KINETIC EQUATIONS FOR NEUTRON DISTRIBUTIONS

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Abstract—Kinetic equations for neutrons and some of the other kinds of particles important to neutron balance in reactors and reactor-like systems are derived from a quantum Liouville equation. In particular, equations for some of the relevant singlet and doublet densities in phase space are deduced for the purpose of studying neutron density fluctuations.

The equations describing neutron densities, ignoring delayed neutron precursors, are then singled out for special attention. These equations are reduced to 'one-speed' or mono-energetic analogues, and are then approximated by equations for the zeroth and first angular moments of these densities. During the process, points of connexion with the descriptions of neutron density fluctuations by various other authors are made.

Formal solutions of the equations in phase space are displayed, along with an explicit solution of the reduced equations. Properties of these solutions are discussed, with particular interest paid to the implications for the 'critical limit.'

1. INTRODUCTION

FLUCTUATIONS in neutron distributions in reactors have been the subject of intensive study—both theoretical and experimental—since the inception of the fission reactor technology.† The keen interest in this subject is generated by two quite different aspects of the problem.

In the first place, it is surmised that the information obtainable from these fluctuations—whenever they can be observed and interpreted—might be of considerable practical importance. Parameters which are peculiarly influential in the determination of the kinetic behaviour of reactors may be inferred from steady-state measurements. Monitoring and interpreting fluctuations in highly subcritical systems may provide valuable information about reactors during shut-down. Continuous observation of fluctuations in power systems might provide an important contribution in the problem of control. For these reasons among others, the problems associated with the measurement and analysis of reactor noise have received considerable attention from those concerned with the advancement of the fission power technology.

There is a second aspect to these problems, however, which broadens and deepens interest in them considerably. This one stems from the fact that neutron distributions in reactors may be an unusually convenient special case of the general many-body problem which has come under ever more intensive scrutiny during the past few years. The suggestion as to its unusual convenience follows from the obverse considerations of the potential experimental and theoretical accessibility of the system. On the one side we have the possibility of observing refinements of neutron distributions which may not be so readily and directly seen in solids, liquids, gases or plasmas. In particular, one is provided with an enormous density range (from about 10^9 particles/ cm^3 on down) over which experiments may be performed, and which is perhaps

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† The bibliography, and present state of the matter are quite thoroughly reviewed and discussed in the recent monograph on reactor noise prepared by THIE (1963).

ideally suited to the study of the higher-order stochastic characteristics of the system because of the fact that at low density (a first-order stochastic characteristic) they presumably become relatively more significant.

On the other side we have the suggestion (which it is the burden of this paper to develop) of clearer interpretation of experimental results when and if obtained. In many respects, neutron distributions in reactors are among the simplest of examples of many-body systems attainable. Because of the extremely short range of nuclear forces, they share with neutral gases some of the simplicity of mathematical characterization that is allowed by the binary collision approximation in the treatment of interactions. Furthermore, because the neutron–neutron collision cross section is probably comparable to the neutron–proton cross section (which is the order of 20 barns or less in the energy range of interest), and because of the very low neutron density compared to the density of the ambient nuclei in realizable systems, these distributions lend themselves to an even simpler description than do neutral gases since it is linear. At least it is linear up to the point that the neutrons begin to modify the distributions of the surrounding nuclei—a point which will be the subject of some discussion later in this investigation. Thus in the sense that the binary collision treatment of interactions is more justifiable and the self-interactions more ignorable than in other realizable systems, neutron distributions represent the simplest examples of many-body problems. Thus, by implication, they represent an important class of systems from the point of view of the search for experimental verification of theoretical descriptions of higher (than the first) order stochastic quantities, such as the doublet density or the closely related variance of the distribution. [It should be noted in passing that, under appropriate circumstances, photon distributions may share an equal and similar simplicity (OSBORN and KLEVANS, 1961; KLEVANS, 1962), and perhaps are equally accessible to experiment.]

Of course, there are respects in which the description of neutron distributions appears somewhat more complicated than that for, say, the neutral gas. The most obvious of these arises from the almost inescapable necessity to take explicit account of reactions in which the particles of interest are not conserved. In particular, in multiplying systems, detailed accounting of both destruction and creation of neutrons must be kept. Furthermore, again in multiplying media, neutrons may be created and destroyed in the same reaction—but not necessarily at effectively the same time, as is exemplified by the production of delayed neutrons from neutron-induced fission. Such delay phenomena can significantly influence the temporal evolution of the neutron distribution, and hence must be dealt with explicitly. Nevertheless, complicating features such as these may be more than offset by the simplifications alluded to above. At least the possibility that this is so should be explored.

If indeed neutron distributions do represent a potentially important class of many-body systems in which to search for experimental comparisons with theoretical refinements, then it seems imperative to obtain predictions from first principles as deductively as possible. It is the purpose of this paper to describe such an attempt. It is not suggested that the approach employed in the present attempt is the only one, merely that it is a feasible one which we feel has advanced the study significantly and perhaps should be explored further.

Until recently (OSBORN and YIP, 1963), the theory of neutron distributions has rested solely upon phenomenological bases. One of the more elaborate treatments of

this kind was presented by MATHES (1962), and it serves well as a summary of the phenomenological approach. Densities are defined as appropriate averages taken with respect to a fundamental probability distribution—which, however, is itself defined by an empirically deduced equation. The point of departure of the investigation described herein is the observation that this probability distribution is in fact rigorously described by the Liouville equation, and our initial effort will be to establish a sense in which working equations for the various, relevant singlet and doublet densities for the reactor may be deduced by approximation from the rigorous description. Of course, the above observation is of the nature of the obvious, and hence it is a bit surprising that there has been no previous attempt to exploit it. Particularly is this so in view of the wide-spread upsurge of interest in problems of this kind; especially as they pertain to neutral and charged gases.

In Section 2 we will sketch a derivation of a sufficient set of equations to enable an attack on the problem of interpreting a certain class of fluctuation measurements in reactors. The equations that we will need are those for the singlet and doublet densities for neutrons, a single kind of delayed neutron precursor, the particles actually counted in a detector (illustratively we will consider them to be α-particles accumulated in a Boron–Trifluoride detector), and all relevant cross densities. The derivation will proceed from the quantum Liouville equation for the reactor. Of course, we do not anticipate any explicit quantum effects to influence our description of the system at the working level. Nevertheless, in dealing with systems in which particles are not conserved, we find the logic of derivation greatly simplified and clarified by a quantum mechanical formulation. When, in the course of the argument, it is convenient to do so, we will pass to the classical limit.

In Sections 3 and 4 we will concentrate exclusively on the equations which describe the neutron singlet and doublet densities, neglecting delayed neutrons. Section 3 will be devoted to an examination of various reduced forms of these equations for the primary purpose of establishing points of connexion and comparison with the work of others in this field. In particular, the diffusion theory analogues will be obtained by integrating the phase-space equations over the angular variables in velocity space; the monoenergetic transport analogues will be obtained by integrating over energy variables; the steady state, consistent P_1 equations will be displayed; and finally the 'point-reactor' kinetic equations will be obtained by integrating over all the phase variables.

In Section 4 we will sketch a formal solution of the phase-space equations for the purpose of extracting a comment on the implications of the linear theory of the doublet density for critical system. We will also develop on explicit solution of the P_1 equation in order to explore the space dependence of the doublet in certain simplified but non-trivial situations.

2. THE PHASE-SPACE DENSITIES

In this section we sketch the derivation of the balance relations for the relevant densities. Manipulative detail will be largely omitted, since it has been presented elsewhere in one form or another in connexion with the derivation of transport equations for neutral gases (ONO, 1954) and plasmas (OSBORN, 1963). However, sufficient discussion of the method of attack and of the approximations made will be included in the effort to make this exposition reasonably self-contained.

