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An Analysis of Errors in Finite Automata*

PHILIP S. DAUBER

The University of Michigan, Information Systems Laboratory
Ann Arbor, Michigan

This paper studies errors in finite automata. An error is defined as a pair of states and errors are then classified according to their probability of being corrected (i.e., being taken into the same state). Various results are then given on the partitioning properties of a particular type of error called a finite error.

I. INTRODUCTION

This problem arose from an attempt to make a general study of reliability in computerlike machines. The classic results of von Neumann (von Neumann, 1956) deal only with networks which do not have any feedback. Thus a malfunction only causes the network to be in the incorrect state for a fixed length of time.

However, a malfunction in the general case with feedback can cause an error which persists forever. Fortunately, not all errors are of this type. Some errors are of the type that can persist only for a bounded time. Some, although they can persist infinitely long, have a probability of being corrected which approaches one as the tapes get longer. Thus "almost all" of the "long" tapes correct the error.

This is the phenomenon which will be studied in this paper. It will be shown that errors of the latter type induce a partition on the set of states. The possibility of adding states to the machine in order to get a more reliable one will also be discussed. In the next section we will formalize the problem in terms of the theory of automata.

II. FORMALIZATION OF THE PROBLEM

In order to clarify the notation and to make the problem more formal,

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we will begin by defining a finite automaton and an error in a finite automaton.

DEFINITION 2.1. A *finite automation* M is a triple

$$M = (M', \Sigma, \delta).$$

M' is a finite set with elements m_i (set of states);
Σ is a finite set with elements σ_i (input alphabet);
δ is a function from $M' \times \Sigma \rightarrow M'$ (next state functions).

Later we will use M both to denote the finite automaton and its set of states. We will also extend δ to Σ^*, the set of sequences of symbols from Σ, in the natural manner with sequences read from left to right.

DEFINITION 2.2. a. An *error*, E, in a finite automaton, M, is a pair of states (m_i, m_j).

b. An error, (m_i, m_j), is *corrected* by a tape t ("tape" is synonymous with "sequence") if and only if

$$\delta(m_i, t) = \delta(m_j, t).$$

We can think of an error (m_i, m_j) as the situation when, due to a previous malfunction, the automaton is in state m_i and should be in state m_j, or is in state m_j and should be in state m_i. We can see from the definition of an error being corrected that these are both equivalent.

In this work we will consider sequences being generated by a random source. We will say that a random source with output alphabet Σ has *property P* if and only if it is stationary and there is a number k, greater than zero, such that the probability of the symbol σ following an arbitrary sequence x is greater than k.

DEFINITION 2.3. Let S be a random source with property P and output symbols Σ, and let $M = (M, \Sigma, \delta)$ be a finite automaton driven by S. For an error $E = (m_i, m_j)$ we define the following:

a. $\gamma_l^S(m_i, m_j)$ = probability that (m_i, m_j) is corrected by a tape of length l.

b. $\gamma^S(m_i, m_j) = \lim_{l\rightarrow\infty} \gamma_l^S(m_i, m_j)$ if the limit exists.

It is easy to see that for any source S and any error (m_i, m_j), $\gamma^S(m_i, m_j)$ exists.

LEMMA 2.1.

$$\lim_{l\rightarrow\infty} \delta_l^S(m_i, m_j) \textit{ always exists.}$$

Proof:

$$1 \geqq \gamma_{l+1}^S(m_i, m_j) \geqq \gamma_l^S(m_i, m_j).$$

Since the limit of a monotonic bound sequence always exists, the theorem is proved.

Now let us consider the following classification of errors in a finite automaton M being driven by a source S as above.

DEFINITION 2.4. An error $E = (m_i, m_j)$ is

a. *definite* if and only if there is an l such that $\gamma_l^S(E) = 1$.

b. *finite* if and only if $\gamma^S(E) = 1$.

c. *correctable* if and only if $\gamma^S(E) > 0$.

d. *non-correctable* if and only if $\gamma^S(E) = 0$.

III. FUNDAMENTAL RESULTS

In this section we will derive some fundamental properties of errors and will show the connection between the concepts of correctable and finite errors.

THEOREM 3.1. *If* $\gamma(m_1, m_2) = g_1$ *and* $\gamma(m_2, m_3) = g_2$, *then* $1 - | g_1 - g_2 | \geqq \gamma(m_1, m_3) \geqq (g_1 + g_2) - 1$.

