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PAIRING MODELS OF Pb^{206}, Pb^{204} AND Pb^{202}

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Abstract: Exact eigenstates of the pairing-force Hamiltonian are used to study paring models of Pb^{206}, Pb^{204} and Pb^{202}. The properties of the eigenstates are discussed as functions of the interaction strength g and numerical results are given for $g = 0.146$ MeV. This interaction strength is stronger than the value, $g = 0.111$ MeV, adopted in a previous study of these models which used the approximate methods of the theory of superconductivity. However, using the exact eigen states and the stronger pairing interaction, the experimentally observed spectra of these three nuclei are predicted with an average error of 0.1 MeV. This error is $\frac{1}{3}$ as large as that of the approximate calculation in which the weaker pairing interaction was used.

1. Introduction

In recent years many authors [1-6]) have studied pairing models of the nucleus. In these models the residual interaction between those neutrons or protons not in closed shells is approximated by a pairing interaction and the remaining nucleons are assumed to be non-interacting. The study of these models has led to a better understanding of such properties of the nucleus as the odd-even mass difference, the moments of inertia of deformed nuclei and the existence of an energy gap in the spectra of even nuclei [7]). In most cases, however, the approximate methods of the BCS theory of superconductivity [8]) or the Bogoliubov-Valatin canonical transformation [9,10]) have been used to study the eigenstates of these models. While these approximate methods can be justified in the theory of superconductivity, they cannot be justified when they are applied to the nucleus. For example, Bayman [11]) has shown that the criterion for the validity of the approximate ground state which results from the use of these methods is $N^{\frac{1}{2}} \gg 1$, where N is the number of interacting pairs of particles. This criterion is satisfied for superconductors but it is not satisfied for nuclei for which N may be a small number. We call the model of the nucleus that uses these approximations to obtain the eigenstates of the model Hamiltonian the "superfluid model".

In this paper, we use the method developed in the preceding paper [12]) (to be referred to as I in the following) to study the exact eigenstates of pairing models of Pb^{206}, Pb^{204} and Pb^{202}. These eigenstates will be shown to belong to the restricted class of eigenstates treated in subsect. 3.4 of I. In studying these eigenstates, we have

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two purposes in mind. On the one hand, we investigate the properties of the equations derived in I (see eqs. (1.4) through (1.7) below) for a specific single-particle spectrum ε_f and for specific numbers of particles. (The single-particle spectrum is given in table 1 below and the numbers of particles that we consider are 2, 4 and 6.) On the other hand, we evaluate the applicability of the pairing-force Hamiltonian as model-Hamiltonians for Pb^{206}, Pb^{204} and Pb^{202}, by comparing the excitation spectra of the models with the experimentally observed spectra. We have chosen these three even isotopes of lead because they provide examples of one, two and three pair systems for which the pairing interaction is a good approximation to the residual nucleon-nucleon interaction and for which the single-particle (hole) spectrum is well defined by the levels of Pb^{207}. We have not included the odd-mass isotopes in our treatment since the pairing interaction is not as good an approximation to the residual interaction for these isotopes [2]) as it is for the even isotopes and their inclusion would not contribute significantly to the understanding of the model.

Since the eigenstates of these models belong to the restricted class treated in I, the wave functions and energies of the states can be given in terms of a few parameters, the pair-energies. These pair-energies can be obtained for each state with arbitrary accuracy by solving the coupled system of algebraic equations which they satisfy. The ease with which these pair-energies (and consequently the wave functions and energies of the states) can be obtained is the principal advantage of our methods over existing numerical diagonalizations of the model Hamiltonian [13-15]).

The model Hamiltonian that describes the neutrons of the isotopes under consideration is taken to be (see eq. (2.1) of I)

$$H = \sum_f 2\varepsilon_f \hat{N}_f - g \sum_f{}' \sum_{f'}{}' b_f^+ b_{f'}, \tag{1.1}$$

where f denotes the set of single-particle quantum numbers $(nlj|m|)$, ε_f are the single-particle energy levels of an external single-particle potential well, and the double summation is taken over the set of states S (see sect. 2 of I). We ignore the protons since they form a closed shell and are assumed to be non-interacting. In eq. (1.1), the single-particle levels ε_f are assumed to contain as many of the effects of the nucleon-nucleon interaction as can be included in an independent-particle model, i.e., in the sense of a Hartree-Fock treatment of the problem. The pairing interaction is then used to approximate the effects of the residual interaction between the neutrons (actually neutron holes) in the unfilled neutron shell. This model has been applied to these isotopes of lead by Kisslinger and Sorensen [2]), who used the approximations of the superfluid model, and by Kerman *et al* [13]), who performed a numerical diagonalization of the Hamiltonian. In both these papers a pairing-interaction strength of $g = 0.111$ MeV was used.

In sect. 2, we determine the parameters of the model. These parameters are the single-particle spectrum ε_f and the pairing-interaction strength g. For the single-particle spectrum ε_f, we follow Kisslinger and Sorensen [2]) and use the observed states

of Pb^{207}. For the pairing-interaction strength g, we derive a relation between the observed pairing energy of a nucleus and the value of g in our model of that nucleus. Using this relation and the pairing energies obtained from the mass data given by Everling *et al.* [16]) we find that any value of g between 0.135 and 0.165 MeV will reproduce the pairing energies of these three nuclei, within the experimental errors. We further determine g by requiring that our model give the same average excitation energy as that of the observed states of these three nuclei which can be identified as neutron excitations. This leads to the value $g = 0.146$ MeV, which is considerably stronger than the value 0.111 MeV used by Kisslinger and Sorensen.