As indicated above, the derivation of the balance relations from the Liouville equation for the system will be carried through from the quantum point of view, in spite of the fact that we do not anticipate any explicitly quantum effects to influence the results we finally settle for. There are at least three strong reasons for doing this. In the first place, the quantum formalism provides a convenient context within which to describe systems in which the particles of interest are being created and destroyed. In the second place, the quantum mechanics enables a precise definition of positive definite densities in phase space, and in the third place, a quantum treatment of reactions is required if transitions among bound states play a significant role in the description of the system. Actually this last reason will not be exploited in this paper, since detailed formulae for reaction rates are not necessary for present purposes.

The densities we need here are defined by

$$F^a(\mathbf{X}, \mathbf{K}, t) \equiv L^{-3} Tr \sum_s \rho^a(\mathbf{X}, \mathbf{K}, s) D(t),$$ (1)

and

$$F^{ab}(\mathbf{X}, \mathbf{K}, \mathbf{X}', \mathbf{K}', t) \equiv L^{-6} Tr \sum_{ss'} \rho^a(\mathbf{X}, \mathbf{K}, s) \rho^b(\mathbf{X}', \mathbf{K}', s') D(t).$$ (2)

The singlet density, $F^a(\mathbf{X}, \mathbf{K}, t)$, defined in equation (1) represents the expected number of particles per cubic centimetre of kind a located in a cubic configuration space cell of volume L^3 having momentum $\hbar\mathbf{K}$ at time, t. The wave vectors, $\mathbf{K}$, are also discretely distributed; the distance between successive values of K_1 for example being $2\pi/L$. We are here following the prescription of Ono for coarse-graining phase space. Accordingly it has been divided up into non-overlapping hypercells of volume $(2\pi)^3$, if the centres are parametrized by $(\mathbf{X}, \mathbf{K})$, or of volume (h^3) if parametrized by $(\mathbf{X}, \mathbf{P} = \hbar\mathbf{K})$. Since all of the particles in a given cell are assigned the co-ordinates of the centre of the cell, it is evident that the uncertainty in the simultaneous specification of the position and momentum of these particles is as it should be according to the uncertainty principle. The operator, $\rho^a(\mathbf{X}, \mathbf{K}, s)$, is a number operator whose eigenvalues in a diagonalizing representation are the numbers of particles of kind a in the cell centred at $(\mathbf{X}, \mathbf{K})$ having spin, s. Keeping an explicit account of particle spins is actually superfluous here, and will be dispensed with in the following. The density matrix, D, presumably satisfies the equation

$$\frac{\partial D}{\partial t} = \frac{i}{\hbar} [D, H],$$ (3)

where H is the Hamiltonian for the system. Of course, because nuclear interactions among others must be included in H, this Hamiltonian can hardly be regarded as fully known. However, for the purpose of deriving balance relations for the densities defined in equations (1) and (2), less than full knowledge proves sufficient. Finally, the doublet density $L^6 F^{ab}(\mathbf{X}, \mathbf{K}, \mathbf{X}', \mathbf{K}', t)$ represents the expected number of particles of kind a to be found in the hypercell centred at $(\mathbf{X}, \mathbf{K})$ and of kind b in the hypercell at $(\mathbf{X}', \mathbf{K}')$ at time, t. Since ρ^a and ρ^b commute, F^{ab} is real.

In order to derive equations for these densities, we observe that

$$\frac{\partial F^a}{\partial t} \simeq \frac{1}{\tau} [F^a(t + \tau) - F^a(\tau)],$$ (4)

provided that F^a does not vary too rapidly over the time interval, τ. Of course, equation (4) is rigorously true by definition if τ can be made limitingly small; but such a limit is not meaningful. The density cannot be fruitfully compared at two times closer together than the longest interaction time of a reaction expected to influence the balance relation. Thus we find it necessary to coarse-grain in time as well as in phase space.

Introducing

$$\eta^a(\mathbf{X}, \mathbf{K}) = \sum_s \rho^a(\mathbf{X}, \mathbf{K}, s), \tag{5}$$

we find from equations (1) and (4),

$$\frac{\partial F^a}{\partial t} \simeq \tau^{-1} L^{-3} Tr\eta^a [D(t + \tau) - D(t)]. \tag{6}$$

According to equation (3), we have

$$D(t + \tau) = u(\tau)D(t)u^+(\tau), \tag{7}$$

where

$$u(\tau) = \exp(-i\tau H/\hbar). \tag{8}$$

To proceed from here, it is desirable to discuss H in a little more detail. We subdivide it according to

$$H = \sum_a T^a + \mathscr{H}, \tag{9}$$

where T^a is the kinetic energy of the a-type particles. In this paper, we will regard all particles except the neutrons, delayed neutron precursors, and the detected particles as being in known distributions. Consequently only the kinetic energy of these kinds of particles requires our explicit attention. The symbol, $\mathscr{H}$, is intended to represent all other contributions to the energy of the system.

For illustrative purposes we consider the kinetic energy of the neutrons, i.e.

$$T^n = \frac{\hbar}{2m} \int d^3x \, \boldsymbol{\nabla}\psi_j^+ . \boldsymbol{\nabla}\psi_j, \tag{10}$$

where the sum over repeated spinor indices is understood. We expand the operators, ψ_j, as

$$\psi_j(\mathbf{x}) = L^{-3/2} \sum_{\mathbf{XK}s} \exp(i\,\mathbf{K}{\cdot}\mathbf{x})U_j(s)E(\mathbf{X}, \mathbf{x})a(\mathbf{X}, \mathbf{K}, s), \tag{11}$$

where $U_j(s)$ is the unit spinor, i.e.

$$U_j{}^*(s)U_j(s') = \delta_{ss'}, \quad \text{and} \quad \sum_s U_j{}^*(s)U_{j'}(s) = \delta_{jj'}, \tag{12}$$

and where $E(\mathbf{X}, \mathbf{x})$ is the three-dimensional step-function equal to unity for all $\mathbf{x}$ in the cell of volume L^3 centred at $\mathbf{X}$ and equal to zero for all $\mathbf{x}$ outside that cell. These functions were first employed in problems of transport theory by ONO, and they provide the analytical realization for the above comments regarding phase-space coarse-graining. The quantity, $a(\mathbf{X}, \mathbf{K}, s)$ is the destruction operator for neutrons of spin, s, in the cell centred at $(\mathbf{X}, \mathbf{K})$. Furthermore,

$$\rho^n(\mathbf{X}, \mathbf{K}, s) = a^+(\mathbf{X}, \mathbf{K}, s)a(\mathbf{X}, \mathbf{K}, s). \tag{13}$$

Entering equation (11) into equation (10), we find

$$T^n = \sum_{\mathbf{XK}s} \frac{\hbar^2 K^2}{2m} \rho^n(\mathbf{X}, \mathbf{K}, s) + \mathcal{T}^n$$

$$= \mathcal{E}^n + \mathcal{T}^n. \tag{14}$$

The first term, $\mathcal{E}^n$, represents the kinetic energy of the neutrons within cells, and helps define the states between which neutrons jump when they experience interactions. The second term consists of two parts. One is linear in the gradient of the step-function, and describes the transport of neutrons from one cell to another. The other is quadratic in the gradient of the step-function, and to the level of approximation employed herein contributes nothing to the subsequent calculations. Similar comments apply to the other kinetic energy terms appearing in equation (9). Thus now we may write

$$H = \sum_a \mathcal{E}^a + \sum_a \mathcal{T}^a + \mathcal{H}$$

$$= T' + \mathcal{H}', \tag{15}$$

where

$$T' = \sum_a \mathcal{T}^a,$$

and

$$\mathcal{H}' = \sum_a \mathcal{E}^a + \mathcal{H}. \tag{16}$$

The term T' in H describes particle transport, and the term $\mathcal{H}'$ describes interactions.

