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LOCAL TRIVIALITY FOR HUREWICZ FIBERINGS OF MANIFOLDS†

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§1. INTRODUCTION

LET $p: E \rightarrow B$ be a fibering in the sense of Hurewicz, i.e. the map p has the path lifting property or what is the same the covering homotopy property holds for all spaces. As is well known, the fibering need not be locally trivial, i.e. $p: E \rightarrow B$ need not be a fiber bundle. On the other hand, one might suspect that stringent conditions on the nature of E and B might force the map to be locally trivial. The following seems plausible:

CONJECTURE. *Let $p: E \rightarrow B$ be a fibering in the sense of Hurewicz of a manifold E (with empty boundary) onto a weakly locally contractible paracompact base B. Then the fibering is locally trivial.*

The conjecture would be false if the manifold E were permitted to have non-empty boundary. (Consider the triangle Δ in the plane with vertices $(-1, 1)$, $(0, 2)$ and $(1, 1)$. Let $p: \Delta \rightarrow I$ where $I = \{t \; -1 \leq t \leq 1\}$ denotes the image of the vertical projection into the real axis. This projection is a fiber map in the sense of Hurewicz and obviously not locally trivial.) Also, if the base fails to be locally connected then even when E is the real line the conjecture would be false.

Our principal result is

THEOREM (2). *Let $p: E \rightarrow B$ be a fibering of a connected separable metric ANR onto a weakly locally contractible base B. Suppose that E is a (generalized) manifold (over a principal ideal domain) and some fiber has a component which is compact and of dimension ≤ 2. Then the fibering is locally trivial.*

Theorem (1) will, in this case, imply that all the fibers are locally Euclidean as well as being homeomorphic. A result of Dyer and Hamstrom is then used to conclude the local triviality of p. In addition to providing the key for the proof of Theorem (2), Theorem (1) also lends some credence to the conjecture stated above. In order to motivate this recall [11; Theorem (6)] which states, in particular, that if $p: E \rightarrow B$ is a *locally trivial* fibering of a (generalized) manifold E onto a Hausdorff space B then the fiber is a generalized k-manifold

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(k-gm) and the base is a $(n-k)$-gm. In other words, although the factors of a locally Euclidean space may even fail to be locally Euclidean they must, nevertheless, be indistinguishable from a manifold on the homology level both locally and globally. (n-gms are the class of spaces for which Poincaré duality holds both locally and globally.) Theorem (1) is the analogue of [11; Theorem (6)] for fiberings in the sense of Hurewicz, (cf. also [10; 3.19]).

THEOREM (1). *Let $p: E \to B$ be a fibering of a connected separable metric ANR onto a weakly locally contractible paracompact base. Suppose that E is an n-gm over L (a field or the integers). Then*

(a) *each fiber F_b is a k-gm over L*

(b) *if some component of some fiber is compact, then B is a $(n-k)$-gm over L.*

Actually we obtain more general results in terms of singular homology manifolds than we have stated here and the reader is referred to the remarks at the end of sections 4 and 5. By a generalized manifold we mean what Wilder calls a *locally orientable* generalized manifold or cohomology manifold. For the pertinent facts and definitions as we shall use them the reader is referred to [11]. Section 2 develops the tools and facts concerning fiber spaces while section 3 develops the necessary material on homology manifolds. The last section is devoted to an application.

I wish to express my thanks to Edward Fadell and M. W. Hirsch for many stimulating discussions concerning this material and again to Edward Fadell for reading a first draft and making several suggestions which simplified the preparation of this paper.

§2. PRELIMINARIES ON FIBER SPACES

Unless specified otherwise fiber spaces will always mean fiber spaces in the sense of Hurewicz [6] as defined by Hurewicz and Curtis. We shall summarize now the results needed from [3].

Let the map $p: E \to B$ denote a fiber space. We shall always assume that p is onto (equivalent to B being 0-connected if E is 0-connected). Then each fiber has the same homotopy type and if E is 0-connected any two arc components of a given fiber are of the same homotopy type.

Recall that a space B is *weakly locally contractible*, wlc, if each point $b \in B$ lies in an open set U which is contractible to b in B. Now suppose that E is (separable) metric and B is wlc and paracompact. Then the fibering is regular (i.e. there exists a regular lifting function λ which lifts the constant path in B into a constant path in E), B is (separable) metric and the map p is an open map. Moreover, if in addition, E is a ANR and B is 0-connected then B as well as all the fibers are separable metric ANR's.

PROPOSITION (2.1) (Fadell [3]). *If (E, B, p) is a fiber space with B wlc, then for each $b \in B$ there is an open set U containing b such that $(p^{-1}(U), U, p)$ is fiber homotopy equivalent to $U \times F_b$ where $F_b = p^{-1}(b)$.*

If during the contraction of u to b in B the point b stays fixed and a regular lifting function is used then (following the notation of [3]) the fiber homotopy equivalence $\phi : p^{-1}(U) \rightleftarrows U \times F_b : \psi$ is the 'identity' on $F_b = p^{-1}(b) = b \times F_b$. In particular, *if A is any subset of F_b, then*

(2.2) $$\phi : (p^{-1}(U), p^{-1}(U) - A) \rightleftarrows (U \times F_b, U \times F_b - b \times A) : \psi$$

is a homotopy equivalence.

If B is a ANR it can be imbedded in a locally convex linear space, L. Select $b \in B$ and take U a neighborhood of b in L. Choose a small convex neighborhood V within U such that V can be deformed to B while staying within U. The line segments of V define a uniform contraction of $V \cap B$ to any point of $V \cap B$. Each contraction of $V \cap B$ stays within $U \cap B$. Namely, there exists

$$D : V \times V \to U^I$$

defined by

$$D(v_1, v_2)(t) = r(tv_1 + (1 - t)v_2)$$

where r denotes the retraction of V into B (staying within U) and $tv_1 + (1 - t)v_2$ denotes the unique line segment joining v_1 to v_2.

