Discrete Applied Mathematics 48 (1994) 163-174
North-Holland

Random knapsacks with many constraints

Kenneth Schilling

Department of Mathematics, University of Michigan-Flint, Flint, MI 48503, USA

Received 7 August 1990
Revised 27 March 1991

Abstract

We provide new results on asymptotic values for the random knapsack problem. For a very general model in which the parameters are determined by a rather arbitrary joint distribution, we compute the rate of growth as the number of objects increases, the number of constraints being fixed. For a particular model, we find strong bounds on the asymptotic value as the numbers of objects and constraints increase together.

This paper is a continuation of the work in [3, 4] on estimating the values of random knapsack problems with many decision variables. It consists of two independent parts. In Section 1, we show how to estimate the growth rate of the value of a random knapsack when the parameters are determined by a very general class of joint distributions. In Section 2, we concentrate on a particular random knapsack model, and give rather sharp new bounds on its asymptotic value. In more detail:

In Section 1, we first settle a question left open in [3] related to a single-constraint random knapsack problem, then apply this new result to a multiconstraint problem. Consider the problem

$$V_n = \max \sum_{j=1}^{n} X_j\delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} W_j\delta_j \le K, \quad \delta_j \in \{0,1\}$$

where the random variable pairs (W_j, X_j) are independent, identically distributed draws from any one of a very wide class of joint distributions F_{WX}. (In particular, we do not assume that W and X are independent.) For $t > 0$, let $F(t) = E(W 1_{\{X \ge tW\}})$ and $G(t) = E(X 1_{\{X \ge tW\}})$.

Correspondence to: Professor K. Schilling, Department of Mathematics, University of Michigan-Flint, Flint, MI 48502-2186, USA.

SSDI 0166-218X(92)00125-5

In [3], we proved that V_n is asymptotically equal to $nG \circ F^{-1}(K/n)$ as $n \to \infty$. However, to carry out this proof we needed a seemingly unnatural extra hypothesis on F_{WX}, namely that the function $G \circ F^{-1}$ is concave on some interval $(0, t)$. In Theorem 1.2, we prove this hypothesis. As an application of Theorem 1.2, we obtain (Theorem 1.3) nice bounds on the asymptotic growth rate of the m-constraint extension of this general problem, and show (Theorem 1.4) that these bounds are essentially the best possible.

In Section 2, we extend and improve our results in [4] on a particular random knapsack model. Consider the problem

$$V_{mn} = \max \sum_{j=1}^{n} X_j \delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} W_{ij}\delta_j \leq 1 \quad \text{for } i = 1, \ldots, m, \ \delta_j \in \{0, 1\}$$

where the random variables X_j, W_{ij} are mutually independent, and all uniformly distributed on the interval $(0, 1)$.

In [4], we showed that, for fixed m, V_{mn}/α_{mn} converges to 1 in probability as $n \to \infty$, where $\alpha_{mn} = (m+1)(n/(m+2)!)^{1/(m+1)}$. In Theorem 2.2, we obtain a rather sharp bound on $P(|(V_{mn}/\alpha_{mn}) - 1| > \varepsilon)$, which will allow us to infer (Corollary 2.3)

(1) V_{mn}/α_{mn} converges to 1 completely (so, a fortiori, almost surely), and

(2) complete convergence holds even if the number of constraints m is allowed to grow with n, provided $m = m_n \leq (\log n)^\eta$ for some $\eta < 1$.

This bound on the growth rate of m is essentially best possible, as we show (Theorem 2.4) that if $m_n \geq \gamma \log n$ for some $\gamma > 0$, then V_{mn} is almost surely uniformly bounded.

We do not assume familiarity with [3, 4]. The few results from those papers needed here are stated in full.

I would like to thank the referee for several most helpful suggestions.

1.

We first consider the single-constraint random knapsack problem

$$V_n = \max \sum_{j=1}^{n} X_j \delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} W_j\delta_j \leq K, \quad \delta_j \in \{0, 1\}. \tag{I}$$

We assume that the pairs (W_j, X_j) are independent draws from a joint distribution F_{WX} which satisfies the properties: $W > 0$, $0 < X < 1$, and the random variable X/W

is absolutely continuous with density $f_{X/W}(t)$ which is positive for all sufficiently large t. Define, for $t > 0$,

$$F(t) = E(W1_{\{X \geq tW\}}) \quad \text{and} \quad G(t) = E(X1_{\{X \geq tW\}}).$$

In [1], we proved

Theorem 1.1. $P(|V_n/(nG \circ F^{-1}(K/n)) - 1| \leq o(1)) \to 1$ *as* $n \to \infty$.