The time evolution operator in equation (8) may be written as

$$u(\tau) = \exp\left[-i\tau(T' + \mathcal{H}')/\hbar\right] \tag{17}$$

$$= \exp\left(-i\tau T'/\hbar\right) \exp\left(-i\tau \mathcal{H}'/\hbar\right) J(\tau).$$

where

$$\frac{\partial J}{\partial \tau} = U(\tau)J, \qquad J(O) = I, \tag{18}$$

and

$$U(\tau) = \frac{i}{\hbar}\left[\mathcal{H}' - \exp\left(i\tau\mathcal{H}'/\hbar\right)\exp\left(i\tau T'/\hbar\right)\mathcal{H}'\exp\left(-i\tau T'/\hbar\right)\exp\left(-\tau\mathcal{H}'/\hbar\right)\right]. \tag{19}$$

Note that for small τ,

$$U(\tau) \sim \frac{\tau}{\hbar^2}\left[T', \mathcal{H}'\right], \tag{20}$$

so that

$$J(\tau) \sim I + \frac{\tau^2}{2\hbar^2}\left[T', \mathcal{H}'\right]. \tag{21}$$

In estimating the right-hand side of equation (6), we retain only those terms which are explicitly or implicitly independent of τ, obtaining

$$\frac{\partial F^a}{\partial t} \simeq -L^{-3}Tr\frac{i}{\hbar}\left[\eta^a, T'\right]D(t) + L^{-3}Tr\eta^a\tau^{-1}$$

$$\times \left\{\exp\left(-i\tau\mathcal{H}'/\hbar\right)D(t)\exp\left(i\tau\mathcal{H}'/\hbar\right) - D(t)\right\}. \tag{22}$$

In a representation which diagonalizes the number operator, η^a, with eigenvalues, $N^a(\mathbf{X}, \mathbf{K})$, equation (22) may be displayed as

$$\frac{\partial F^a}{\partial t} + L^{-3} Tr \frac{i}{\hbar} [\eta^a, T'] D(t) = L^{-3} \sum_{nn'} \{N^{a'}(\mathbf{X}, \mathbf{K}) - N^a(\mathbf{X}, \mathbf{K})\} W_{n'n} D_{nn}(t) \quad (23)$$

$$+ \text{(Terms involving the off-diagonal elements of } D),$$

where

$$W_{n'n} = \frac{1}{\tau} \left| [\exp(-i\tau\mathcal{H}'/\hbar)]_{n'n} \right|^2 \quad (24)$$

is the transition probability per unit time, and will be evaluated for large τ (large compared to the longest interaction time of any interaction which significantly influences the distribution of particles of the a-th kind) for which it is independent of τ. Henceforth we will neglect the terms involving the off-diagonal elements of D in Equation (23) and in other similar equations. Such a similar equation is the one for a doublet density, which, at this point, would read

$$\frac{\partial F^{ab}}{\partial t} + L^{-6} Tr \frac{i}{\hbar} [\eta^a, T']\eta^b D(t) + L^{-6} Tr \frac{i}{\hbar} \eta^a[\eta^b, T'] D(t)$$

$$+ L^{-6} \sum_{nn'} \{N^{a'}(\mathbf{X}, \mathbf{K})N^{b'}(\mathbf{X}', \mathbf{K}') - N^a(\mathbf{X}, \mathbf{K})N^b(\mathbf{X}', \mathbf{K}')\} W_{n'n} D_{nn}(t). \quad (25)$$

Further reduction of the streaming terms (those involving the commutators with T') is straightforward, but approximate. An important approximation is that the various densities vary sufficiently slowly from cell to cell so that, for example,

$$\frac{F^a(X_i + L) - F^a(X_i)}{L} \simeq \frac{\partial F^a}{\partial X_i}. \quad (26)$$

Then equations (23) and (25) become

$$\frac{\partial F^a}{\partial t} + \frac{\hbar K_i}{m_a} \frac{\partial F^a}{\partial X_i} = L^{-3} \sum_{nn'} \{N^{a'}(\mathbf{X}, \mathbf{K}) - N^a(\mathbf{X}, \mathbf{K})\} W_{n'n} D_{nn}, \quad (27a)$$

$$\frac{\partial F^{ab}}{\partial t} + \frac{\hbar K_i}{m_a} \frac{\partial F^{ab}}{\partial X_i} + \frac{\hbar K_i'}{m_b} \frac{\partial F^{ab}}{\partial X_i'}$$

$$= L^{-6} \sum_{nn'} \{N^{a'}(\mathbf{X}, \mathbf{K})N^{b'}(\mathbf{X}', \mathbf{K}') - N^a(\mathbf{X}, \mathbf{K})N^b(\mathbf{X}', \mathbf{K}')\} W_{n'n} D_{nn}. \quad (27b)$$

The last task to be accomplished in this section is that of expressing the interaction terms in equations (27a) and (27b) as explicit functionals of the densities of interest. Thus it becomes necessary to decide what interactions are important to these balance relations, and how much of the information about them, implicit in the transition probabilities W, requires explicit attention. To obtain a realistic, but at the same time

not too unwieldy, description we will restrict consideration to scattering, radiative capture, and fission so far as the neutrons are concerned. We will neglect the scattering of neutrons by neutrons. In the fission process, we will assume both the production of prompt neutrons and of precursors of delayed neutrons (fission fragments which, after one or more β-decays, achieve a nucleonic configuration capable of de-exciting by neutron emission). The delayed neutron precursors will be presumed created by fission and destroyed by a β-decay which simultaneously produces a neutron. We will ignore the possibility that the precursor might be transmuted by neutron capture. The detected particles will be considered to be α-particles produced in the $^{10}B(n, \alpha)^7Li$ reaction, and it will be assumed that upon production they accumulate. We will also include a neutron source other than fission so that we may broaden our discussion to include subcritical systems.

For present purposes, it turns out to be unnecessary to give much attention to the details of these transition probabilities, except for their dependence upon occupation numbers. This dependence is easily elicited, since, for any process in which a particle of kind a is destroyed at the phase point $(\mathbf{X}, \mathbf{K})$, we anticipate the factor, $N^a(\mathbf{X}, \mathbf{K})$. Conversely, if a similar particle is created, we expect the factor, $[1 \pm N^a(\mathbf{X}, \mathbf{K})]$, where the plus or minus sign goes with bosons and fermions respectively. However, it can be readily shown that the retention of the term N compared to unity in the factor that goes with particle creation leads to peculiarly quantum effects—quantum statistics vs. classical statistics. Since we do not expect that the systems to which the present theory will be applied will be sensitive to such a distinction, we henceforth ignore the occupation number dependence arising from creation processes.

With these remarks in mind, the final reduction of equation (27a) and (27b) to useful form is essentially nothing but a task in accounting—albeit a large and tedious one. After it is completed it is convenient to assume that the discrete points in phase space are actually sufficiently closely spaced that we may treat them as continuously distributed. Points in this continuous domain will be labelled $(\mathbf{x}, \mathbf{v})$, where $\mathbf{v}$ is velocity and is given by $\mathbf{v} = \hbar\mathbf{K}/m$. We will also use lower case symbols to represent the densities defined over this domain. We then find that a set of working equations for the study of neutron fluctuations is:

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + L(v)\right] f_1^N(\mathbf{x}, \mathbf{v}, t) - \lambda\beta(\mathbf{v})\int d^3v' f_1^A(\mathbf{x}, \mathbf{v}', t) = S(\mathbf{x}, \mathbf{v}, t), \quad (28a)$$

where

$$L(\mathbf{v})f_1^N(\mathbf{x}, \mathbf{v}, t) = v\Sigma_t(\mathbf{v})f_1(\mathbf{x}, \mathbf{v}, t) - \int d^3v' v' \Sigma_s(\mathbf{v}' \to \mathbf{v})f_1^N(\mathbf{x}, \mathbf{v}', t)$$

$$- \int d^3v' v' \Sigma_f(\mathbf{v}') \sum_{j\alpha} \alpha B_\alpha^j(\mathbf{v}', \mathbf{v})f_1^N(\mathbf{x}, \mathbf{v}', t), \quad (28b)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \lambda\right] f_1^A(\mathbf{x}, \mathbf{v}, t) = \mu(\mathbf{v})\int d^3v' v' \Sigma_f(\mathbf{v}')f_1^N(\mathbf{x}, \mathbf{v}', t), \quad (29)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right] f_1^D(\mathbf{x}, \mathbf{v}, t) = \int d^3v' v' \Sigma_D(\mathbf{v}' \to \mathbf{v})f_1^N(\mathbf{x}, \mathbf{v}', t), \quad (30)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{v}' \cdot \boldsymbol{\nabla}' + L(\mathbf{v}) + \lambda\right] f_2^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