In sects. 3, 4 and 5 we obtain some of the eigenstates of the models of Pb^{206}, Pb^{204} and Pb^{202} for the above choice of parameters. In choosing which states to discuss we have, somewhat arbitrarily, restricted ourselves to some of the low-lying seniority-zero and seniority-two states. These states have been chosen either because they correspond to the experimentally observed states or because they exemplify the properties of the equations being solved. We also compare the excitation energies of these states with the experimentally observed excitation energies of these nuclei in these sections. Here, we find that our treatment of the model predicts the experimentally observed levels of these nuclei with three times the accuracy of those of the Kisslinger and Sorensen treatment. Sects. 4 and 5 also include rather long digressions on the existence and interpretation of complex pair-energies.

For easy reference, we now summarize the results of I that we shall use in our study of the isotopes of lead. The wave functions and energies of the restricted class of N-pair eigenstates of (1.1) are

$$\varphi(f_1 \ldots f_N) = C \sum_P P\{\prod_{k=1}^{N} (2\varepsilon_{f_k} - E_{p_k})^{-1}\}, \tag{1.2}$$

$$E = E_{p_1} + \ldots + E_{p_N}, \tag{1.3}$$

where C is a normalization constant and $\sum_P P$ is a sum over the $N!$ permutations of the indices $p_1 \ldots p_N$. The E_{p_i} are roots of the N coupled equations

$$F(E_{p_i}) = g_i^{-1}, \tag{1.4}$$

where

$$g_i^{-1} = g^{-1} + 2 \sum_{j \neq i} (E_{p_j} - E_{p_i})^{-1}, \tag{1.5}$$

$$F(E) = \sum_f (2\varepsilon_f - E)^{-1}. \tag{1.6}$$

The restriction on this class of eigenstates is the requirement that the pair-energies E_{p_i} must be distinct, i.e., they must satisfy

$$E_{p_i} \neq E_{p_j}; \qquad \text{all } i \neq j, \tag{1.7}$$

in addition to (1.4).

2. Determination of the Parameters of the Model

In discussing a pairing-model of a specific system, we must first determine the parameters of the model ε_f and g. For the even isotopes of lead that we are studying, we follow Kisslinger and Sorensen [2]) and use the observed single-particle (hole) spectrum of Pb^{207} to determine the levels ε_f contained in the set S. These levels, together with $(lj)^\pi$ and the pair degeneracy $\Omega_n = j+\frac{1}{2}$, are given in table 1. We use the index n defined in this table to denote the levels, instead of $(lj)^\pi$.

TABLE 1

The single-particle levels in S in MeV

n	$(lj)^\pi$	Ω_n	ε_n
1	$(p\frac{1}{2})^-$	1	0.00
2	$(f\frac{5}{2})^-$	3	0.57
3	$(p\frac{3}{2})^-$	2	0.90
4	$(i\frac{13}{2})^+$	7	1.63
5	$(f\frac{7}{2})^-$	4	2.35

In order to determine g, we consider the neutron pairing energy defined by

$$P(2\mathcal{N}) = 2E(2\mathcal{N}-1)-E(2\mathcal{N})-E(2\mathcal{N}-2), \tag{2.1}$$

where $E(2\mathcal{N}-\mu)$, $\mu = 0, 1, 2$, is the observed ground-state energy of the nucleus with $2\mathcal{N}-\mu$ neutrons and z protons. We suppress the z-dependence of all the quantities in (2.1) since we are only considering the isotopes of lead. We assume that the three model ground states in (2.1) belong to the restricted class of eigenstates of the pairing-force Hamiltonian described in sect. 1. The model expressions for the energies $E(2\mathcal{N}-\mu)$ are then

$$E(2\mathcal{N}) = \sum_{i=1}^{\mathcal{N}} E_{p_i}(2\mathcal{N})+\{\text{the energy of the } z \text{ protons}\},$$

$$E(2\mathcal{N}-1) = \sum_{i=1}^{\mathcal{N}-1} E_{p_i}(2\mathcal{N}-1)+\varepsilon_{p_{\mathcal{N}}}+\{\text{the energy of the } z \text{ protons}\},$$

$$E(2\mathcal{N}-2) = \sum_{i=1}^{\mathcal{N}-1} E_{p_i}(2\mathcal{N}-2)+\{\text{the energy of the } z \text{ protons}\},$$

where $E_{p_i}(2\mathcal{N}-\mu)$ is the pair-energy of the ith pair of neutrons in the ground state of $2\mathcal{N}-\mu$ neutrons. Substituting these expressions into (2.1), we obtain the model expression for P

$$P_{\text{mod}}(2\mathcal{N}) = 2\varepsilon_{p_{\mathcal{N}}}-E_{p_{\mathcal{N}}}(2\mathcal{N}) + \sum_{i=1}^{\mathcal{N}-1} 2E_{p_i}(2\mathcal{N}-1)-E_{p_i}(2\mathcal{N})-E_{p_i}(2\mathcal{N}-2)\}. \tag{2.2}$$

Note that the expression for P (2.1) reflects only the effects of the residual nucleon-

nucleon interaction when the contribution of the self-consistent field to $E(2\mathcal{N}-\mu)$ can be approximated by a linear function of μ, i.e., all contributions in a "Taylor series" expansion of this part of $E(2\mathcal{N}-\mu)$ proportional to $\mu^2, \mu^3, \ldots$ are ignored. We also note that the last term of (2.2) also vanishes if quadratic and higher powers of μ are ignored in expansions of $E_{p_i}(2\mathcal{N}-\mu)$. It is therefore consistent to ignore the last term in (2.2) if we are to identify P with P_{mod}. We now determine g by requiring $P(2\mathcal{N}) = P_{\text{mod}}(2\mathcal{N})$. This leads to the relation

$$P(2\mathcal{N}) = 2\varepsilon_{p_{\mathcal{N}}} - E_{p_{\mathcal{N}}}(2\mathcal{N}). \tag{2.3}$$

Thus, the pairing energy determines $E_{p_{\mathcal{N}}}$, the pair-energy of the $\mathcal{N}$th pair in the ground state of $2\mathcal{N}$ particles. The $\mathcal{N}$ equations (1.9) then determine the remaining $(\mathcal{N}-1)$ pair-energies $E_{p_1} \ldots E_{p_{\mathcal{N}-1}}$ of the ground state and the interaction strength g.