As usual, o(1) denotes a sequence which converges to 0. To carry out this proof, we required the additional hypothesis (called (A2) in [3]) that the function $G \circ F^{-1}$ is concave (that is, lies above its chords) on the interval $(0, t_1)$, for some $t_1 > 0$. Our first task here is to prove hypothesis (A2).

Theorem 1.2. *There exists* $t_1 > 0$ *such that* $\mathrm{d}/\mathrm{d}t(G \circ F^{-1}(t)) = F^{-1}(t)$ *for* $0 < t < t_1$. *In particular, the function* $G \circ F^{-1}(t)$ *is concave on* $(0, t_1)$.

Proof. It is clear from our hypotheses that $F(t)$ decreases monotonically to 0 and is continuous for sufficiently large t. Thus there exists t_1 such that $F^{-1}(t)$ exists and is monotone decreasing on $(0, t_1)$. Therefore once we have shown that $\mathrm{d}/\mathrm{d}t(G \circ F^{-1}(t)) = F^{-1}(t)$ for t in $(0, t_1)$, it will follow that $G \circ F^{-1}(t)$ is concave there.

To this end, for $0 < t < t_1$ let A_t denote the area of the set $\{(x, y) \in \mathbb{R}^2 \colon x \geq 0$ and $0 \leq y \leq \min\{t, F(x)\}\}$. By ordinary integration,

$$A_t = \int_0^t F^{-1}(y)\,\mathrm{d}y = tF^{-1}(t) + \int_{F^{-1}(t)}^{\infty} F(x)\,\mathrm{d}x. \qquad (*)$$

Now, by Fubini's theorem, $\int_{F^{-1}(t)}^{\infty} F(x)\,\mathrm{d}x = E(\int_{F^{-1}(t)}^{\infty} W1_{\{X \geq xW\}}\,\mathrm{d}x)$. For fixed ω,

$$\int_{F^{-1}(t)}^{\infty} W1_{\{X \geq xW\}}\,\mathrm{d}x = \begin{cases} W\left(\dfrac{X}{W} - F^{-1}(t)\right), & \text{if } X \geq F^{-1}(t)W, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} E\left(\int_{F^{-1}(t)}^{\infty} W1_{\{X \geq xW\}}\,\mathrm{d}x\right) &= E\left(W\left(\frac{X}{W} - F^{-1}(t)\right)1_{\{X \geq F^{-1}(t)W\}}\right) \\ &= E(X1_{\{X \geq F^{-1}(t)W\}}) - F^{-1}(t)E(W1_{\{X \geq F^{-1}(t)W\}}) \\ &= G \circ F^{-1}(t) - tF^{-1}(t). \end{aligned}$$

Thus by $(*)$ $\int_0^t F^{-1}(y)\,\mathrm{d}y = G \circ F^{-1}(t)$. By the fundamental theorem of calculus, the proof of the theorem is complete. $\square$

We now show how Theorems 1.1 and 1.2 can be applied to a multiconstraint knapsack problem. Consider the problem

$$V_n = \max \sum_{j=1}^{n} X_j\delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} W_{ij}\delta_j \le 1 \quad \text{for } i = 1, 2, \dots, m,\ \delta_j \in \{0, 1\}. \tag{II}$$

We shall compute to within a multiplicative constant the asymptotic value of V_n as $n \to \infty$, for fixed m.

Let

$$\bar{W}_j = \frac{1}{m}(W_{1j} + W_{2j} + \cdots + W_{mj})$$

and

$$\underline{W}_j = \max\{W_{1j}, W_{2j}, \dots, W_{mj}\}.$$

Consider the two single-constraint problems

$$\bar{V}_n = \max \sum_{j=1}^{n} X_j\delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} \bar{W}_j\delta_j \le 1, \quad \delta_j \in \{0, 1\}, \tag{II*}$$

and

$$\underline{V}_n = \max \sum_{j=1}^{n} X_j\delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} \underline{W}_j\delta_j \le 1, \quad \delta_j \in \{0, 1\}. \tag{II$_*$}$$

