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Singular Perturbations and a Theorem of Kisyński*

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1. INTRODUCTION

In a recent paper, Kisyński [1] has studied the solutions of the abstract Cauchy problem

$$\epsilon x''(t) + x'(t) + Ax(t) = 0, \qquad x(0) = x_0\,, \qquad x'(0) = x_1\,, \tag{1.1}$$

where $t \geqslant 0$, $\epsilon > 0$ is a small parameter and A is a nonnegative self-adjoint (not necessarily bounded) operator in a Hilbert space H. With the aid of the functional calculus of the operator A, he has shown that the solutions of (1.1) converge, as $\epsilon \to 0$, to the solution of the unperturbed Cauchy problem

$$x'(t) + Ax(t) = 0, \qquad x(0) = x_0\,. \tag{1.2}$$

The purpose of this paper is twofold. First we shall extend Kisyński's result to third order equations. More precisely, we shall show that if the initial data is taken from a suitable dense subset of H, then the solutions of the Cauchy problem

$$\epsilon x'''(t) + x'(t) + Ax(t) = 0, \qquad x(0) = x_0\,, \qquad x'(0) = x_1\,, \qquad x''(0) = x_2\,, \tag{1.3}$$

where $t \geqslant 0$, $\epsilon > 0$ is a small parameter, and A is a nonnegative self-adjoint (not necessarily bounded) operator in H, converge, as $\epsilon \to 0$, to the solution of (1.2). While we borrow Kisyński's idea of using the functional calculus of the operator A in order to construct a solution of (1.3), our approach is different from Kisyński's in that we do not employ the techniques of the theory of semigroups. Secondly, we shall show that, in general, one cannot expect higher order perturbations of (1.2) to converge to a solution of (1.2). To this end, we shall show the following: If $H = R_1$, the real line, there is no dense subset $D \subset R_1$ for which the solutions of the Cauchy problem

$$\epsilon x^{(4)}(t) + x'(t) + \lambda x(t) = 0, \qquad x^{(i)}(0) = x_i\,, \qquad i = 0, 1, 2, 3, \tag{1.4}$$

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where $\epsilon > 0$ is a small parameter, λ is a positive real number, and $x_i \in D$, converge as $\epsilon \to 0$ to the solution of the associated unperturbed Cauchy problem

$$x'(t) + \lambda x(t) = 0, \qquad x(0) = x_0 . \tag{1.5}$$

2. THE PROBLEM (1.3) WHEN $H = R_1$

Before considering (1.3) in the general case, it is necessary to consider (1.3) in the case where $H = R_1$, the real line. Thus we consider the Cauchy problem

$$\epsilon x'''(t) + x'(t) + \lambda x(t) = 0,$$

$$x(0) = x_0 , \qquad x'(0) = x_1 , \qquad x''(0) = x_2 \tag{2.1}$$

where $t \geqslant 0$, $\epsilon > 0$, and $\lambda \geqslant 0$. By considering the equivalent system of first order equations, we obtain for the solution of this problem, the formulas

$$\begin{aligned} x(t) &= s_{00}(t, \epsilon, \lambda)\, x_0 + s_{01}(t, \epsilon, \lambda)\, x_1 + s_{02}(t, \epsilon, \lambda)\, x_2 \\ x'(t) &= s_{10}(t, \epsilon, \lambda)\, x_0 + s_{11}(t, \epsilon, \lambda)\, x_1 + s_{12}(t, \epsilon, \lambda)\, x_2 \\ x''(t) &= s_{20}(t, \epsilon, \lambda)\, x_0 + s_{21}(t, \epsilon, \lambda)\, x_1 + s_{22}(t, \epsilon, \lambda)\, x_2 , \end{aligned} \tag{2.2}$$

where the $s_{ij}(t, \epsilon, \lambda)$ are defined by

$$\begin{pmatrix} s_{00}(t, \epsilon, \lambda) & s_{01}(t, \epsilon, \lambda) & s_{02}(t, \epsilon, \lambda) \\ s_{10}(t, \epsilon, \lambda) & s_{11}(t, \epsilon, \lambda) & s_{12}(t, \epsilon, \lambda) \\ s_{20}(t, \epsilon, \lambda) & s_{21}(t, \epsilon, \lambda) & s_{22}(t, \epsilon, \lambda) \end{pmatrix} = \exp \left[t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda/\epsilon & -1/\epsilon & 0 \end{pmatrix} \right] . \tag{2.3}$$

