rightarrow \alpha' \vdash_{\underline{A}} \beta_0.$$

[1] In an extended version of (Scott, 1982), Scott calls a finite set α_0 with the property $(\forall \alpha \in X . \alpha_0 \cap \alpha \neq \varnothing)$ a *choice set*.

Let $a_0 \in \alpha_0$. For any $\alpha \in X$, there is a $b_\alpha \in \alpha$ for which $\{a_0\} \vdash_{\underline{A}} \{b_\alpha\}$. Clearly $\{b_\alpha \mid \alpha \in X\} \cap \alpha \neq \emptyset$ for all $\alpha \in X$. Therefore,

$$\{b_\alpha \mid \alpha \in X\} \vdash_{\underline{A}} \beta_0.$$

However, $\{a_0\} \vdash^* \{b_\alpha \mid \alpha \in X\}$. Thus $\{a_0\} \vdash_{\underline{A}} \beta_0$, which implies that $\{a_0\} \vdash_{\underline{A}} \{b_0\}$ for some $b_0 \in \beta_0$, and this is true for any a_0 in α_0, which means that $\{\alpha_0\} \vdash \{\beta_0\}$.

$X \vdash' Y \Rightarrow X \vdash Y$: Suppose $X \vdash' Y$. Rewrite X as $\{\alpha_i \mid i \in I\}$, where I is finite. For any

$$f \in \left\{ g: I \to \bigcup X \;\middle|\; \forall i \in I . g(i) \in \alpha_i \right\},$$

let $\{f(i) \mid i \in I\} \rightleftharpoons_{\underline{A}} Z_f$. Clearly for any c in $\bigcup_f Z_f$, for any $i \in I$, $\{c\} \vdash_{\underline{A}} \alpha_i$. Therefore for any i, $\{\bigcup_f Z_f\} \vdash \{\alpha_i\}$, and hence $\{\bigcup_f Z_f\} \vdash \{\beta_0\}$ for some $\beta_0 \in Y$, since $X \vdash' Y$. Now let $\alpha_0 \in C$ be such that for any α in X, $\alpha_0 \cap \alpha \neq \emptyset$. For each i, select an $a_i \in \alpha_0 \cap \alpha_i$. The collection of such a_i's corresponds to a function h such that $h(i) = a_i$. If $\alpha_0 \vdash_{\underline{A}} \emptyset$ then there is nothing to prove; otherwise we have $\{h(i) \mid i \in I\} \rightleftharpoons_{\underline{A}} Z_h$ and $Z_h \neq \emptyset$. $\{\bigcup_f Z_f\} \vdash \{\beta_0\}$ implies $\{c\} \vdash_{\underline{A}} \beta_0$ for all $c \in Z_h$. Hence $\alpha_0 \vdash_{\underline{A}} \beta_0$.

From the above explanation it can be seen that we can use the following equivalent definition of entailment for the Smyth powerdomain:

$$X \vdash Y \qquad \text{iff} \quad \exists \beta \in Y . \forall \alpha_0 \subseteq \bigcup X . (\forall \alpha \in X . |\alpha_0 \cap \alpha| = 1) \Rightarrow \alpha_0 \vdash_{\underline{A}} \beta.$$

This is much better, since it avoids the use of the universal quantifier over C.

It is suitable to think of $\alpha \in C$ as $\Box \bigvee \alpha$, with the interpretation that a set of processes S satisfies $\Box \bigvee \alpha$ if each process in S satisfies $\bigvee \alpha$. We have, under this interpretation, $\Box \bigvee X \Rightarrow \Box \bigvee Y$ if and only if $\bigvee X \Rightarrow \bigvee Y$, if and only if $\forall a \in X \, \exists b \in Y . a \Rightarrow b$ and

$$\begin{aligned} &(\Box \bigvee X_1) \wedge (\Box \bigvee X_2) \wedge \cdots (\Box \bigvee X_n) \\ &\quad \Leftrightarrow \Box[(\bigvee X_1) \wedge (\bigvee X_2) \wedge \cdots (\bigvee X_n)]. \end{aligned}$$