It is easy to see that $\underline{V}_n \le V_n \le \bar{V}_n$; indeed, any $(\delta_1, \dots, \delta_n)$ feasible in (II$_*$) will be feasible in (II), and any $(\delta_1, \dots, \delta_n)$ feasible in (II) will be feasible in (II*). This turns out to be somewhat useful because $\underline{V}_n$ and $\bar{V}_n$ exhibit the same asymptotic growth rate under the following rather weak hypotheses: The $(m+1)$-tuples $(W_{1j}, \dots, W_{mj}, X_j)$ are independent draws from an absolutely continuous joint distribution $F_{W_1,\dots,W_m,X}$ such that $W_i > 0$ for $i = 1, \dots, m$, $0 < X < 1$, and such that the density $f_{X/\underline{W}}(t)$ of the random variable $X/\underline{W}$ is positive for all large enough t. As before, for $t > 0$ we let

$$\bar{F}(t) = E(\bar{W} 1_{\{X \ge t\bar{W}\}}) \quad \text{and} \quad \bar{G}(t) = E(X 1_{\{X \ge t\bar{W}\}})$$

and similarly define $\underline{F}$ and $\underline{G}$. Then we have

Theorem 1.3. *$P(n\underline{G} \circ \underline{F}^{-1}(1/n)(1 - o(1)) \le V_n \le n\bar{G} \circ \bar{F}^{-1}(1/n)(1 + o(1))) \to 1$ as $n \to \infty$. This computes the asymptotic value of V_n to within a multiplicative constant, because $\overline{\lim}_{n\to\infty}\, \bar{G} \circ \bar{F}^{-1}(1/n)/\underline{G} \circ \underline{F}^{-1}(1/n) \le m$.*

Proof. By Theorem 1.1,

$$P\left(\bar{V}_n \le n\bar{G}\circ\bar{F}^{-1}\left(\frac{1}{n}\right)(1+\mathrm{o}(1))\right)\to 1$$

and

$$P\left(n\underline{G}\circ\underline{F}^{-1}\left(\frac{1}{n}\right)(1-\mathrm{o}(1))\le \underline{V}_n\right)\to 1 \quad \text{as } n\to\infty$$

and since $\underline{V}_n \le V_n \le \bar{V}_n$, the first part of Theorem 1.3 is proved.

To prove the second part, first note that $\underline{W}_j \le m\bar{W}_j$ for all j, so $\bar{V}_n \le \max\sum_{j=1}^n X_j\delta_j$, subject to $\sum_{j=1}^n \underline{W}_j\delta_j \le m$, $\delta_j \in \{0,1\}$. Thus, by another use of Theorem 1.1,

$$P\left(\bar{V}_n \le n\underline{G}\circ\underline{F}^{-1}\left(\frac{m}{n}\right)(1+\mathrm{o}(1))\right)\to 1 \quad \text{as } n\to\infty.$$

Note that $\underline{G}\circ\underline{F}^{-1}(0) = \underline{G}(\infty) = 0$. Using Theorem 1.2, we have

$$\underline{G}\circ\underline{F}^{-1}\left(\frac{m}{n}\right) = \underline{G}\circ\underline{F}^{-1}\left(\frac{1}{n}\right) + (m-1)\frac{\underline{G}\circ\underline{F}^{-1}\left(\frac{m}{n}\right)-\underline{G}\circ\underline{F}^{-1}\left(\frac{1}{n}\right)}{\frac{m}{n}-\frac{1}{n}}\cdot\frac{1}{n}$$

$$\le \underline{G}\circ\underline{F}^{-1}\left(\frac{1}{n}\right) + (m-1)\frac{\underline{G}\circ\underline{F}^{-1}\left(\frac{1}{n}\right)}{1/n}\cdot\frac{1}{n}$$

(because $\underline{G}\circ\underline{F}^{-1}$ is concave)

$$= m\underline{G}\circ\underline{F}^{-1}\left(\frac{1}{n}\right)$$

so in fact $P(\bar{V}_n \le mn\underline{G}\circ\underline{F}^{-1}(1/n)(1+\mathrm{o}(1)))\to 1$. But since $P(\bar{V}_n \ge n\bar{G}\circ\bar{F}^{-1}(1/n) \times(1-\mathrm{o}(1)))\to 1$ as $n\to\infty$, the proof of the theorem is complete. □

We conclude this section by observing that the bounds on V_n in Theorem 1.3 are in a sense best possible; that is, there exists a class of joint distributions on $(W_1,\ldots,W_m,X)$ under which V_n is asymptotic to $n\bar{G}\circ\bar{F}^{-1}(1/n)$, and another class of joint distributions under which V_n is asymptotic to $n\underline{G}\circ\underline{F}^{-1}(1/n)$.