The main idea in the proof of the convergence of solutions of (1.3) and (2.1) in the general case of an abstract Hilbert space or in the particular case $H = R_1$ is to obtain favorable estimates on the s_{ij}'s. In our third order case, in contrast to the one in [1], we do not have the energy inequalities and therefore we need to write out explicitly the solution of (2.1) and obtain our estimates on the s_{ij}'s from this. To this end, we consider the associated polynomial

$$f(m) = f_\epsilon(m) = \epsilon m^3 + m + \lambda. \tag{2.4}$$

Using Descartes rule of signs, we see that this polynomial has one negative root $\theta = \theta_\epsilon$, and the complex roots $a \pm ib = a_\epsilon \pm ib_\epsilon$, $b > 0$. It follows from Hurwitz' theorem [2, Theorem 1.5] that as $\epsilon \to 0$, $\theta \to -\lambda$. Moreover, since

$$m^3 + \frac{1}{\epsilon} m + \frac{\lambda}{\epsilon} = (m - \theta)(m^2 - 2am + a^2 + b^2),$$

we see that $0 = \theta + 2a$, and $\lambda/\epsilon = -\theta(a^2 + b^2)$ so that $a \to \lambda/2$ and $b \to \infty$ as $\epsilon \to 0$. Furthermore, since $a + ib$ is a root of the polynomial (2.4), we obtain from the imaginary part of $f(a + ib) = 0$ that $\epsilon b^2 = 3\epsilon a^2 + 1$. Therefore, $b^2 \geqslant 1/\epsilon$, $-\lambda/\theta\epsilon = a^2 + b^2 \geqslant b^2 \geqslant 1/\epsilon$, so that $\lambda \geqslant -\theta$ and $0 \leqslant a = -\theta/2 \leqslant \lambda/2$.

Now let the solution of (2.1) be written as

$$x(t) = x_\epsilon(t) = c_1 e^{\theta t} + c_2 e^{at} \cos bt + c_3 e^{at} \sin bt,$$

where $c_i = c_i^\epsilon$. The initial conditions yield

$$\begin{aligned} x_0 &= c_1 + c_2 + c_3 \\ x_1 &= \theta c_1 + a c_2 + b c_3 \\ x_2 &= \theta^2 c_1 + (a^2 - b^2)\, c_2 + 2ab c_3 \,. \end{aligned}$$

These three equations enable us to explicitly compute the s_{ij}'s and straightforward calculations yield

$$\begin{aligned} s_{00}(t, \epsilon, \lambda) &= \frac{a^2 + b^2}{(a - \theta)^2 + b^2}\, e^{\theta t} \\ &\quad + e^{at} \left[\frac{\theta^2 - 2a\theta}{(a - \theta)^2 + b^2} \cos bt + \frac{(a^2 - b^2)\,\theta - a\theta^2}{b(a - \theta)^2 + b^3} \sin bt \right] \\ s_{01}(t, \epsilon, \lambda) &= \frac{-2a}{(a - \theta)^2 + b^2}\, e^{\theta t} \\ &\quad + e^{at} \left[\frac{2a}{(a - \theta)^2 + b^2} \cos bt + \frac{\theta^2 + b^2 - a^2}{b(a - \theta)^2 + b^3} \sin bt \right] \\ s_{02}(t, \epsilon, \lambda) &= \frac{1}{(a - \theta)^2 + b^2}\, e^{\theta t} \\ &\quad + e^{at} \left[\frac{-1}{(a - \theta)^2 + b^2} \cos bt + \frac{a - \theta}{b(a - \theta)^2 + b^3} \sin bt \right] \end{aligned} \tag{2.5}$$