Proof: Let $T0$ be the set of tapes that do not correct (m_1, m_2) or (m_2, m_3); $T1$ be the set that corrects (m_1, m_2) or (m_2, m_3) but not both; $T2$ be the set that corrects (m_1, m_2) and (m_2, m_3); and $T3$ be the set that corrects (m_1, m_3). We know that $T0$, $T1$, and $T2$ are mutually disjoint and that $T2 \subset T3 \subset T2 \cup T0$. We will use $\Pr_l(T)$ to mean the probability that a tape t of length l is in T. Therefore, we have

$$g_1 + g_2 = \lim_{l \to \infty} (\Pr_l(T1) + 2\Pr_l(T2))$$

$$= \lim_{l \to \infty} \Pr_l(T1) + 2 \lim_{l \to \infty} \Pr_l(T2).$$

But we also have for all l, $\Pr_l(T1) + \Pr_l(T2) \leqq 1$. Therefore

$$g_1 + g_2 \leqq 1 + \lim_{l \to \infty} \Pr_l(T2) \leqq 1 + g_3$$

where $g_3 = \gamma(m_1, m_3)$. Hence $g_3 \geqq g_1 + g_2 - 1$. Likewise, letting $T11$ be the set of tapes which corrects (m_1, m_2) and not (m_2, m_3), and $T12$ be the set which corrects (m_2, m_3) and not (m_1, m_2), we have

$$\gamma_l(m_1, m_2) = \Pr_l(T_2) + \Pr_l(T11)$$

and

$$\gamma_l(m_2, m_3) = \Pr_l(T2) + \Pr_l(T12).$$

Thus

$$| \gamma_l(m_1, m_2) - \gamma_l(m_2, m_3) | = | \Pr_l(T11) - \Pr_l(T12) |.$$

But

$$|\Pr_l(T11) - \Pr_l(T12)| \leqq \Pr_l(T1) \leqq 1 - (\Pr_l(T0) + \Pr_l(T2))$$

Now taking limits as l goes to infinity we get

$$|g_1 - g_2| \leqq 1 - g_3$$

$$g_3 \leqq 1 - |g_1 - g_2|.$$

COROLLARY 3.1. *The set of finite errors in an automaton M driven by a source with property P induces a partition on the set of states. That is, there is a partition π_F on the states of M so that $E = (m_i, m_j)$ is finite if and only if $m_i \equiv m_j(\pi_F)$.*

Proof: It is obvious that if $\gamma(m_i, m_j) = 1$, then $\gamma(m_j, m_i) = 1$ by the symmetry of the definition of being corrected. Likewise $\gamma(m_i, m_i) = 1$. Now by Theorem 3.1 we have that if $\gamma(m_i, m_j) = 1$ and $\gamma(m_j, m_k) = 1$, then $\gamma(m_i, m_k) = 1$. Hence, the finiteness relation is an equivalence relation and partitions the set of states.

THEOREM 3.2. *Let $C \subset M \times M$ be the relation $(m_i, m_j) \in C$ if and only if (m_i, m_j) is a correctable error. Then an error $E = (m_i, m_j)$ is finite if and only if $(m_i, m_j) \in C$ and for all tapes t, $(\delta(m_i, t), \delta(m_j, t)) \in C$.*

Proof: If (m_i, m_j) is finite then obviously (m_i, m_j) is correctable. If there is a tape t such that $(\delta(m_i, t), \delta(m_j, t))$ is not correctable, then for all $t'(\delta(m_i, tt'), \delta(m_j, tt'))$ is not correctable. Hence, $\gamma(m_i, m_j) \leqq 1 - (k)^{lg(t)} < 1$ where $lg(t)$ is the length of the tape t, and k is the constant greater than zero associated with the source. Therefore (m_i, m_j) is not a finite error. Conversely, let us assume that for all t $(\delta(m_i, t), \delta(m_j, t)) \in C$. Let $A = \{(m_k, m_l) \mid \text{for some } t\, \delta(m_i, t) = m_k$ and $\delta(m_k, t) = m_l\}$. Then, for each $(m_k, m_l) \in A$, pick a t' which corrects (m_k, m_l). Let $p = k^r$ where $r = \max lg(t')$. Then $\gamma_l(m_i, m_j) \geqq 1 - (1 - p)^{[l/r]}$ where $[l/r]$ is the greatest integer less than l/r. Hence

$$\lim_{l \to \infty} \gamma_l(m_i, m_j) \geqq 1 - \lim_{l \to \infty} (1 - p)^{[l/r]}.$$

Since $p > 0$ we have $\gamma(m_i, m_j) = 1$.

From this theorem we can get some idea of the connection of C and Π_F. We can also see that since the concept of an error being corrected is not dependent upon a source, the property of it being finite is also independent of the source. (This is true only as long as we are only dealing with a source with property P.) The next theorem is a stronger characterization of π_F with respect to the relation C.