Let $p : E \to B$ be a fibering such that B is a ANR; $b \in B$ and V a neighborhood of the form $V \cup B$ of above. Define a *slicing function*

$$\theta_V : V \times p^{-1}(V) \to p^{-1}(V) \quad \text{by}$$

$$\theta_V(v, e) = \lambda(e, D(v, p(e)))(1).$$

For each $b' \in V$, define

$$\theta_{V,b'} : p^{-1}(V) \rightleftarrows V \times p^{-1}(b') : \Psi_{V,b'}$$

by

$$\theta_{V,b'}(e) = (p(e), \theta_V(b'e))$$

and

$$\Psi_{V,b'}(v, y) = \theta_V(v, y), \; y \in F_{b'}.$$

Note that $\theta_{V,b} : p^{-1}(V) \rightleftarrows V \times p^{-1}(b) : \psi_{V,b}$ is identical with (2.2) of Fadell.

Thus we have obtained

PROPOSITION (2.3). (Hurewicz [6].) *If B is a ANR then the fibering $p : E \to B$ is a fibering in the sense of* Hu *with slicing functions* $\{\theta_V\}$.

DEFINITION (2.4). *An open mapping $p : E \to B$ of a metric space E onto another space B is called* homotopically n-regular (Hamstrom and Dyer [4]) *if given e_0 of E and $S(e_0, \epsilon)$, an ϵ-sphere about e_0, there exists a $\delta > 0$ such that each map $f : S^k \to S(e_0, \delta) \cap p^{-1}(b)$ is homotopic to* 0 *in $S(e_0, \epsilon) \cap p^{-1}(b)$, for all $k \leq n$, $b \in B$, where S^k denotes the k-sphere.* (Note that we do not require the map p to be proper as is required in [4].) We shall say that the *map p is locally contractible* if $S(e_0, \delta) \cap p^{-1}(b)$ is contractible in $S(e_0, \epsilon) \cap p^{-1}(b)$ to a point in $S(e_0, \epsilon) \cap p^{-1}(b)$ and the *map p is uniformly locally contractible* if $S(e_0, \delta) \cap p^{-1}(b)$

is contractible in $S(e_0, \epsilon) \cap p^{-1}(b)$ to each point in $S(e_0, \delta) \cap p^{-1}(b)$ with the given point left fixed during the contraction.

PROPOSITION (2.5). *Let $p: E \to B$ be a fibering in the sense of* Hu *with E metric.*

(a) *E is locally k-connected in the homotopy sense, $k \leq n$ or,*

(b) *E is locally contractible or,*

(c) *E is uniformly locally contract ible (e.g. if E is an ANR) then*

(a) *p is homotopically n-regular or,*

(b) *p is locally contractible or,*

(c) *p is uniformly locally contractible, respectively.*

Proof. Let $\theta_V: V \times p^{-1}(V) \to p^{-1}(V)$ be a given slicing function. Choose $e_0 \in p^{-1}(V)$. Then $\theta_V(p(e_0), e_0) = e_0$. Given $\epsilon > 0$, there exists by continuity a neighborhood $W \times U$ of $(p(e_0), e_0)$ in $V \times p^{-1}(V)$ such that $\theta_V(W \times U) \subset S(e_0, \epsilon)$. Choose $\delta_1 > 0$ such that $S(e_0, \delta_1) \subset U$, $p(S(e_0, \delta_1)) \subset W$. By hypothesis there exists $\delta > 0$, such that every map f of a k-sphere S^k, $k \leq n$ into $S(e_0, \delta)$ can be contracted to a point within $S(e_0, \delta_1)$. In particular, let $F: D^{k+1} \to S(e_0, \delta_1)$ be an extension of $f: S^k \to p^{-1}(b) \cap S(e_0, \delta)$. The composition $\theta_{V,b} \circ F$ sends D^{k+1} into $S(e_0, \epsilon) \cap p^{-1}(b)$ and extends f. This completes (a). Parts (b) and (c) are proved very similarly.

PROPOSITION (2.6) (Fadell). *Let $p: E \to B$ be a regular fibering with B locally o-connected (o-lc), then p is an open map.*

PROPOSITION (2.7) *Let $p: E \to B$ be a fibering with E, B metric E locally compact, and $Bo - lc$. Then the set of points U of B having compact fibers is open and the fiber map $p: p^{-1}(U) \to U$ is a proper map.*

Proof. Let $F_b = p^{-1}(b)$ be compact. Let U_1 and U_2 be two open sets with compact closures such that $F_b \subset U_1 \subset \overline{U}_1 \subset U_2$. Then there exists a neighborhood V of b such that $p^{-1}(V) \cap (U_2 - U_1)$ is empty. If not, there exists a sequence of distinct points $\{b_i\}$ with $b_i \to b$, and points $\{y_i\}$ such that $p(y_i) = b_i$ and $y_i \in U_2 - U_1$. We may find a convergent subsequence $\{y_{i_j}\}$, $y_{i_j} \to y \in \overline{U}_2 - U_1$. But, $p(y_{i_j}) \to p(y)$ which leads to a contradiction. Choose an arcwise connected neighborhood W of b within V. If $b' \in W$, $y \in p^{-1}(b')$, then a path within W from b' to b can be lifted to E starting at y. The path can never enter $\overline{U}_2 - U_1$ thus y must lie in U_1.

We shall not need the next proposition for the proofs of Theorems (1) and (2). It is included here because it eliminates the necessity for appealing to a regular convergence theorem for an important case of Theorem (1) and because the light it sheds upon fiberings with non-empty boundary about which we expect to say something at a later date.