LEMMA 2.1. *The following estimates on the* $s_{ij}(t, \epsilon, \lambda)$ *are valid for fixed* $\epsilon > 0$, $t \geqslant 0$, $\lambda \geqslant 0$:

$$| s_{00}(t, \epsilon, \lambda) | \leqslant 1 + e^{(\lambda/2)t} \left[2\lambda\epsilon + \lambda \sqrt{\epsilon} \left(1 + \frac{\lambda^2}{2} \epsilon \right) \right] \tag{2.6}$$

$$| s_{01}(t, \epsilon, \lambda) | \leqslant \lambda\epsilon + \epsilon e^{(\lambda/2)t} \left[\lambda + \sqrt{\epsilon} \left(\frac{5\lambda^2}{4} + \epsilon \right) \right] \tag{2.7}$$

$$| s_{02}(t, \epsilon, \lambda) | \leqslant \epsilon + \epsilon e^{(\lambda/2)t} [1 + \tfrac{3}{2} \lambda \sqrt{\epsilon}]. \tag{2.8}$$

Furthermore, the following equations are also valid for the $s_{ij}(t, \epsilon, \lambda)$:

$$s_{10}(t, \epsilon, \lambda) = \frac{-\lambda}{\epsilon} s_{02}(t, \epsilon, \lambda), \qquad s_{11}(t, \epsilon, \lambda) = s_{00}(t, \epsilon, \lambda) - \frac{1}{\epsilon} s_{02}(t, \epsilon, \lambda),$$

$$s_{12}(t, \epsilon, \lambda) = s_{01}(t, \epsilon, \lambda), \qquad s_{20}(t, \epsilon, \lambda) = \frac{-\lambda}{\epsilon} s_{12}(t, \epsilon, \lambda),$$

$$s_{21}(t, \epsilon, \lambda) = s_{10}(t, \epsilon, \lambda) - \frac{1}{\epsilon} s_{12}(t, \epsilon, \lambda),$$

$$s_{22}(t, \epsilon, \lambda) = s_{11}(t, \epsilon, \lambda). \tag{2.9}$$

PROOF. Since $\theta < 0$, $a = -\theta/2$ and $b^2 > 1/\epsilon$, we have at once that

$$\frac{a^2 + b^2}{(a - \theta)^2 + b^2} e^{\theta t} \leqslant 1 \qquad \text{and} \qquad \frac{\theta^2 - 2a\theta}{(a - \theta)^2 + b^2} \leqslant \frac{2\theta^2}{b^2} \leqslant 2\lambda\epsilon.$$

Furthermore

$$|(a^2 - b^2)\,\theta| = -\theta\,|a^2 - b^2| \leqslant -\theta(a^2 + b^2) = \frac{\lambda}{\epsilon}, \qquad |a\theta^2| \leqslant \frac{\lambda^3}{2},$$

so that we get

$$\left| \frac{(a^2 - b^2)\,\theta - a\theta^2}{b(a - \theta)^2 + b^3} \right| \leqslant \frac{(\lambda/\epsilon) + (\lambda^3/2)}{b^3} \leqslant \lambda\sqrt{\epsilon} + \frac{\lambda^3}{2}\epsilon\sqrt{\epsilon},$$

from which (2.6) follows. Also

$$\left| \frac{-2a}{(a - \theta)^2 + b^2} e^{\theta t} \right| \leqslant \frac{\lambda}{b^2} \leqslant \lambda\epsilon,$$

$$|\theta^2 + b^2 - a^2| \leqslant \theta^2 + b^2 + a^2 \leqslant \frac{5\lambda^2}{4} + b^2,$$

so that

$$\left| \frac{\theta^2 + b^2 - a^2}{b(a - \theta)^2 + b^3} \right| \leqslant \frac{(5\lambda^2/4) + b^2}{b^3} \leqslant \frac{5\lambda^2}{4}\epsilon\sqrt{\epsilon} + \sqrt{\epsilon},$$

from which (2.7) follows. The proof of (2.8) is similar.