Since we are dealing with the interactions between holes in the 126-neutron shell in the isotopes of lead, eq. (2.3) requires a small modification. If we let $2N$ be the number of neutron holes in the 126-neutron shell, i.e., $2N = 126-2\mathcal{N}$ and let $P(2N) = P(2\mathcal{N})$ then a repetition of the steps that led from eqs. (2.1) to (2.3) shows that

$$P(2N) = 2\varepsilon_{p_{N+1}} - E_{p_{N+1}}(2N+2). \tag{2.4}$$

Therefore, the pairing energy defined by (2.1) determines the pair-energy $E_{p_{N+1}}(2N+2)$ of the $(N+1)$th pair of holes in the ground state of $2N+2$ holes.

We use the mass data given by Everling *et al.* [16]) to obtain the pairing energies 0.65 ± 0.20, 1.39 ± 0.22 and 1.44 ± 0.39 MeV for Pb^{208}, Pb^{206} and Pb^{204}, respectively. Eqs. (2.4) and (1.4) then relate these energies to the pairing-interaction strengths in our models of Pb^{206}, Pb^{204} and Pb^{202}. Using the solutions of eqs. (1.4) described in sects. 3, 4 and 5, we obtain the result that any interaction strength between 0.135 and 0.165 MeV will reproduce the three pairing energies within the experimental errors.

We further choose the value of g by requiring our model to give the same average excitation energy for the observed excited states that are identified as neutron excitations as is given by experiment [17]). This leads to the value $g = 0.146$ MeV. Note that this value of g lies within the range determined by the observed pairing energies. Therefore, with this value of g our model predicts the observed pairing energies and it predicts the correct average excitation energy for the observed ecxited states of the neutrons of these three nuclei.

In sects. 3, 4 and 5, we consider the solution of eqs. (1.4) for Pb^{206}, Pb^{204} and Pb^{202} using table 1 for the levels contained in S and $g = 0.146$ MeV. We label the states by the configurations to which they correspond in the limit $g \to 0$. Thus, $(1)^2$ and $(1)^2(2)^2$ will denote the ground states of Pb^{206} and Pb^{204} (even though, for $g \neq 0$, these states are not pure configurations).

3. The One-Pair System Pb^{206}

For the seniority-zero states of the one-pair system Pb^{206}, eqs. (1.4) reduce to the one equation

$$g^{-1} = \sum_n \Omega_n (2\varepsilon_n - E_{p_1})^{-1} = F(E_{p_1}) \tag{3.1}$$

for the pair energy E_{p_1}. The restrictions (1.7) play no role since we only have one pair. We shall define the index p_1 by requiring

$$\lim_{g=0^+} E_{p_1} = 2\varepsilon_{p_1}. \tag{3.2}$$

Eq. (3.1) may be solved by tabulating or plotting $F(E)$, using ε_n and Ω_n given in table 1, and then picking those values of E for which (3.1) is satisfied.

The energies of the seniority-zero states are the pair-energies E_{p_1}. The energies of the seniority-two states are just the independent-particle energies $\varepsilon_l + \varepsilon_m$, where l and m are the levels occupied by the two unpaired particles.

TABLE 2
The excitation spectrum of Pb^{206} in MeV

	Model				Experiment	
Config.	$(\text{spin})^\pi$	ν	Pair energies	Exc. energy	$(\text{spin})^\pi$	Exc. energy
$(1)^2$	0^+	0	$E_1 = -0.59$	0.00	0^+	0.00
(1)(2)	2^+	2		1.16	2^+	0.80
$(2)^2$	0^+	0	$E_2 = 0.44$	1.03	(0^+)	1.19
(1)(2)	3^+	2		1.16	3^+	1.34
(1)(3)	2^+	2		1.49	(2^+)	1.47
$(2)^2$	4^+	2		1.73	4^+	1.68
(1)(3)	1^+	2		1.49	(1^+)	1.72
$(2)^2$	2^+	2		1.73	(2^+)	1.85
(2)(3)	$1^+ \ldots 4^+$	2		2.06	4^+	2.00
(1)(4)	7^-	2		2.22	(7^-)	2.20
(1)(4)	6^-	2		2.22	(6^-)	2.38
(2)(4)	$4^- \ldots 9^-$	2		2.79	(5^-)	2.78
(1)(5)	$3^+, 4^+$	2		2.94	(4^+)	2.95
(3)(4)	$5^- \ldots 8^-$	2		3.12	(5^-)	3.02

The pair-energies E_1 and E_2 and the excitation energies of some states of our model of Pb^{206} are given in table 2. In this table the excitation energies are compared with the experimentally observed excitation energies. (We have identified each experimentally observed level with the level of the model which has the same spin and parity and whose energy lies closest to that of the experimentally observed level.) From this table, we see that the one level which does not fit into our model is the lowest 2^+ state. This is a general feature of the model since the lowest 2^+ states of even nuclei are collective levels [2]) and they are not described by our model Hamiltonian (1.1). Excluding this lowest 2^+ state from consideration, we see that our model reproduces the observed excitation spectrum of Pb^{206} with an average error of 0.1 MeV.

4. The Two-Pair System Pb^{204}

Eqs. (1.4) for the two pair energies of the seniority-zero states of Pb^{204} are

$$g^{-1}+\frac{2}{E_{p_2}-E_{p_1}}=F(E_{p_1}),$$
$$g^{-1}-\frac{2}{E_{p_2}-E_{p_1}}=F(E_{p_2}). \tag{4.1}$$

These equations are valid whenever the restriction $E_{p_1} \neq E_{p_2}$ is also satisfied. Due to the coupling of these two equations, they are vastly more complicated than eq. (3.1) for the one-pair system. Therefore, we first discuss the behaviour of the roots of (4.1) as functions of g and then return to our model of Pb^{204}.