PROPOSITION (2.8). *Let $p: E \to B$ be a fibering in the sense of* Hu *with E locally compact separable metric. Then the set of fibers of dimension $\leq k$ form a closed subset of E.*

Proof. Let B_k be the set of points of B for which $f^{-1}(b)$ has dimension $\leq k$. Let b be a limit point of B_k. Arbitrarily near F_b, there exists fibers of dimension $\leq k$. Let C be a compact subset of F_b. We shall show that dimension $C \leq k$. Let α be any finite covering of C. The covering α has a Lebesque number ϵ. Thus any subset of C of diameter $< \epsilon$ lies in an element of α. Let V be a neighborhood of b and θ_V a given slicing function. By continuity, since $\theta_V(b, e) = e$ for all $e \in F_b$, and compactness of C choose a neighborhood W of b so that $d(\theta_V(b', e), e) < \epsilon/2$, $b' \in W$, $e \in C$ where d denotes the metric in E. For each $b' \in B_k \cap W$, there exists an $\epsilon/2$ map $m_{b'}$ of $\theta_V(b', C)$ into a polyhedron $P_{b'}$ of dimension $\leq k$, by Alexandroff's theorem [7, p. 71]. The diameter $m_{b'}^{-1}(t) < \epsilon/2$, $t \in P_{b'}$. Hence $\theta_V^{-1}(m_{b'}^{-1}(t)) \cap C$ has diameter $< \epsilon$. Therefore the set lies within some element of α. Again using Alexandroff's theorem this implies that dimension $C \leq k$. Now dimension of F_b is the maximum dimension of all compact subsets C of F_b. Hence, we have shown that $\dim F_b \leq \dim F_{b'}$ for all b' in some neighborhood of b. In particular, B_k is a closed subset of B.

Remark (2.9). The theorem is also true for fiberings in the sense of Hurewicz with E locally compact separable metric, B locally 0-connected and metric.

The following proposition will enable us to reduce many arguments to the case of a 0-connected fiber.

PROPOSITION (2.10). *Given a regular fibering $p: E \to B$ where E is o-connected and o-lc and where B is locally o-connected and semi 1-connected (small loops may be shrunk in all of B) then there exists a o-connected covering space B' and maps $p': E \to B'$, $q: B' \to B$ such that is commutative. Moreover p' is a regular fibering such that $p'^{-1}(b')$ is an arc component of $p^{-1}(q(b'))$.*

$$\begin{tikzcd} E \arrow[rr, "p'"] \arrow[dr, "p"'] & & B' \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Proof. We shall actually prove a more general theorem than that which is stated above. Choose base points $b_0 \in B$ and $e_0 \in E$ such that $p(e_0) = b_0$. Let G denote $(p_{\#}\pi_1(E, e_0)) \subset \pi_1(B, b_0)$. Construct a covering space (B', b_0) such that the natural projection $q_{\#}: \pi_1(B', b_0') \to \pi_1(B, b_0)$ is precisely G. Then by covering space theory there exists a unique lifting $p': E \to B'$ such that $p'(e_0) = b_0$ and $qp' = p$. We shall now show that $p': E \to B'$ is a regular fibering. Let λ be a regular lifting function for the fiber map p. Let $\alpha \in B'^I$, $e \in E$, such that $\alpha(0) = p'(e)$. Define $\tilde{\lambda}(\alpha) = \tilde{\alpha} \in E^I$ by

$$\tilde{\lambda}(\alpha) = \lambda(q(\alpha), e).$$

The map q induces a continuous map from $B'^I \times E$ into $B^I \times E$ and $\tilde{\lambda}$ is induced by the composition of λ with that induced by q. Clearly, $\tilde{\lambda}(\alpha)(0) = e$. Since $p(\tilde{\alpha}) = q(\alpha) = qp'(\tilde{\alpha})$ the uniqueness of lifting a path from B to B' implies that $p'(\tilde{\alpha}) = \alpha$. Thus $p': E \to B'$ is a regular fibering such that $qp' = p$.

Let $b' \in B$. Then $p'^{-1}(b') \subset p^{-1}(q(b'))$. Let C be an arc component of $p^{-1}(q(b'))$ that meets $p'^{-1}(b')$. Since $p(C) = b$, it follows that $p'(C) = b'$.

So far we have only used the fact that $G \subset q_{\#}(\pi_1(B', b_0'))$ to construct the unique map d'. Continuing in this more general situation consider the commutative diagram

$$\begin{tikzcd}
\pi_1(E, e_0) \arrow[r, "p'_*"', "p'_*"] \arrow[d, "1_*"] & \pi_1(B', b_0') \arrow[r, "\partial_*"] \arrow[d, "q_*"] & \pi_0(F_{b_0}', e_0) \arrow[d, "1'_*"] \\
\pi_1(E, e_0) \arrow[r, "p'_*"] & \pi_1(B, b_0) \arrow[r, "\partial_*"] & \pi_0(F_{b_0}, e_0).
\end{tikzcd}$$

As $q_{\#}$ is an isomorphism into, $p'_{\#}(\pi_1(E, e_0)) = G$. Thus kernel $\partial_{\#} = G$, which implies that $\pi_0(F_b, e_0)$ has cardinality equal to the index of G in $\pi_1(B', b_0)$ and $\pi_0(F, e_0)$ has cardinality equal to the index of G in $\pi_1(B, b_0)$. Thus $p'^{-1}(b_0)$ consists of a certain number of components of F_{b_0}. The most significant case is where $G = \pi_1(B', b_0')$ and therefore $p^{-1}(b_0')$ is precisely one component of F_{b_0} and each point of $q^{-1}(b_0)$ corresponds to a component of F_{b_0}.