$$= \mu(\mathbf{v}') \int d^3 v'' f_2^{NN}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}'', t) v'' \Sigma_f(v'')$$

$$+ S(\mathbf{x}, \mathbf{v}, t) f_1^{A}(x', v', t) + \lambda\beta(\mathbf{v}) \int d^3 v'' f_2^{AA}(\mathbf{x}, \mathbf{v}'', \mathbf{x}', \mathbf{v}', t) - \delta(\mathbf{x} - \mathbf{x}')$$

$$\times \left[\lambda\beta(\mathbf{v}) f_1^{A}(\mathbf{x}, \mathbf{v}', t) + \mu(\mathbf{v}') v \Sigma_f(\mathbf{v}) f_1^{N}(\mathbf{x}, \mathbf{v}, t) \right.$$

$$\left. - \mu(\mathbf{v}') \int d^3 v'' v'' \Sigma_f(\mathbf{v}'') \sum_{j\alpha} \alpha B_\alpha^{j}(\mathbf{v}'', \mathbf{v}) f_1^{N}(\mathbf{x}, \mathbf{v}'', t) \right]. \qquad (31)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{v}' \cdot \boldsymbol{\nabla}' + L(\mathbf{v})\right] f_2^{ND}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

$$= \int d^3 v'' v'' \Sigma_D(\mathbf{v}'' \to \mathbf{v}') f_2^{NN}(\mathbf{x}, \mathbf{v}, \mathbf{x}, \mathbf{v}'', t)$$

$$+ \lambda\beta(\mathbf{v}) \int d^3 v'' f_2^{AD}(\mathbf{x}, \mathbf{v}'', \mathbf{x}', \mathbf{v}', t) + S(\mathbf{x}, \mathbf{v}, t) f_1^{D}(\mathbf{x}', \mathbf{v}', t)$$

$$- \delta(\mathbf{x} - \mathbf{x}') v \Sigma_D(\mathbf{v} \to \mathbf{v}') f_1^{N}(\mathbf{x}, \mathbf{v}, t). \qquad (32)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{v}' \cdot \boldsymbol{\nabla}' + \lambda\right] f_2^{AD}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

$$= \int d^3 v'' v'' \Sigma_D(\mathbf{v}'' \to \mathbf{v}') f_2^{NA}(\mathbf{x}', \mathbf{v}'', \mathbf{x}, \mathbf{v}, t)$$

$$+ \mu(\mathbf{v}) \int d^3 v'' v'' \Sigma_f(\mathbf{v}'') f_2^{ND}(\mathbf{x}, \mathbf{v}'', \mathbf{x}, \mathbf{v}, t). \qquad (33)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{v}' \cdot \boldsymbol{\nabla}' + L(\mathbf{v}) + L(\mathbf{v}')\right] f_2^{NN}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

$$= \lambda\beta(\mathbf{v}') \int d^3 v'' f_2^{NA}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}'', t)$$

$$+ \lambda\beta(\mathbf{v}) \int d^3 v'' f_2^{NA}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}'', t) + S(\mathbf{x}', \mathbf{v}', t) f_1^{N}(\mathbf{x}, \mathbf{v}, t)$$

$$+ S(\mathbf{x}, \mathbf{v}, t) f_1^{N}(\mathbf{x}', \mathbf{v}', t) + \delta(\mathbf{x} - \mathbf{x}')$$

$$\times \left[\Lambda(\mathbf{v}, \mathbf{v}', \mathbf{x}, t) + \delta(\mathbf{v} - \mathbf{v}') \lambda\beta(\mathbf{v}) \int d^3 v'' f_1^{A}(\mathbf{x}, \mathbf{v}'', t) + \delta(\mathbf{v} - \mathbf{v}') S(\mathbf{x}, \mathbf{v}, t) \right],$$

$$(34a)$$

where

$$\Lambda(\mathbf{v}, \mathbf{v}', \mathbf{x}, t) = \delta(\mathbf{v} - \mathbf{v}')\Big\{ v\Sigma_t(\mathbf{v})f_1^N(\mathbf{x}, \mathbf{v}, t) + \int d^3v''v''$$

$$\times \left[\Sigma_s(\mathbf{v}'' \to \mathbf{v})\right] f_1^N(\mathbf{x}, \mathbf{v}'', t)\Big\}$$

$$-v\left[\Sigma_s(\mathbf{v} \to \mathbf{v}') + \Sigma_f(\mathbf{v})\sum_{j\alpha}\alpha B_\alpha{}^j(\mathbf{v}, \mathbf{v}')\right]f_1^N(\mathbf{x}, \mathbf{v}, t)$$

$$-v'\left[\Sigma_s(\mathbf{v}' \to \mathbf{v}) + \Sigma_f(\mathbf{v}')\sum_{j\alpha}\alpha B_\alpha{}^j(\mathbf{v}', \mathbf{v})\right]f_1^N(\mathbf{x}, \mathbf{v}', t)$$

$$+\int d^3v''v''\Sigma_f(\mathbf{v}'')\sum_{j\alpha\beta}\alpha\beta B_{\alpha\beta}{}^j(\mathbf{v}'' \mid \mathbf{v}, \mathbf{v}')f_1^N(\mathbf{x}, \mathbf{v}'', t), \tag{34b}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{\nabla} + \mathbf{v}'\cdot\mathbf{\nabla}' + 2\lambda\right]f_2^{AA}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$$

$$= \mu(\mathbf{v})\int d^3v''v''\Sigma_f(\mathbf{v}'')f_2^{NA}(\mathbf{x}, \mathbf{v}'', \mathbf{x}', \mathbf{v}', t)$$

$$+ \mu(\mathbf{v}')\int d^3v''v''\Sigma_f(\mathbf{v}'')f_2^{NA}(\mathbf{x}', \mathbf{v}'', \mathbf{x}, \mathbf{v}, t) + \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}')$$

$$\times \left[\mu(\mathbf{v})\int d^3v''v''\Sigma_f(\mathbf{v}'')f_1^N(\mathbf{x}, \mathbf{v}'', t) + \lambda f_1^A(\mathbf{x}, \mathbf{v}, t)\right]. \tag{35}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{\nabla} + \mathbf{v}'\cdot\mathbf{\nabla}'\right]f_2^{DD}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) = \int d^3v''v''\Sigma_D(\mathbf{v}'' \to \mathbf{v})f_2^{ND}(\mathbf{x}, \mathbf{v}'', \mathbf{x}', \mathbf{v}', t)$$

$$+ \int d^3v''v''\Sigma_D(\mathbf{v}'' \to \mathbf{v}')f_2^{ND}(\mathbf{x}', \mathbf{v}'', \mathbf{x}, \mathbf{v}, t)$$

$$+ \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}')\int d^3v''v''\Sigma_D(\mathbf{v}'' \to \mathbf{v})f_1^N(\mathbf{x}, \mathbf{v}'', t). \tag{36}$$

A table follows in which previously undefined quantities, appearing in the above equations, are defined.

(i) The superscripts N, A and D, on the densities refer to the various particles as follows: N, neutrons; A, delayed neutron precursors; D, detected particles (alphas).

(ii) $\beta(\mathbf{v})\, d^3v$ is the probability that a delayed neutron will be born with velocity $\mathbf{v}\varepsilon\, d^3v$.

(iii) λ is the precursor β-decay constant.

(iv) $S(\mathbf{x}, \mathbf{v}, t)\, d^3x\, d^3v$ is the expected number of neutrons produced per second at time t, in d^3x about $\mathbf{x}$, and with velocity $\mathbf{v}\varepsilon\, d^3v$, by means other than the fission process.

(v) $\mu(\mathbf{v})\, d^3v$ is the expected number of precursors, with velocity $\mathbf{v}\varepsilon\, d^3v$, produced in a fission.

