these measurements the following conclusions can be drawn:

1. The value of the surface anisotropy obtained from our measurements is in good agreement with the theoretically computed value [10] and with the exchange surface anisotropy [11] and has the order of K_S observed experimentally by means of thin films spin-wave resonance [12].

2. The irregularities of the surface play a minor role in line broadening as shown by the small difference between whiskers and electrolytically polished crystals having largely differing surface states.

3. The line broadening due to an increase of the measured area indicates that here the inhomogeneity of the demagnetising field begins to assert itself.

4. The agreement between the theory referred to above and the experimental results on bulk silicon iron single crystals is good. It is proof that, at least in some metallic ferromagnets having few imperfections, the intrinsic resonance linewidth (without skin effect and surface anisotropy broadening) is under 10 Oe.

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A RESTRICTED CLASS OF EXACT EIGENSTATES OF THE PAIRING-FORCE HAMILTONIAN *

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In this note we will show that there exists a restricted class of eigenstates of the pair-force Hamiltonian which contain N pairs of particles and which are natural generalisations of those states which contain one pair. The wave function of these N-pair states has the form of an antisymmetrised product of N 1-pair wave functions and the energy is a sum of N 1-pair energies. These 1-pair wave functions and energies are obtained from a 1-pair Schrödinger equation in which the pairing interaction has been replaced by an effective pairing interaction which in turn depends upon the N 1-pair energies. These eigenstates may be used to study the effects of including a pairing interaction in the Nilsson model of the atomic nucleus [1] and also in some nuclei near closed shells. They may also be used to evaluate the accuracy of the methods of the theory of superconductivity which have been used by many authors [2,3] to include pairing forces in an independent particle model of the nucleus.

We split the pairing-force Hamiltonian into that part which describes unpaired (i.e., noninteracting) particles and that part which describes the paired particles. Thus

$$H = H_1 + H_2 \,, \tag{1}$$

where

$$H_1 = \sum_{(f\in S_1)} 2\epsilon_f N_f \,, \tag{2}$$

$$H_2 = \sum_{(f\in S_2)} 2\epsilon_f N_f - g \sum_{(f\in S_2)} \sum_{(f'\in S_2)} b_f^+ b_{f'} \,, \tag{3}$$

and where ϵ_f is the energy of the single-particle state (f,σ) (where $\sigma = \pm$ denotes states which are conjugate with respect to time reversal), S_1 is the set of states outside a finite energy interval about the Fermi energy (which is specified by the interaction) plus those states in this interval that are occupied by unpaired particles, S_2 is the finite set of states not included in S_1,

$$N_f = \tfrac{1}{2}\,(a_{f+}^+ a_{f+} + a_{f-}^+ a_{f-}) \,, \tag{4}$$

$$b_f = a_{f-}\, a_{f+} \,, \tag{5}$$

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and the a^+'s and a's are fermion creation and annihilation operators.

Since H_1 and H_2 refer to different dynamical variables of the system, the eigenfunctions of H are products of an eigenfunction of H_1 with one of H_2 and the energy is a sum of the corresponding energy eigenvalues. Since H_1 is already diagonal, we will only consider H_2 and its eigenfunctions and we will restrict f to those values contained in S_2.

The operators N_f and b_f satisfy the commutation relations

$$[b_f, N_{f'}] = \delta_{ff'} b_f \tag{6}$$

$$[b_f, b^+_{f'}] = \delta_{ff'}(1 - 2N_f) . \tag{7}$$

If we could neglect the term $2N_f$ on the right hand side of (7), we would have boson commutation relations and the diagonalisation of H_2 would be straightforward. The presence of the term, $2N_f$, reflects the fact thet the b's are products of Fermi operators which must obey the Pauli principle.

We now formulate a method in which the complicating term, $2N_f$, in (7) gives no contribution to the calculation of the matrix elements of H_2.

i. We first expand an eigenstate of H_2 as

$$|\psi\rangle = (N!)^{-\frac{1}{2}} \sum_{f_1 \cdots f_N} \psi(f_1 \ldots f_N)\, b^+_{f_1} \ldots b^+_{f_N} |0\rangle , \tag{8}$$

where $|0\rangle$ is the vacuum state.

ii. We can impose two requirements on $\psi(f_1 \ldots f_N)$:

1. Since $[b_f, b^+_{f'}] = 0$, we require $\psi(f_1 \ldots f_N)$ to be a symmetric function of its arguments.
2. Since $b_f^2 = 0$, the values of $\psi(f_1 \ldots f_N)$ with two or more arguments equal are without physical significance. We are therefore free to choose these values of ψ in any convenient way. A convenient choice is zero.