Finally, (2.9) follows from the equations

$$\frac{d}{dt}\begin{pmatrix} s_{00} s_{01} s_{02} \\ s_{10} s_{11} s_{12} \\ s_{20} s_{21} s_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda/\epsilon & -1/\epsilon & 0 \end{pmatrix} \begin{pmatrix} s_{00} s_{01} s_{02} \\ s_{10} s_{11} s_{12} \\ s_{20} s_{21} s_{22} \end{pmatrix}$$

$$= \begin{pmatrix} s_{00} s_{01} s_{02} \\ s_{10} s_{11} s_{12} \\ s_{20} s_{21} s_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda/\epsilon & -1/\epsilon & 0 \end{pmatrix}, \tag{2.10}$$

which in turn follows from (2.3). The proof of the lemma is complete.

We note that the first equation in (2.2) along with (2.7) and (2.8) shows that $x(t)$ converges to the solution $x_0(t) = x_0 \exp(-t\lambda)$ of (1.5) as $\epsilon \to 0$. This follows since we have the estimate

$$| s_{00}(t, \epsilon, \lambda) - e^{-t\lambda} | \leqslant \left| \frac{a^2 + b^2}{(a-\theta)^2 + b^2} e^{\theta t} - e^{-t\lambda} \right| + e^{(\lambda/2)t} \left[2\lambda\epsilon + \lambda\sqrt{\epsilon} \left(1 + \frac{\lambda^2}{2} \epsilon \right) \right]. \tag{2.11}$$

The second term on the right clearly converges to zero as $\epsilon \to 0$ and since $\theta \to -\lambda$ and

$$\lim_{\epsilon \to 0} \frac{a^2 + b^2}{(a-\theta)^2 + b^2} = 1,$$

we see that the first term on the right likewise converges to zero.

3. The Problem (1.3) in Abstract Hilbert Space

We shall now consider the Problem (1.3) in any Hilbert space H with norm $\| \cdot \|$. Since A is a nonnegative self-adjoint (not necessarily bounded) operator in H, there is a resolution of the identity $\{E_\lambda\}$ such that A has the spectral representation

$$A = \int_0^\infty \lambda dE_\lambda .$$

We shall next use the functional calculus of the operator A. For fixed $\epsilon > 0$, $t \geqslant 0$, we define the operator $S_{ij}(t, \epsilon)$ on H by

$$S_{ij}(t, \epsilon) = \int_0^\infty s_{ij}(t, \epsilon, \lambda)\, dE_\lambda , \qquad i, j = 0, 1, 2, \tag{3.1}$$

where the $s_{ij}(t, \epsilon, \lambda)$ are defined by (2.3). If we let D denote the (dense) domain of the operator $\exp(A^2)$, then our estimates (2.6) through (2.8) along with (2.9) imply that D is contained in the domain of $S_{ij}(t, \epsilon)$ for every $i, j = 0, 1, 2$.

For x_0, x_1, and x_2 in D, we put

$$x_\epsilon(t) = x(t) = S_{00}(t, \epsilon)\, x_0 + S_{01}(t, \epsilon)\, x_1 + S_{02}(t, \epsilon)\, x_2 , \tag{3.2}$$

and we see, from (2.6) through (2.8), that $x(t)$ is in the domain of A for every $t \geqslant 0$, and $\epsilon > 0$. We are now in a position to state the main theorem.

THEOREM 3.1. *Let $x(t)$ be defined as in (3.2) where x_0, x_2, x_1 are in D. Then $x(t)$ is the unique solution of the Cauchy problem (1.3) and $x(t)$ converges to the solution of (1.2) as $\epsilon \to 0$.*

In order to prove this theorem we first establish three lemmas.