Theorem 1.4. (a) *If $W_1 \ge W_2,\ldots,W_m$ a.s., then $P(V_n \le n\underline{G}\circ\underline{F}^{-1}(1/n)(1+\mathrm{o}(1)))\to 1$.*

(b) *If $X, W_1,\ldots,W_m$ are mutually independent and $W_1,\ldots,W_m$ are identically distributed, then $P(V_n \ge n\bar{G}\circ\bar{F}^{-1}(1/n)(1-\mathrm{o}(1)))\to 1$.*

Proof. Part (a) follows immediately from Theorem 1.1 once we observe that, under the hypotheses of (a), $V_n = \underline{V}_n$.

The proof of (b) seems to require repetition of part of the proof of [3, Theorem 1]. By [3, Lemma 2], there exists a sequence $\{t_n\}$ of real numbers such that

$$n\bar{F}(t_n) < 1 \quad \text{for all } n, \quad \text{and} \quad n\bar{F}(t_n) \to 1, \qquad t_n(1 - n\bar{F}(t_n))^2 \to 0,$$

$$\bar{G}(t_n) \to \infty \quad \text{and} \quad \left(\bar{G}(t_n)/n\bar{G} \circ \bar{F}^{-1}\left(\frac{1}{n}\right)\right) \to 1 \quad \text{as } n \to \infty. \tag{$*$}$$

Let $\delta_j^n = 1_{\{X_j \geq t_n \bar{W}_j\}}$. Since the W's are i.i.d., we have, for $i = 1, \ldots, m$,

$$E\left(\sum_{j=1}^n W_{ij}\delta_j^n\right) = E\left(\sum_{j=1}^n \bar{W}_j\delta_j^n\right) = n\bar{F}(t_n)$$

and

$$\operatorname{Var}\left(\sum_{j=1}^n W_{ij}\delta_j^n\right) \leq nE(W_{11}^2\delta_1^n) \leq \frac{mn}{t_n}E(W_{11}\delta_1^n) = \frac{mn\bar{F}(t_n)}{t_n}.$$

(The second inequality holds because, on $\{X_1 \geq t_n \bar{W}_{11}\}$, $1 \geq X_1 \geq t_n(W_{11} + \cdots + W_{m1})/m$, so $W_{11} \leq m/t_n$.)

Now, by Chebyshev's inequality,

$$P\left(\sum_{j=1}^n W_{ij}\delta_j^n > 1\right) = P\left(\sum_{j=1}^n W_{ij}\delta_j^n - n\bar{F}(t_n) > 1 - n\bar{F}(t_n)\right)$$
$$\leq \frac{mn\bar{F}(t_n)}{t_n(1 - n\bar{F}(t_n))^2}.$$

By $(*)$ we have $P((\delta_1^n, \delta_2^n, \ldots, \delta_n^n)$ is feasible in (II)$) \to 1$ as $n \to \infty$ and so

$$P\left(V_n \geq \sum_{j=1}^n X_j\delta_j^n\right) \to 1 \quad \text{as } n \to \infty. \tag{$**$}$$

Now

$$E\left(\sum_{j=1}^n X_j\delta_j^n\right) = n\bar{G}(t_n)$$

and

$$\operatorname{Var}\left(\sum_{j=1}^n X_j\delta_j^n\right) \leq nE(X_1^2\delta_1^n) \leq nE(X_1\delta_1^n) = n\bar{G}(t_n),$$

so by another use of Chebyshev's inequality,

$$P\left(\sum_{j=1}^n X_j\delta_j^n < n\bar{G}(t_n)(1 - \varepsilon_n)\right) \leq 1/(n\bar{G}(t_n)\varepsilon_n^2) \to 0,$$

where we take ε_n to be, say, $(n\bar{G}(t_n))^{-1/3}$. By $(**)$ we have

$$P(V_n \geq n\bar{G}(t_n)(1 - \mathrm{o}(1))) \to 1,$$

and by the last part of $(*)$, the proof of (b) is complete. $\square$

2.

There seems to have been increasing interest in recent years in providing tighter bounds on the values of random combinatorial problems. In this section we shall do this for a particular random knapsack model.

For the rest of this paper we shall consider the problem

$$V_{mn} = \max \sum_{j=1}^{n} X_j \delta_j,$$

$$\text{subject to } \sum_{j=1}^{n} W_{ij}\delta_j \leq 1 \quad \text{for } i = 1, 2, \ldots, m,\ \delta_j \in \{0, 1\} \tag{III}$$

where the random variables X_j and W_{ij} are mutually independent, and all uniformly distributed on the interval (0, 1).