(vi) $\Sigma_s(\mathbf{v})$ is the probability per unit path for small paths that a neutron with velocity $\mathbf{v}$ will be scattered. The rest of the sigmas are defined in the same manner,

with the subscripts indicating the specific interaction of interest, i.e. s, scattering; f, fission; a, absorption; c, capture $(\Sigma_c = \Sigma_a - \Sigma_f)$, and D, detection process.

(vii) $\Sigma_s (\mathbf{v}' \to \mathbf{v}) \, d^3v$ is the probability per unit path for small paths that a neutron with initial velocity $\mathbf{v}'$ will be scattered into a final velocity $\mathbf{v}\varepsilon \, d^3v$.

(viii) $\Sigma_D (\mathbf{v}' \to \mathbf{v}) \, d^3v$ is the probability per unit path for small paths that a neutron with velocity $\mathbf{v}'$ will experience a detection interaction to produce a detection (α-) particle with velocity $\mathbf{v}\varepsilon \, d^3v$.

(ix) $B_\alpha^{\ j}(\mathbf{v}', \mathbf{v}) \, d^3\mathbf{v}$ is the probability that a fission induced by a neutron with velocity $\mathbf{v}'$ will produce j prompt neutrons of which α will have velocities in d^3v about $\mathbf{v}$. Similarly $B_{\alpha\beta}^{\ \ j}(\mathbf{v}' \, | \, \mathbf{v}, \mathbf{v}'') \, d^3v \, d^3v''$ is the probability that a fission induced by a neutron with velocity $\mathbf{v}'$ will produce j prompt neutrons of which α will have $\mathbf{v}\varepsilon \, d^3v$ and β have $\mathbf{v}''\varepsilon \, d^3v''$.

3. DISCUSSION OF THE NEUTRON EQUATIONS

The full set of equations, (28) through (36), is probably a minimal description of particle distributions required for an interpretation of experiments designed to measure neutron fluctuations. However, it is evident that this set of equations poses a rather complicated mathematical problem—one in fact which has received little or no attention to date. In fact, even the far simpler problem of studying the description of prompt neutron distributions only, provided by equations (28a) and (34a) when delayed neutrons are neglected, has been given only limited and preliminary consideration. Equations similar to the ones referred to above have been presented by GOVORKOV (1962) and derived by PÁL (1957), BORGWALDT and SANITZ (1963), and BELL (1965) (though by arguments quite different from those employed here). In each case some specific implications of the equation for the neutron doublet were explored to some extent; nevertheless it seems to us that more remains to be done than has yet been done. Thus, for the remainder of this work, we will concentrate on the simplified (by the elimination of delayed neutrons) versions of (28a) and (34a) in an effort to examine points of connexion with other work in this field and to explore some of their implications.

For the purposes of clarification and ease of reference, we rewrite these equations here in a changed and compacted notation. Introducing $f(\mathbf{x}, \mathbf{v}, t)$ to represent the neutron singlet density, and $F(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t)$ for the doublet density, we display (28a) as

$$\frac{\partial f}{\partial t} + Bf = S, \tag{37}$$

and (34a) as

$$\frac{\partial F}{\partial t} + (B + B')F = Sf' + S'f + \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}')S + \delta(\mathbf{x} - \mathbf{x}')\Gamma f. \tag{38}$$

Here we have introduced the symbol B to stand for the Boltzmann operator which acts on the unprimed or primed phase point depending upon whether it itself is unprimed or primed. As is readily established by reference to equation (28a), it is defined by

$$B\psi(\mathbf{x}, \mathbf{v}) \equiv \mathbf{v} \cdot \nabla \psi + v\Sigma_t \, \psi - \int d^3v'v'\Sigma_s (v' \to v)\psi(v')$$

$$- \int d^3v'v' \, \Sigma_{f'} \sum_{j\alpha} \alpha B_\alpha^{\ j}(\mathbf{v}', \mathbf{v})\psi(\mathbf{v}'). \tag{39}$$

The operator, Γ, appearing in the inhomogeneous terms in equation (38), is defined by [as can be seen from (34a) and (34b)]

$$\Gamma\psi(\mathbf{x}, \mathbf{v}) = \delta(\mathbf{v} - \mathbf{v}')\{v\Sigma_t\,\psi + \int d^3v''v'' \Sigma_s(\mathbf{v}'' \rightarrow \mathbf{v})\psi(\mathbf{v}'')\}$$

$$- v\,[\Sigma_s(\mathbf{v} \rightarrow \mathbf{v}') + \Sigma_f \sum_{j\alpha} \alpha B_\alpha{}^j(\mathbf{v}, \mathbf{v}')]\psi(\mathbf{v})$$

$$- v'[\Sigma_s(\mathbf{v}' \rightarrow \mathbf{v}) + \Sigma_f' \sum_{j\alpha} \alpha B_\alpha{}^j(\mathbf{v}', \mathbf{v})]\psi(\mathbf{v}')$$

$$+ \int d^3v''v'' \Sigma_f'' \sum_{j\alpha\beta} \alpha\beta B_{\alpha\beta}{}^j(\mathbf{v}''\,|\,\mathbf{v}, \mathbf{v}')\psi(\mathbf{v}''). \tag{40}$$

It is sometimes more convenient to study a correlation function rather than the neutron doublet density itself. Thus we define

$$G(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) = F(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) - f(\mathbf{x}, \mathbf{v}, t)f(\mathbf{x}', \mathbf{v}', t), \tag{41}$$

and show by an obvious exploitation of equations (37) and (38) that G satisfies

$$\frac{\partial G}{\partial t} + (B + B')G = \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{v} - \mathbf{v}')S + \delta(\mathbf{x} - \mathbf{x}')\Gamma f. \tag{42}$$

In the next section we will examine some interesting aspects of formal solutions to these equations; however in this section we emphasize comparison between equations employed by others in the study of neutron fluctuations and various reduced forms of (37) and (38). To this end we introduce some notation for the representation of angle-integrated densities and currents, energy-integrated densities, and energy-and-angle-integrated densities, i.e.

$$\psi(\mathbf{x}, E, t) = \int d\Omega\, f(\mathbf{x}, E, \mathbf{\Omega}, t), \tag{43a}$$

$$\mathbf{j}(\mathbf{x}, E, t) = \int d\Omega\, v\mathbf{\Omega} f(\mathbf{x}, E, \mathbf{\Omega}, t), \tag{43b}$$

$$\Phi(\mathbf{x}, E, \mathbf{x}', E', t) = \int d\Omega\, d\Omega' F(\mathbf{x}, E, \mathbf{\Omega}, \mathbf{x}', E', \mathbf{\Omega}', t), \tag{43c}$$

$$\mathbf{J}(\mathbf{x}, E, \mathbf{x}', E', t) = \int d\Omega\, d\Omega' v\mathbf{\Omega} F(\mathbf{x}, E, \mathbf{\Omega}, \mathbf{x}', E', \mathbf{\Omega}', t), \tag{43d}$$

$$\mathbf{J}'(\mathbf{x}, E, \mathbf{x}', E', t) = \int d\Omega\, d\Omega' v'\mathbf{\Omega}' F(\mathbf{x}, E, \mathbf{\Omega}, \mathbf{x}', E', \mathbf{\Omega}', t), \tag{43e}$$

$$\phi(\mathbf{x}, \mathbf{\Omega}, t) = \int dE f(\mathbf{x}, E, \mathbf{\Omega}, t), \tag{43f}$$

$$\Phi(\mathbf{x}, \boldsymbol{\Omega}, \mathbf{x}', \boldsymbol{\Omega}', t) = \int dE \, dE' F(\mathbf{x}, E, \boldsymbol{\Omega}, \mathbf{x}', E', \boldsymbol{\Omega}', t), \tag{43g}$$

$$n(\mathbf{x}, t) = \int d\Omega \, dE f(\mathbf{x}, E, \boldsymbol{\Omega}, t), \tag{43h}$$

and

$$N(\mathbf{x}, \mathbf{x}', t) = \int dE \, d\Omega \, dE' \, d\Omega' F(\mathbf{x}, E, \boldsymbol{\Omega}, \mathbf{x}', E', \boldsymbol{\Omega}', t). \tag{43i}$$