Remarks. It is a simple matter to determine which components of F_{b_0} map onto a given point b_0' under p'. This is determined of course by the action of the fundamental group on B.

It is easily seen that if in addition to the hypothesis of (2.10) it is assumed that each fiber is totally disconnected, then $p: E \to B$ is a covering map in the classical sense.

In the special case where it is desired that $p'^{-1}(b_0')$ be an arc component of $p^{-1}(b_0)$ the description can also be given directly although admittedly in a much more complicated manner. Namely for the fibering $p: E \to B$ one collapses each arc component of $p^{-1}(b)$ to a single point and takes the resulting decomposition space for E'. In fact, it was this method that we originally had in mind but with more restrictions on the fiber and the base. To see that the construction that we used for the proof of the theorem agrees with that just described it suffices to check that B' has the decomposition topology. But, this is clear as p' is open and continuous.

If instead of having been given a regular fibering in the sense of Hurewicz we were given a fibering in the sense of Hu with a given slicing structure then the fibering p', constructed as above, has a naturally induced slicing structure. Similarly, if p were actually locally trivial the resulting fibering map p' would also be locally trivial.

§3. PRELIMINARIES ON HOMOLOGY MANIFOLDS

Let G be an L-module where L is a principal ideal domain. Singular homology of a topological space E and Čech cohomology (or equivalently Alexander–Spanier cohomology) with compact supports of a locally compact space E will be denoted by $H_r^s(E; G)$ and $H_c^r(E; G)$ respectively. The coefficients will be omitted if no confusion as to what is meant can arise.

A space E is said to be lc_n^s *over* G (*locally connected in the singular sense with respect to* G) if given $e \in E$, k an integer $\leq n$, U a neighborhood of e there exists a neighborhood V of e such that $i_*: H_k^s(V; G) \to H_k^s(U; G)$ is trivial. A locally compact space E is said to have *cohomology dimension with respect to* $G \leq n$, $\dim_G E \leq n$, if $H_c^{n+1}(U; G) = 0$, for all open subset U of E. It is well known that $\dim_G E \leq \dim_L E \leq \dim_Z E \leq$ covering dimension E. In fact, if covering dimension $E < \infty$ then $\dim_Z E =$ covering dimension E.

PROPOSITION (3.1). *Let E be a locally compact space which is lc_n^s over L; let V, U be open sets such that $\bar{V} \subset U$, $\bar{U}$ is compact. Then the images of the homomorphisms*

$$i_* : H_p^s(V; L) \to H_p^s(U; L)$$

and

$$j_* : H_p^s(E, E - U; L) \to H_p^s(E, E - V; L)$$

are finitely generated for all $p \leq n$.

Proof. The following direct argument has been suggested by C. N. Lee. One first proves the absolute case by paralleling the argument for Čech theory and making use of the fact that the Mayer–Vietoris sequence for singular homology is exact when open subsets are used. The relative case then follows from the absolute case by diagram chasing.

A general theory of singular homology manifolds is developed in [10]. We recall the pertinent facts and definitions.

LEMMA (3.2) [10; 3.9]. *Let U be an open set of a Hausdorff space E, Γ a non-trivial submodule of $H_n^s(E, E - \bar{U}; G)$ such that the natural inclusion $j : (E, E - \bar{U}) \subset (E, E - y)$ induces an isomorphism of Γ onto $H_n^s(E, E - y; G)$, for all $y \in U$. Then U is locally a Peano space* (i.e. *U is locally compact, locally separable metric, and locally arc-wise connected subspace*) *with compact closure.*

DEFINITION (3.3) [10]. A *singular homology n-manifold over G*, hereafter referred to by n-s-hm, is a Hausdorff space E such that

$$(1)\ \ H_r(E, E - y; G) = \begin{cases} G, & r = n \\ 0, & r \neq n \end{cases} \quad \text{for all } y \in E;$$

(2) There exists a covering of E by open sets $\{U_\alpha\}$ such that $j_* : H_n(E, E - \bar{U}_\alpha; G) \to H_n(E, E - y; G)$ is an isomorphism onto, for all $y \in U_\alpha$;

(3) $\dim_L E < \infty$.

Condition (2) and Lemma (3.2) show that condition (3) makes sense. Furthermore an n-s-hm must be a reasonable space. In particular, if a component of E is paracompact then the component is a Peano space. An n-s-hm is what I called in [10] a locally orientable singular homology n-manifold over G. Condition (2) can be weakened to having only some submodule $\Gamma \subset H_n(E, E - \bar{U}_\alpha)$ mapped bijectively. Since *Poincaré duality holds* both *locally and globally* (possibly with twisted coefficients) it is easy to see that in condition (2) each U_α must be connected and have compact closure. (Furthermore, a n-s-hm where $n \leq 2$ is locally Euclidean if an additional, perhaps redundant, condition weaker than lc_1^s is added to the definition.) We shall say that an n-s-hm E is *orientable* if every homomorphism

$$j_* : H_n(E, E - \bar{U}; G) \to H_n(E, E - y; G),\ y \in U$$

is an isomorphism where U is an open connected subset with compact closure. If E is a compact connected n-s-hm, E is orientable if and only if $H_n^s(E; G) \approx G$. Similarly, if E is paracompact and connected E is orientable if and only if $h_n^s(E; G) = G$, where h_n^s denotes the homology theory derived from locally finite singular chains.

We shall need to compare n-s-hms with n-gms.

PROPOSITION (3.4). *Let E be lc_n^s over L. Then E is an* n-gm *over L (respectively; orientable* n-gm *over L), if and only if, E is an* n-s-hm *over L (respectively; orientable* n-s-hm *over L).*

Proof. If E is an n-s-hm over L, then Poincaré duality (locally) together with lc_n^s condition trivially implies that E is an n-gm over L. The other direction is more difficult.