Let $\alpha_{mn} = (m+1)(n/(m+2)!)^{1/(m+1)}$. In [4], we showed that, for fixed m, V_{mn}/α_{mn} converges to 1 in probability, i.e.,

Theorem 2.1. *For fixed m, $P(|V_{mn}/\alpha_{mn} - 1| \leq o(1)) \to 1$ as $n \to \infty$.*

(In fact, this is an instance of the present Theorem 1.4.) We shall improve this as follows:

Theorem 2.2. *There exist constants h and K such that, for all m and n,*

$$P\left(\left|\frac{V_{mn}}{\alpha_{mn}} - 1\right| \geq \varepsilon\right) \leq (2m+6)\exp\left(-K\left(\frac{\varepsilon - hn^{-1/(m+1)}}{m}\right)^2 \cdot n^{1/(m+1)}\right).$$

Corollary 2.3. *Suppose that, for some $\eta < 1$, $m = m_n < (\log n)^\eta$ for all sufficiently large n. Then V_{mn}/α_{mn} converges to 1 completely, i.e., $\sum_{n=1}^{\infty} P(|V_{mn}/\alpha_{mn} - 1| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In particular, this holds if m is fixed.*

Furthermore, the bound on the growth of m_n in Corollary 2.3 is essentially best possible. We have

Theorem 2.4. *If, for some $\gamma > 0$, $m = m_n \geq \gamma \log n$ for all sufficiently large n, then for some $r > 0$, $\sum_{n=1}^{\infty} P(V_{mn} > r) < \infty$. In particular, V_{mn} is a.s. uniformly bounded.*

In the proof of Theorem 2.2, we shall repeatedly use two standard probabilistic bounds.

Chernoff's bounds (cf. [1]). Let Y be a binomial random variable, with parameters n and p. If $\varepsilon > 0$, then

$$P(Y - np \leq -\varepsilon) \leq \exp(-\varepsilon^2/2np)$$

and

$$P(Y - np \geq \varepsilon) \leq \exp(-\varepsilon^2/3np).$$

Hoeffding's bound (cf. [2]). Suppose that $Y_1, \ldots, Y_n$ are independent random variables each with mean μ such that $a \leq Y_i \leq b$ for $i = 1, \ldots, n$. Then

$$P\left(\sum_{i=1}^{n} Y_i - n\mu \geq \varepsilon\right) \leq \exp(-2\varepsilon^2/n(b-a)^2).$$

We also require the following lemma from [4]:

Lemma 2.5. *Let $t_1, \ldots, t_n$ be positive numbers. Suppose that*

$$\sum_{j=1}^{n} W_{ij} 1_{\{X_j \geq t_1 W_{1j} + \cdots + t_m W_{mj}\}} \geq 1$$

for $i = 1, \ldots, m$. Then

$$V_{mn} \leq \sum_{j=1}^{n} X_j 1_{\{X_j \geq t_1 W_{1j} + \cdots + t_m W_{mj}\}}.$$

We now proceed to prove Theorem 2.2. For the remainder of this proof, let m and n be fixed.

Let $I_j(t) = 1_{\{X_j \geq t(W_{1j} + \cdots + W_{mj})\}}$. A computation shows that, for $t \geq 1$,

$$P(I_1(t) = 1) = \frac{1}{(m+1)!\, t^m},$$

$$E(W_{11} I_1(t)) = \frac{1}{(m+2)!\, t^{m+1}}, \tag{1}$$

and

$$E(X_1 I_1(t)) = \frac{1}{(m+2)m!\, t^m}.$$

Let $\tau = \tau(t) = (nt/(m+2)!)^{1/(m+1)}$. τ was chosen so that $nE(W_{11} I(\tau)) = 1/t$; we shall show that, in fact, $\sum_{j=1}^{n} W_{ij} I_j(\tau)$ is usually near $1/t$. A direct use of Hoeffding's bound seems not to work, so we proceed somewhat indirectly.

Let $Y_{ij}(t) = W_{ik}$, where k is the jth positive integer with the property that $I_k(t) = 1$. We have, for any positive integer r,

(a) if $\sum_{j=1}^{n} I_j(t) \geq r$, then $0 \leq \sum_{j=1}^{n} W_{ij} I_j(t) - \sum_{j=1}^{r} Y_{ij}(t) \leq (\sum_{j=1}^{n} I_j(t) - r)/t$;

(b) if $\sum_{j=1}^{n} I_j(t) \leq r$, then $0 \leq \sum_{j=1}^{r} Y_{ij}(t) - \sum_{j=1}^{n} W_{ij} I_j(t) \leq (r - \sum_{j=1}^{n} I_j(t))/t$.