4.1. THE ROOTS OF THE TWO-PAIR EQUATIONS AS FUNCTIONS OF g.

In discussing the roots of (4.1) as functions of g, the first question to be answered is whether these are any values of g for which the restrictions $E_1 \neq E_2$ are not satisfied (for simplicity, we use E_1 and E_2 to denote E_{p_1} and E_{p_2}). For, our formalism does not apply at these values of g. Eqs. (4.1) are not in a form which allows the easy calculation of these "singular" values of g for which $E_1 = E_2$. However, in I, we mentioned that E_1 and E_2 could be complex conjugates of each other. We may therefore determine the singular points by first letting $E_1 = E_2^*$ and then calculating the values of g for which Im $(E_1) = 0$. We shall show that this approach gives explicit expressions for the singular values of g. After calculating the singular values of g we shall give physical interpretations of the complex roots and draw qualitative pictures of $E_1(g)$ and $E_2(g)$ for the ground state and the lowest seniority-zero excited state.

In I we asserted that when the roots of (1.4) are complex, they occur in complex conjugate pairs. For the two-pair systems, this assertion may be proved by using (4.1) and the requirement that the total energy of the state be real, i.e., Im $(E_1) = -\text{Im}(E_2)$. The proof is accomplished by showing that Re $(E_1) \neq$ Re (E_2) is inconsistent with E_1 and E_2 satisfying eqs. (4.1) and Im $(E_1) = -$Im (E_2). For if we assume that

$$E_1 = \xi_1 + i\eta, \qquad E_2 = \xi_2 - i\eta,$$

with $\xi_1 \neq \xi_2$, then we have three unknowns ξ_1, ξ_2 and η to solve for. Equating the real and imaginary parts of the two equations of (4.1) gives four equations for these three unknowns. It is then a simple matter to show that these four equations cannot be satisfied for $\xi_1 \neq \xi_2$ and $\eta \neq 0$. However, if

$$E_1 = \xi + i\eta, \qquad E_2 = \xi - i\eta, \tag{4.2}$$

then the second equation of (4.1) is just the complex conjugate of the first and we have two equations for the two unknowns ξ and η, which may then be solved. Thus, if

E_1 and E_2 are complex, they have the form (4.2), where ξ and η satisfy the equations

$$g^{-1} = \sum_n \frac{\Omega_n(2\varepsilon_n - \xi)}{(2\varepsilon_n - \xi)^2 + \eta^2}, \tag{4.3a}$$

$$1 = \eta^2 \sum_n \frac{\Omega_n}{(2\varepsilon_n - \xi)^2 + \eta^2}, \tag{4.3b}$$

which are obtained by substituting the expressions (4.2) into eq. (4.1). Note that the sign of η is undetermined since only η^2 appears in (4.3). Thus the choice of the plus and minus signs that was made in (4.2) is not significant.

We now determine the singular values of g, for which $E_1 = E_2$ by determining those values of g for which (4.3) implies $\eta = 0$. We first show that (4.3b) implies that $\eta = 0$ if $\xi = 2\varepsilon_m$. For, let us assume that $\xi = 2\varepsilon_m$ and $\eta \neq 0$, then (4.3b) becomes

$$1 = \Omega_m + \eta^2 \sum_{n \neq m} \frac{\Omega_n}{(2\varepsilon_n - 2\varepsilon_m)^2 + \eta^2},$$

which cannot be satisfied since $\Omega_m \geqq 1$. Thus, we seek those values of g for which $\xi = 2\varepsilon_m$. It can easily be shown that these values of g are all the possible singular points of (4.1). For, if $E_1 = E_2 = \xi$, the left hand sides of eqs. (4.1) are singular and ξ must be some value for which $F(\xi)$ is singular. But, this is just the set of points $\xi = 2\varepsilon_m$. A direct substitution of $\xi = 2\varepsilon_m$ and $\eta = 0$ into (4.3a) yields the result $g = 0$. But this is a misleading result and one must consider instead the limit of eqs. (4.3) as $\xi \to 2\varepsilon_m$. For example, let us consider the case in which $m = 1$. We let $\xi = 2\varepsilon_1 - \Delta$ and keep only the lowest order terms in Δ; then we let $\Delta \to 0$ at the end of the calculation. Using $\xi = 2\varepsilon_1 - \Delta$, $\Omega_1 = 1$, and the fact that η^2 is of order Δ or Δ^2, eq. (4.3b) becomes

$$1 = \frac{\eta^2}{\Delta^2 + \eta^2} + \eta^2 \sum_{n=2}^{5} \frac{\Omega_n}{(2\varepsilon_n - 2\varepsilon_1)^2}$$

to the lowest order in Δ. The solution of this equation, to first order in Δ, is

$$\eta^2 = |\Delta|/A, \tag{4.4}$$

where

$$A^2 = \sum_{n=2}^{5} \frac{\Omega_n}{(2\varepsilon_n - 2\varepsilon_1)^2}. \tag{4.5}$$

Substituting $\xi = 2\varepsilon_1 - \Delta$, $\Omega_1 = 1$ and the expression (4.4) into eq. (4.3a) gives

$$g^{-1} = \frac{\Delta}{|\Delta|} A + \sum_{n=2}^{5} \frac{\Omega_n}{(2\varepsilon_n - 2\varepsilon_1)}. \tag{4.6}$$

Thus, $g \neq 0$ for $\xi = 2\varepsilon_1$ and g as a function of ξ is discontinuous at $\xi = 2\varepsilon_1$. If we

use the values of ε_n and Ω_n given in table 1 in eqs. (4.5) and (4.6), we obtain the results

$$\lim_{\Delta \to 0^+} g = 0.115 \text{ MeV}, \qquad \lim_{\Delta \to 0^-} g = 0.209 \text{ MeV}. \tag{4.7}$$

If we consider the other values of m, $m = 2 \ldots 5$, then we have for $\xi = 2\varepsilon_m - \Delta$ that $\eta^2 \alpha \Delta^2$ and $g \alpha \Delta$. This leads to the qualitative picture of ξ as a function of g given in fig. 1 and the result that the singular points of eqs. (4.1) are located at $g = 0$, 0.115 and 0.209 MeV. The critical role played by the fact that $\Omega_1 = 1$ in the discontinuous behaviour of ξ at $\xi = 2\varepsilon_1$ should be pointed out. For if $\xi = 2\varepsilon_1 - \Delta$ and $\Omega_1 > 1$, then $\eta^2 \alpha \Delta^2$, there would be no discontinuity at $\xi = 2\varepsilon_1$, and the value of g for which $\xi = 2\varepsilon_1$ would be $g = 0$. The existence of the discontinuity for $\Omega_1 = 1$ reflects the fact that the Pauli principle does not allow two pairs to occupy level 1 for $g = 0$.