LEMMA 3.1. *For $x \in D$, $(d/dt)\, S_{ij}(t, \epsilon)\, x$ exists and*

$$\frac{d}{dt} S_{ij}(t, \epsilon)\, x = \int_0^\infty \frac{d}{dt} s_{ij}(t, \epsilon, \lambda)\, dE_\lambda x, \qquad i, j = 0, 1, 2. \tag{3.3}$$

PROOF. We shall show that $(d/dt)\, S_{00}(t, \epsilon)\, x = S_{10}(t, \epsilon)\, x$. Then if we use (2.10), this will imply that (3.3) holds for $i = j = 0$. Since the proofs for the other cases are similar, they will be omitted.

For $x \in D$, and $t \geqslant 0$ fixed, we have

$$\left\| \frac{S_{00}(t + \Delta t, \epsilon) - S_{00}(t, \epsilon)}{\Delta t}\, x - S_{10}(t, \epsilon)\, x \right\|^2$$

$$= \int_0^\infty \left[\frac{s_{00}(t + \Delta t, \epsilon, \lambda) - s_{00}(t, \epsilon, \lambda)}{\Delta t} - s_{10}(t, \epsilon, \lambda)\right]^2 d\, \| E_\lambda x \|^2$$

$$= \int_0^\infty [s_{10}(t', \epsilon, \lambda) - s_{10}(t, \epsilon, \lambda)]^2\, d\, \| E_\lambda x \|^2,$$

where $t \leqslant t' \leqslant t + \Delta t$, using the theorem of the mean and (2.10). Now there is a T such that $t + \Delta t \leqslant T$ for all Δt sufficiently small so that if we use (2.9) and (2.10) we see that

$$| s_{10}(t', \epsilon, \lambda) - s_{10}(t, \epsilon, \lambda) | \leqslant 2\lambda + 2\lambda e^{(\lambda/2)T}(1 + \tfrac{3}{2}\lambda \sqrt{\epsilon}\,) \leqslant K_{T,\epsilon} e^{\lambda^2}$$

where $K_{T,\epsilon}$ is a constant depending only on T and ϵ. Therefore, the function $[s_{10}(t', \epsilon, \lambda) - s_{10}(t, \epsilon, \lambda)]^2$ is summable with respect to the measure $d\, \| E_\lambda x \|^2$ if Δt is sufficiently small. Furthermore,

$$\lim_{\Delta t \to 0} [s_{10}(t', \epsilon, \lambda) - s_{10}(t, \epsilon, \lambda)]^2 = 0,$$

so that the Lebesgue dominated convergence theorem gives

$$\lim_{\Delta t \to 0} \int_0^\infty [s_{10}(t', \epsilon, \lambda) - s_{10}(t, \epsilon, \lambda)]^2\, d\, \| E_\lambda x \|^2 = 0.$$

This completes the proof.

LEMMA 3.2. *For $x \in D$ and $t \geqslant 0$, we have*

$$\lim_{\epsilon \to 0} \| S_{00}(t, \epsilon)\, x - \exp(-\, tA)\, x \| = 0 \tag{3.4}$$

$$\lim_{\epsilon \to 0} \| S_{01}(t, \epsilon)\, x \| = 0 \tag{3.5}$$

$$\lim_{\epsilon \to 0} \| S_{02}(t, \epsilon)\, x \| = 0. \tag{3.6}$$

PROOF. From (2.11) we have

$$\| S_{00}(t, \epsilon) x - \exp(-tA) x \|^2 = \int_0^\infty (s_{00}(t, \epsilon, \lambda) - e^{-\lambda t})^2 \, d \| E_\lambda x \|^2$$

$$\leqslant 2 \int_0^\infty \left| \frac{a^2 + b^2}{(a - \theta)^2 + b^2} e^{\theta t} - e^{-t\lambda} \right|^2 d \| E_\lambda x \|^2$$

$$+ 2 \int_0^\infty e^{\lambda t} \left[2\lambda\epsilon + \lambda \sqrt{\epsilon} \left(1 + \frac{\lambda^2}{2} \epsilon \right) \right]^2 d \| E_\lambda x \|^2,$$

Now the second integral converges to zero as $\epsilon \to 0$ since $x \in D$. Also the first integrand is bounded by 4 which is summable with respect to the measure $d \| E_\lambda x \|^2$ and as seen previously, the integrand converges pointwise to zero. We apply the Lebesgue dominated convergence theorem to conclude that the first integral likewise converges to zero as $\epsilon \to 0$. This proves (3.4). Relations (3.5) and (3.6) follow at once from (2.7) and (2.8).