(a) follows from the observation that $\sum_{j=1}^{n} I_j(t) - r$ counts the number of j's among $1, \ldots, n$ which satisfy $I_j(t) = 1$, excluding the first r such j's, and $\sum_{j=1}^{n} W_{ij} I_j(t) - \sum_{j=1}^{r} Y_{ij}(t)$ is the sum of W_{ij} over those same j's. But if $I_j(t) = 1$, then $1 \geq X_j \geq t W_{ij}$, so $W_{ij} \leq 1/t$. The proof of (b) is similar. From (a) and (b) we have, for $A, B > 0$,

$$P\left(\max_{i=1,\ldots,m} \sum_{j=1}^{n} W_{ij} I_j(t) \geq A + B\right)$$
$$\leq P\left(\max_{i=1,\ldots,m} \sum_{j=1}^{r} Y_{ij}(t) \geq A\right) + P\left(\sum_{j=1}^{n} I_j(t) - r \geq Bt\right). \tag{2}$$

Now let $\beta(t) = nP(I_1(t) = 1)$ (= the expected number of j's among $1, \dots, n$ such that $I_1(t) = 1$). Note that $\beta(\tau)E(Y_{ij}(\tau)) = nP(I_1(\tau) = 1) \cdot E(W_{11} | I_1(\tau) = 1) = nE(W_{11} I_1(\tau)) = 1/t$. Also note that $0 \le Y_{ij}(\tau) \le 1/\tau$. Therefore

$$
\begin{aligned}
&P\left(\sum_{j=1}^{\lceil \beta(\tau) \rceil} Y_{ij}(\tau) \ge \frac{1}{t} + \varepsilon/2\right) \\
&= P\left(\sum_{j=1}^{\lceil \beta(\tau) \rceil} Y_{ij}(\tau) - \lceil \beta(\tau) \rceil E(Y_{11}(\tau)) \ge \varepsilon/2 - \left(\lceil \beta(\tau) \rceil E(Y_{11}(\tau)) - \frac{1}{t}\right)\right) \\
&\le \exp(-2(\varepsilon/2 - f)^2 \tau^2 / \lceil \beta(\tau) \rceil) \quad \text{(by Hoeffding; we have put} \\
&\qquad\qquad f = \lceil \beta(\tau) \rceil E(Y_{11}(\tau)) - 1/t \\
&\qquad\qquad\;\; = (\lceil \beta(\tau) \rceil - \beta(\tau)) E(Y_{11}(\tau)) \\
&\le \exp(-(\varepsilon/2 - f)^2 \tau^2 / \beta(\tau)).
\end{aligned} \tag{3}
$$

Also

$$
\begin{aligned}
P\left(\sum_{j=1}^{n} I_j(\tau) - \lceil \beta(\tau) \rceil \ge \tau\varepsilon/2\right) &\le P\left(\sum_{j=1}^{n} I_j(\tau) - \beta(\tau) \ge \tau\varepsilon/2\right) \\
&\le \exp(-(\varepsilon/2)^2 \tau^2 / 3\beta(\tau)) \\
&\qquad \text{by Chernoff's bound.}
\end{aligned} \tag{4}
$$

By (1), (2), (3), and (4) we have

$$
\begin{aligned}
&P\left(\max_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(\tau) \ge \frac{1}{t} + \varepsilon\right) \\
&\le (m+1)\exp(-(\varepsilon/2 - f)^2 \tau^2 / 3\beta(\tau)) \\
&= (m+1)\exp\left(\frac{-(\varepsilon - 2f)^2}{12}\left(\frac{nt^{m+2}}{(m+2)^{m+2}(m+1)!}\right)^{1/(m+1)}\right),
\end{aligned} \tag{5}
$$

where

$$
f = (\lceil \beta(\tau) \rceil - \beta(\tau)) E(Y_{11}(\tau)) \le E(Y_{11}(\tau)) = \left(\frac{(m+1)!}{nt(m+2)^m}\right)^{1/(m+1)}.
$$

By the same methods, we also have the corresponding lower bound

$$
\begin{aligned}
&P\left(\min_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(\tau) \le \frac{1}{t} - \varepsilon\right) \\
&\le (m+1)\exp(-(\varepsilon/2 - f)^2 \tau^2 / 2\beta) \\
&= (m+1)\exp\left(\frac{-(\varepsilon - 2f)^2}{8}\left(\frac{nt^{m+2}}{(m+2)^{m+2}(m+1)!}\right)^{1/(m+1)}\right).
\end{aligned} \tag{5'}
$$

Next we shall show that $\sum_{j=1}^{n} X_j I_j(\tau)$ is usually relatively near $\alpha t^{-m/(m+1)}$. (Recall that $\alpha = \alpha_{mn} = (m+1)(n/(m+2)!)^{1/(m+1)}$.) We use the device of (a) and (b) again.