THEOREM 3.3. *π_F is the largest partition with the substitution property such that $m_i \equiv m_j(\pi_F) \Rightarrow (m_i, m_j) \in C$.*

Proof: Let π be a partition with the substitution property such that $m_i \equiv m_j(\pi) \Rightarrow (m_i, m_j) \in C$. Then, if $m_i \equiv m_j(\pi)$, $(m_i, m_j) \in C$. Also, since π has the substitution property, for all tapes t $\delta(m_i, t) \equiv \delta(m_j, t)(\pi)$ and hence $(\delta(m_i, t), \delta(m_j, t)) \in C$. But by Theorem 3.2, this means that $m_i \equiv m_j(\pi_F)$. Therefore $\pi \leqq \pi_F$.

An immediate consequence of this theorem is a decomposition of the automaton as follows.

COROLLARY 3.2. *If M is a finite automaton with a finite error partition π_F, then M can be state behavior realized by a cascade connection of two automata M/π_F and T, where all errors in T are finite, and M/π_F has no finite errors.*

Proof: By Theorem 3.3, π_F is a partition with the substitution property. Hence we know (Hartmanis, 1962) that we can decompose M into a cascade connection of two automata where the state of the front automaton distinguishes between blocks of the partition and the back machine distinguishes the elements of a single block.

Let us now look at an example in order to demonstrate these theorems. Let $M = (\{a, b, c, d, e\}, \{0, 1\}, \delta)$ where δ is the mapping shown in Table I. It is easy to show that

$$C = \{(a, d), (d, a), (b, c), (c, b), (e, a), (a, e), (e, d), (d, e), (b, e),$$
$$(e, b), (c, e), (e, c), (a, a), (b, b), (c, c), (d, d), (e, e)\}.$$

There are four equivalence relations with the substitution property contained in C.

$$\pi_1 = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}\}$$
$$\pi_2 = \{\overline{a, d}, \bar{b}, \bar{c}, \bar{e}\}$$
$$\pi_3 = \{\bar{a}, \overline{b, c}, \bar{d}, \bar{e}\}$$
$$\pi_4 = \{\overline{a, d}, \overline{b, c}, \bar{e}\}.$$

The greatest one is π_4. Thus the only finite errors are

$$\{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b)\}.$$

Also using Theorem 3.3 we can get a simple proof of a special case of a

TABLE I
AUTOMATON A

δ	0	1
a	b	d
b	a	b
c	a	b
d	b	d
e	a	d

theorem which was proved by Winograd (1964), and also in another context by Gilbert and Moore (1959).

COROLLARY 3.3. *All errors in an automaton M are finite if and only if M has a reset tape.* (*A tape t is a reset tape if $\delta(m_i, t)$ is independent of m_i.*)

Proof: From Theorem 3.3 we find that all errors in an automaton M are finite if and only if all errors are correctable. Define a tape $t = t_1 t_2 \cdots t_{k-1}$($k$ = number of states of M) as follows:

$$t_1 \text{ corrects } (m_1, m_2)$$

$$t_{i+1} \text{ corrects } (\delta(m_1, t_1 \cdots t_i), \delta(m_{i+2}, t_1 \cdots t_i)).$$

If it is possible to construct such a t, then t is a reset sequence. It is not possible to construct such a tape if and only if for some i, $(\delta(m_1, t_1 \cdots t_i), \delta(m_{i+2}, t_1 \cdots t_i))$ is not a correctable error. But then, this $(\delta(m_1, t_1 \cdots t_i), \delta(m_{i+2}, t_1 \cdots t_i))$ is not finite. Hence we can construct t if and only if all the errors are finite.

Let us now look at another example to show the use of this theorem. Let $M = (\{a, b, c, d\}, \{0, 1\}, \delta)$ where δ is shown in Table II. It is easy to see that all the errors are correctable. Hence $\pi_F = \{\overline{a, b, c, d}\}$ and all errors are finite. Upon examination it can be seen that the tape 000 is a reset tape since $\delta(m_i, 000) = d$ regardless of m_i.

IV. ERRORS IN EXPANDED AUTOMATA

This section will discuss the possibility of adding states to a finite automaton so that the new automaton has, in some sense or another, better error properties and still is a realization of the original automaton. It will be shown that for one sense of "nicer" this is not possible and that the reduced automaton has the best error properties. We will use $E(M)$ for $M \times M$, the set of ordered pairs of states of M, and $E(A) \leqq E(B)$ for the concept, which has not been made precise yet, of the error properties of an automaton A being better than those of an automaton B.

TABLE II

AUTOMATON B

δ	0	1
a	b	c
b	c	d
c	d	b
d	d	b

There are three properties which we intuitively require for a notion of $\leqq$:

1. It must be source independent.
2. The comparison of errors in A and in B must be a total comparison. That is, every error in A must be compared to at least one error in B and vice-versa.
3. For any source, an error in A must have at least as high a probability of correction as that of those in B to which it is compared.