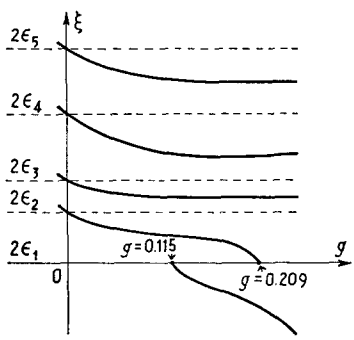

Fig. 1. The qualitative behaviour of $\xi(g)$ for the states of Pb^{204}.

There remains the question of the physical interpretation of the various branches of $\xi(g)$ shown in fig. 1. The interpretation of the upper four branches is clear from their $g = 0$ limits. They correspond to the four seniority-zero states $(2)^4$, $(3)^4$, $(4)^4$ and $(5)^4$. However, the lowest branch does not have a $g = 0$ limit. In order to interpret this lowest branch of $\xi(g)$, one must consider the real roots of (4.1) for $g < 0.115$ MeV. One then sees that the lowest branch of $\xi(g)$ is connected to two real roots, $E_1(g)$ and $E_2(g)$, of (4.1) which satisfy

$$E_1(0) = 2\varepsilon_1, \qquad E_2(0) = 2\varepsilon_2,$$
$$E_1(0.115) = E_2(0.115) = 2\varepsilon_1.$$

Thus, the $g = 0$ limit of the lowest branch of $\xi(g)$ is the configuration $(1)^2(2)^2$ and this branch of ξ corresponds to the ground state $(1)^2(2)^2$. The branch of ξ which is second from the bottom and which corresponds to the state $(2)^4$ seems to end at $g = 0.209$ MeV where $\xi = 2\varepsilon_1$. However, at this point it is connected to two real roots $E_1(g)$ and $E_2(g)$ of (4.1) which satisfy

$$E_1(0.209) = E_2(0.209) = 2\varepsilon_1,$$
$$E_1(\infty) = -\infty, \qquad 2\varepsilon_1 < E_2(\infty) < 2\varepsilon_2.$$

This leads to the qualitative pictures of the pair-energies for the states $(1)^2(2)^2$ and $(2)^4$ as functions of g given in figs. 2 and 3. We use here and later the convention

$$\text{Re}\,(E_i) \leqq \text{Re}\,(E_j), \text{ for } i < j,$$

to number the pair-energies.

Thus the singular points of (4.1) are located at $g = 0.115$ MeV for the ground state, $g = 0$ and $g = 0.209$ MeV for the state $(2)^4$, and $g = 0$ for the states $(3)^4$, $(4)^4$ and $(5)^4$. The remaining seniority-zero states have no singular points. These singular points can be interpreted as the values of g for which the roots of (4.1) change from being real to being complex or vice versa.

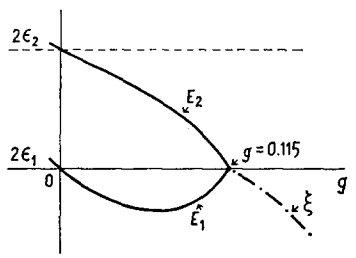

Fig. 2. The qualitative behaviour of the pair-energies in the ground state of Pb^{204}.

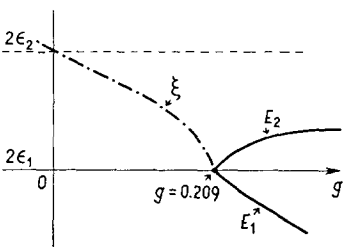

Fig. 3. The qualitative behaviour of the pair-energies in the state $(2)^4$ of Pb^{204}.

4.2. THE LEVELS OF Pb^{204}

Having discussed the structure of the roots of (4.1), we are now able to apply these equations to our model of Pb^{204} for which $g = 0.146$ MeV. The results of our calculations are given in table 3. In addition to the observed states of this nucleus, we have included in our calculations the two lowest excited 0^+ states as examples of the roots of eqs. (4.1).

From figs. 2 and 3, we see that the pair-energies for the two seniority-zero states, $(1)^2(2)^2$ and $(2)^4$, are complex for $g = 0.146$ MeV. They are given by (4.2), where ξ and η satisfy (4.3). The calculated values of ξ and η^2 for $g = 0.146$ MeV are listed in table 3. The remaining seniority-zero state that we treat, $(1)^2(3)^2$, has real pair-energies which we label according to (3.2). The energies of these states are given by (1.3).