The equivalence when L is a field can be found in [12]. This equivalence used the equivalence of Čech homology with compact carriers and the singular homology theory in locally compact lc_n^s spaces, see [8]. To obtain the general case when L is a principal ideal domain the following facts are used.

(a) *If E is an* n-gm *(respectively;* n-s-hm) *over L, then it is an* n-gm *(respectively;* n-s-hm) *over L', where L' is a principal ideal domain and also an L-module (respectively; where L' is an L-module).* This is a direct consequence of the universal coefficient theorems and diagram chasing.

Let L_o and $\{L_q\}$ denote the field of quotients of L and the set of all fields formed by taking L modulo a prime ideal of L.

(b) *For any locally compact Hausdorff space E,* $\dim_L E \leq \max_{o,q}\{\dim_{L_0} E, \dim_{L_q} E\} + 1$. The proof again is a consequence of the universal coefficient theorems.

(c) *Let E be cohomology locally connected* (clc) *over L, (respectively; lc_n^s over L). Then E is an (orientable)* n-gm *(respectively; an (orientable)* n-s-hm) *over L if E is an (orientable)* n-gm *(respectively; (orientable)* n-s-hm) *over L_o and all L_q.* The proof, which will appear elsewhere, is obtained by careful and rather delicate use of the universal coefficient theorems. However, a sketch of what we have in mind can be found for the case of E an n-gm and $L = Z$ in KWUN and RAYMOND: Generalized cells in generalized manifolds, *Proc. Amer. Math. Soc.* **11** (1960), 135–139.

If E is an n-gm over L, then by (a) it is an n-gm over L_o and each L_q. Since L_o and L_q are fields E is an n-s-hm over L_o and L_q by [12]. Restrict E to an orientable part, if necessary, then by (c) E is an n-s-hm over L. We remark that in [9; §4] a proof can be found for separable metric E and $L = Z$; however, the method used doesn't seem to extend to the general case.

(3.5). One of the forms of *Poincaré duality* for an n-s-hm E over G which we shall repeatedly use is that

$$H_p^s(E, E - F; G) \approx H_c^{n-p}(F; G)$$

for any closed subset F contained within an orientable part of E. Furthermore, the *isomorphism is natural with respect to inclusions* [10].

Proposition (2.10) is a technical device which often permits one to replace an arbitrary fibering by one with a connected fiber. In a similar vein, if $p: E \to B$ is a fibering where E is a n-s-hm or an n-gm it is often useful to replace this fibering by $p': E' \to B$ where E' is orientable. In general one can do this by taking the universal covering E' of E and then composing the covering map with the fiber map p to get p'. However we would like to obtain a minimal such E' for application in §4.

PROPOSITION (3.6). *Let $\mathscr{S}$ denote a local system of algebraic objects (say groups or L-modules) on a 0-connected space E which admits a universal covering space. Then there exists a minimal covering space E^* for which the induced system is simple.*

Proof. Let $q: E' \to E$ be the projection of the universal covering E' of E onto E such that $q(e'_0) = e_0$, where $e'_0 \in E'$, and $e_0 \in E$. The group of covering transformations $\pi_1(E, e_0)$ acts as a group of homeomorphisms of E' and hence as a group of automorphisms on the simple system $\mathscr{S}'$. $\mathscr{S}'$ is the local system on E' induced from $\mathscr{S}$ on E. Let $h: \pi_1(E, e_0) \to \mathrm{Aut}(\mathscr{S}'_{e'_0})$ be the induced representation and let $K \subset \pi_1(E, e_0)$ be the kernel of h. Clearly K is the set of elements of $\pi_1(E, e_0)$ which induce the identity automorphism on $\mathscr{S}'$. The *regular* covering space $E^* = E'/K$ has K as fundamental group, i.e. $\pi_1(E^*, e_0^*) = K$ and $\pi_1(E, e_0)/K$ as its group of covering transformations. Clearly, E^* is minimal with respect to the induced system $\mathscr{S}^*$ being simple. Furthermore, note that $\pi_1(E, e_0)/K \cong h_*(\pi_1(E, e_0)) \subset \mathrm{Aut}(\mathscr{S}_{e'_0})$.

As an application, let E be a 0-connected, locally arcwise connected and semi 1-connected n-gm or n-s-hm over any L. Then the *orientation sheaf* $\mathscr{S}$, i.e. the sheaf of local homology groups in dimension n, is locally constant and under the conditions given forms a local system of L-modules where each module is isomorphic to L. Let E^* be the minimal covering space constructed above for the local system $\mathscr{S}$. Then E^* is orientable and is called the *minimal orientable covering of E*. $E^* = E$ if and only if E is already orientable. If L is the integers Z, then as $\mathrm{Aut}Z = Z_2$, E^* is the orientable double covering of E if E is not orientable. If L is a field of characteristic p, $p \neq 0$, then $q^*: E^* \to E$ has at most $(p-1)$-sheets. This follows immediately from above and the fact:

*A space E is a n-*gm *or a n-s-*hm *over L (respectively; orientable) where L is a field of characteristic p, if and only if E is an n-*gm *or a n-s-*hm *over Z_p (respectively; orientable) where Z_p denotes the field of integers modulo p, $p \neq 0$, and Z_0 denotes the rational numbers.*

§4. PROOF OF THEOREM (1)

Let E be a 0-connected (separable metric) ANR and also an n-s-hm over a principal ideal domain L; let $p: E \to B$ be a fibering of E onto B where B is paracompact and wlc. We have seen in §2 that the fibers F_b and base B are all (separable metric) ANR's, that there exists a regular lifting function and in fact give rise to slicing functions $\{\theta_V\}$ which make the map p into a fibering in the sense of Hu. Choose $b_0 \in B$, U a neighborhood of b_0 and V a neighborhood of b_0 such that V contracts to any point $b \in V$, b remaining fixed, and the contraction staying within U. Let $\theta_V: V \times p^{-1}(V) \to p^{-1}(V)$ be the corresponding slicing function.