PROPOSITION 5.5. *If $\underline{A}$ is a strongly finite sequent structure then so is $P_S \underline{A}$.*

Proof. The only non-trivial part is finite closure. Let V be a finite set of propositions of $P_S \underline{A}$. $\bigcup V$ is a finite set of propositions of $\underline{A}$. Therefore there is a finite set P of $\underline{A}$, $\rightleftharpoons_{\underline{A}}$-closed and contains $\bigcup V$. Then $\{\alpha \mid \alpha \subseteq P\}$ is a $\rightleftharpoons$-closed set of $P_S \underline{A}$. In fact, let $X \subseteq \{\alpha \mid \alpha \subseteq P\}$. Rewrite X as $\{\alpha_i \mid i \in I\}$, where I is finite. For any

$$f \in \left\{ g: I \to \bigcup X \;\middle|\; \forall i \in I . g(i) \in \alpha_i \right\},$$

let $\{f(i) \mid i \in I\} \rightleftharpoons_{\underline{A}} Z_f$. Clearly for any c in $\bigcup_f Z_f$, for any $i \in I$, $\{c\} \vdash_{\underline{A}} \alpha_i$. Therefore for any i, $\{\bigcup_f Z_f\} \vdash \{\alpha_i\}$. According to our definition of entailment for $P_S\underline{A}$, $X \rightleftharpoons \{\bigcup_f Z_f\}$. ∎

From the proof we can see that $P_S\underline{A}$, too, is deterministic. Similarly we have the following proposition, whose proof is omitted.

PROPOSITION 5.6. *$|P_S\underline{A}|$ is isomorphic to $\mathscr{P}_S\,|\underline{A}|$.*

The Plotkin powerdomain construction is more complicated. One of the reasons for this is that it is this powerdomain construction that does not preserve consistent completeness of Scott domains.

DEFINITION 5.5. Let $\underline{A}$ be an SFSS. The Plotkin powerdomain of $\underline{A}$ is the sequent structure $P_P\underline{A} = (C, \vdash)$, with

- $C = \{B \mid B \subseteq^{\text{fin}} A \;\&\; B \neq \varnothing\}$,
- $X \vdash Y$ iff $\forall \beta \in C.(\forall \alpha \in X.\{\beta\} \vdash \{\alpha\}) \Rightarrow \exists \beta' \in Y.\{\beta\} \vdash \{\beta'\}$, where $\{\alpha\} \vdash \{\beta\}$ iff $\{\alpha\} \vdash_{P_H\underline{A}} \{\beta\} \;\&\; \{\alpha\} \vdash_{P_S\underline{A}} \{\beta\}$.

Again, the universal quantification $\forall \beta \in C$ in the definition of entailment can be avoided. That can be seen after the proof of the following proposition.

PROPOSITION 5.7. *If $\underline{A}$ is an SFSS then so is $P_P\underline{A}$.*

Proof. $P_P\underline{A}$ is clearly a SS. Let $P_P\underline{A} = (C, \vdash)$.

$$V \subseteq^{\text{fin}} C \Rightarrow \bigcup V \subseteq^{\text{fin}} A$$

$$\Rightarrow \exists P \subseteq^{\text{fin}} A.\bigcup V \subseteq P \;\&\; P \text{ is } \rightleftharpoons_{\underline{A}}\text{-closed}$$

$$\Rightarrow \{\alpha \mid \alpha \subseteq P\} \text{ is } \rightleftharpoons_{P_P\underline{A}}\text{-closed.}$$

We check the last step in tha bove implication. Let $X \subseteq \{\alpha \mid \alpha \subseteq P\}$. We have the following algorithm which finds a subset $X' \subseteq \{\alpha \mid \alpha \subseteq P\}$ such that $X \rightleftharpoons_{P_P\underline{A}} X'$.

Step 1. Form a list l with its head being

$$\left(\bigcup_f Z_f, \bigcup X\right),$$

where $\bigcup_f Z_f$ is the set constructed by the procedure described in the proof of Proposition 5.5.