TABLE 3
The excitation spectrum of Pb^{204} in MeV

Model					Experiment	
Config.	$(\text{spin})^\pi$	ν	Pair energies	Exc. energy	$(\text{spin})^\pi$	Exc. energy
$(1)^2(2)^2$	0^+	0	$\xi = -0.30$ $\eta^2 = 0.15$	0.00	0^+	0.00
$(1)^2(2)^2$	2^+	2	$E_1 = -0.39$	1.35	2^+	0.90
$(1)^2(2)^2$	4^+	2	$E_1 = -0.39$	1.35	4^+	1.27
$(2)^4$	0^+	0	$\xi = 0.43$ $\eta^2 = 0.093$	1.46		
$(1)^2(2)(3)$	$1^+ \ldots 4^+$	2	$E_1 = -0.41$	1.66	4^+	1.56
$(1)^2(3)^2$	0^+	0	$E_1 = -0.43$ $E_3 = 1.56$	1.73		
$(1)^2(2)(4)$	$4^- \ldots 9^-$	2	$E_1 = -0.44$	2.36	9^-	2.19

The equation for the pair energy of the seniority-two states of Pb^{204} is the same as (3.1) with Ω_n replaced by $\Omega_n - \delta_{nl} - \delta_{nm}$, where l and m are the levels occupied by the two unpaired particles. Here the two Kronecker deltas express the blocking effect of the two unpaired particles (see I). The energies of these states are

$$E = E_{p_1} + \varepsilon_l + \varepsilon_m .$$

From table 3, we again see that the one experimentally observed level which does not fit into our model is the collective 2^+ level. The three remaining observed excitation energies are predicted with an average error of 0.1 MeV.

5. The Three-Pair System Pb^{202}

Eqs. (1.4) for the three pair-energies of the seniority-zero states of the three-pair system Pb^{202} are

$$g^{-1} + \frac{2}{E_2 - E_1} + \frac{2}{E_3 - E_1} = F(E_1), \tag{5.1a}$$

$$g^{-1} - \frac{2}{E_2 - E_1} + \frac{2}{E_3 - E_2} = F(E_2), \tag{5.1b}$$

$$g^{-1} - \frac{2}{E_3 - E_1} - \frac{2}{E_3 - E_2} = F(E_3), \tag{5.1c}$$

where again we have used E_i to denote E_{p_i}. These equations are valid as long as the E_i are distinct, i.e., the E_i must satisfy the three restrictions $E_i \neq E_j$, for $i \neq j$, in addition to eq. (5.1). In discussing the structure of the pair-energies E_i, we are led to an investigation of the singular points of eqs. (5.1), i.e., those values of g for which some $E_i = E_j$ for $i \neq j$. This may be done in complete analogy with our discussion

of the structure of the roots of eqs. (4.1) for Pb^{204}. However, since we now have three pair-energies instead of two and we have a large number of seniority-zero states (29 to be exact), a general discussion of the singular points of (5.1) is a long and tedious job. We therefore restrict our discussion of the singular points of (5.1) to those that appear in the ground state and the lowest seniority-zero excited state since these are the states that we shall consider in our model of Pb^{202}. After this discussion of the singular points we shall return to our model of Pb^{202}.

5.1. THE ROOTS OF THE THREE-PAIR EQUATIONS AS FUNCTIONS OF g.

We now discuss the behaviour of the roots of (5.1) as functions of g for the two seniority-zero two states, $(1)^2(2)^4$ and $(1)^2(2)^2(3)^2$, that we shall treat in our model of Pb^{202}. This discussion is completely analogous to that given for Pb^{204} and we draw upon many of the ideas developed there.

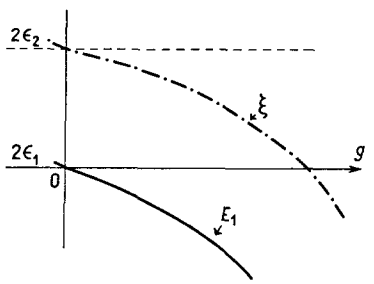

Fig. 4. The qualitative behaviour of the pair-energies in the ground state of Pb^{202}.

The $g = 0$ limits of the pair-energies for the state $(1)^2(2)^4$ are given by

$$E_1(0) = 2\varepsilon_1, \quad E_2(0) = E_3(0) = 2\varepsilon_2.$$

Thus $g = 0$ is a singular point of the equations for the pair-energies of this state. By analogy with the state $(2)^4$ of Pb^{204}, we assume that E_1 is real and E_2 and E_3 are complex as in (4.2) for this state for $g \neq 0$. Eqs. (5.1) then become

$$g^{-1} + \frac{4(\xi - E_1)}{(\xi - E_1)^2 + \eta^2} = F(E_1), \tag{5.2a}$$

$$g^{-1} - \frac{2(\xi - E_1)}{(\xi - E_1)^2 + \eta^2} = \sum_n \frac{\Omega_n(2\varepsilon_n - \xi)}{(2\varepsilon_n - \xi)^2 + \eta^2}, \tag{5.2b}$$

$$1 + \frac{2\eta^2}{(\xi - E_1)^2 + \eta^2} = \eta^2 \sum_n \frac{\Omega_n}{(2\varepsilon_n - \xi)^2 + \eta^2}, \tag{5.2c}$$

for E_1 and for the real and imaginary parts of E_2 and E_3 (ξ and $\pm\eta$). There are no other singular points in the equations for the pair-energies of this state and eqs. (5.2) plus the boundary conditions

$$E_1(0) = 2\varepsilon_1, \quad \xi(0) = 2\varepsilon_2$$

determine the ground-state pair-energies for all $g > 0$. A qualitative picture of $E_1(g)$ and $\xi(g)$ for this state is given in fig. 4. From this picture we see that the value of g for which $\xi = 2\varepsilon_1$ could be a singular point. However, it is not since $\eta \neq 0$ at this point.

For the $g = 0$ limit of the pair-energies of the state $(1)^2(2)^2(3)^2$, we have

$$E_1(0) = 2\varepsilon_1, \quad E_2(0) = 2\varepsilon_2, \quad E_3(0) = 2\varepsilon_3,$$

and therefore $g = 0$ is not a singular point of the equations for the pair energies of this state. However, there may exist singular points for $g > 0$. These would occur at those values of g for which either

$$E_1 = E_2 = 2\varepsilon_1, \quad \text{or} \quad E_2 = E_3 = 2\varepsilon_2.$$

The former possibility is the analogue of the singular point at $g = 0.115$ MeV in the ground state of Pb^{204}. The latter possibility cannot in fact occur. This may be shown by allowing E_2 to approach $2\varepsilon_2$ from below and E_3 to approach it from above and then using (5.1) to try to calculate the value of g that brings this about. However, it turns out that Ω_2 must be one in order that E_2 equal E_3. Since $\Omega_2 = 3$, the latter possibility is excluded and we need only investigate the possibility $E_1 = E_2 = 2\varepsilon_1$.