The densities f and F appearing in the definitions (43a) through (43i) are, of course, related to the densities in equations (37) and (38) according to

$$f(\mathbf{x}, \mathbf{v}, t) \, d^3v = f(\mathbf{x}, E, \boldsymbol{\Omega}, t) \, dE \, d\Omega$$

$$F(\mathbf{v}, \mathbf{v}') \, d^3v \, d^3v' = F(E, \boldsymbol{\Omega}, E', \boldsymbol{\Omega}') \, dE \, d\Omega \, dE' \, d\Omega'. \tag{44}$$

Consider first the equations which describe the reduced densities ψ and Φ. They are obtained from (37) and (38) by first transforming according to (44) and then integrating over directions of motion $\boldsymbol{\Omega}$ (or $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$). Assuming that the scattering frequency depends only on the angle between the initial and final directions of motion of the scattered neutron, i.e.

$$\Sigma_s(E', \boldsymbol{\Omega}' \to E, \boldsymbol{\Omega}) = \Sigma_s(E', E, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}), \tag{45}$$

and that fission can be treated as completely isotropic in the laboratory system so that

$$B_\alpha{}^j(E', \boldsymbol{\Omega}' \to E, \boldsymbol{\Omega}) = (4\pi)^{-1} B_\alpha{}^j(E' \to E),$$

$$B_{\alpha\sigma}{}^j(E'', \boldsymbol{\Omega}'' \mid E, \boldsymbol{\Omega}; \, E', \boldsymbol{\Omega}') = (4\pi)^{-2} \, B_{\alpha\sigma}{}^j(E'' \mid E, E'), \tag{46}$$

we find that (37) reduces to

$$\frac{\partial \psi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} + v \Sigma_t \, \psi - \int dE' v' \Sigma_{s0} \, (E' \to E) \psi(E')$$

$$- \int dE' v' \Sigma_f' \, \psi(E') \sum_{j\alpha} \alpha B_\alpha{}^j(E' \to E) = S_0. \tag{47}$$

In this equation we have introduced the further notation

$$\Sigma_{s0} \, (E' \to E) = \int d\Omega \, \Sigma_s \, (E', \boldsymbol{\Omega}', E, \boldsymbol{\Omega}),$$

$$S_0(\mathbf{x}, E, t) = \int d\Omega S(\mathbf{x}, E, \boldsymbol{\Omega}, t). \tag{48}$$

Equation (47) is, of course, quite conventional and familiar. The reduced equation obtained from (38) is

$$
\frac{\partial \Phi}{\partial t} + \mathbf{\nabla} \cdot \mathbf{J} + \mathbf{\nabla}' \cdot \mathbf{J}' + (v \Sigma_t + v' \Sigma_{t'}) \Phi - \int dE'' v'' \Sigma_{s0} (E'' \to E) \Phi(E'', E')
$$

$$
- \int dE'' v'' \Sigma_{s0} (E'' \to E') \Phi(E, E'') - \int dE'' v'' \Sigma_f{}'' \Phi(E'', E') \sum_{j\alpha} \alpha B_\alpha{}^j (E'' \to E)
$$

$$
- \int dE'' v'' \Sigma_f{}'' \Phi(E, E'') \sum_{j\alpha} \alpha B_\alpha{}^j (E'' \to E')
$$

$$
= S_0'\psi + S_0\psi' + \delta(\mathbf{x} - \mathbf{x}')\delta(E - E')S_0 + \delta(\mathbf{x} - \mathbf{x}')[\delta(E - E')
$$

$$
\times \left\{ v \Sigma_t \psi + \int dE'' v'' \Sigma_{s0} (E'' \to E) \psi(E'') \right\}
$$

$$
- v \Sigma_{s0} (E \to E')\psi(E) - v \Sigma_f \sum_{j\alpha} \alpha B_\alpha{}^j (E \to E')\psi(E) - v' \Sigma_{s0} (E' \to E)\psi(E')
$$

$$
- v' \Sigma_f' \sum_{j\alpha} \alpha B_\alpha{}^j (E' \to E)\psi(E') + \int dE'' v'' \Sigma_f{}'' \, \psi(E'') \sum_{j\alpha\sigma} \alpha\sigma B_{\alpha\sigma}{}^j (E'' \mid EE')]. \qquad (49)
$$

This equation is substantially* the same as the one derived and discussed by MATHES (1962). It does differ in one significant respect, however, in that instead of the terms $\nabla \cdot \mathbf{j}$, $\nabla \cdot \mathbf{J}$, and $\nabla' \cdot \mathbf{J}'$ MATHES obtains $D\nabla^2\psi$, $D\nabla^2\Phi$ and $D\nabla'^2\Phi$ respectively. In the present instance it is clear that these identifications are Fick's rule approximations to the current divergences.

A second reduced form of (37) and (38) is obtained by integrating over all energy variables. However, this process of reduction is seriously complicated by the fact that the cross sections in the operators B and Γ are usually significantly energy-dependent. Conventionally, in dealing with the singlet density, this complication is cut through by defining energy-averaged parameters in such a way that

$$
\int dEBf = \bar{B}\phi, \qquad (50)
$$

where ϕ is defined by (43f). Proceeding analogously for the reduction of the doublet density, we find that

$$
\int dE \, dE'(B + B')F = (\bar{\bar{B}} + \bar{\bar{\bar{B}}})\Phi, \qquad (51)
$$

where $\bar{\bar{B}}$ and $\bar{\bar{\bar{B}}}$ differ from each other and from $\bar{B}$ because they depend upon parameters which have been averaged with respect to different weight functions. The extent and significance of these differences are not known at the present time. In the discussion that follows, we will ignore them and equate

$$
\bar{\bar{\bar{B}}} = \bar{\bar{B}} = \bar{B}. \qquad (52)
$$

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* MATHES' equation describes the variance, $\Psi(E,E')$, rather than the doublet density. The two quantities are simply related as follows:
$$
\Psi(E,E') = \Phi(E,E') - \psi(E)\psi(E') - \delta(x - x')\delta(E - E')\psi(E).
$$

It then follows that (37) and (38) reduce to

$$\frac{\partial \phi}{\partial t} + \bar{B}\phi = \bar{S},$$ (53)

and

$$\frac{\partial \Phi}{\partial t} + (\bar{B} + \bar{B}')\Phi = \bar{S}\phi' + \bar{S}'\phi + \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{\Omega} - \mathbf{\Omega}')\bar{S} + \delta(\mathbf{x} - \mathbf{x}')\bar{\Gamma}\phi,$$ (54)

where

$$\bar{S}(\mathbf{x}, \mathbf{\Omega}, t) = \int dE S(\mathbf{x}, E, \mathbf{\Omega}, t).$$ (55)

The operators $\bar{B}$ and $\bar{\Gamma}$ are the 'one-speed' counterparts of B and Γ and are here defined by

$$\bar{B}\phi = \langle v \rangle \mathbf{\Omega} . \nabla\phi + r_t\phi - r_s \int d\Omega' \phi(\mathbf{\Omega}')\gamma(\mathbf{\Omega}' \to \mathbf{\Omega}) - \frac{\langle j \rangle r_f}{4\pi} \int d\Omega' \phi(\mathbf{\Omega}'),$$ (56)

and

$$\Gamma\phi = \delta(\mathbf{\Omega} - \mathbf{\Omega}') [r_t\phi + r_s \int d\Omega'' \phi(\mathbf{\Omega}'')\gamma(\mathbf{\Omega}'' \to \mathbf{\Omega})$$

$$+ \frac{\langle j^2 \rangle r_f}{4\pi} \int d\Omega'' \phi(\mathbf{\Omega}'')] - r_s\gamma(\mathbf{\Omega}' \to \mathbf{\Omega})\phi(\mathbf{\Omega}')$$

$$- r_s\gamma(\mathbf{\Omega} \to \mathbf{\Omega}')\phi(\mathbf{\Omega}) - \frac{\langle j \rangle r_f}{4\pi} [\phi(\mathbf{\Omega}) + \phi(\mathbf{\Omega}')].$$ (57)