These two conditions may be satisfied by letting

$$\psi(f_1 \ldots f_N) = \theta(f_1 \ldots f_N)\, \varphi(f_1 \ldots f_N) , \tag{9}$$

where φ is a symmetric function of its arguments and

$$\theta(f_1 \ldots f_N) = 1 , \quad \text{if} \quad f_i \neq f_j , \quad \text{all } i \neq j ,$$
$$= 0 , \quad \text{if any } f_i = f_j , \quad i \neq j .$$

In general it may be verified that

$$\theta(f_1 \ldots f_N) = \prod_{i<j} (1 - \delta_{f_i f_j}) . \tag{10}$$

iii. For ψ's of the form (9) the normalisation is

$$\langle\psi|\psi\rangle = \sum_{f_1 \cdots f_N} |\psi(f_1 \ldots f_N)|^2 = 1 \tag{11}$$

and the expectation value of H_2 is

$$\langle\psi|H_2|\psi\rangle = \sum_{f_1 \ldots f_N} (2\epsilon_{f_1} + \ldots + 2\epsilon_{f_N})\, |\psi(f_1 \ldots f_N)|^2$$
$$- g \sum_{i=1}^{N} \sum_{f \ldots f_{i-1} f_{i+1} \ldots f_N f f'} \psi^*(f_1 \ldots f_{i-1} f f_{i+1} \ldots f_N)$$
$$\times \psi(f_1 \ldots f_{i-1} f' f_{i+1} \ldots f_N) . \tag{12}$$

Note that these are the same results that would have been gotten using boson commutation relations for the b's by ignoring the term $2N_f$ in (7).

iv. We now determine the exact eigenstates of H_2 by requiring ψ to satisfy the normalisation condition (11) and the Schrödinger equation derived from the variational equations

$$\frac{\delta \langle\psi|H_2 - E|\psi\rangle}{\delta\varphi^*(f_1 \ldots f_N)} = \frac{\delta \langle\psi|H_2 - E|\psi\rangle}{\delta\varphi(f_1 \ldots f_N)} = 0 . \tag{13}$$

Using (9), (12) and $\theta^2 = \theta$, (13) becomes

$$(2\epsilon_{f_1} + \ldots + 2\epsilon_{f_N} - E)\, \theta(f_1 \ldots f_N)\, \varphi(f_1 \ldots f_N)$$
$$- g \sum_{i=1}^{N} \sum_{f} \theta(f_1 \ldots f_N)\, \theta(f_1 \ldots f_{i-1} f f_{i+1} \ldots f_N)$$
$$\times \varphi(f_1 \ldots f_{i-1} f f_{i+1} \ldots f_N) = 0 . \tag{14}$$

Since we obtain the same equation for φ^*, we can choose φ to be real. From the definition of θ it follows that

$$\theta(f_1 \ldots f_N)\, \theta(f_1 \ldots f_{i-1} f f_{i+1} \ldots f_N)$$
$$= \theta(f_1 \ldots f_N) \Big[1 - \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \delta_{f_j f}\Big] . \tag{15}$$

Therefore φ satisfies (14) if

$$(2\epsilon_{f_1} + \ldots + 2\epsilon_{f_N} - E)\, \varphi(f_1 \ldots f_N)$$
$$- g \sum_{i=1}^{N} \sum_{f} \Big[1 - \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \delta_{f_j f}\Big] \varphi(f_1 \ldots f_{i-1} f f_{i+1} \ldots f_N) = 0 . \tag{16}$$

For $N = 1$, (16) may be solved without any restrictions. For, in this case (16) becomes

$$(2\epsilon_{f_1} - E)\, \varphi(f_1) - g \Sigma_f \varphi(f) = 0 . \tag{17}$$

The solutions of (17) are

$$\varphi_{p_1}(f_1) = g\, C_{p_1} (2\epsilon_{f_1} - E_{p_1})^{-1} , \tag{18}$$

where gC_{p_1} is a normalisation constant and $E = E_{p_1}$, where E_{p_1} is a root of the equation

$$1 = g \Sigma_f (2\epsilon_f - E_{p_1})^{-1} . \tag{19}$$

For $N > 1$, we will show that there exists a restricted class of solutions of (16) which are natural generalisations of (18) and (19), i.e.,