To study the singular point at which $E_1 = E_2 = 2\varepsilon_1$, in the state $(1)^2(2)^2(3)^2$, we follow the same line of thought used in our study of Pb^{204}. In order to locate this singular point, we assume that E_1 and E_2 are complex as in (4.2). We also assume that E_3 is real and

$$2\varepsilon_2 < E_3 < 2\varepsilon_3. \tag{5.3}$$

The equations for the pair-energies are then the same as eqs. (5.2) with E_1 replaced by E_3. We then let $\xi = 2\varepsilon_1 - \Delta$ and calculate g retaining only the lowest order terms in Δ. We first calculate η^2 to lowest order in Δ from (5.2c). Substituting $\xi = 2\varepsilon_1 - \Delta$ and $\Omega_1 = 1$ into (5.2c), we have

$$1 = \frac{\eta^2}{\Delta^2 + \eta^2} + \eta^2 \sum_{n=2}^{5} \frac{\Omega_n}{(2\varepsilon_n - 2\varepsilon_1)^2} - \frac{2\eta^2}{(2\varepsilon_1 - E_3)^2} \tag{5.4}$$

to lowest order in Δ. The solution of this is, see (4.4),

$$\eta^2 = |\Delta|/A', \tag{5.5}$$

where

$$A'^2 = \sum_{n=2}^{5} \frac{\Omega_n}{(2\varepsilon_n - 2\varepsilon_1)^2} - \frac{2}{(2\varepsilon_1 - E_3)^2}. \tag{5.6}$$

Note that A' is a function of E_3 and it is not a constant as is A in (4.4). Of course, A'^2 must be greater than zero and this places restrictions on the allowable values of E_3. Using the values of ε_n and Ω_n given in table 1, we find that this restriction is

$$|E_3| > 0.73 \text{ MeV}.$$

However, our assumption (5.3) insures that $|E_3| > 2\varepsilon_2 = 1.14$ MeV so that this is not an additional restriction on E_3. Our next step is to use eqs. (5.5) and (5.2b) to obtain g as a function of E_3 to lowest order in Δ. This gives

$$g^{-1} = \frac{\Delta}{|\Delta|} A' + \frac{2}{2\varepsilon_1 - E_3} + \sum_{n=2}^{5} \frac{\Omega_n}{2\varepsilon_n - 2\varepsilon_1}. \tag{5.7}$$

We finally use this result and (5.2a) to obtain the single equation

$$\frac{\Delta}{|\Delta|} A' + \frac{6}{2\varepsilon_1 - E_3} + \sum_{n=2}^{5} \frac{\Omega_n}{2\varepsilon_n - 2\varepsilon_1} = F(E_3) \tag{5.8}$$

for E_3. Eq. (5.8) clearly has a root in the range (5.3) since $F(E)$ takes on all values between $+\infty$ and $-\infty$ in this range and the left-hand side of eq. (5.8) is a smoothly varying function of E_3 in this range. A short calculation yields the results

$$\lim_{\Delta \to 0^+} E_3 = 1.52 \text{ MeV}, \qquad \lim_{\Delta \to 0^-} E_3 = 1.46 \text{ MeV}$$

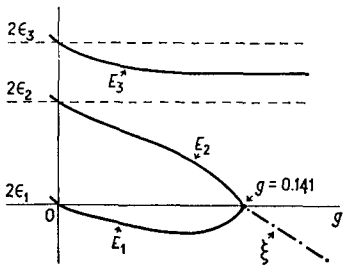

Fig. 5. The qualitative behaviour of the pair-energies in the state $(1)^2(2)^2(3)^2$ of Pb^{202}.

for the roots of eq. (5.8). These values of E_3 can now be used in eq. (5.7) to locate the singular points of eqs. (5.1). The results are

$$\lim_{\Delta \to 0^+} g = 0.141 \text{ MeV}, \qquad \lim_{\Delta \to 0^-} g = 0.272 \text{ MeV}.$$

These singular points can be identified with the states of the system in the same way that the points $g = 0.115$ and 0.209 MeV were identified for Pb^{204}. The results of this identification are that the point $g = 0.141$ MeV is a singular point for the state $(1)^2(2)^2(3)^2$ and the point $g = 0.272$ MeV is a singular point for the state $(2)^4(3)^2$. A qualitative picture of the pair-energies of the state $(1)^2(2)^2(3)^2$ given in fig. 5. The picture of the pair-energies of the state $(2)^4(3)^2$ may be obtained by drawing a third pair-energy between $2\varepsilon_2$ and $2\varepsilon_3$ in fig. 3.

5.2. THE LEVELS OF Pb^{202}

We now apply our equations to our model of Pb^{202} for which $g = 0.146$ MeV. The results of our calculations are given in table 4. In addition to the observed states of

Pb^{202}, we have included in our calculations the lowest excited 0^+ state as an example of the roots of eqs. (5.1).