Equation (53) is the familiar monoenergetic neutron transport equation for the singlet density and, as such, requires no further comment here. Equation (54) does not appear to have been presented explicitly before, although it is, of course, implicit in the work described in GOVORKOV, PAL, BORGWALDT and SANITZ and BELL. These reduced (and approximated) equations still pose a formidable problem. Thus it is desirable to reduce (and approximate) them still further. With respect to equation (53), this is conventionally done by taking its zero-th and first velocity–angular moments (moments of $\mathbf{\Omega}$) and then, in the equation for the first moment, approximating the second moment according to the assumption that ϕ depends at most linearly upon $\mathbf{\Omega}$. This leads to an equation for the zero-th moment of ϕ, i.e. $n(x, t)$, which is second order in both space and time derivatives and is sometimes referred to as a consistent P_1 equation. For present purposes, we carry out the same programme with respect to equation (54). On the whole the required manipulations are straightforward and conventional, so we present no detail here. However, one mildly subtle point should be singled out. The quantity

$$\int d\Omega \, d\Omega' \mathbf{\Omega}\mathbf{\Omega}'\Phi$$ (58)

is not generally zero* and hence must be duly considered if the resultant equation for $N(x, x', t)$ is to be consistent in the same sense that the equation for $n(\mathbf{x}, t)$ is consistent. Since it is our intention here to discuss only some of the steady-state characteristics of the doublet density, N, we eschew the presentation of the full

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* We are indebted to Dr. F. SHURE for bringing this fact to our attention and for valuable discussions illuminating its significance.

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equation governing its temporal evolution as well as its spatial variation. The equation for the latter is

$$\left[2a - D(\nabla^2 + \nabla'^2) + \frac{bD^2}{2(a+b)^2}(\nabla^2 - \nabla'^2)^2 \right] N$$

$$= \left[1 - \frac{D}{2(a+b)}(\nabla^2 + \nabla'^2) \right] \mathscr{H} + \frac{3D}{a+b} \nabla_j \nabla_k' \mathscr{H}_{jk} - \frac{\langle v \rangle}{a+b}$$

$$\times (\nabla \cdot \underline{\mathscr{H}} + \nabla' \cdot \underline{\mathscr{H}}') + \frac{\langle v \rangle D}{2(a+b)^2}(\nabla^2 - \nabla'^2)(\nabla \cdot \underline{\mathscr{H}} - \nabla' \cdot \underline{\mathscr{H}}'). \quad (59)$$

The definitions of the new symbols appearing in this equation are:

$$a = r_a - \langle j \rangle r_f = r_c + r_f - \langle j \rangle r_f, \tag{60a}$$

$$D = \frac{\langle v \rangle^2}{3b} \tag{60b}$$

$$b = r_t - \bar{\mu} r_s = r_a + r_s - \bar{\mu} r_s, \tag{60c}$$

$$\bar{\mu} = \int d\Omega \mathbf{\Omega} \cdot \mathbf{\Omega}' \gamma(\mathbf{\Omega}' \to \mathbf{\Omega}), \tag{60d}$$

$$\mathscr{H} = \int d\Omega \, d\Omega' \bar{H}, \tag{60e}$$

$$\underline{\mathscr{H}} = \int d\Omega \, d\Omega' \mathbf{\Omega} \bar{H}, \qquad \underline{\mathscr{H}}' = \int d\Omega \, d\Omega' \mathbf{\Omega}' \bar{H}, \tag{60f}$$

$$\mathscr{H}_{jk} = \int d\Omega \, d\Omega' \Omega_j \Omega_k' \bar{H}, \quad \text{and} \tag{60g}$$

$$\bar{H} = S\phi' + S'\phi + \delta(\mathbf{x} - \mathbf{x}')\delta(\mathbf{\Omega} - \mathbf{\Omega}')\bar{S} + \delta(\mathbf{x} - \mathbf{x}')\Gamma\phi. \tag{60h}$$

Equation (59) does not seem to have appeared elsewhere before, though an approximation to it was presented in Osborn and Yip. It is interesting to note that an integration of equation (49) over energy accompanied by the Fick's rule identifications of the current divergences leads to

$$[2a - D(\nabla^2 + \nabla'^2)]N = \mathscr{H}. \tag{61}$$

As a final comment in this section, we note that the conventional, so-called 'point-reactor,' kinetic equations are immediately obtainable from equations (53) and (54). This may be done by integrating these equations over all space and angle variables, neglecting the streaming terms in the operator $\bar{B}$, and defining

$$\phi(t) = \int d^3x \, d\Omega \phi(x, \Omega, t),$$

$$\Phi(t) = \int d^3x \, d^3x' \, d\Omega \, d\Omega' \Phi(\mathbf{x}, \mathbf{\Omega}, \mathbf{x}', \mathbf{\Omega}', t),$$

$$\bar{S}(t) = \int d^3x \, d\Omega \bar{S}(\mathbf{x}, \mathbf{\Omega}, t). \tag{62}$$

One finds immediately that

$$\frac{d\phi(t)}{dt} + a\phi(t) = \bar{S}(t) \tag{63}$$

and

$$\frac{d\Phi}{dt} + 2a\Phi(t) = 2\bar{S}(t)\phi(t) + \bar{S}(t) + C\phi(t), \tag{64}$$

where

$$C = r_c + \langle (j-1)^2 \rangle r_f. \tag{65}$$

Recalling that, in this section, we have neglected delayed neutrons, it is seen that equations (64) and (65) are the same as those extensively employed in studies of neutron fluctuations (COURANT and WALLACE, 1947; HARRIS, 1958; SOODAK, 1961; PLUTA, 1961).

4. SOME IMPLICATIONS OF THE DOUBLET EQUATIONS

In this section we examine a few of the characteristics of some rather formal solutions of the various equations presented for the description of the neutron singlet and doublet densities in the previous section. In particular, we examine an implicit solution of the unreduced equations (37) and (38) in an effort to extract some information about the temperal behaviour of the doublet density, and then display an explicit solution to the one-speed, P_1 equation (59), in order to reveal some aspects of its approximate space dependence.

The formal solutions to equations (37) and (38) are

$$f(t) = e^{-tB}f(0)$$
$$+ \int_0^t dt' e^{-(t-t')B} S(t') \tag{66}$$

and

$$F(t) = e^{-t(B+B')}F(0) + \int_0^t dt' e^{-(t-t')(B+B')}H(t'), \tag{67}$$

where we have introduced $H(t)$ to represent the inhomogeneous terms in equation (38). Note that

$$\bar{H} = \int dE \, dE' H, \tag{68}$$

where $\bar{H}$ has been given in (60h). These solutions are too implicit to be of much practical use in general, although they do provide us with a framework within which an interesting observation about the probable temporal behaviour of the doublet density in a critical system can be developed. To this end we first look for the stationary, asymptotic singlet density in the source-free case, i.e.

$$f(t) \sim e^{-tB}f(0). \tag{69}$$

We assume a complete set of eigenfunctions of B and of its adjoint. The existence of such sets has not yet been demonstrated so the following argument is suggestive rather

than conclusive. The eigenfunctions and eigenvalues satisfy

$$B\psi_j = \lambda_j \psi_j, \tag{70}$$

where the eigenvalues may be discretely and/or continuously distributed. Let the eigenvalues be ordered, i.e.