For the sake of completeness we state and prove the following known lemma.

LEMMA 3.3. *Let B be a bounded operator in H. If $x'(t) + Bx(t) = 0$, $0 \leqslant t < \infty$ and $x(0) = 0$, then $x(t) \equiv 0$.*

PROOF. Choose α such that $\alpha | B | < 1$ where $| B |$ denotes the norm of B. The hypotheses imply that we can write

$$x(t) = \int_0^t Bx(t) \, dt,$$

so that

$$\sup_{0 \leqslant t \leqslant \alpha} \| x(t) \| \leqslant \alpha | B | \sup_{0 \leqslant t \leqslant \alpha} \| x(t) \|,$$

and therefore $\| x(t) \| = 0, \quad 0 \leqslant t \leqslant \alpha$. Then writing

$$x(t) = \int_\alpha^t Bx(t) \, dt,$$

we get $\| x(t) \| = 0$, $\alpha \leqslant t \leqslant 2\alpha$, and so continuing in this way we see that $x(t) \equiv 0$.

PROOF OF THEOREM 3.1. That $x(t)$ defined by (3.2) is a solution of (1.3) follows at once from (2.10) along with Lemma 3.1 by direct verification. The uniqueness of $x(t)$ follows from Lemma 3.3 just as in [1]. In fact, if $x(t)$ is a solution of (1.3) for $x_0 = x_1 = x_2 = 0$, then if we put

$$A_n = \int_0^n \lambda dE_\lambda$$

we have that $y_n(t) = E_n x(t)$ satisfies (1.3) with A_n replacing A and

$$y_n(0) = y_n'(0) = y_n''(0) = 0$$

so that by Lemma 3.1, $y_n(t) = 0$, $n = 1, 2, 3, \cdots$. Hence

$$x(t) = \lim_{n\to\infty} E_n x(t) = 0.$$

Finally, since $\exp(-tA)\, x_0$ is the solution of (1.2), Lemma 3.2 shows that

$$\lim_{\epsilon\to 0} \| x(t) - \exp(-tA)\, x_0 \| = 0.$$

We note that if we fix $T > 0$, then we can easily show that $x(t)$, as defined by (3.2), converges to the solution of (1.3) uniformly for all t in the interval $[0, T]$. This follows since estimates analagous to (2.6) through (2.8) can be made to hold uniformly for t in the interval $[0, T]$.

4. The Nonconvergence of Higher Order Perturbations

In this section, we shall show that the solutions of

$$\epsilon x^{(4)}(t) + x'(t) + \lambda x(t) = 0, \qquad x^{(i)}(0) = x_i\,, \qquad i = 0, 1, 2, 3, \tag{4.1}$$

where $\epsilon > 0$, $t \geqslant 0$, $\lambda > 0$ and $x_1 \neq -\lambda x_0$ or $x_2 \neq \lambda^2 x_0$, do not converge as $\epsilon \to 0$ to the solution of

$$x'(t) + \lambda x(t) = 0, \qquad x(0) = x_0\,. \tag{4.2}$$

Consider the associated polynomial

$$g(m) = g_\epsilon(m) = \epsilon m^4 + m + \lambda.$$

This polynomial has exactly two negative real roots if ϵ is taken small enough ($g'(m) = 0$ if $m = m_0 = -(4\epsilon)^{-1/3}$ and $g''(m_0) > 0$ while $g(m_0) < 0$). Let $\alpha, \beta(\beta < \alpha)$ be the real negative roots and $a \pm ib$ the complex roots of g. Hurwitz' theorem shows that $\beta \to -\infty$ and $\alpha \to -\lambda$ as $\epsilon \to 0$. We also have