Step 2. Add to l the element

$$\left(S, S \cup \left(\bigcup X\right)\right)$$

for each $S \subseteq \bigcup_f Z_f$, and remove the head $((\bigcup_f Z_f, \bigcup X)$ from l.

Step 3. Do this step for l until each member of it is of the form (S, T), where $S \supseteq T$: Pick up an element (S, T) in l for which $S \not\supseteq T$ (After Step 2 we must have $S \subseteq T$, obviously). Let a be in T but not in S. For each $s \in S$, add to l the element

$$(S \cup S', \{s'\} \cup (T \backslash \{a\}))$$

for each $s' \in S'$ if S' is not empty, where $\{s, a\} \rightleftharpoons_{\underline{A}} S'$ and $S' \subseteq P$. If S' is empty then skip. Remove the current element (S, T) from l.

Step 4. Replace each element (S, T) of l with $S \neq T$ by the elements (T, T), and (U_a, U_a) for each $a \in (S \backslash T)$ and $U_a \subseteq (S \backslash \{a\})$.

The algorithm must terminate. To show this it is enough to check that Step 3 terminates. But

$$|(s' \cup (T \backslash \{a\})) \backslash (S \cup S')| = |(T \backslash \{a\}) \backslash (S \cup S')| < |T \backslash S|,$$

since $s' \in S'$ and $a \in T \backslash S$. The conclusion is then clear.

We claim that after the algorithm stops, we have

$$X \rightleftharpoons_{P_P \underline{A}} \{U \mid (U, U) \text{ is a member of the list } l\},$$

where each U is clearly a subset of P. Each member (S, T) of the list l corresponds to a logical formula $(\Box \bigvee S) \wedge \bigwedge \{\Diamond b \mid b \in T\}$, i.e., the first element S takes care of the order of the Smyth powerdomain while the second element T takes care of the order of the Hoare powerdomain. At each stage, the whole list corresponds to a disjunction of all its members, and we need only make sure that at each step the algorithm maintains the equivalence of the big formula the list represents. More precisely, note those lemmas are still valid if we replace the 'prime assertions' by the propositions in A of $\underline{A}$. The equivalence should be rephased as $l \Leftrightarrow l'$ iff for any $\alpha \in C$,

$$\begin{aligned} &\exists (S, T) \in l. \{\alpha\} \vdash_{P_S \underline{A}} \{S\} \;\&\; \{\alpha\} \vdash_{P_H \underline{A}} \{T\} \\ &\quad \Leftrightarrow \exists (S, T) \in l'. \{\alpha\} \vdash_{P_S \underline{A}} \{S\} \;\&\; \{\alpha\} \vdash_{P_H \underline{A}} \{T\} \end{aligned}$$

where we borrowed the notation $(S, T) \in l$ to mean (S, T) is an element in the list.

The following sequence of observations, which are routine to check, finish the proof of Proposition 5.7.

Observation 1. $\forall \alpha \in X.\{\beta\} \vdash_{P_P\underline{A}} \{\alpha\}$ iff $\{\beta\} \vdash_{P_S\underline{A}} \{\bigcup_f Z_f\}$ and $\{\beta\} \vdash_{P_H\underline{A}} \{\bigcup X\}$, where $\bigcup_f Z_f$ is the set introduced in the proof of Proposition 5.5.

Observation 2. $\{\beta\} \vdash_{P_S\underline{A}} \{S\}$ & $\{\beta\} \vdash_{P_H\underline{A}} \{T\}$ iff

$$\exists S' \subseteq S.\{\beta\} \vdash_{P_S\underline{A}} \{S'\} \;\&\; \{\beta\} \vdash_{P_H\underline{A}} \{S' \cup T\}.$$

Observation 3. Suppose $S \subseteq T$ and $a \in T \backslash S$. Then $\{\beta\} \vdash_{P_S\underline{A}} \{S\}$ & $\{\beta\} \vdash_{P_H\underline{A}} \{T\}$ iff $\exists s' \in S'$, $\{\beta\} \vdash_{P_S\underline{A}} \{S \cup S'\}$ & $\{\beta\} \vdash_{P_H\underline{A}} \{\{s'\} \cup (T \backslash \{a\})\}$. Here we reused all the notations used in Step 3 of the algorithm.