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots \tag{71}$$

and assume that at least λ_0 belongs to the discrete set. Again, this is an untested assumption, but seems justified if the critical state is physically realizable in a finite time. If we display

$$f(0) = \sum_j a_j(0)\psi_j, \tag{72}$$

where we sum or integrate depending upon whether λ_j belongs to a discrete or continuous distribution, we find

$$f(t) = a_0(0)e^{-\lambda_0 t}\psi_0 + \sum_{j \geq 1} a_j(0)e^{-\lambda_j t}\psi_j. \tag{73}$$

The critical state is now defined by the requirement that $\lambda_0 = 0$. Then in the limit of large time, we find

$$f(t) \sim a_0(0)\psi_0. \tag{74}$$

Now use these eigenfunctions of B for a series representation of $F(0)$ and H (which, in the present instance, may be considered to be time-independent). We find that

$$F(t) = A_{00}(0)\psi_0\psi_0' + \sum_{ij} A_{ij}(0) \exp\left[-(\lambda_i + \lambda_j)t\right]\psi_i\psi_j$$

(not both zero)

$$+ th_{00}\psi_0\psi_0' + \sum_{i,j \geq 1} h_{ij} \frac{1 - \exp\left[-(\lambda_i + \lambda_j)t\right]}{\lambda_i + \lambda_j} \psi_i\psi_j, \tag{75}$$

(not both zero)

where the A_{ij}'s and h_{ij}'s are the coefficients in the expansion of $F(0)$ and H respectively. Evidently, asymptotically we have

$$F(t) \sim th_{00}\psi_0\psi_0', \tag{76}$$

and hence no stationary doublet density exists (unless, of course, $h_{00} = 0$ which would seem to be purely fortuitous if ever the case).

Since this result is not obtained by well-defended arguments, it is perhaps unwise to struggle too hard to interpret it. However, we will demonstrate a similar conclusion from an explicit argument below, and we note that this conclusion is either explicit or implicit in the point-reactor-kinetics studies of the doublet density—even if delayed neutrons are taken into consideration (see, for example, HARRIS, and also PLUTA. Thus it is tempting to argue that the critical state of a reactor is a mathematical fiction (which seems intuitively reasonable, since, if it is realized at an instant it is not

so immediately thereafter). Of course, these remarks are premised upon the applicability of the present theory which has explicitly neglected the influence of the neutrons upon the distribution of ambient atoms. If, in fact, this influence were not neglected and the response of the atomic distribution, as well as its correlation with the neutron distribution, were properly accounted for, these issues including the very notion of the critical state itself might be radically altered.

The second illustrative solution for the doublet density will be obtained from equation (59)—the steady-state, P_1 equation. To find a fairly general solution which is of more than purely formal interest, we imagine a cubical region of edge length l interior to a materially homogeneous region. The eigenfunctions of the Laplacian, subject to periodic boundary conditions on the surface of the cube, are a complete, orthonomal set suitable for the series representation of arbitrary functions defined within the cube. In terms of these eigenfunctions we may readily construct the Green's function for the differential operator in (59) which acts on the doublet density, N, i.e.

$$\left[2a - D(\nabla^2 + \nabla'^2) + \frac{bD^2}{2(a+b)^2}(\nabla^2 - \nabla'^2)^2\right]G = \delta(x - x'')\delta(x' - x'''), \quad (77)$$

and

$$G = \frac{1}{l^6}\sum_{\mathbf{k}\mathbf{k}'}\frac{\exp\left[i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}'') + i\mathbf{k}'\cdot(\mathbf{x}' - \mathbf{x}''')\right]}{\left[2a + D(k^2 + k'^2) - \frac{bD^2}{2(a+b)^2}(k^2 - k'^2)^2\right]}. \quad (78)$$

The vectors $\mathbf{k}$ and $\mathbf{k}'$ have components which take on the discrete values

$$k_j = \frac{2\pi n_j}{l}, \qquad n_j = 0, \pm1, \pm2, \ldots \quad (79)$$

If we designate the inhomogeneous term on the right-hand side of equation (59) by $Q(\mathbf{x}, \mathbf{x}')$, then

$$N(\mathbf{x}, \mathbf{x}') = \int d^3x'' \, d^3x''' G(\mathbf{x}, \mathbf{x}', \mathbf{x}'', \mathbf{x}''')Q(\mathbf{x}'', \mathbf{x}'''), \quad (80)$$

where the integration runs over the volume of the cube. It is important to note that the Green's function (78) does not exist for $a = 0$, the critical condition in the infinite, homogeneous medium.

In order to express (80) in more explicit terms, we assume that our cube is embedded in a sufficiently large system that both the neutron source and singlet density may be presumed constant over the volume of the cube. Designate these constants by S and n respectively. In this case we find that

$$Q(\mathbf{x}, \mathbf{x}') \rightarrow 2nS + (S + Cn)\delta(\mathbf{x} - \mathbf{x}')$$
$$- \frac{D}{a+b}\left(S + \frac{C+C'}{2}n\right)(\nabla^2 + \nabla'^2)\delta(\mathbf{x} - \mathbf{x}'), \quad (81)$$

where

$$C' = 2b - r_c + \langle j^2 - 1\rangle r_f.$$

If we further assume that the edge length of our cube is very large compared to any characteristic neutron length (mean-free paths, diffusion lengths, etc.), we may approximate sums over $\mathbf{k}$ by integrals. The solution (80) now becomes

$$N(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^6} \int \frac{d^3x''\, d^3x'''\, d^3k\, d^3k'\, \exp\left[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'') + i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}''')\right]}{2a + D(k^2 + k'^2) - \dfrac{bD^2}{2(a+b)^2}(k^2 - k'^2)^2}$$

$$(x)[2nS + (S + Cn)\delta(\mathbf{x}'' - \mathbf{x}''')$$

$$- \frac{D}{a+b}\left(S + \frac{C+C'}{2}n\right)(\nabla''^2 + \nabla'''^2)\delta(\mathbf{x}'' - \mathbf{x}''')]. \quad (82)$$

Introducing the new variable of integration $\mathbf{r} = \mathbf{x}'' - \mathbf{x}'''$ and approximating

$$\iint d^3x''\, d^3x''' \simeq \iint d^3r\, d^3x''', \tag{83}$$

we find after some straightforward manipulations that

$$N(\mathbf{x}, \mathbf{x}') = \frac{nS}{a} + \frac{2S + (C+C')n}{2(a+b)}\,\delta(\mathbf{x} - \mathbf{x}')$$

$$+ \left[\frac{S+Cn}{2D} - \frac{a}{D}\frac{2S+(C+C')n}{2(a+b)}\right]\frac{\exp\left[-|\mathbf{x}-\mathbf{x}'|\sqrt{(a/D)}\right]}{4\pi|\mathbf{x}-\mathbf{x}'|}. \quad (84)$$

Throughout the region in which this solution is valid we have $S = na$. Also, quite generally, $b \gg a$. Consequently, equation (84) may be well approximated by

$$N(\mathbf{x}, \mathbf{x}') \simeq n^2 + n\delta(\mathbf{x} - \mathbf{x}') + \frac{\langle j(j-1)\rangle r_f n}{2D}\frac{\exp\left[-|\mathbf{x}-\mathbf{x}'|\sqrt{(a/D)}\right]}{4\pi|\mathbf{x}-\mathbf{x}'|}. \quad (85)$$

Another convenient way to display this result is in terms of a correlation function defined as

$$G(x, x') = \frac{N(x, x') - n^2}{n^2}, \tag{86}$$

which, by virtue of (85) is

$$G = \frac{1}{n}\delta(\mathbf{x} - \mathbf{x}') + \frac{\langle j(j-1)\rangle r_f}{2Dn}\frac{\exp\left[-|\mathbf{x}-\mathbf{x}'|\sqrt{(a/D)}\right]}{4\pi|\mathbf{x}-\mathbf{x}'|}. \quad (87)$$

A few aspects of (85) or (87) deserve comment. In the first place these solutions may not be presumed applicable to the infinite, homogeneous, critical system for which $a = 0$. This point was made explicitly above when it was observed that the Green's function (78), and hence the solution (80) [or (84)], does not exist in that case. This is an explicit realization of the earlier surmise that the doublet density has no meaning in a critical system (or that the critical system itself has no physical meaning in the dynamic sense). Secondly we observe the appearance of a characteristic length, $\sqrt{(D/a)}$, which measures the range of correlation in some sense. This length may be rewritten according to

$$\sqrt{(D/a)} = L/\sqrt{(1-k)}, \tag{88}$$

where L is the conventional diffusion length and k is the infinite medium multiplication constant in the absence of fast fission and fast absorption. Thirdly we note that spatial correlation is proportional to the factor r_f, and hence depends strongly upon the multiplying properties of the medium. Also, as is perhaps to be expected intuitively, it is seen that this correlation decreases as the singlet density increases.

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