Let $Z_k(t) = X_k$, where k is the jth positive integer such that $I_k(t) = 1$. Then for any positive integer r,

(c) if $\sum_{j=1}^{n} I_j(t) \geq r$, then $0 \leq \sum_{j=1}^{n} X_j I_j(t) - \sum_{j=1}^{r} Z_j(t) \leq \sum_{j=1}^{n} I_j(t) - r$;

(d) if $\sum_{j=1}^{n} I_j(t) \leq r$, then $0 \leq \sum_{j=1}^{n} Z_j(t) - \sum_{j=1}^{n} X_j I_j(t) \leq r - \sum_{j=1}^{n} I_j(t)$.

Therefore for $A, B > 0$,

$$P\left(\sum_{j=1}^{n} X_j I_j(t) \geq A + B\right) \leq P\left(\sum_{j=1}^{n} Z_j(t) \geq A\right) + P\left(\sum_{j=1}^{n} I_j(t) - r \geq B\right). \tag{6}$$

Note that $\beta(\tau)E(Z_j(\tau)) = nP(I_1(\tau) = 1) \cdot E(X_1 \mid I_1(\tau) = 1) = \alpha t^{-m/(m+1)}$ and $0 \leq Z_j \leq 1$, so

$$\begin{aligned} &P\left(\sum_{j=1}^{\lceil\beta(\tau)\rceil} Z_j(\tau) \geq \alpha(t^{-m/(m+1)} + \varepsilon/2)\right) \\ &= P\left(\sum_{j=1}^{\lceil\beta(\tau)\rceil} Z_j(\tau) - \lceil\beta(\tau)\rceil E(Z_1(\tau)) \right. \\ &\qquad \left. \geq \alpha\varepsilon/2 - \lceil\beta(\tau)\rceil E(Z_1(\tau) - \alpha t^{-m/(m+1)})\right) \\ &\leq \exp(-2(\varepsilon/2 - g)^2\alpha^2/\lceil\beta(\tau)\rceil) \\ &\qquad \text{(by Hoeffding; we have put } g = \lceil\beta(\tau)\rceil E(Z_1(\tau) - \alpha t^{-m/(m+1)})/\alpha \\ &\qquad\qquad = (\lceil\beta(\tau)\rceil - \beta(\tau))E(Z_1(\tau))/\alpha) \\ &\leq \exp(-(\varepsilon/2 - g)^2\alpha^2/\beta(\tau)). \end{aligned} \tag{7}$$

Also

$$\begin{aligned} &P\left(\sum_{j=1}^{n} I_j(\tau) - \lceil\beta(\tau)\rceil \geq \alpha\varepsilon/2\right) \\ &\leq P\left(\sum_{j=1}^{n} I_j(\tau) - \beta(\tau) \geq \alpha\varepsilon/2\right) \\ &\leq \exp(-(\varepsilon/2)^2\alpha^2/3\beta(\tau)) \quad \text{by Chernoff's bound.} \end{aligned} \tag{8}$$

By (1), (6), (7) and (8), we have

$$\begin{aligned} &P\left(\sum_{j=1}^{n} X_j I_j(\tau) \geq \alpha(t^{-m/(m+1)} + \varepsilon)\right) \\ &\leq 2\exp(-(\varepsilon/2 - g)^2\alpha^2/3\beta(\tau)) \\ &= 2\exp\left(\frac{-(\varepsilon - 2g)^2}{12}(m+1)^2\left(\frac{nt^m}{(m+2)^{m+2}(m+1)!}\right)^{1/(m+1)}\right), \end{aligned} \tag{9}$$

where

$$g = (\lceil \beta(\tau) \rceil - \beta(\tau))\, E(Z_1(\tau))/\alpha \le E(Z_1(\tau))/\alpha = \left(\frac{(m+1)!}{n(m+2)^m}\right)^{1/(m+1)}.$$

By the same method, we also have the corresponding lower bound

$$P\left(\sum_{j=1}^{n} X_j I_j(\tau) \le \alpha(t^{-m/(m+1)} - \varepsilon)\right)$$
$$\le 2\exp(-(\varepsilon/2 - g)^2 \alpha^2 / 2\beta(\tau))$$
$$= 2\exp\left(\frac{-(\varepsilon - 2g)^2}{8}(m+1)^2 \left(\frac{nt^m}{(m+2)^{m+2}(m+1)!}\right)^{1/(m+1)}\right). \tag{9'}$$