We are thus led to the following definition which satisfies these three properties.

DEFINITION 4.1. Let $A = (M_1, \Sigma, \delta_1)$ and $B = (M_2, \Sigma, \delta_2)$ be two finite automata with the same input alphabet. We will say errors in A are less than errors in B, $E(A) \leqq E(B)$, if and only if there is a mapping of pairs of states of B onto pairs of states of A, $h: M_2 \times M_2 \rightarrow M_1 \times M_1$ onto, with the property that if $(m_i, m_j) \in M_2 \times M_2$ is corrected by a tape t, then t also corrects $h(m_i, m_j) \in M_1 \times M_1$.

We can see that the definition fits our intuitive notion of the set of errors in one automaton being better than the set of errors in another since for any source S if $\gamma(m_i, m_j) = c$, then $\gamma(h(m_i, m_j)) \geqq c$. Thus, h takes finite errors into finite errors and correctable errors into correctable errors. Also, h^{-1} assigns to each error in A at least one error in B with the same or a lower probability of being corrected for any source.

We should note at this point that the relation $\leqq$ is not an ordering on the set of finite automata. As an example of two automata which are not isomorphic and yet for which both $E(A) \leqq E(B)$ and $E(B) \leqq E(A)$, let A be the two input, mod 4 clock and let B be the two input, mod 4 counter of ones. The two are not isomorphic and yet since all the errors in both are not correctable and they both have the same number of errors, $E(A) \leqq E(B)$ and $E(B) \leqq E(A)$.

We will now show some of the properties of this relation.

LEMMA 4.1. *If M_1 is a submachine of M_2, then $E(M_1) \leqq E(M_2)$.*

Proof: Let h be the identity mapping on $M_1 \times M_1$ and let it map all other pairs in $M_2 \times M_2$ into (m_1, m_1), $m_1 \in M_1$. Now if $(m_i, m_j) \in M_1 \times M_1$, then the set of tapes which correct it in M_1 is the same as the set of tapes which correct it in M_2. However, if (m_i, m_j) is not in $M_1 \times M_1$, then $h(m_i, m_j) = (m_1, m_1)$ and is thus corrected by all tapes. Therefore h has the desired properties.

LEMMA 4.2. *If a finite automaton M_1 is a homomorphic image of a finite automaton M_2, then $E(M_1) \leqq E(M_2)$.*

Proof: Let g be a homomorphism of M_2 onto M_1 and then let $h(m_i, m_j) = (g(m_i), g(m_j))$. Also, let t be a tape that corrects (m_i, m_j). We have $\delta_1(g(m_i), t) = g(\delta_2(m_i, t))$ since g is a homomorphism of M_2 onto M_1. Also, $g(\delta_2(m_i, t)) = g(\delta_2(m_j, t))$ since t corrects (m_i, m_j). Thus we have $\delta_1(g(m_i), t) = g(\delta_2(m_i, t)) = g(\delta_2(m_j, t)) = \delta_1(g(m_j), t))$. Therefore t corrects $(g(m_i), g(m_j)) = h(m_i, m_j)$. Thus h has the required property.

THEOREM 4.1. *If A is a reduced finite automaton and B is any other automaton which realizes A, then $E(B) \geqq E(A)$.*

Proof: If B realizes A, then B is a homomorphic image of a submachine of A (Hartmanis and Stearns, 1964). By using Lemmas 4.1 and 4.2 the theorem is proven.

COROLLARY 4.1. *If R is a regular set of tapes, a finite automata with minimum errors that recognizes R is the minimum automata which recognizes R.*

Thus if we are interested in obtaining an automaton which realizes a given automaton (or recognizes a given regular set) and which has minimum errors under our definition of the ordering, we should use the reduced automaton (or the minimum one associated with the set of tapes) since any state splitting, or adding states, makes a new automaton whose errors are no less than those of the original one.

The results of this section can be easily misinterpreted. It appears to claim that techniques such as triplicating and multiplexing are not effective since they increase the number of states. Hence, the errors in the multiplexed automaton are greater than those in the original one. However, the benefit of multiplexing and triplicating lies in that they reduce the probability of a malfunction causing an error between states which are not *behaviorally* equivalent. Since we consider automata without outputs the concept of behaviorally equivalent states does not arise.

If we want to use our theory to handle such cases, we must consider the automaton modulo the relation of behavior equivalence. However, even after doing this the multiplexed or triplicated automaton has errors which are not less than the original. Thus the advantages of these methods, like those of increasing the reliability of components in a physical realization of the automaton, do not show up in the theory. On the other hand, a possible disadvantage does.

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