Recall that the Cartesian product of the pairs $(X, A) \times (Y, B)$ is defined to be the pair $(X \times Y, A \times Y \cup X \times B)$. The relative Künneth theorem for singular homology states that the sequence

$$(4.1) \quad 0 \to \textstyle\sum_{i+j=r} H_i^s(X, A) \otimes H_j^s(Y, B) \to H_r^s(X \times Y, A \times Y \cup X \times B) \to \sum_{i+j=r-1} H_i^s(X, A) * H_j^s(Y, B) \to 0$$

is exact (and *splits*) provided that the triad $(A \times Y \cup X \times B;\ A \times Y,\ X \times B)$ is *excisive*. Note that if A and B are open subsets of X and Y, the triad is excisive.

Let $e \in F_{b_0} = F; A$ a closed subset of F. The triad $((V - b_0) \times F \cup V \times (F - A);\ (V - b_0) \times F,\ V \times (F - A))$ is excisive.

(a) The fiber

Let C be a compact subset of F which is the closure of an open (*in* F) connected neighborhood of e and contained within an orientable part of E. Consider the commutative diagram

$$
(4.2)\quad
\begin{array}{ccccc}
\sum_{i+j=r} H_i^s(V,V-b_0) \otimes H_j^s(F,F-C) & \rightarrow & H_r^s(V\times F, V\times F-(b_0\times C)) & \overset{\psi_{V,b_0}}{\approx} & H_r^s(p^{-1}(V), p^{-1}(V)-(b_0\times C)) \\
\downarrow & & \downarrow & & \downarrow \\
\sum_{i+j=r} H_i(V,V-b_0) \otimes H_j^s(F,F-e) & \rightarrow & H_r^s(V\times F, V\times F-(b_0\times e)) & \overset{\psi_{V,b_0}}{\approx} & H_r^s(p^{-1}(V), p^{-1}(V)-(b_0\times e))
\end{array}
$$

The vertical maps are induced by inclusions, the first horizontal maps are injections given by the Künneth sequence, the second horizontal maps are bijections given by (2.2). Consider now the second row since $p^{-1}(V)$ is a n-s-hm over L, the local groups $H_r^s(p^{-1}(V), p^{-1}(V) - (b_0 \times e); L)$ are 0 except for $r = n$, and there it is isomorphic to L. Using the splitting of the Künneth sequence it follows that the Tor term must be trivial and hence the first injection must be bijective. Thus it is easy to see that there exists an integer k such that

$$H_{n-k}(V,\ V - b_0;\ L) \otimes H_k(F,\ F - e;\ L) \approx L$$

and 0 for any other combination. In dimension n,

$$j_* : H_n^s(p^{-1}(V);\ p^{-1}(V) - (b_0 \times C);\ L) \rightarrow H_n^s(p^{-1}(V),\ p^{-1}(V) - (b_0 \times e);\ L)$$

is bijective ($\approx L$), see (3.5). Thus we have by a similar argument and the commutativity of (4.2) the existence of the bijection ($\approx L$),

$$
\begin{array}{c}
H_{n-k}(V,\ V - b_0;\ L) \otimes H_k(F,\ F - C;\ L) \\
\downarrow \\
H_{n-k}(V,\ V - b_0;\ L) \otimes H_k(F,\ F - e;\ L).
\end{array}
$$

However, $j_* \colon H_k(F,\ F - C;\ L) \rightarrow H_k(F,\ F - e; L)$ must be finitely generated by (3.1). Using facts about tensor products and that L is a principal ideal domain, it follows that $H_{n-k}(V,\ V - b_0; L) \approx L$ and therefore that $j_* \colon H_k(F,\ F - C;\ L) \rightarrow H_k(F,\ F - e; L)$ is bijective $\approx L$. By Poincaré duality (3.5), $j_* \colon H_n(p^{-1}(V),\ p^{-1}(V) - C; L) \rightarrow H_n^s(p^{-1}(V),\ p^{-1}(V) - C'$; L is bijective for any closed connected subset C' of C. From (4.2) and the facts just deduced above, $j_* \colon H_k(F,\ F - C;\ L) \rightarrow H_k(F,\ F - y;\ L)$ is bijective for all $y \in C$. Thus F is a k-s-hm over L and by (3.4) a k-gm over L. It is not too hard to see that any component C of F is orientable, if and only if C is contained within an orientable open subset of E.

We shall now show that the integer k does not depend upon the point b_0. Let us assume for the moment that E is orientable and F is connected. Let b be any point in B and $\{b_i\}$ a sequence of points converging to b. Since p is uniformly locally contractible (2.5), the sequence $\{F_{b_i}\}$ certainly *converges regularly* to F_b *over* L in the sense used by Floyd in [2; Chap. VI, section 2]. We now apply Theorem (2.3) of [2; Chap. VI] and conclude that for sufficiently large $i, \dim_L F_{b_i} = \dim_L F_b = k(b)$. Since B is connected, this implies that the integer k is independent of the points of B. If E is not orientable choose E' to be any connected covering of E which is orientable. By (2.10) one can choose a connected covering space B' of B such that the fibering $E' \to B'$ has connected fibers. Since $\dim_L F_b = \dim_L F_{b'}$, and $\dim_L F_{b'}$ does not depend upon the point $b' \in B'$, we have found that the integer k is independent of the points in B.