We now find probabilistic bounds on V_{mn}. To find an upper bound, first note that if $\max_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(t) \le 1$, then the assignment $\delta_j = I_j(t)$ is feasible in problem (III), so $V_{mn} \ge \sum_{j=1}^{n} X_j I_j(t)$. Thus, for $A > 0$,

(e) $P(V_{mn} < A) \le P(\max_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(t) > 1) + P(\sum_{j=1}^{n} X_j I_j(t) < A)$.
In particular, given $0 < \varepsilon < 1$, let $t = 1/(1-\varepsilon)$, so $1/t + \varepsilon = 1$. Since $t^{-m/(m+1)} - \varepsilon = (1-\varepsilon)^{m/(m+1)} - \varepsilon \ge 1 - 2\varepsilon$, we have

$$P(V_{mn} < \alpha(1-2\varepsilon)) \le P(V_{mn} < \alpha(t^{-m/(m+1)} - \varepsilon))$$
$$\le P\left(\max_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(\tau(t)) > 1\right)$$
$$+ P\left(\sum_{j=1}^{n} X_j I_j(\tau(t)) < \alpha(t^{-m/(m+1)} - \varepsilon)\right)$$
$$= P\left(\max_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(\tau(t)) > \frac{1}{t} + \varepsilon\right)$$
$$+ P\left(\sum_{j=1}^{n} X_j I_j(\tau(t)) < \alpha(t^{-m/(m+1)} - \varepsilon)\right). \tag{10}$$

To establish the corresponding lower bound V_{mn}, note that, by Lemma 2.5, if $\min_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(t) \ge 1$, then $V_{mn} \le \sum_{j=1}^{n} X_j I_j(t)$. Thus, for $A > 0$,

(f) $P(V_{mn} > A) \le P(\min_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(t) < 1) + P(\sum_{j=1}^{n} X_j I_j(t) > A)$.
Given $\varepsilon > 0$, let $t = 1/(1+\varepsilon)$, so $1/t - \varepsilon = 1$. Since, $t^{-m/(m+1)} + \varepsilon = (1+\varepsilon)^{m/(m+1)} + \varepsilon \le 1 + 2\varepsilon$, we have

$$P(V_{mn} > \alpha(1+2\varepsilon)) \le P(V_{mn} > \alpha(t^{-m/(m+1)} + \varepsilon))$$
$$\le P\left(\min_{i=1,\dots,m} \sum_{j=1}^{n} W_{ij} I_j(\tau(t)) < 1\right)$$
$$+ P\left(\sum_{j=1}^{n} X_j I_j(\tau(t)) > \alpha(t^{-m/(m+1)} + \varepsilon)\right)$$

$$= P\left(\min_{i=1,\ldots,m} \sum_{j=1}^{n} W_{ij} I_j(\tau(t)) < \frac{1}{t} + \varepsilon\right)$$

$$+ P\left(\sum_{j=1}^{n} X_j I_j(\tau(t)) > \alpha(t^{-m/(m+1)} + \varepsilon)\right). \tag{10'}$$

Theorem 2.2 now follows from (5), (9) and (10).

The proof of Theorem 2.4 is a bit easier. It is known that, for any positive integer r, $P(W_{11} + \cdots + W_{1r} \le 1) = 1/r!$, so we have $P(V_{mn} \ge r) \le P($ there exists $j_1 < j_2 < \cdots < j_r$ such that $W_{ij_1} + \cdots + W_{ij_r} \le 1$ for $i = 1, \ldots, m) \le \binom{n}{r}(1/r!)^m \le n^r/(r!)^m \le n^r/(r!)^{\gamma \log n} = n^{r - \gamma \log(r!)}$. Thus if r is chosen large enough that $\gamma \log(r!) > r + 1$, then $\sum_{n=1}^{\infty} P(V_{mn} \ge r) < \infty$, as required.

References

[1] P. Erdös and J. Spencer, Probabilistic Methods in Combinatorics (Academic Press, New York, 1974).

[2] W. Hoeffding, Probability inequalities for sums of bounded random variables, Amer. Statist. Assoc. J. 58 (1963) 13–30.

[3] J. Mamer and K. Schilling, On the growth of random knapsacks, Discrete Appl. Math. 28 (1990) 223–230.

[4] K. Schilling, The growth of m-constraint random knapsacks, European J. Oper. Res. 46 (1990) 109–112.