$$m^4 + \frac{1}{\epsilon} m + \frac{\lambda}{\epsilon} = (m^2 - (\alpha + \beta)\, m + \alpha\beta)\,(m^2 - 2am + a^2 + b^2)$$

so that, equating coefficients gives

$$2a + \alpha + \beta = 0 \tag{4.3}$$

$$a^2 + b^2 + 2a(\alpha + \beta) + \alpha\beta = 0 \tag{4.4}$$

$$-(\alpha + \beta)\,(a^2 + b^2) - 2a\alpha\beta = 1/\epsilon \tag{4.5}$$

$$\alpha\beta(a^2 + b^2) = \lambda/\epsilon. \tag{4.6}$$

From (4.3) we see that $a \to +\infty$ as $\epsilon \to 0$. We next eliminate ϵ from (4.5) and (4.6) to get three algebraic equations in the quantities α, β, a, b. Therefore we see that α, β, b are algebraic functions of a.

Now the solution of the problem (4.1) can be written as

$$x(t) = c_1 e^{\beta t} + c_2 e s^t + e^{at}[c_3 \sin bt + c_4 \cos bt],$$

where the c_i's are algebraic functions of α, β, a, b so that they are algebraic functions of a. Hence for fixed t, $e^{at}(c_3 \sin bt + c_4 \cos bt)$ goes to infinity like $a^s e^{at}$ [1] for $\epsilon_n \to 0$, (where ϵ_n is chosen so that $bt = 2n\pi$) while

$$c_1 e^{\beta t} + c_2 e^{\alpha t} = 0(a^r) \qquad \text{as} \qquad \epsilon \to 0,$$

where r and s are rational numbers. Hence $x(t)$ does not converge as $\epsilon \to 0$.

In a similar way, one can easily show that the solutions of the Cauchy problem

$$\epsilon x^{(5)}(t) + x'(t) + \lambda x(t) = 0\,, x^{(i)}(0) = x_i\,, \qquad i = 0, 1, 2, 3, 4, \tag{4.7}$$

behave in a similar manner. A more careful analysis of these two examples suggests that any perturbations of higher order do not in general converge.

5. Concluding Remarks

The first remark we wish to make is that it is not too hard to extend Theorem 3.1 to the following Cauchy problem:

$$\epsilon x'''(t) + K(\epsilon)\, x''(t) + x'(t) + Ax(t) = 0, \qquad x^{(i)}(0) = x_i\,, \qquad i = 0, 1, 2,$$

where A again is a nonnegative self-adjoint (not necessarily bounded) operator on a Hilbert space H; $K(\epsilon) \geqslant 0$, $[K(\epsilon)]^2 < 3\epsilon$, $\lim_{\epsilon \to 0} K(\epsilon) = 0$, x_0 is the initial value in the Problem (1.2), and x_i is in D for $i = 0, 1, 2$.

We do not know whether Theorem 3.1 is valid if the initial data are allowed to be chosen from a larger set then D. A desirable situation would be if we could take x_0 in the domain of A but this appears unlikely due to the presence of e^{at} in the first of the Equations (2.5).

The question of perturbing an nth order Cauchy problem (i.e., an nth order differential equation with Cauchy data) by an $(n+1)$th order Cauchy

[1] The hypothesis $x_0 \neq 0$ is implicitly assumed here, therefore

$$c_3 \sin bt + c_4 \cos bt = 0$$

for a sequence $\epsilon_n \to 0$ would imply that $x_1 = -\lambda x_0$, $x_2 = \lambda^2 x_0$, by considering the resulting solution of (4.1). Hence, once x_0 is chosen, we can find an open interval I about x_0 for which not both $-\lambda x_0$ and $\lambda^2 x_0$ are in I; thus there is no dense subset from which data can be taken in order to get convergence.

problem appears still to be open. The question of relaxing the hypotheses on A as well as allowing A to depend on t for even second order perturbations also seems to be unanswered. Finally, there is also the problem of considering such questions in a Banach space. The author hopes to consider some of these problems in a future paper.

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