(b) The base

A consequence of the paragraph above is that B now satisfies all but condition (2) of the definition of a $(n-k)-s$-hm. In particular, by a variation of (3.4), it is a Wilder $(n-k)$-manifold over L, i.e. B is a $(n-k)$-gm without assuming anything about local orientability. Hence if $(n-k) \leq 2$, or if $n-k=3$ and B is triangulable, then B is *locally Euclidean*. Similar statements, of course, hold for each fiber if $k \leq 2$, or if $k \leq 3$ and each fiber is triangulable.

A connected k-gm or a $k-s$-hm F is compact and orientable if and only if $H^k(F; L) \approx L$ (closed supports) or if $H_k^s(F; L) \approx L$. Therefore, for the fibering $p: E \to B$, if some component of some fiber is compact and orientable then every component of each fiber must also be compact and orientable since they all belong to the same homotopy type.

The existence of some fiber with a compact orientable component implies that B satisfies condition (2) of (3.3), i.e. *B is a $(n-k)-s$-hm*. First, by (2.10), we may assume without loss of generality each fiber is connected. Since the fibers are compact and orientable each fiber is contained within an orientable neighborhood of E. Hence by (2.7) the map p is proper, and for each $b \in B$ there exists an open connected neighborhood V with compact closure such that $p^{-1}(V)$ is orientable. Furthermore choose V such that $p^{-1}(V)$ is fiber homotopically equivalent to $V \times F_b$, for any $b \in V$. Choose U an open connected neighborhood of b such that $\overline{U} \subset V$. Let W be any connected open subset of U. Consider the commutative diagram

$$\begin{array}{ccccccc}
H^s_{n-k}(V, V-\overline{U}) \otimes H^s_k(F_b) & \xrightarrow{\approx} & H^s_n(V \times F_b, (V-\overline{U}) \times F_b) & \overset{\psi_{V,b}}{\approx} & H^s_n(p^{-1}(V), p^{-1}(V-\overline{U})) & \approx & H^0_c(p^{-1}(\overline{U})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^s_{n-k}(U, U-\overline{W}) \otimes H^s_k(F_b) & \xrightarrow{\approx} & H^s_n(V \times F_b, (V-\overline{W}) \times F_b) & \overset{\psi_{V,b}}{\approx} & H^s_n(p^{-1}(V), p^{-1}(V-\overline{W})) & \approx & H^0_c p^{-1}(\overline{W}))
\end{array}$$

The first horizontal isomorphism is the Künneth isomorphism, the second is induced by the fiber homotopy equivalence and the last by Poincaré duality. The vertical homomorphisms are induced by inclusion. However, the last vertical map is bijective with the image isomorphic to L. Hence,

$$j_* H^s_{n-k}(V, V-\overline{U}) \xrightarrow{\approx} H^s_{n-k}(U, U-\overline{W}) \approx L,$$

which implies that B is a $(n-k)-s$-hm.

For the general case, if some fiber has a compact component and it is not orientable then by the technique of passing to the minimal orientable covering (see 3.6) we may conclude that B is a $(n-k)-s$-hm over L provided that L is the integers or a field of characteristic $\neq 0$. Otherwise it is necessary to appeal to a forthcoming paper of G. E. Bredon in which it is proved that every connected non-orientable n-gm has a connected orientable double covering. This concludes the proof of Theorem (1).

Remarks. Theorems 1 and 2 are stated in terms of generalized manifolds. Since E, B and F are all ANR's we may interchangeably use singular homology manifolds by virtue of (3.4).

In case L is a field, the hypothesis that E is a ANR is much stronger than is needed. All that is needed in addition to E being a n-s-hm over L, B being wlc and paracompact is that during *the contraction (in B) of a small neighborhood of a given point b is that the point b should remain fixed.* Since (2.2) still holds our method yields that each F_b is a $k-s$-hm over L. We are not able to use Floyd's theorem to show that k is independent of b. Nevertheless, we can avoid this difficulty by assuming some component of each fiber is compact. Since the components of all the fibers belong to the same homotopy type our method now yields the independence of k from b and the fact that B is a $(n-k)-s$-hm.

It would be interesting to establish that the homotopies between each fiber are proper. This fact, if true, would give a simple proof of the constancy of the integer k as well as the local orientability of B without any assumption on F. It appears to be difficult and it is false already for fiberings where E is a 2-manifold with *non-empty* boundary.

It seems very likely that B is a $(n-k)-s$-hm over L without any special hypothesis of compactness on the fibers. In fact, it is an unsolved problem whether conditions (1) and (3) together with lc_n^s over L implies condition (2) of the definition of n-s-hm (3.3). (Without the lc_n^s condition (1) and (3) does not imply condition (2).) On the other hand, if a space does satisfy conditions (1) and (3) and is triangulable it is easy to see that it must also satisfy condition (2) of (3.3). Hence if B is triangulable it must be a $(n-k)-s$-hm and in particular if $k \geq n-3$, B must be locally Euclidean.

§5. PROOF OF THEOREM (2)

Let $p: E \to B$ be a fiber map of a connected separable metric ANR E which is also an n-s-hm over a principal ideal domain L onto a wlc paracompact base B. Assume also that the covering dimension of B is finite (which is certainly true if the covering dimension of E is finite). By Theorem (1), each fiber F_b is a $k-s$-hm over L. If $k \leq 2$, then F_b is a locally Euclidean k-manifold. If in addition, some component of some fiber F_b is compact then all the fibers are *homeomorphic* since compact k-manifolds are classified by homotopy type if $k \leq 2$. In order to apply a theorem of Dyer and Hamstrom we need to know that the map p is homotopically 0-regular, open and proper. We actually verified that p is uniformly locally contractible in (2.5). Factor the map p into p' and q as in (2.10). The fiber map p' is proper (see (2.7)). Hence by [4; Corollary (2)], the fibering $p': E \to B'$ is *locally trivial.* The composition $q \circ p'$ is also locally trivial since q is a covering map and the base B is wlc.