Observation 4. Suppose $S \supseteq T$ but $S \neq T$. Then $\{\beta\} \vdash_{P_S\underline{A}} \{S\}$ & $\{\beta\} \vdash_{P_H\underline{A}} \{T\}$ iff for some U, $\{\beta\} \vdash_{P_S\underline{A}} \{U\}$ & $\{\beta\} \vdash_{P_H\underline{A}} \{U\}$, where $U = T$ or $U \subseteq (S \backslash \{a\})$ for some $a \in S \backslash T$. ∎

Let us write $\Xi(V)$ for the finite, $\rightleftharpoons$-closed set $\{\alpha \mid \alpha \subseteq P\}$ offered by the previous proof. The following is a different definition for entailment in the Plotkin powerdomain:

$$X \vdash Y \qquad \text{iff} \quad \forall \beta \in \Xi(X).(\forall \alpha \in X.\{\beta\} \vdash \{\alpha\}) \Rightarrow \exists \beta' \in Y.\{\beta\} \vdash \{\beta'\}.$$

Although it is equivalent to Definition 5.5, the advantage is that it avoids the use of unconfined universal quantification. Thus we also have an effective definition of entailment for the Plotkin powerdomain construction. More precisely, we should say that the entailment preserves effectiveness because it certainly depends on the effectiveness of the entailment on the components.

Finally, we have the following expected result, whose proof is omitted.

PROPOSITION 5.8. *$|P_P\underline{A}|$ is isomorphic to $\mathscr{P}_P\,|\underline{A}|$.*

6. A CPO OF SFSS

This section introduces a complete partial order of SFSSs following the idea described in (Larsen and Winskel, 1984). This provides a basis on which recursively defined structures can be given. We show that all the constructions induce continuous functions on the cpo. Because sequent structures and the substructure relation on them are based concretely on sets and relations, we can derive solution to equations in such systems up to equality.

DEFINITION 6.1. Let $\underline{A} = (A, \vdash_{\underline{A}})$, $\underline{B} = (B, \vdash_{\underline{B}})$ be SFSSs. $\underline{A} \trianglelefteq \underline{B}$ if

- $A \subseteq B$,
- $X \vdash_{\underline{A}} Y \Leftrightarrow X \cup Y \subseteq A \;\&\; X \vdash_{\underline{B}} Y$.

When $\underline{A} \trianglelefteq \underline{B}$ we call $\underline{A}$ a substructure of $\underline{B}$.

PROPOSITION 6.1. *Let $\underline{A}$ and $\underline{B}$ be SFSSs. If $\underline{A} \trianglelefteq \underline{B}$ then there is an embedding-projection pair between $|\underline{A}|$ and $|\underline{B}|$.*

Proof. Define

$$\theta\colon |\underline{A}| \to |\underline{B}| \qquad x \mapsto \{b \mid \exists a \in x . a \vdash_{\underline{B}} b\}$$

and

$$\phi\colon |\underline{B}| \to |\underline{A}| \qquad y \mapsto y \cap A.$$

Let $x \in |\underline{A}|$. We check $\theta(x) \in |\underline{B}|$. Suppose that $Z \subseteq \theta(x)$ and $Z \vdash_{\underline{B}} H$. By definition $\forall c \in Z \, \exists a \in x . a \vdash_{\underline{B}} c$. Let X be the collection of such a's. Clearly $X \subseteq x . X \vdash^* Z$, which implies that $X \vdash_{\underline{B}} H$. By the finite closure axiom there is ome $X' \subseteq A$ such that $X \rightleftharpoons_{\underline{A}} X'$. As $X \subseteq x$ and $X \vdash_{\underline{A}} X'$, there is some $a' \in x \cap X'$. But $\underline{A}$ is a substructure of $\underline{B}$. We have $X \rightleftharpoons_{\underline{B}} X'$, and hence $a' \vdash a$ for each $a \in X$. Thus $a' \vdash_{\underline{B}} b$ for some $b \in H$. That is, $b \in \theta(x)$.