$$\varphi_{p_1 \ldots p_N}(f_1 \ldots f_N) = g^N C_{p_1 \ldots p_N} \sum_P P \left(\prod_{k=1}^{N} (2\epsilon_{f_k} - E_{p_k})^{-1} \right) , \quad (20)$$

where $g^N C_{p_1 \ldots p_N}$ is a normalisation constant, $\Sigma_P P$ is a sum over the $N!$ permutations of the indices $(p_1 \ldots p_N)$, and

$$E = \sum_{i=1}^{N} E_{p_i} . \quad (21)$$

The E_{p_i} $(i = 1 \ldots N)$ are N *distinct* roots of the coupled equations

$$1 = g_i \Sigma_f (2\epsilon_f - E_{p_i})^{-1} , \quad i = 1 \ldots N , \quad (22)$$

where

$$g_i = g \left(1 + 2g \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \frac{1}{E_{p_j} - E_{p_i}} \right)^{-1} . \quad (23)$$

This may be verified by direct substitution of (20) into (16). However, the restriction

$$E_{p_i} \neq E_{p_j} , \quad \text{all } i \neq j , \quad (24)$$

is essential to the derivation of (22) and (23).

Given a single-particle spectrum, ϵ_f, and an interaction strength, g, the solution of eq. (22) proceeds in two steps:

1. One must first decide whether the restrictions (24) on the E_{p_i} are compatible with the E_{p_i} satisfying (22). This must be carefully checked for each particular problem. However, for the Nilsson model of the atomic nucleus [1] (whose single-particle spectrum satisfies $\epsilon_f \neq \epsilon_{f'}$ for $f \neq f'$), we may choose the indices p_i so that

$$\lim_{g \to 0^+} E_{p_i} = 2\epsilon_{p_i} ,$$

and therefore (24) is satisfied in this limit. Study of eqs. (22) then indicates that there is a range of g, $g > 0$, for which (22) and (24) are compatible. These same considerations hold true for any state of a system for which

$$\lim_{g \to 0^+} E_{p_i} \neq \lim_{g \to 0^+} E_{p_j} , \quad i \neq j .$$

Thus, there exists a large class of systems and interaction strengths for which (22) and (24) are compatible.

2. For systems for which (22) and (24) are compatible, eqs. (22) may be solved numerically for the E_{p_i}. The solution of eqs. (22) is greatly facilitated by the presence of the same function,

$$F(E) = \Sigma_f (2\epsilon_f - E)^{-1} ,$$

evaluated for different values of its argument in each equation. This function may be calculated once and then eqs. (22) may be solved by iteration.

Some of the advantages of this method over existing work (for example, see ref. [4]) on the exact diagonalisation of the pairing-force Hamiltonian are:

1. The computational effort needed to solve (22) is in general much less than that needed to perform a direct diagonalisation of H_2.
2. General properties of the eigenvalues of H_2 may be studied by considerings eqs. (21) and (22). For example, the pairing energy can be related directly to g. The pairing energy is defined as

$$P(2N) = 2E(2N - 1) - E(2N) - E(2N - 2) , \quad (25)$$

where $E(2N - \nu)$ is the ground state energy of $2N - \nu$ particles. Note that this definition of $P(2N)$ assumes that $E(2N - \nu)$ is a linear function of ν plus a discontinuous pairing term and that it neglects quadratic terms in ν. If we assume that the energies of the paired particles in the three ground states in (25) are given by (21), then we obtain the total energies of the states by adding the energy of the unpaired particles to that of the paired particles, i.e.,

$$E(2N-2) = \sum_{i=1}^{N-1} E_{p_i}(2N-2) ,$$

$$E(2N-1) = \sum_{i=1}^{N-1} E_{p_i}(2N-1) + \epsilon_F ,$$

$$E(2N) = \sum_{i=1}^{N} E_{p_i}(2N) .$$

where ϵ_F is the energy of the last filled level in the ground state of $2N$ noninteracting particles, $E_{p_i}(2N - \nu)$ is the energy of the i-th pair in the ground state of $2N - \nu$ interacting particles, and $p_N = F$. Substituting these expressions into (25) and again neglecting terms quadratic in ν, we have

$$P(2N) = 2\epsilon_F - E_F(2N) .$$

Thus, if $P(2N)$ is given, then E_F is known and the N unknowns in eqs. (22) are g and E_{p_i}, $i = 1 \ldots N-1$.

3. The wave function is given as a rational function of the N pair energies E_{p_i} which may be calculated as accurately as is desired.

We are now performing detailed applications of these equations and the results will be reported at a later date.

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