To extend Theorem (2) to higher dimensions by *this* method would probably be very difficult. First one would have to know that all the fibers are homeomorphic. Then one would have to extend the Theorem of Dyer and Hamstrom. Hamstrom [5] has been successful for a certain class of 3-manifolds but the extension to higher dimensions is tied up with the study of the group of homeomorphisms of higher dimensional manifolds—of which little is known.

However, Hamstrom's result is strong enough for a very special situation of our conjecture. If $p: E \to B$ is as usual and $\dim_L F_b = 3$, and each F_b is triangulable then F_b is a 3-manifold. Now suppose each component of F_b is compact and simply connected. Then each component of F_b is a homotopy 3-sphere. Now if the Poincaré conjecture is true in dimension 3, each component is a 3-sphere. Assuming the truth of the Poincaré conjecture we may apply [5; Theorem (6.1)] to obtain the local triviality of the fibering $p: E \to B$.

Related to the situation for compact *triangulated 3-dimensional* fibers would be an attempt to show that each fiber has the same *simple homotopy type*.

Another special situation that seems amenable to attack occurs when E and B are triangulated and the fiber map p is simplicial. Then, if s^r denotes the interior of a r-dimensional simplex of B it is easy to see that $p^{-1}(s^r)$ is homeomorphic to $s^r \times F_b$, where b is any point in s^r. If E also happens to be a combinatorial manifold, then it seems likely that F_b for any point b not in the $(n-k-1)$-skeleton would be a combinatorial k-manifold. Obviously, one must decide whether the covering homotopy property is strong enough to imply compatibility on the boundary of each s^r.

§6. SOME APPLICATIONS

For *locally trivial* fiberings the vanishing of the homology of the total space imposes drastic restrictions on the nature of the fibering. In particular, there are no locally trivial fiberings of Euclidean space with compact fiber, [1]. The purpose of this section is to prove the analogous results for *Hurewicz fiberings*.

LEMMA (6.1). *Let $p: E \to B$ be a fibering such that $\pi_1(B)$ acts trivially upon the homology of the fiber F*
and

$$H_p^s(F; L) = 0, \quad p > k, \quad H_k^s(F; L) \neq 0,$$

finitely generated and torsion free,

$$H_q^s(B; L) = 0, \quad q > m, \quad H_m^s(B; L) \neq 0$$

then

$$H_{p+q}^s(E; L) = 0, \quad p + q > k + m, \quad \text{and} \quad H_{k+m}^s(E; L) \neq 0.$$

Proof. A simple spectral sequence argument.

COROLLARY (1). *Let $p: E \to B$ be a fibering of a connected separable metric* ANR *E onto a* wlc *paracompact base B; let E be an n-s-*hm *over a principal ideal domain L with $H_i(E; L) = 0, l \leq i \leq n$, and m the largest integer such that $H_m(B; L) \neq 0$. If either E is* *simply connected and some component of F is compact, or $\pi_1(B, b)$ operates trivially upon the homology of a compact and orientable fiber F, then B is a $(n - k) - s$-*hm *where $n - k > n + m - l$.*

Proof. If $\pi_1(B, b)$ acts trivially upon the homology of F then F must be connected. In case F is not connected take the fibering $p' : E \to B'$, where B' is the universal covering space of B (see (2.10)). Therefore we may as well assume F_b is connected, orientable and $\pi_1(B, b)$ acts trivially upon $H^s_*(F_b; L)$. By Theorem (1), F_b is a compact orientable $k - s$-hm and hence $H_k(F; L) \approx L$. Let m denote the largest integer so that $H^s_m(B; L) \neq 0$, m, of course, is bounded by $n - k$. By the lemma it follows that $H^s_{k+m}(E; L) \neq 0$ and $H_i(E; L) = 0$, for all $i > k + m$. In fact, because F is orientable $H^s_{k-1}(F; L)$ is torsion free (by Poincaré duality) and so $H_{k+m}(E; L) \approx H_m(B; L)$, (or $H_m(B; L)$ as the case may be). Thus $k + m < l$; hence $n - k > (n + m) - l \geq n - l$. In particular we have

COROLLARY (2). *If $l = 0$, p is a covering map and hence a homeomorphism if F is connected.*

Proof. B must be an n-s-hm, hence the fiber F must be a discrete set of points. Therefore the map $p' : E \to B'$ is a homeomorphism and the composition $q \circ p' = p$ a covering map.

COROLLARY (3). *Let E be a separable metric* AR *and an n-s-*hm *over L. Then any fibering $p: E \to B$ onto a paracompact* wlc *base is*

(i) *fiber homotopically equivalent to the product $B \times F_b$, if some fiber F_b is connected;*
(ii) *a homeomorphism if some fiber is compact;*
(iii) *an infinite sheeted covering space with no elements of $\pi_1(B)$ of finite order, if there exists a compact component of some fiber and no fiber is compact.*

Proof. It follows easily that B and F are AR's if F is connected. Hence by (2.1) E is fiber homotopically equivalent to $B \times F_b$. Corollary (2) implies that in case (ii) and (iii), $p: E \to B$ is a covering map. But as E is simply connected, E is the universal covering space of B and hence the group $\pi_1(B, b)$ operates freely as a transformation group on E. Smith's theorem implies that $\pi_1(B)$ can not have any elements of finite order since E is a AR.

Many other results for locally trivial fiberings of manifolds remain valid in the setting of Hurewicz fiberings. For example, using these techniques it is not difficult to extend known results on singular fiberings.

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