Let $y \in |\underline{B}|$. We check $y \cap A \in |\underline{A}|$. Suppose $Y \subseteq y \cap A$ and $Y \vdash_{\underline{A}} Z$. Then $\exists c \in Z . c \in y$. But $Z \subseteq A$, so $c \in y \cap A$.

The proof that θ, ϕ form an embedding–projection pair, i.e., $\phi \circ \theta = \mathrm{id}_{|\underline{A}|}$ and $\theta \circ \phi \sqsubseteq \mathrm{id}_{|\underline{B}|}$, is then straightforward. ∎

Note that the collection of SFSSs do not form a set but rather a class. Therefore they do not form a complete partial order in the ordinary sense. We could say that they form a *large* cpo $\mathbf{CPO}_{\mathrm{SFSS}}$. Nevertheless, the standard theory of fixed points of continuous functions still works for $\mathbf{CPO}_{\mathrm{SFSS}}$, and that is all we need.

THEOREM 6.1. *The relation $\trianglelefteq$ on $\mathbf{CPO}_{\mathrm{SFSS}}$ is a partial order with the least element $\bot = (\varnothing, \varnothing)$. If $\underline{A}_0 \trianglelefteq \underline{A}_1 \trianglelefteq \cdots \trianglelefteq \underline{A}_i \trianglelefteq \cdots$ is an increasing chain of SFSSs, where $\underline{A}_i = (A_i, \vdash_i)$, then their least upper bound is*

$$\bigcup_{i \in \omega} \underline{A}_i = \Big(\bigcup_{i \in \omega} A_i, \bigcup_{i \in \omega} \vdash_i\Big).$$

Proof. It is routine to check that

$$\bigcup_{i \in \omega} \underline{A}_i = \Big(\bigcup_{i \in \omega} A_i, \bigcup_{i \in \omega} \vdash_i\Big)$$

is a SFSS. For each i, $\underline{A}_i \trianglelefteq \bigcup_{k\in\omega} \underline{A}_k$ because of the following:

1. $A_i \subseteq \bigcup_{k\in\omega} A_k$.
2. If $X \cup Y \subseteq A_i$ and $X \vdash_{\bigcup_{k\in\omega} \underline{A}_k} Y$ then $X \vdash_j Y$ for some $j \geqslant i$ because $\vdash_{\bigcup_{k\in\omega} \underline{A}_k} = \bigcup_{k\in\omega} \vdash_k$. Therefore $X \vdash_i Y$.

It is also the least upper bound of the chain. Suppose $\underline{B}$ is an upper bound of the chain. Then $\bigcup_{i\in\omega} \underline{A}_i \trianglelefteq \underline{B}$ since $\bigcup_{i\in\omega} A_i \subseteq B$ and

$$X \vdash_{\bigcup_{k\in\omega} \underline{A}_k} Y \Leftrightarrow X \cup Y \subseteq \bigcup_{k\in\omega} A_k \;\&\; \exists i. X \vdash_i Y$$
$$\Leftrightarrow X \cup Y \subseteq \bigcup_{k\in\omega} A_k \;\&\; X \vdash_{\underline{B}} Y. \quad \blacksquare$$

The substructure relation $\trianglelefteq$ can be extended to $(n+1)$-tuples of sequent structure coordinatewise. More precisely we require that

$$(\underline{A}_0, \underline{A}_1, ..., \underline{A}_n) \trianglelefteq (\underline{B}_0, \underline{B}_1, ..., \underline{B}_n)$$

if and only if, for each $0 \leqslant i \leqslant n$, $\underline{A}_i \trianglelefteq \underline{B}_i$. For convenience write $\vec{\underline{A}}$ for $(\underline{A}_0, \underline{A}_1, ..., \underline{A}_n)$.