TABLE 4
The excitation spectrum of Pb^{202} in MeV

Model					Experiment	
Config.	$(\text{spin})^\pi$	ν	Pair energies	Exc. energy	$(\text{spin})^\pi$	Exc. energy
$(1)^2(2)^4$	0^+	0	$\xi = 0.05$ $\eta^2 = 0.51$ $E_1 = -0.31$	0.00	0^+	0.00
$(1)^2(2)^4$	2^+	2	$E_1 = -0.13$ $E_2 = 0.23$	1.45	2^+	0.96
$(1)^2(2)^4$	4^+	2	$E_1 = -0.13$ $E_2 = 0.23$	1.45	4^+	1.38
$(1)^2(2)^2(3)^2$	0^+	0	$\xi = -0.04$ $\eta^2 = 0.023$ $E_3 = 1.52$	1.65		
$(1)^2(2)^3(3)$	$1^+ \ldots 4^+$	2	$\xi = -0.01$ $\eta^2 = 0.0063$	1.66	4^+	1.62
$(1)^2(2)^3(4)$	$4^- \ldots 9^-$	2	$\xi = -0.06$ $\eta^2 = 0.035$	2.29	5^-	2.04
$(1)^2(2)^3(4)$	$4^- \ldots 9^-$	2	$\xi = -0.06$ $\eta^2 = 0.035$	2.29	9^-	2.17

The two seniority-zero states that we treat are the states $(1)^2(2)^4$ and $(1)^2(2)^2(3)^2$. For the ground state $(1)^2(2)^4$, the pair-energies are roots of eqs. (5.2). For the excited state $(1)^2(2)^2(3)^2$, the pair-energies are also roots of eqs. (5.2) but with E_1 replaced by E_3. Owing to an error in calculation, this state was incorrectly reported in ref. [18]).

For the seniority-two states of Pb^{202}, eqs. (1.4) are the same as the eqs. (4.1) given for Pb^{204} but with Ω_n replaced by $\Omega_n - \delta_{nl} - \delta_{nm}$, where l and m are the levels occupied by the two unpaired particles. However, the values of g at the singular points must be determined for each different set of values of l and m. This can be done by retracing the steps of subsect. 4.1 with the above replacement for Ω_n. In this way it is found that the singular points for the seniority-two states $(1)^2(2)^4$, $(1)^2(2)^3(3)$ and $(1)^2(2)^3(4)$ are located at $g = 0.154$, 0.143, and 0.137 MeV, respectively, and the pair-energies for these states are complex for interactions stronger than these values. Thus, for $g = 0.146$ MeV, the pair-energies for the 2^+ and 4^+ states $(1)^2(2)^4$ are real and are determined by (4.1) with Ω_n replaced by $\Omega_n - 2\delta_{n_2}$. The pair-energies for the remaining states are complex and are determined by (4.3) with Ω_n replaced by $\Omega_n - \delta_{n_2} - \delta_{nm}$, where $m = 3$ or 4.

From table 4, we see that the excitation energies of the observed states of Pb^{202} (excluding the collective 2^+ level) are predicted by our model with an aveage error of 0.1 MeV.

6. Conclusion

By considering pairing models of Pb^{206}, Pb^{204} and Pb^{202} the one, two and three pair eigenstates of the pairing force Hamiltonian have been discussed as functions of the pairing-interaction strength g for a fixed single-particle spectrum. These eigenstates belong to the restricted class of eigenstates treated in ref. [12]) and the N-pair wave functions and energies can be given in terms of N pair-energies. In discussing the behaviour of these eigenstates as functions of g we have shown that the pair-energies change from being real to being complex or vice versa at those values of g for which the restrictions are violated, i.e., those values of g for which two pair-energies are equal. This behaviour is discussed in detail for some low-lying states of Pb^{204} and Pb^{202} in subsects. 4.1 and 5.1.

Numerical results for some eigenstates of Pb^{206}, Pb^{204} and Pb^{202} are given in sects. 3, 4.2 and 5.2. These eigenstates have been calculated using a pairing-interaction strength of $g = 0.146$ MeV. This value of g was chosen because it reproduces the observed pairing energies (see sect. 2) and the average excitation energy of the observed states of these three nuclei. It is considerably stronger than the 0.111 MeV value used by Kisslinger and Sorensen [2]) in their approximate treatment of these models. However, our use of the exact eigenstates and the stronger pairing interaction predicts the excitation energies of the observed states of these nuclei with an average error of 0.1 MeV which is $\frac{1}{3}$ as large as the average error of ref. [2]). In addition to the excitation energies, we have given the pair-energies which may be used to construct the wave functions of the states. These wave functions may be used to calculate the matrix elements of operators using the exact eigenstates of the pairing-force Hamiltonian.

References

1) S. T. Belyaev, Mat. Fys. Medd. Dan. Vid. Selsk. **31**, No. 11 (1959)
2) L. S. Kisslinger and R. A. Sorensen, Mat. Fys. Medd. Dan. Vid. Selsk. **32**, No. 9 (1960)
3) S. G. Nilsson and O. Prior, Mat. Fys. Medd. Dan. Vid. Selsk. **32**, No. 16 (1961)
4) V. G. Soloviev, Thesis, Joint Institute for Nuclear Research, Dubna, U.S.S.R., (1961)
5) J. J. Griffin and M. Rich. Phys. Rev. Lett. **3** (1959) 342
6) C. J. Gallagher, Jr. and V. G. Soloviev, Mat. Fys. Skr. Dan. Vid. Selsk. **2**, No. 2 (1962)
7) A. Bohr, B. R. Mottelson and D. Pines, Phys. Rev. **110** (1958) 936
8) J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108** (1957) 1175
9) N. N. Bogoliubov, Nuovo Cim. **7** (1958) 794
10) J. G. Valatin, Nuovo Cim. **7** (1958) 843
11) B. F. Bayman, Nuclear Physics **15** (1960) 33
12) R. W. Richardson and N. Sherman, Nuclear Physics **52** (1964) 221
13) A. K. Kerman, R. D. Lawson and M. H. MacFarlane, Phys. Rev. **124** (1961) 162
14) J. Høgaasen-Feldman, Nuclear Physics **28** (1961) 258
15) A. Pawlikowski and V. Rybarska, JETP (Soviet Physics) **16** (1963) 388
16) F. Everling *et al.*, Nuclear Physics **18** (1960) 529
17) K. Way *et al.*, in Landolt-Bornstein numerical data and functional relationships in science and technology, New Series, Group I- nuclear physics and technology (Springer-Verlag, Berlin, 1961) Vol. 1
18) R. W. Richardson, Phys. Lett. **5** (1963) 82