The least upper bound of an ω-increasing chain of n-tuples of sequent structures is then just the n-tuple of sequent structures consisting of the least upper bounds on each component; i.e., if

$$\vec{\underline{A}}_1 \trianglelefteq \vec{\underline{A}}_2 \cdots \trianglelefteq \vec{\underline{A}}_i \trianglelefteq \cdots$$

then

$$\pi_j\left(\bigsqcup_{i\in\omega} \vec{\underline{A}}_i\right) = \bigcup_{i\in\omega} \pi_j(\vec{\underline{A}}_i).$$

An operation F from n-tuples of sequent structures to m-tuples of sequent structures is said to be *continuous* if it is monotonic; i.e., $\vec{\underline{A}} \trianglelefteq \vec{\underline{B}}$ implies $F(\vec{\underline{A}}) \trianglelefteq F(\vec{\underline{B}})$, and it preserves ω-increasing chains of sequent structures, i.e.,

$$\vec{\underline{A}}_1 \trianglelefteq \vec{\underline{A}}_2 \cdots \trianglelefteq \vec{\underline{A}}_i \trianglelefteq \cdots$$

implies

$$\bigcup_{i\in\omega} F(\vec{\underline{A}}_i) = F\left(\bigcup_{i\in\omega} \vec{\underline{A}}_i\right).$$

It is well known that functions on tuples of cpos are continuous if and only if, by changing any argument while fixing others, the induced function is continuous.

Larsen and Winskel have a useful lemma which concludes that an operation F is continuous if and only if it is monotonic with respect to $\trianglelefteq$ and continuous on proposition sets in the sense that for any ω-increasing chain

$$\vec{\underline{A}}_i \trianglelefteq \vec{\underline{A}}_2 \cdots \trianglelefteq \vec{\underline{A}}_i \trianglelefteq \cdots,$$

each proposition of $F(\bigcup_{i\in\omega} \vec{\underline{A}}_i)$ is a proposition of $\bigcup_{i\in\omega} F(\vec{\underline{A}}_i)$. Generalized to SFSSs we have the following lemma.

LEMMA 6.1. *Let F be a function on* $\mathbf{CPO}_{\mathrm{SFSS}}$. *$F$ is continuous if and only if it is monotonic and continuous on proposition sets.*

The following theorem implies the existence of recursively defined systems using construction of least fixed points.

THEOREM 6.2. *$\rightarrow$, P_{H}, P_{S}, and P_{P} are all continuous.*

Proof. We illustrate the proof for $\rightarrow$. Proofs for other cases follow the same pattern, hence omitted.

We have to show that $\rightarrow$ is a continuous operation from pairs of sequent structures to sequent structures. $\rightarrow$ is monotonic in its first argument. Suppose $\underline{A} \trianglelefteq \underline{A}'$ and $\underline{B}$ is a sequent structure. Write

$$\underline{C} = (C, \vdash) = \underline{A} \rightarrow \underline{B}$$

and

$$\underline{C}' = (C', \vdash') = \underline{A}' \rightarrow \underline{B}.$$

We check 1 and 2 in Definition 6.1 to show that $\underline{C} \trianglelefteq \underline{C}'$.

1. Assume $X \in C$. It is easy to see that $X \in C'$.

2. Clearly $X \vdash_{\underline{C}} Y$ implies $X \vdash_{\underline{C}'} Y$. Assume $X \subseteq C$, $Y \subseteq C$ and $X \vdash_{\underline{C}'} Y$. Because in this case each entailment with subscript $\underline{A}'$ is an entailment with subscript $\underline{A}$, by the assumption that $\underline{A} \trianglelefteq \underline{A}'$. Hence we have $X \vdash_{\underline{C}} Y$.

Now we show that $\rightarrow$ is continuous on proposition sets. Let

$$\underline{A}_0 \trianglelefteq \underline{A}_1 \trianglelefteq \cdots \trianglelefteq \underline{A}_i \trianglelefteq \cdots$$

be a chain of SFSSs. Let X be a proposition of $(\bigcup_{i\in\omega} \underline{A}_i) \rightarrow \underline{B}$. Then $\pi_1 X \subseteq^{\mathrm{fin}} \bigcup_{i\in\omega} A_i$. Hence $\pi_1 X \subseteq A_j$ for some j, which means that X is a proposition of $\underline{A}_j \rightarrow \underline{B}$. Thus X is a proposition of $\bigcup_{i\in\omega} (\underline{A}_i \rightarrow \underline{B})$. By Lemma 6.1, $\rightarrow$ is continuous in its first argument. Similarly we can prove that $\rightarrow$ is continuous in its second argument, and hence continuous. ∎

To show that the theory developed in this section is applicable, we illustrate how to find a solution to the equation

$$X = X_{\uparrow} \to P_{\mathrm{P}}(X)$$

within strongly finite sequent structures. Here $(\ )_{\uparrow}$ is the lifting construction specified as follows. Its use here makes sure that a non-trivial solution is obtained.

Let $\underline{A} = (A, \vdash)$ be an SFSS. Define the *lift* of $\underline{A}$ to be $\underline{A}_{\uparrow} = (A', \vdash')$, where

- $A' = (\{0\} \times A) \cup \{0\}$,
- $X \vdash' Y \Leftrightarrow [0 \in Y \text{ or } \{c \mid (0, c) \in X\} \vdash_{\underline{A}} \{b \mid (0, b) \in Y\}]$.

Lifting is an operation which, given an SFSS, produces a new one by joining a new proposition weaker than all the old ones. One can easily check that it gives a continuous operation.

Since the composition of continuous functions remains continuous, and any continuous function $F(x, y)$ of two variables gives rise to a continuous function $F(x, x)$ of one variable, the operation

$$X \mapsto X_{\uparrow} \to P_{\mathrm{P}}(X)$$

is a continuous operation on SFSSs. It has a least fixed point

$$\underline{A} = (\underline{A})_{\uparrow} \to P_{\mathrm{P}}(\underline{A}).$$

By the same token, Plotkin's resumptions (Plotkin, 1976) can also be constructed in this way.

7. Discussion

We believe that the results presented in this paper strongly indicate that SFSSs are the right kind of "information systems" for SFP domains. The only thing that might seem not too pleasing is the construction of function space, where "molecules" are used as propositions. However, the use of a finite set of propositions as a proposition of a higher type has been adopted for the construction of powerdomains, too, which seemed unavoidable. Also, if one were to use the atomicity axiom in Definition 4.1 and to interpret the entailment topologically (see Definition 4.2), propositions should correspond to finite elements. Yet there does not seem to be a simpler way to characterize the finite elements in the function space (see the example after Definition 5.2). Therefore, if one were to try to find a "better" representation which allows a simpler construction for function space, one probably should not take the atomicity axiom too seriously. (Nevertheless, entailment should be interpreted topologically.)

There are other works related to information systems in general. In (Abramsky, 1991), domain pre-locales are adopted for representing SFP domains. One advantage of domain pre-locales is that propositions of higher types can be constructed more naturally by allowing explicit conjunctions. However, that treatment is not close to the spirit of "information systems" in that it allows the explicit use of logical operators and employs heavy machinery (however, it suits the purpose of that paper well).

In both the sequent structure and the domain pre-locale treatment of SFP domains, an inevitable technical point is the characterization of continuous functions as some joinable sets. The authors believes that the characterization introduced here (see Definition 3.1) is simpler and more elegant. This is because it provides the right definition of approximable mappings on strongly finite sequent structures (see Definition 4.3). In contrast, the definition of morphisms on domain pre-locales is clumsy, and doesn't appear to have much to do with the characterization of finite elements in the function space given in (Abramsky, 1991).

The other related work is (Droste and Göbel, 1990). Their purpose was to give an axiomatization of the domains determined by *non-deterministic information systems*. Because entailments such as $\varnothing \vdash \varnothing$ are allowed in a non-deterministic information system, the resulting domain need not be a cpo. Apart from this, non-deterministic information systems and sequent structures determine the same class of cpos. Therefore, their examples imply that the domains determined by sequent structures need not be algebraic.

SFP domains form one of the largest Cartesian closed categories of algebraic cpos. The other prominent maximal Cartesian closed category is the category of L-domains (Coquand, 1989; Jung, 1990). It turns out that a nice, information-system-like representation can also be given to L-domains. For this purpose a notion of *a disjunctive system* is appropriate. Results on disjunctive systems are reported in a separate paper (Zhang, 1992), which the reader may find interesting as well.

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